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On the phase dependence of the soliton collisions in the Dyachenko–Zakharov envelope equation

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Abstract. We study soliton collisions in the Dyachenko–Zakharov equation for the envelope of gravity waves in deep water. The numerical simulations of the soliton interactions revealed several fundamentally different effects when compared to analytical two-soliton solutions of the nonlinear Schrodinger equation. The relative phase of the solitons is shown to be the key parameter determining the dynamics of the interaction. We find that the maximum of the wave field can significantly exceed the sum of the soliton amplitudes. The solitons lose up to a few percent of their energy during the collisions due to radiation of incoherent waves and in addition exchange energy with each other. The level of the energy loss increases with certain synchronization of soliton phases. Each of the solitons can gain or lose the energy after collision, resulting in increase or decrease in the amplitude. The magnitude of the space shifts that solitons acquire after collisions depends on the relative phase and can be either positive or negative.

1 Introduction

The existence and interactions of coherent structures like solitons and breathers on the surface of a deep water are a remarkably rich and fascinating subject for both experimental and theoretical studies. The exact mathematical model describing gravity waves in the ocean is the Euler equation, yet it is often rather complicated to study it by analytic or numerical means. Instead, various reduced models for water waves have demonstrated good agreement with the experimental data and have been widely adopted in the fluid dynamics and geophysics communities.

The most prominent and widely used model for weakly nonlinear surface waves in deep water is the nonlinear Schrödinger (NLS) equation. It describes time evolution of the envelope of a quasi-monochromatic wave train (Zakharov, 1968) and is integrable via the inverse scattering transform (IST) in 1-D (Zakharov and Shabat, 1972). Other models for weakly nonlinear waves include the Dysthe equation (Dysthe, 1979), and the compact Dyachenko–Zakharov equation (DZ) (Dyachenko and Zakharov, 2011), neither of which is known to be integrable by the IST.

By means of the IST one can find NLS soliton solutions and track their evolution in time until their collision and beyond analytically. The collision of the NLS solitons is perfectly elastic; that is, no loss of the energy occurs. The equations which are not integrable by the IST may have exact stationary solitary solutions interacting inelastically. For example, the Dysthe equation is known to admit solitary solutions whose existence has been demonstrated by other approaches unrelated to the IST (see Akylas, 1989; Zakharov and Dyachenko, 2010).

Both the NLS and Dysthe equations are formulated to describe the evolution of the envelope function. They require that the steepness of the wave train is small and that it is modulated weakly, i.e., that there are sufficiently many carrier wavelengths in the characteristic scale of the modulation. In terms of the Fourier transform of the surface elevation this is equivalent to having a sufficiently narrow band concentrated in the vicinity of the carrier wave number. The DZ equation is formulated for the wave train itself and is free from the assumptions of the weak nonlinearity and narrow bandness (Dyachenko and Zakharov, 2011, 2012). More precisely, the DZ equation describes the evolution of the review by Zakharov and Kuznetsov, 2012). Here we have found that the dynamics of a single collision is not universal: the direction of energy swap is determined by the soliton phases.

Furthermore, we have studied space shifts that solitons acquire after the collision. Solitons of the NLS equation always acquire a positive constant shift δx to their space position after interaction with another soliton moving with a different velocity. The value of δx is defined only by the amplitudes and velocities of the colliding solitons. The interaction of solitons in the DZe equation also leads to the appearance of the space shifts. We show that the character of this effect is not universal (δx can be positive or negative) and is determined in addition by the soliton phases.

The inelasticity of soliton collisions in nonintegrable models may destroy the initially coherent wave groups. However, as we have demonstrated here the total energy loss for interactions described by Eq. (1) does not exceed a few percent of energy of the solitons and we expect that observation of several subsequent soliton collisions will be possible. The study of the influence of the relative phase of the colliding solitons in the fully nonlinear model is of fundamental interest. As was shown by Dyachenko et al. (2016b), the DZ equation quantitatively describes strongly nonlinear phenomena at the surface of deep fluid. Thus we believe that the effects reported here for the solitons of the DZe equation can also be observed for the fully nonlinear Euler equations.

Pairwise collisions of solitons (or breathers) is an important elementary process that can be observed in the wave dynamics of an arbitrarily disturbed fluid surface. For example, the recent numerical simulations of the DZe equation demonstrate that an ensemble of interacting solitons can appear as a result of modulation instability driven by random perturbations of an unstable plane wave (Dyachenko et al., 2017a). Another important field of studies is the turbulence of rarified soliton gas where pairwise collision processes play the key role in the formation of wave field statistics (see the recent works of Pelinovsky et al., 2013; Shurgalina and Pelinovsky, 2016). We believe that the results presented here can serve as a starting point in the analytical description of such processes. Moreover, the reported dependence of soliton interaction dynamics on the relative phase is to be verified in laboratory experiments.

Data availability. The data used to generate figures in the article are accessible upon request to Dmitry Kachulin.

Author contributions. Both authors (DK and AG) proposed key ideas, performed numerical simulations and contributed equally to this work.

Competing interests. The authors declare that they have no conflict of interest.

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Analytic theory of a wind-driven sea

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Abstract

A self-sustained analytic theory of a wind-driven sea is presented. It is shown that the wave field can be separated into two ensembles: the Hasselmann sea that consists of long waves with frequency $\omega < \omega_H$, $\omega_H \sim 4 - 5\omega_p$ (ω_p is the frequency of the spectral peak), and the Phillips sea with shorter waves. In the Hasselmann sea, which contains up to 95 % of wave energy, a resonant nonlinear interaction dominates over generation of wave energy by wind. White-cap dissipation in the Hasselmann sea in negligibly small. The resonant interaction forms a flux of energy into the Phillips sea, which plays a role of a universal sink of energy. This theory is supported by massive numerical experiments and explains the majority of pertinent experimental facts accumulated in physical oceanography.

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Keywords: Kinetic (Hasselmann) equation; wave turbulence; Kolmogorov-Zakharov spectra; self-similarity of wave spectra; wind-wave forecasting.

1. Introduction

We will start with the taken-for-granted aphorism that "there is nothing more practical than a good theory." Since the time of Galileo, physicists have tried to develop theoretical models of natural phenomena. They have succeeded for phenomena of very different scales: from the scale of elementary particles to the scale of the Universe. Geophysical phenomena - weather forecasting, prediction of earthquakes or origin of hurricanes - are intermediate in scale but not in complexity. As a rule, these phenomena are very difficult for theoretical investigation because there are too many factors involved. Creation of a theoretically justified analytic theory of wind-driven sea looks, at first glance, to be "mission impossible." Waves are generated by turbulent winds; these waves break, forming white caps, sprays, bubbles, etc. Nevertheless, the development of an adequate analytic theory of wind-driven sea is possible. The purpose of this paper is to demonstrate this possibility.

It is obvious that a wind-driven sea needs a statistical description. In the system under consideration, such a description can be performed efficiently if we have at least one small parameter. The absence of a small parameter makes

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we see very good qualitative coincidence but large quantitative differences. The Snyder model overestimates the rate of energy growth with fetch by almost an order of magnitude. Because of the limited length of this article we cannot discuss an extremely important question: the shape of spectra in the universal spectral range $1 < \sigma < 5$. Eq. (48) does not preserve energy that leaks from the Hasselmann sea to the Phillips sea, forming an energy flux P. Thus the solution of Eq. (48) must have asymptotic behavior

$$G(\xi) \to \beta \frac{P^{1/3}}{\sigma^4} \tag{57}$$

Because $\gamma_0 \ll 1$, β is a small number. This implies the inevitable formation of Zakharov-Filonenko spectral tails $F(\omega) \sim 1/\omega^4$. Such tails are routinely observed in numerous field and laboratory experiments, see for example [42], [43]. This important subject deserves a special consideration.

7. Conclusions

Let us summarize the results. We claim that the majority of data obtained in field and numerical experiments can be explained in a framework of a simple model

$$\frac{d\epsilon}{dt} = S_{nl} + \gamma_{in}(\omega, \phi)\epsilon$$

Moreover, most of the facts can be explained by the assumption that $\gamma_{in}(\omega, \phi)$ is a powerlike function on frequency, $\gamma_{in}(\omega, \phi) = \gamma_0\, \omega^{1+s} f(\phi)$. Here $1 < s < 2.3$ and $f(\phi), \gamma_0$ are tunable. This model pertains only to the description of the Hasselmann sea, $0 < \omega < \omega_H$, $\omega_H \simeq (4-5)\omega_p$.

In fact, this model is a simplification of the widely accepted model in oceanography (1). What is the difference between these models? The main difference is obvious: we excluded from our consideration any mention of wave energy dissipation. This does not mean that we deny a crucial role of wave-breaking in the dynamics of ocean surface. But, from the spectral viewpoint, the wave-breaking takes place outside the Hasselmann sea. It is going into the Phillips sea, in the spectral area of short scales. This very important statement is supported by experimental data and by numerical solutions of dynamical phase-resolving equations for a free surface.

What we offer could be called ”poor man's oceanography.” A ”poor man” refuses attempts to derive the equation for S_{in} from ”first principles,” but has in his possession powerful analytic and computer models to use as test beds for compatibility of models for $\gamma_{in}(\omega, \phi)$ with experimental data. The Snyder model does not pass this test and should be excluded from operational models.

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Comparison of Different Models for Wave Generation of The Hasselmann Equation

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Abstract

We compare two recently developed sets of source terms, based on different assumptions of wave energy input and dissipation, for Hasselmann equation. The numerical simulation, performed for limited fetch conditions with the constant wind speed shows the difference in total energy and mean frequency distributions along the fetch as well as in wave energy spectra. Possible reasons of such deviations are offered.

Keywords:
Hasselmann equation; wind and dissipation source terms; self-similar solutions; Kolmogorov-Zakharov spectra; wave-breaking dissipation; magic relation

1. Introduction

The physical oceanography community consents on the fact [5] that deep water ocean gravity surface wave forecasting models are described by Hasselmann equation (hearafter HE) [10, 11], also known as kinetic equation for waves, or energy balance equation:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial \omega_k}{\partial \vec{k}} \frac{\partial \varepsilon}{\partial \vec{r}} = S_{nl} + S_{in} + S_{diss} \tag{1}$$

where $\varepsilon = \varepsilon(\omega_k, \theta, \vec{r}, t)$ is the wave energy spectral density, as the function of wave frequency $\omega_k = \omega(k)$, angle θ, two-dimensional real space coordinate $\vec{r} = (x, y)$ and time t. S_{nl}, S_{in} and S_{diss} are the nonlinear, wind input and wave-breaking dissipation source terms, respectively. Hereafter, only the deep water case, $\omega = \sqrt{gk}$ is considered, where g is the gravity acceleration and $k = |\vec{k}|$ is the absolute value of the vector wavenumber $\vec{k} = (k_x, k_y)$.

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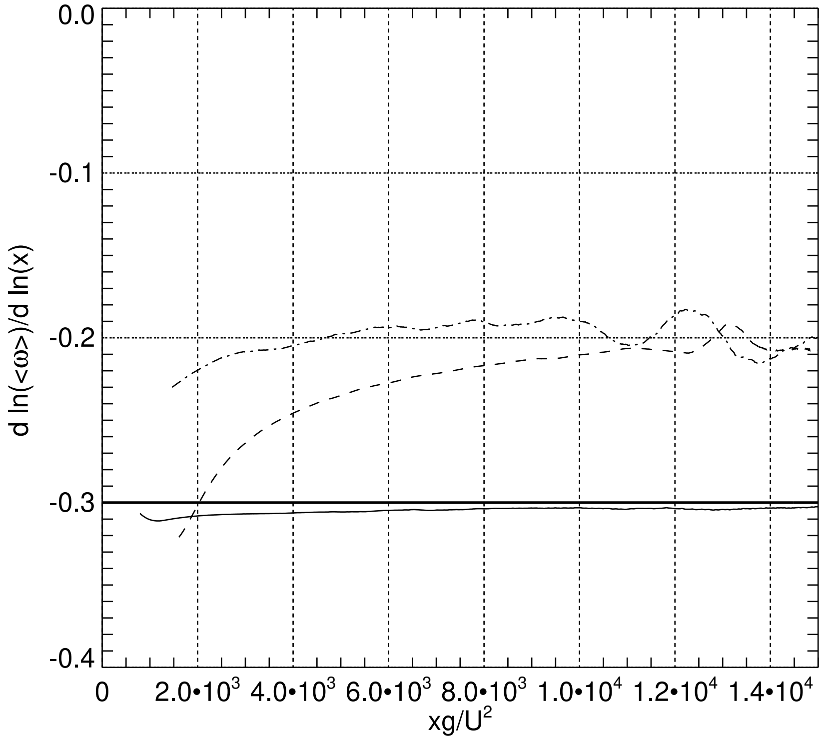

Fig. 4. Local mean frequency exponent $-q = \frac{d \ln <\omega>}{d \ln x}$ as the function of dimensionless fetch xg/U^2 for $U = 10$ m/sec limited fetch case. ZRP case - solid line; dashed line - MD1 case; dash-dotted line - MD2 case. Thick horizontal solid line - target value of the self-similar exponent $q = 0.3$.

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ON THE TUNING-FREE STATISTICAL MODEL OF OCEAN SURFACE WAVES

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ABSTRACT

Absence of mathematically justified criteria during development of the wind energy input and wave breaking energy dissipation source terms in Hasselmann equation (HE), used as the framework of modern operational wave forecasting models, lead to creation of plethora of parameterizations, having enormous scatter, disconnected from the physical background and obeying dozens of tuning parameters to adjust the HE model to the specific situation. We show that it's possible, based on analytical analysis and experimental observation data, to create the new set of source terms, reproducing experimental observations with minimal number of tuning parameters. We also numerically analyze six historically developed and new wind input source terms for their ability to hold specific invariants, related to HE self-similar nature. The degree of preservation of those invariants could be used as their selection tool. We hope that this research is the step toward the creation of physically justified tuning-free operational models.

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1 INTRODUCTION

The statistical theory of wind driven gravity waves on the surface of water has been started with the invention of Hasselmann kinetic equation [1,2]

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial \omega_k}{\partial \vec{k}} \frac{\partial \varepsilon}{\partial \vec{r}} = S_{nl} + S_{in} + S_{diss} \tag{1}$$

where $\varepsilon = \varepsilon(\omega_k, \theta, \vec{r}, t)$ is the wave energy spectrum, $\omega(k) = \sqrt{gk}$ is the deep water wave frequency as the function of the absolute value of the vector wavenumber $\vec{k} = (k_x, k_y)$; θ is the angle, $\vec{r} = (x, y)$ is the real space coordinate and t is the time. S_{nl}, S_{in} and S_{diss} are the nonlinear, wind input and wave-breaking dissipation source terms, correspondingly.

Eq.(1) is the basis of modern operational wave forecasting models, which use DIA-type surrogates of S_{nl} nonlinear interaction term for computational capacity reasons, and the plethora of parameterizations for the wind input S_{in} and wave energy S_{diss} dissipation source terms. This approach a-priori erodes the hope for universal tuning-free operational model, because of the leading role of S_{nl} term [3,4] in Eq.(1). In other words, due to distortion of S_{nl} first approximation in Eq.(1), subsequent attempts

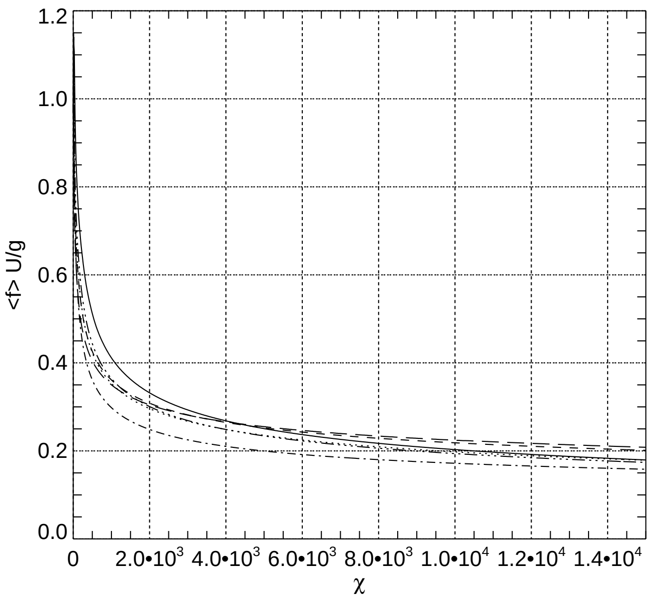

FIGURE 3. Dimensionless mean frequency $\sigma/2\pi$, as the function of the dimensionless fetch $\chi = xg/U^2$, calculated as $< f >= \frac{1}{2\pi}\frac{\int \omega n d\omega d\theta}{\int n d\omega d\theta}$, where $n(\omega,\theta) = \frac{\varepsilon(\omega,\theta)}{\omega}$ is the wave action spectrum, for wind speed 10 m/sec. Solid line - ZRP case, dotted line - Plant case, short-dashed line - Donelan case, dashed-dotted line - Snyder case, dash-triple-dotted line - "Synthetic" case, long dashed line - Hsiao-Shemdin case.

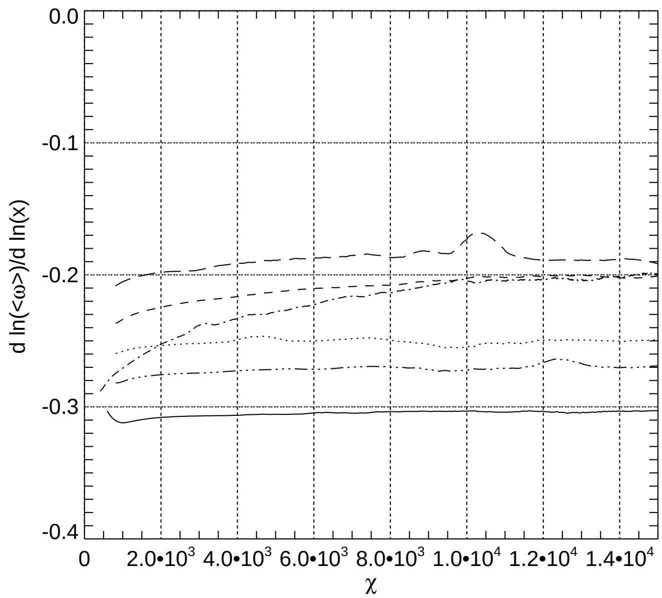

FIGURE 4. Mean frequency local power index $-q = \frac{d\ln<\omega>}{d\ln x}$ as the function of dimensionless fetch $\chi = xg/U^2$ for $U = 10$ m/sec limited fetch case. Solid line - ZRP case, dotted line - Plant case, short-dashed line - Donelan case, dashed-dotted line - Snyder case, dash-triple-dotted line - "Synthetic" case, long dashed line - Hsiao-Shemdin case.

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Strongly interacting soliton gas and formation of rogue waves

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We study numerically the properties of (statistically) homogeneous soliton gas depending on soliton density (proportional to number of solitons per unit length) and soliton velocities, in the framework of the focusing one-dimensional nonlinear Schrödinger (NLS) equation. To model such gas we use N-soliton solutions (N-SS) with $N \sim 100$, which we generate with specific implementation of the dressing method combined with 100-digits arithmetics. We examine the major statistical characteristics, in particular the kinetic and potential energies, the kurtosis, the wave-action spectrum and the probability density function (PDF) of wavefield intensity. We show that in the case of small soliton density the kinetic and potential energies, as well as the kurtosis, are very well described by the analytical relations derived without taking into account soliton interactions. With increasing soliton density and velocities, soliton interactions enhance, and we observe increasing deviations from these relations leading to increased absolute values for all of these three characteristics. The wave-action spectrum is smooth, decays close to exponentially at large wavenumbers and widens with increasing soliton density and velocities. The PDF of wave intensity deviates from the exponential (Rayleigh) PDF drastically for rarefied soliton gas, transforming much closer to it at densities corresponding to essential interaction between the solitons. Rogue waves emerging in soliton gas are multisoliton collisions, and yet some of them have spatial profiles very similar to those of the Peregrine solutions of different orders. We present example of three-soliton collision, for which even the temporal behavior of the maximal amplitude is very well approximated by the Peregrine solution of the second order.

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I. INTRODUCTION

Statistical behavior of nonlinear integrable systems, called in general *integrable turbulence* [1], is a rapidly developing area of theoretical and experimental studies, as illustrated by the recent publications [2–5]. On the one hand, up to a certain degree of accuracy many physical systems can be described with nonlinear integrable mathematical models. In comparison with nonintegrable models, the corresponding integrable equations demonstrate significantly different statistical properties; see, e.g., Refs. [6,7]. On the other hand, an integrable system allows transformation to the so-called *scattering data*, which is in one-to-one correspondence with the wavefield and, similarly to the Fourier harmonics in the linear wave theory, changes trivially during the motion. With numerical methods, see, e.g., Refs. [8,9], the scattering data can be partly analyzed, that may bring some insights into the dynamical behavior. Another distinctive feature of an integrable system is the conservation of infinite series of invariants, so that different types of initial conditions are characterized by different sets of integrals of motion and, during the evolution, demonstrate different statistical properties; see, e.g., Refs. [3–5].

In the present paper we examine integrable turbulence using controlled initial conditions, in the sense that we construct these initial conditions from known scattering data. In contrast to other studies, this gives us exact knowledge which nonlinear objects interact during the evolution, for instance, when a rogue wave appear. As a model, we consider one-dimensional nonlinear Schrödinger (NLS) equation of the focusing type with initial conditions in the form of N-soliton solutions (N-SS), with N of order 100. Our methods allow generation of sufficiently dense N-SS with essential interaction between the solitons, in contrast to rarefied multi-soliton solutions analyzed, e.g., in Refs. [10–12] for KdV and mKdV equations. We believe that our approach can also be used to examine turbulence governed by other integrable equations and developing from other types of initial conditions, e.g., containing nonlinear dispersive waves and different types of breathers; see Refs. [13,14].

For spatially localized wavefield, the scattering data consists of discrete (solitons) and continuous (nonlinear dispersive waves) parts of eigenvalue spectrum, which is calculated for specific auxiliary linear system. At the first step in our study, we generate an ensemble of multiple realizations of scattering data, with each realization containing N discrete eigenvalues and N complex coefficients. Such scattering data corresponds to N-SS. Then, we find the wavefield for N-SS from this data, that for $N \sim 100$ is made possible by specific implementation of the dressing method applied numerically with 100-digits precision. To our knowledge, multisoliton solutions containing so many solitons were not generated by anyone else before. The generating procedure is very expensive from the computational point of view and returns

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very well described by analytical relations Eq. (25) derived without taking into account soliton interactions. For larger density ρ and characteristic soliton velocity V_0, we observe increasing next-order corrections leading to increased absolute values for all these three characteristics. These next-order corrections come from enhanced soliton interactions due to decreased spacing and more frequent collisions, respectively. The wave-action spectrum for soliton gas is smooth, decays close to exponentially at large wave numbers and widens with increasing ρ and V_0.

The PDF of relative wave intensity has the form of a composition of PDFs representing a singular soliton and soliton interactions. Compared to the cnoidal wave initial conditions, the PDF deviates from the exponential (Rayleigh) distribution Eq. (1) much more pronouncedly, especially at the region of soliton interactions where it exceeds the exponential PDF by orders of magnitude. This excess is larger for soliton gas with larger velocities, that corresponds to more frequent soliton collisions. For rarefied soliton gas $\rho \ll 1$, the average amplitude of the wavefield is much smaller than the soliton amplitude and the PDF deviates from the exponential PDF drastically. For larger densities, solitons interact stronger and the PDF transforms closer to the exponential distribution. We think that for dense soliton gas $\rho \gg 1$ the PDF may match the exponential one, that is supported by the behavior of the kurtosis approaching to 2 with increasing density. Soliton gas containing solitons of different amplitudes demonstrate the similar properties, except that the regions of soliton interactions on the PDF are less pronounced.

Rogue waves emerging in soliton gas are collisions of solitons, and some of these collisions have spatial profiles very similar to those of the (scaled) Peregrine solutions of different orders. In particular, we present specifically designed examples of two- and three-soliton collisions, which have almost the same spatial profiles as the Peregrine solutions of the first and the second orders. In the case of the three-soliton collision, even the temporal dependency of the maximal amplitude is very well approximated by that of the Peregrine solution of the second order. When soliton parameters are far from the "ideal" sets, the emerging large waves differ significantly from the rational breathers. In our opinion, these results highlight that the similarity in spatial and/or temporal behavior cannot be used to draw conclusions on rogue waves' composition and origin.

For a statistical study, it is crucial to define the ensemble of initial conditions. In this paper, we have used initial conditions with fixed value of wave action (average intensity) and with zero momentum, while the integrals of higher order were not fixed; for instance, the total energy could change significantly from one realization to another. To check the influence of this effect, we examined soliton gas for which—in addition to the wave action and the momentum—the value of the total energy was also fixed, and came to the identical results.

We suggest that our methods for generation of initial conditions from known scattering data can be used to examine turbulence governed by other integrable equations and developing from other types of initial conditions, e.g., containing nonlinear dispersive waves and different types of breathers [13,14]. We believe that, in general, our approach can be promising, as it allows to study turbulence with controlled initial conditions, i.e., with exact knowledge which nonlinear objects interact during the evolution. Our methods can also be used in optical fibre communications, where strongly interacting N-SS were recently proposed as information carrier [13].

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Development of high vorticity structures and geometrical properties of the vortex line representation

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The incompressible three-dimensional Euler equations develop very thin pancake-like regions of increasing vorticity. These regions evolve with the scaling $\omega_{max} \propto l^{-2/3}$ between the vorticity maximum and the pancake thickness, as was observed in the recent numerical experiments [D. S. Agafontsev *et al.*, "Development of high vorticity structures in incompressible 3D Euler equations," Phys. Fluids **27**, 085102 (2015)]. We study the process of pancakes' development in terms of the vortex line representation (VLR), which represents a partial integration of the Euler equations with the explicit conservation of the Cauchy invariants and describes the compressible dynamics of continuously distributed vortex lines. We present, for the first time, the numerical simulations of the VLR equations with high accuracy, which we perform in adaptive anisotropic grids of up to 1536^3 nodes. With these simulations, we show that the vorticity growth is connected with the compressibility of the vortex lines and find geometric properties responsible for the observed scaling $\omega_{max} \propto l^{-2/3}$. *Published by AIP Publishing.* https://doi.org/10.1063/1.5049119

I. INTRODUCTION

The mechanism of vorticity growth in the incompressible 3D Euler equations was intensively studied over the last decades because of its relation to a possible finite-time blowup and subsequent transition to turbulence. Several analytical blowup and no-blowup criteria were established; see the reviews in Refs. 1 and 2. The central result is the Beale–Kato–Majda theorem,[3] which states that at a singular point (if it exists), the time integral of maximum vorticity must explode. In parallel, a large effort was made with numerical analysis. One of the early numerical studies[4] examined the evolution of periodic flows starting from random initial conditions and the symmetric Taylor–Green vortex. In both cases, maximum of vorticity was growing nearly exponentially with time and the regions of high vorticity represented pancake-like structures (thin vortex sheets) compressing in the transverse direction. The tendency toward a vortex sheet should suppress the three-dimensionality of the flow and, hence, the formation of a finite-time singularity since the dynamics within 2D Euler equations are known to be regular; see, e.g., Refs. 5–9. Thus, further numerical studies were mainly focused on specific initial conditions providing enhanced vorticity growth, e.g., antiparallel or orthogonal vortices; we refer to Refs. 2 and 10 for a brief review and to Refs. 11–15 for examples of recent numerical studies. Despite these efforts, the existence of a blowup (unless it is triggered by a physical boundary[16]) remains a highly controversial question.

In our previous papers,[10,17,18] we returned to the problem of vorticity growth from generic large-scale initial conditions. We carried out several simulations in anisotropic grids of up to 2048^3 total number of nodes and observed in detail the evolution of high-vorticity regions. We confirmed that these regions represent pancake-like structures and found that the flow near the pancake is described locally by a novel exact self-similar solution of the Euler equations combining a shear flow with an asymmetric straining flow. The maximum vorticity growth $\omega_{max}(t) \propto e^{\beta_2 t}$ and the pancake compression in the transverse direction $l(t) \propto e^{-\beta_1 t}$ are characterized by significantly different exponents $\beta_2/\beta_1 \approx 2/3$, leading to the Kolmogorov-type scaling law

$$\omega_{max}(t) \propto l(t)^{-2/3} \tag{1}$$

observed numerically during the pancake evolution. On the other hand, the pancake model solution allows for an arbitrary scaling between the vorticity maximum and the pancake thickness, and our observation of the 2/3-scaling remained unexplained. Note that rewritten for the velocity variation, the relation (1) has the same form as the 1/3-Hölder velocity continuity necessary for the energy cascade in developed turbulence.[19,20] The pancake structures emerge in increasing number with time and provide the leading contribution to the energy spectrum, where, for some initial conditions,[10,17] we observed the gradual formation of the Kolmogorov spectrum, $E(k) \propto k^{-5/3}$, in a fully inviscid flow.

In the present paper, we study the pancake vorticity structures from the point of view of the vortex line representation (VLR). The VLR is the transformation from the Eulerian coordinates of the fluid to the Lagrangian markers of the vortex lines,[21] which is compressible so that its Jacobian may take arbitrary values. In gas dynamics, the similar in spirit transformation from the Eulerian to the Lagrangian coordinates of the flow can be used to completely characterize the areas

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gradual dependence of the pancake model parameters with the pancake segment together with the finiteness of the pancake in the **a**-space is connected with the finiteness of the Hessian elements γ_{ij}; however, more study is necessary to clarify this phenomenon in detail.

The results discussed above are related to the high-vorticity structure corresponding to the global vorticity maximum and are obtained from the simulation of the initial condition I_2. We verified that several other pancakes from the same simulation corresponding to other local maxima of vorticity, as well as pancakes from the simulation of the initial flow I_1 discussed in Appendix C, demonstrate the same properties for the Jacobi and the Hessian matrices and follow the 2/3-scaling (1).

VII. CONCLUSIONS

In the present paper, we have studied high-vorticity structures developing in the incompressible 3D Euler equations in terms of the vortex line representation (VLR). The VLR is the transformation from the Eulerian coordinates of the flow to the Lagrangian markers of the vortex lines and represents a partial integration of the Euler equations with the explicit conservation of the Cauchy invariants. The latter means that a numerical simulation of the VLR equations must conserve the Cauchy invariants along the vortex line trajectories with the round-off accuracy. This property may be very important in the sense of the accuracy and control of the 3D Euler simulations while approaching sharp gradients. We have developed a new numerical method for the Euler equations in terms of the VLR and performed high-resolution simulations for two initial flows.

As the first result, we have demonstrated that the growth of vorticity is related to the smallness of the Jacobian of the VLR, with the inverse-proportional relation between the two, $\omega_{\max} \propto 1/J_{\min}$. This agrees with the pancake model solution of Ref. 18 and the relation (12) derived under the assumption of unidirectional vorticity. Thus, a high-vorticity region for which the vorticity direction changes sharply with the coordinate may feature a different relation between the vorticity and the Jacobian. The pancake model solution turns out to be degenerate in terms of the VLR so that all time-dependency for the vorticity comes from the denominator of Eq. (8), i.e., the Jacobian. The inverse of the Jacobian has the meaning of the density of vortex lines so that the vorticity within the pancake grows proportionally to this density. The latter is the manifestation of the compressibility of the vortex lines.

A developing pancake structure affects the VLR mapping, which we examine with the singular value decomposition (SVD) of the Jacobi matrix. As indicated by the singular values, the mapping is strongly compressed along one direction with the rate proportional to the pancake thickness $\sigma_1 \propto l_1 \propto e^{-\beta_1 t}$. Assuming that such behavior persists in the limit $t \to +\infty$, this may be seen as touching of the vortex lines, with the vorticity growing unboundly, $\omega_{\max}(t) \to +\infty$. Along the other two directions, the mapping either does not change substantially, $\sigma_2 \propto 1$, or stretches as $\sigma_3 \propto \omega_{\max}^{-1} l_1^{-1} \propto e^{\beta_3 t}$. In the local coordinate system of the pancake, the rotation matrix of the SVD in the **x**-space is close to unity, $\mathbf{U} \simeq \mathbf{1}$, while the rotation matrix in the **a**-space $\mathbf{V}$ approaches to a constant matrix that depends on the initial parameters. The main characteristic size of the pancake—its exponentially decaying thickness—corresponds in the **a**-space to the distance of unity order. Thus, in the space of Lagrangian markers **a**, the high-vorticity region does not shrink, in contrast to the previously made assumption[25] made by analogy with the gas dynamics case. These results also follow analytically from the VLR written for the pancake model solution of Ref. 18, what confirms the applicability of this model.

In simulations, the Jacobian changes sharply along the pancake perpendicular direction x_1. This property can only come from the next-order corrections to the pancake model solution since the Jacobian for the model does not depend on spatial coordinates. Assuming that these corrections are present, we demonstrate that the Hessian $\boldsymbol{\gamma}$ for the Jacobian in the basis induced by the rotation matrix $\mathbf{V}$ must be close to the diagonal and the sharp dependency for the Jacobian along the x_1-axis comes from small but finite element γ_{11}.

The pancake model solution allows for an arbitrary power-law scaling between the vorticity maximum and the pancake thickness, given by the ratio of the exponents β_2/β_1. For the first time, we discovered numerically that this ratio is close to 2/3 in Ref. 10; however, we were not able to explain this observation. With the present VLR study, we identify that the 2/3-scaling (1) comes from the finite Hessian element $\gamma_{33} \propto 1$ and the finite lateral pancake size $\ell_3 \propto 1$. We think that the finiteness of γ_{33} is connected with the two properties of the pancake structures, namely, the gradual dependence of the pancake model parameters with the pancake segment and the finiteness of the pancake thickness in the **a**-space. However, more study is necessary to clarify this connection in detail.

Our approach utilizes the general properties of the frozen-in-fluid fields and, potentially, can be generalized to a wider group of physical phenomena far beyond the scope of this paper. For instance, the compressibility of magnetic field lines[26] should play an essential role in the generation of magnetic filaments in the convective zone of the Sun and in the magnetic dynamo theories in space plasma; see, e.g., Refs. 28 and 31. As shown in Refs. 29, 32, and 33, the compressible character of the frozen-in-fluid divorticity field is an important factor in the formation of a direct Kraichnan cascade in 2D hydrodynamic turbulence.

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УДК 532.59

РОСТ ВЕТРОВОГО ВОЛНЕНИЯ В ТЕРМИНАХ СПЕКТРАЛЬНЫХ ПОТОКОВ

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С позиций теории волновой турбулентности описаны основные режимы развития ветрового волнения, отвечающие постоянству потоков к волнам волнового импульса, энергии и действия. На основе экспериментальных данных о росте волнения даны оценки величины потока энергии в область больших масштабов (обратный каскад в теории волновой турбулентности). Интенсивность прямого каскада энергии в область коротких волн оценена по экспериментальным параметризациям спектров ветрового волнения, отвечающим спектру Колмогорова-Захарова $E(\omega) \sim \omega^{-4}$. Проведенные оценки показывают, что интенсивность прямого каскада на два порядка превышает интенсивность обратного. Приближенное решение для прямого каскада энергии получено как возмущение классического решения Захарова–Заславского для обратного каскада с нулевым потоком энергии. Результаты обсуждаются в их связи с развитием спектральных моделей ветрового волнения.

ков отмечалась ранее [30, 13] и качественно согласуется с нашими результатами. В то же время, количественные различия требуют серьезного анализа.

Мы представили решение для обратного каскада энергии, использовав в качестве начального приближения классическое решение Захарова–Заславского [6]. Подробности вывода соотношений (17,18) будут представлены в отдельной работе. Здесь отметим лишь, что введение в рассмотрение решений, отличных от классических Колмогорова-Захарова является необходимым в задаче развития волнения, рассматриваемой в терминах потоков энергии. Именно в терминах потоков исследования ветрового волнения способны дать необходимые оценки энергетического баланса в системе океан-атмосфера.

Таким образом, представленные оценки и теоретические результаты предлагают дополнительные возможности для целенаправленной настройки используемых моделей ветрового волнения в терминах прямых и обратных потоков энергии по масштабам волновых движений.

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Formation of rogue waves from a locally perturbed condensate

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The one-dimensional focusing nonlinear Schrödinger equation (NLSE) on an unstable condensate background is the fundamental physical model that can be applied to study the development of modulation instability (MI) and formation of rogue waves. The complete integrability of the NLSE via inverse scattering transform enables the decomposition of the initial conditions into elementary nonlinear modes: breathers and continuous spectrum waves. The small localized condensate perturbations (SLCP) that grow as a result of MI have been of fundamental interest in nonlinear physics for many years. Here, we demonstrate that Kuznetsov-Ma and superregular NLSE breathers play the key role in the dynamics of a wide class of SLCP. During the nonlinear stage of MI development, collisions of these breathers lead to the formation of rogue waves. We present new scenarios of rogue wave formation for randomly distributed breathers as well as for artificially prepared initial conditions. For the latter case, we present an analytical description based on the exact expressions found for the space-phase shifts that breathers acquire after collisions with each other. Finally, the presence of Kuznetsov-Ma and superregular breathers in arbitrary-type condensate perturbations is demonstrated by solving the Zakharov-Shabat eigenvalue problem with high numerical accuracy.

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I. INTRODUCTION

The formation of extreme-amplitude waves is among the most remarkable phenomena in the physics of wave processes. In the linear case, these events may appear only as a result of simple wave interference, whereas the interactions of nonlinear waves exhibit a wide range of nontrivial mechanisms, such as the development of modulation instability (MI) and nonlinear wave focusing [1,2]. The localized extreme-amplitude events, so-called rogue waves, are of special interest as they are observed more frequently than predicted by Gaussian statistics and can appear from relatively weak perturbations of a calm background [1,3]. This phenomenon being studied first in oceanography has been observed experimentally in different nonlinear media, such as optical fiber with Kerr nonlinearity, Bose-Einstein condensate, surface of a fluid, and plasmas, which demonstrates its universal nature [2,4–7].

The one-dimensional focusing nonlinear Schrödinger equation (NLSE)

$$i\psi_t + \tfrac{1}{2}\psi_{xx} + (|\psi|^2 - A^2)\psi = 0 \qquad (1)$$

is the fundamental mathematical model describing weakly nonlinear wave propagation. Here, $\psi(t,x)$ is the complex-valued envelope of the physical wave field, and t and x are the time and space coordinates. Zakharov and Shabat found that the NLSE can be completely integrated using the inverse scattering transform (IST) [8]. Here we study solutions of the NLSE (1) on the so-called condensate background—a simple quasimonochromatic plane wave. The condensate solution of Eq. (1) is $\psi_0(t,x) = A$, where A is the background amplitude,

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which we assume to be real without loss of generality. T
condensate is unstable with respect to long-wave perturbatic
(MI phenomena, see, e.g., [9]) with the following growth ra

$$\Gamma(k) = k\sqrt{A^2 - k^2/4},$$

where k is the perturbation wave number. In the region $0 < k$
2, the amplitude of these perturbations grows $\sim e^{\Gamma t}$ in the init
(linear) stage. The nonlinear stage of MI is of fundamen
interest and may lead to the formation of rogue waves [1,2

The NLSE describes only the first-order nonlinear effec
However, its universality and integrability allows us to c
ture the fundamentally important features of MI and to fi
analytical rogue wave solutions. Indeed, the IST links
initial NLSE wave field with the so-called scattering da
which play the role of elementary nonlinear modes, simila
Fourier harmonics in linear wave theory. In the case of spatia
localized NLSE solutions, the scattering data are represen
by the discrete (solitons) and continuous (nonlinear dispers
waves) parts of the eigenvalue spectrum of the Zakhar
Shabat auxiliary linear system (ZH system).

The IST for the spatially localized wave field and z
background ($A = 0$) was developed in [8], where the
soliton solutions were found analytically and the gene
Cauchy problem was solved implicitly via the integral Gelfa
Levitan-Marchenko equations (GLME). In 1977, Kuznets
[10] and later Kawata and Inoue (in 1978, [11]) and
(in 1979, [12]) generalized this theory to the case of
condensate background. In this model the discrete spectr
solutions, interacting with condensate, transform from solit
to the oscillating structures—breathers. The family of NL
breathers includes the well-known solutions of Peregrine [1
Kuznetsov-Ma [10–12], and Akhmediev [14].

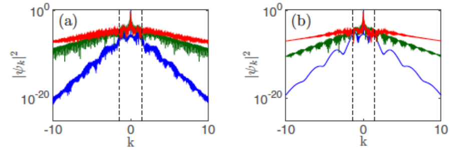

FIG. 9. Fourier spectra for analytical scenarios of rogue wave formation from the condensate locally perturbed by (a) superregular and (b) one Kuznetsov-Ma and one superregular breathers presented in Fig. 6. Black dotted lines correspond to the value of $k_{\max}$. Colors of the lines of the spectra match with the notation of Fig. 6.

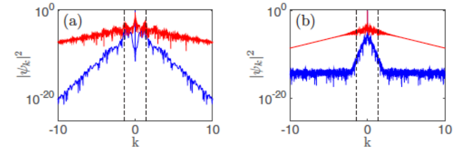

FIG. 10. Fourier spectra for scenarios of rogue wave formation from the condensate locally perturbed by random distributions of (a) superregular and (b) Kuznetsov-Ma breathers presented in Fig. 7. Black dotted lines correspond to the value of $k_{\max}$. Colors of the lines of the spectra match with the notation of Fig. 7.

that superregular breathers with $\mu \approx A/\sqrt{2}$ are responsible for the fastest perturbation growth.

VIII. DISCUSSION AND CONCLUSION

The eigenvalue spectrum does not reveal the impact of the continuous spectrum waves (we also need to study the *reflection coefficient*, see, e.g., [10,12]). The numerical simulation of the evolution of the perturbations presented in Fig. 8 shows complicated wave patterns that are apparently driven by nonlinear interaction between discrete and continuous spectrum solutions. Another important task is to study the combinations of imaginary and real perturbations or perturbations based on broadband random noise. For the latter case, Kibler *et al.* found strong signatures of superregular breathers at the intermediate stage of MI development [25]. All these questions demand separate consideration.

Another fundamental problem is the development of MI from spatially periodic condensate perturbations (see, e.g., the monograph [34]). In 2015, Agafontsev and Zakharov found that in this case, MI driven by small-amplitude perturbations ($\sim 10^{-5}A$) leads to the formation of Gaussian wave-field statistics [35]. In the same year Walczak *et al.* demonstrated (numerically and by experiment with laser fields in optical fiber) that in the case of strong initial condensate disturbances, the tail of the probability density function of wave amplitudes increases [36] (in other words they observed the emergence of *heavy-tailed* wave-field statistics). In 2016, Soto-Crespo *et al.* numerically investigated the transition between the cases of low- and high-amplitude condensate perturbations and studied the influence of the perturbation amplitude both on the wave-field statistics and on the distribution of eigenvalues of the ZS system [37]. They concluded that initial perturbations of significant amplitude can contain spatially localized breathers, which is consistent with the theory suggested here. Indeed the minimal amplitude of SLCP generated by breathers is determined by the perturbation width; see Sec. II. Moreover, in 2016 the precise wave-field dynamics of strongly fluctuating condensate was experimentally recorded and reported [38].

Meanwhile Randoux *et al.* have suggested that the key role in the development of periodic perturbations should be attributed to the *finite-band* solutions of the NLSE [33] (see also [39]). The problems of localized and periodic condensate perturbations are complimentary and both have fundamental importance. The understanding of the link between the localized and periodic IST description of MI is an important task. The obtained pictures of eigenvalues for real and imaginary SLCP correlate with results presented earlier for the periodic perturbations [31], which can be a starting point for such study.

The found space-phase shifts in Eq. (14) are critically important to describing the interactions of NLSE breathers when their number $N \geqslant 3$. They can be used for further studies, among which the experimental realization of complicated multibreather dynamics is of special interest. Similar breathers describe the dynamics of unstable backgrounds in different integrable models (see, e.g., [40–43]), which allows us to generalize our results.

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Stability criterion for solitons of the Zakharov–Kuznetsov-type equations

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ABSTRACT

Early results concerning the linear stability of the solitons in equation of the KDV-type [1] are generalized to solitons describing by the Zakharov–Kuznetsov-type equation. The linear stability criterion for ground solitons in the Vakhitov–Kolokolov form is derived for such equations with arbitrary nonlinearity. For the power nonlinearity the instability criterion coincides with the condition of the Hamiltonian unboundedness from below. The latter represents the main feature for appearance of collapse in such systems.

soliton will be stable and unstable in the opposite case. This criterion represents the analog of the Vakhitov–Kolokolov criterion[11] for the NLS equations. In the case of power law nonlinearity $f(u) = u^p$, the dependence of momentum P on V turns out to be power law: $P \propto V^\gamma$ where

$$\gamma = \frac{2}{p-2} - \frac{d}{2}.$$

Hence one can see that *the 3D solitons for the ZK equation* ($p = 3, d = 3$) *are stable, in a full agreement with boundedness of the Hamiltonian proved in [3]*. The instability criterion for solitons with $p - 2 > 4/d$ coincides with the unboundedness condition of the Hamiltonian. Like for the NLS-type equations we can state that the nonlinear stage of this instability should result in the wave collapse.

3. Conclusion

Thus, we have found the linear stability criterion for ground soliton solutions in the ZK-type equation. This criterion is necessary and sufficient: if $\partial P/\partial V > 0$ the solitons are stable and unstable in the opposite case. This criterion is analogous to the Kolokolov–Vakhitov criterion for soliton stability in the NLS-type equations. For power law nonlinearity this criterion demonstrates different behavior of the system. In the stable region solitons realize minimum of the Hamiltonian with fixed momentum P, i.e. they are stable in the Lyapunov sense. But it does not mean that scattering of solitons will be elastic. While scattering of such solitons it is energetically favorable to form solitons with higher amplitude. This process will be accompanied by radiation of small amplitude waves which play the role of friction in the system. For the systems with Hamiltonians unbounded from below the nonlinear stage of the soliton instability should result in the formation of singularity, probably, in a finite time.

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Формирование складок в двумерной гидродинамической турбулентности

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рПУФХРЙМБ Ч ТЕДБЛГЙА 26.11.2018

Для вырождающейся двумерной гидродинамической турбулентности поле ротора завихренности $\mathbf{B} = \mathrm{rot}\,\omega$ благодаря тенденции к опрокидыванию сосредоточено в окрестности линий, соответствующих положению квази-шоков завихренности. На стадии формирования квази-шоков максимальное значение ротора завихренности B_{max} растет экспоненциально во времени, при этом толщина $\ell(t)$ максимальной области в поперечном направлении к вектору $\mathbf{B}$ сужается во времени также экспоненциально. Численно показано, что $B_{max}(t)$ зависит от толщины степенным образом: $B_{max}(t) \sim \ell^{-\alpha}(t)$, где показатель $\alpha \approx 2/3$. Такое поведение свидетельствует в пользу формирования складок для бездивергентного векторного поля ротора завихренности.

1. Введение

В двумерной развитой гидродинамической турбулентности (при числах Рейнольдса $Re \gg 1$), как было выяснено в работах [1, 2, 3], формирование прямосчитать, что указанное соотношение можно рассматривать как универсальное

для якобиана отображения J (как меры изменения дифференциально малого элемента площади):

$$\frac{dJ}{dt} = \mathrm{div}\,\mathbf{v_n} \cdot J.$$

гидродинамики, где магнитное поле в бездиссипативном пределе является вмороженным.

3. Заключение

Основной вывод этой работы состоит в том, что формирование степенной зависимости B_{max} от ширины ℓ - закон 2/3 - можно рассматривать как проявление формирования складки, подчеркнем, для бездивергентного векторного поля - ротора завихренности. Как уже отмечалось во введении, такой же

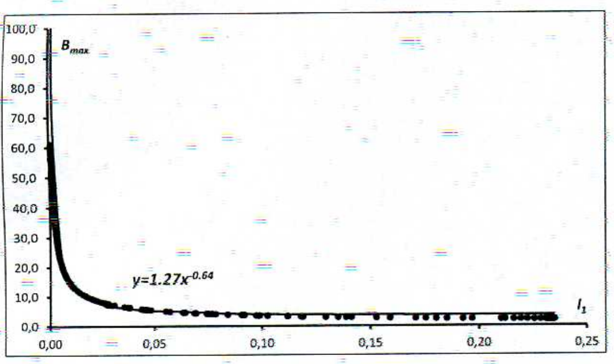

Рис.5. Зависимость максимума ротора завихренности от толщины ℓ. Точки соответствуют численным результатам, а линия - степенной зависимости $B_{max} \sim \ell^{-2/3}$.

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