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On ST6 Source Terms Model Assessment and Alternative

Andrei Pushkarev [1,2,†], Vladimir Geogjaev [1,3,*,†] and Vladimir Zakharov [1,2,3,†]

[1] Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, bld. 1, Moscow 121205, Russia
[2] Lebedev Physical Institute Russian Academy of Sciences, Leninsky 53, Moscow 119991, Russia
[3] Shirshov Institute of Oceanology of Russian Academy of Sciences, Nakhimovsky pr. 36, Moscow 117997, Russia
* Correspondence: vvg@mail.geogjaev.ru
† These authors contributed equally to this work.

Abstract: We present the study of the ST6 balanced set of wind energy input and wave energy dissipation due to wave breaking source terms, offered as the option in operational wave forecasting models and based on theoretical self-similarity analysis and numerical simulation of the wave energy radiative transfer equation. The study relies on the classical limited fetch wind wave excitation problem with constant wind blowing orthogonally to the shoreline toward the open ocean. It is found that the ST6 model exhibits asymptotic quasi self-similar behavior for fetches exceeding $\simeq 25$ km, as well as non-universal wave energy growth for shorter fetches, depending on the shoreline wave energy levels. We construct the alternative model PGZ-2 from a self-similar consideration, which reproduces field experimental data almost in the whole fetch span and reduces wave energy evolution dependence on the shoreline energy level. We assert that the PGZ-2 model is more accurate than the ST6 model. Moreover, it has a wider area of applicability.

Keywords: water waves; wave spectra; limited fetch; wind wave pumping; self-similarity; kinetic equation; forecasting; modeling

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1. Introduction

The study of ocean waves is significant in many ways. First, waves are one of the main mechanisms of interaction between air and water. As such, they determine the effect of the ocean on the weather (both locally and globally) as well as the weather effect on the ocean motion. The impact of the ocean on the Earth's climate cannot be overestimated. Another important field is the study of the waves themselves. This is of great value for ship navigation, coastal construction, and other fields.

It is widely accepted nowadays that [1,2] the wave kinetic equation based on slowly changing wave amplitude (WKB approximation), also known in physical oceanography as the radiative transfer equation [3], statistically describes the evolution of nonlinear ocean surface waves:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial \omega}{\partial \mathbf{k}} \frac{\partial \varepsilon}{\partial \mathbf{r}} = S_{\rm nl} + S_{\rm in} + S_{\rm ds} \tag{1}$$

where $\varepsilon(k, \theta, \mathbf{r}, t)$ is the wave energy spectral density, $\mathbf{r} = \mathbf{r}(x, y)$ and $\mathbf{k} = (k, \theta)$ are the real space vector coordinate and Fourier space wave vector, k and θ are the modulus of the wave number and the angle of wave propagation, $\omega = \sqrt{kg}$ is deep water dispersion relation, and g is the gravity acceleration. $S_{\rm nl}$, $S_{\rm in}$, and $S_{\rm ds}$ are the nonlinear wave interaction, the wind energy input, and the dissipation source terms, respectively.

For more than 40 years, significant efforts have been devoted to the construction of different forms of $S_{\rm in}$ and $S_{\rm ds}$. In the current research, we are focusing on the assessment of recently developed ST6 wind energy input $S_{\rm in}^{ST6}$ and wave energy dissipation $S_{\rm ds}^{\rm ST6}$ source terms, recently included as a package in the operational wave forecasting model WAVEWATCH III [4].

We formulated the new model of the balanced wind energy input and dissipation source terms with two unknown tunable parameters, the amplitude and index of the power-like wind source term, and call it PGZ-2 . Those unknown values are obtained with the help of matching the PGZ-2 numerical results with the field experimentally observed energy growth curve KC1992 and theoretically predicted PGZ-2 self-similar relations.

The formulated PGZ-2 model significantly improves the correspondence of the total wave energy behavior for smaller fetches as well as larger ones with experimental curve KC1992.

Both ST6 and PGZ-2 models exhibit dependence of the wave energy growth along the fetch on the total wave energy value at the shore. The scatter of this dependence is lower for the PGZ-2 model.

One should note that the alternative PGZ-2 model should be construed as the demonstrational one and not for practical purposes since the KC1992 experiment analysis presents not quite typical behavior for the pool of the multiple experimental observations, studied in Badulin et al. [12], where the characteristic power-like total wave energy index value was $p = 1$. In this relation, in the authors' opinion, though close in spirit, the different ZRP approach to the formulation of the balanced source terms presents the better alternative.

The tuning of the PGZ-2 model was performed to be in agreement with the experimental data. More fine tuning may be performed on the basis of more numeric experiments.

We believe that the PGZ-2 model is a basis for further study. In the near future, we plan to study its properties in more detail and continue the tuning of the model.

Author Contributions: Conceptualization, V.Z.; methodology, V.Z.; software, V.G.; formal analysis, A.P.; investigation, A.P.; writing—original draft preparation, A.P.; writing—review and editing, V.G.; project administration, V.Z. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: Data analyzed in this study were obtained through numerical simulation of the limited fetch code, which calculated nonlinear interaction through WRT algorithm. These data are available upon request sent to dr.push@gmail.com.

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Conflicts of Interest: The authors declare no conflict of interest.

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ORIGINAL ARTICLE

Deep Water Waves from Oscillating Elliptic Source

Vladimir Gnevyshev[1] · Sergei Badulin[1,2]

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Abstract
Asymptotic solutions describing linear waves generated by oscillating elliptic source are constructed employing the recently developed "Reference Solution Approach" (RSA). The source is assumed to be in rest. The resulting wave pattern exhibits pronounced anisotropy of the solutions for elongated sources. The classic Kelvin angles of the ship wave patterns determine specific distinguished directions. The analytical results within the RSA are shown to agree remarkably well with the exact linear solutions of the problem.

Keywords Cauchy–Poisson problem · Surface gravity waves · The Kelvin wake · Kelvin's angle · Stationary phase approach (SPA)

1 Introduction

There is a vast anecdotal evidence that in the presence of swell ship wakes exhibit St. Andrew's cross-shaped patterns superimposed on the classical Kelvin wake. Here we explain these observations following Sir Lighthill [17, § 6] as "surface gravity waves generated by a traveling disturbance that is not steady but oscillatory with frequency ω_0". The new family of cross-shaped waves does not vanish when ship is at rest but still oscillating. In this paper we focus on this simplified problem statement. The shape and finite size of the oscillating wave source turn out essential factors. Their effect is found to be counterintuitive, at the first glance: an oscillating elliptic source

Vladimir Gnevyshev and Sergei Badulin have contributed equally to this work.

✉ Sergei Badulin
badulin.si@ocean.ru

Vladimir Gnevyshev
avi9783608@gmail.com

1 Ocean Physics Division, Shirshov Institute of Oceanology, Russian Academy of Sciences, 36, Nakhimovskii pr., Moscow, Russia 117997

2 Laboratory of Integrable Systems and Turbulence, Skolkovo Institute of Science and Technology, 30, bld. 1, Bolshoy Boulevard, Moscow, Russia 121205

The price for the convenience of RSA for the special problem is a necessity of careful accounting for the physical scale hierarchy. The wave envelope should be much longer than the carrier wavelength to imitate the oscillating source. Additionally, this envelope should be short enough as compared with the distance (time) where the effect of strong anisotropy of the wave field occurs. It is shown that the latter scale is rather short and cannot be captured by conventional asymptotic methods like SPA operating in a far zone [18]. The Reference Solution Approach is used as an alternative which is capable to describe the intermediate asymptotic regime of the wave evolution. In the above examples, the wave envelopes contain approximately 4–10 wavelengths, and the pronounced anisotropy of wave amplitudes is observed at distances less than 100 wave periods.

An experimental demonstration of the wave pattern anisotropy in a wave tank would be highly desirable but, probably, hardly feasible for deep water waves with these spatial scaling. Laboratory experiments or careful analysis of ship wakes [21, 25] at low speeds (low Froude numbers) would be helpful for verification of the phenomenon presented in this paper.

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Data Availability Not applicable.

Declarations

Conflict of Interest The authors declare no conflict of interest.

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Author for correspondence:
A. A. Gelash
e-mail: Andrey.Gelash@u-bourgogne.fr

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Numerical direct scattering transform for breathers

I. I. Mullyadzhanov[1,2], A. S. Gudko[3,4], R. I. Mullyadzhanov[3,4], and A. A. Gelash[5]

[1]Institute of Automation and Electrometry SB RAS, Novosibirsk 630090, Russia
[2]Skolkovo Institute of Science and Technology, Moscow, 121205, Russia
[3]Institute of Thermophysics SB RAS, Novosibirsk 630090, Russia
[4]Novosibirsk State University, Novosibirsk 630090, Russia
[5]Laboratoire Interdisciplinaire Carnot de Bourgogne (ICB), UMR 6303 CNRS – Université Bourgogne Franche-Comté, 21078 Dijon, France

We consider the model of the focusing one-dimensional nonlinear Schrödinger equation (fNLSE) in the presence of an unstable constant background, which exhibits coherent solitary wave structures – breathers. Within the inverse scattering transform (IST) method, we study the problem of the scattering data numerical computation for a broad class of breathers localized in space. Such direct scattering transform (DST) procedure requires a numerical solution of the auxiliary Zakharov–Shabat system with boundary conditions corresponding to the background. To find the solution we compute the transfer matrix using the second-order Boffetta–Osborne approach and recently developed high-order numerical schemes based on the Magnus expansion. To recover the scattering data of breathers, we derive analytical relations between the scattering coefficients and the transfer matrix elements. Then we construct localized single- and multi-breather solutions and verify the developed numerical approach by recovering the complete set of the scattering data with the built-in accuracy providing the information about the amplitude, velocity, phase, and position of each breather. To combine the conventional IST approach with the efficient dressing method for multi-breather solutions, we derive the exact relation between the parameters of breathers in these two frameworks.

into the formulas obtained in [47] for the soliton scattering data. Our derivation is based on the asymptotic behavior of the wave function (2.8) for spatially localized KM and TW breathers. To evaluate the scattering matrix numerically, we use the standard second-order approach from [47] and the recently developed advanced high-order numerical schemes based on the Magnus expansion [37,48]. Our numerical tests demonstrate accurate recovery of the breather's parameters and confirm the declared convergence orders of the developed schemes. Note that identifying the complete discrete scattering data set $\{\lambda_n, \rho_n\}$ always requires consideration of the full scattering problem as we present here. Meanwhile computing solely the breathers eigenvalues $\{\lambda_n\}$ can be performed using other techniques [60] such as for example Fourier collocation method [61]. In our approach, we consider the general case of arbitrary asymptotic condensate phases $\Theta_\pm$ and thereby cover the whole general class of the localized breathers, except the P breather and its high-order analogue [62]. The first and high-order P breathers being a degenerate case of the KM class require a separate consideration.

Extending the frontiers of the numerical DST applications by considering the constant amplitude boundary conditions, we highlight that the problem of characterization of arbitrary wave fields remains open. An accurate treatment of the data obtained in experiments or numerical simulations requires further development of periodic and quasi-periodic DST procedures, particularly for the norming constants of the A breathers. In addition, the existence of the variety of data analysis approaches, such as windowing or periodization methods and finite gap theory or theta-functions framework rises the question of finding direct links between different wave field sacttering data interpretations [21,55,63–65].

Our work contributes to the domain of numerical DST tools and suggests an efficient way to analyze the nonlinear dynamics of wave fields containing breathers. The latter is of special importance in light of the problem of modulation instability development, where breathers play one of the critical roles [2,12,66]. In particular, our DST algorithm can be used for a comprehensive analysis of breathers embedded into localized arbitrary-shaped background perturbations [60]. With the complete set of scattering data, one can predict the picture of wave field evolution and also compute the pure breather's part of the wave field, elucidating their impact to the initial condition similar to as it was recently done with solitons [55,67]. In addition one can use our approach to compute the reflection coefficients $\{r^{\mathrm{s}}, r^{\mathrm{MI}}\}$, which is important for problems where continuous spectrum waves dominate in the development of modulation instability or are in interplay with breathers [60,68]. Another important direction represents the rapidly growing field of integrable turbulence, where the DST algorithms can be used to study phase correlations of individual breathers as well as phase correlations in gases of breathers [55,69–75]. Note that both IST and DST algorithms face rapid growth of numerical errors when increasing the number of discrete spectrum components, which can be coped with using high-precision arithmetic as recently suggested in [55,57,76]. We highlight that the complete characterization of the breather's parameters benefits the IST analysis of experimental data on coherent structures in optics, hydrodynamics, and other nonlinear waveguides described by nearly integrable models [21,32,34,77,78]. Also our approach can be generalized to the case of vector breathers of the Manakov system [79–83] and other integrable systems with constant amplitude boundary conditions.

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8. Appendix

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Моделирование волновой турбулентности поверхности жидкости на основе метода динамических конформных преобразований

Е. А. Кочурин [1)]

Институт электрофизики Уральского отделения РАН, 620016 Екатеринбург, Россия

Сколковский институт науки и технологий, 121205 Москва, Россия

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В работе впервые обобщен метод динамических конформных преобразований для численного моделирования волновой капиллярной турбулентности поверхности жидкости в плоско-симметричной анизотропной геометрии. Модель является сильно нелинейной и учитывает эффекты поверхностного натяжения, а также диссипации и накачки энергии. Результаты моделирования показывают, что система нелинейных капиллярных волн может переходить в режим квазистационарного хаотического движения (волновая турбулентность). Вычисленные показатели спектров не совпадают с классическим спектром Захарова–Филоненко для изотропной капиллярной турбулентности, но хорошо согласуются с оценкой, полученной в предположении о доминирующем влиянии резонансных пятиволновых взаимодействий.

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Введение. Хорошо известно, что нелинейные волновые системы могут переходить в квазистационарное хаотическое состояние (режим волновой турбулентности) в результате резонансных волновых взаимодействий [1]. Явление волновой турбулентности может возникнуть в произвольной нелинейной волновой системе. Так, например, известна оптическая турбулентность [2], магнито- и электрогидродинамическая волновая турбулентность [3–7], акустическая турбулентность [8–10] и турбулентность диспересионных капиллярных и гравитационных волн на свободной поверхности жидкости [11–13]. В работах Захарова и соавторов [11, 8] были впервые получены точные решения кинетических уравнений для функции распределения квазичастиц-волн, описывающие стационарный перенос энергии (или других интегралов движения) вдоль различных масштабов. Такие решения получили название спектров турбулентности Колмогорова–Захарова по аналогии с классической гидродинамической турбулентностью [1].

В настоящий момент времени спектр изотропной капиллярной турбулентности на поверхности жидкости (также известный как спектр Захарова–Филоненко [11]) подтвержден с высокой точностью как экспериментально [14, 15], так и численно [16–18]. Спектр Захарова–Филоненко обычно записывается в терминах Фурье спектров функции $\eta(\mathbf{r},t)$, определяющей форму поверхности жидкости, $S_\eta(k)=|\eta_\mathbf{k}|^2$ и $S_\eta(\omega)=|\eta_\omega|^2$:

$$S_\eta(\omega)=C_{3w}^{\omega}P^{1/2}(\sigma/\rho)^{1/6}\omega^{-17/6},$$

$$S_\eta(k)=C_{3w}^{k}P^{1/2}(\sigma/\rho)^{-3/4}k^{-15/4},\quad k=|\mathbf{k}|, \qquad (1)$$

где $\mathbf{k}$ – волновой вектор, ω – угловая частота, C_{3w}^k и C_{3w}^ω – безразмерные константы, P – скорость диссипации энергии на единицу площади поверхности, σ и ρ – поверхностное натяжение и массовая плотность жидкости, соответственно. Пространственный и частотный спектры (1) связаны между собой законом сохранения энергии в пространстве Фурье, т.е. $S_\eta(k)d\mathbf{k}=S_\eta(\omega)d\omega$. Степенная зависимость от P в спектрах (1) с показателем степени $1/2$ отражает резонансный характер трехволновых взаимодействий:

$$\omega=\omega_1+\omega_2,\qquad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2, \qquad (2)$$

где ω связана с волновым числом k законом дисперсии, $\omega=(\sigma/\rho)^{1/2}k^{3/2}$.

Таким образом, справедливость спектра (1) для описания изотропной капиллярной турбулентности в настоящее время не подлежит сомнению. Ситуация меняется в случае анизотропных возмущений поверхности, когда рассматриваемые волны распространяются в одном направлении, т.е. являются коллинеарными. В такой ситуации условия трехволнового резонансного взаимодействия (2) перестают выполняться. Турбулентность коллинеарных ка-

1)e-mail: kochurin@iep.uran.ru

бодной поверхности жидкости. Вычислительная модель является полностью нелинейной и учитывает эффекты поверхностного натяжения, накачки и диссипации энергии. Результаты моделирования показывают, что система взаимодействующих нелинейных капиллярных волн может переходить в квазистационарное состояние, когда действие внешней силы полностью компенсируется диссипативными эффектами. В этом режиме движение жидкости приобретает сложный и нерегулярный характер, а плотность вероятности амплитуды границы становится близкой к нормальному распределению Гаусса. Измеренный спектр поверхностных возмущений в квазистационарном состоянии приобретает степенную зависимость с показателем, близким к аналитическому спектру, полученному в предположении о доминирующем влиянии резонансных пятиволновых взаимодействий в анизотропной плоско-симметричной геометрии. Анализ пространственно-временного преобразования Фурье также свидетельствует о слабо нелинейном характере эволюции волн. Следует также отметить, что численные результаты хорошо согласуются с экспериментальными исследованиями, проведенными для жидкой ртути [20].

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Eigenvalue Problem for a Reduced Dynamo Model in Thick Astrophysical Discs

Evgeny Mikhailov [1,2,3,*] and Maria Pashentseva [1]

[1] Faculty of Physics, M. V. Lomonosov Moscow State University, 119991 Moscow, Russia; pashentceva.mv17@physics.msu.ru
[2] P. N. Lebedev Physical Institute, 119991 Moscow, Russia
[3] Center for Advanced Studies, Skolkovo Institute of Science and Technology, 121205 Moscow, Russia
* Correspondence: ea.mikhajlov@physics.msu.ru

Abstract: Magnetic fields of different astrophysical objects are generated by the dynamo mechanism. Dynamo is based on the alpha-effect and differential rotation, which are described using a system of parabolic equations. Their solution is an important problem in magnetohydrodynamics and mathematical physics. They can be solved assuming exponential growth of the solution, which leads to an eigenvalue problem for a differential operator connected with spatial coordinates. Here, we describe a system of equations connected with the generation of magnetic field in discs, which are associated with galaxies and binary systems. For an ideal case of an infinitely thin disc, the eigenvalue problem can be precisely solved. If we take into account the finite thickness of the disc, the problem becomes more difficult. The solution can be found using asymptotical methods based on perturbations of the eigenvalues. Here, we present two different models which describe field evolution for different cases. For the first, we find eigenvalues taking into account linear and quadratic terms for the perturbations in the eigenvalue problem. For the second, we find eigenvalues using only linear terms; this is quite sufficient. Results were verified through numerical modeling, and basic computational tests show proper correspondence between different methods.

Keywords: eigenvalue problem; perturbations; eigenfunctions; dynamo; magnetic field; differential operator

MSC: 35Q85; 35P15

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1. Introduction

Magnetic fields of different astrophysical objects play an important role in astronomy. They have been studied for a long time, beginning with research regarding sunspots, which are the basic tracers of magnetic fields of the Sun [1]. It has been shown that the generation of such fields is described by the dynamo mechanism [2]. Dynamo is an important process in magnetohydrodynamics, which is based on the helicity of turbulent motions and non-uniform rotation of the medium (differential rotation) [3–5]. Such mechanisms take place not only on the Sun; similar processes also describe magnetic fields in other celestial bodies, such as stars [6], galaxies [7–9], accretion discs [10,11], planets [12], etc. Usually, three-dimensional equations for magnetic fields are reduced to simpler problems based on the properties of symmetry. One of the most important cases is connected with objects that have the shape of a disc [13–15]. Such models are connected with spiral galaxies and accretion discs surrounding compact astrophysical objects (black holes, neutron stars and white dwarfs) in binary systems.

First models for the dynamo process in discs were devoted to the thin objects and connected with so-called no-z approximation, constructed first by Mestel, Brandenburg [13] and Moss [14]. Such models used a specific law for the z-dependence of the magnetic field and assumed that the vertical component of magnetic field could be neglected. Thickness values. The value of the critical dynamo number, which characterizes the possibility of generating a magnetic field, was obtained; it was shown to be higher (reaching values up to $D_{cr} \approx 10..12$) than that of the dynamo model of a thin disk. Thus, it was shown that vertical flows can complicate the conditions for field generation, although they provide alpha-effect, which is necessary for the dynamo. So, we should have some balance between different factors.

We also considered the problem of generating a magnetic field within the framework of a model that takes into account a more complex structure of the field. In this case, it was necessary to solve the eigenvalue problem for an operator acting in the space of two-dimensional functions. In this case, a linear approximation was constructed. Note that quadratic approximation is extremely cumbersome. In addition, numerical simulations show that linear approximation provides much more accurate results than the case of a one-dimensional model does. Note that in this case, taking vertical fluxes into account also led to an increase in the generation threshold of the magnetic field.

This approach is not the only possible way to describe magnetic field generation. First, we should take into account the model for axially-symmetric galactic dynamos developed by Henriksen and colleagues [30–32]. There are exact solutions for some cases for alpha–alpha dynamo and classical dynamo with zero and non-zero diffusion. These results correspond to modern observational data connected with edge-on galaxies [33]. Mathematically, this work describes cases when equations cannot be solved exactly, but we can take into account special effects using perturbation theory.

Thus, it can be summarized that in the case of thick disks, the process of generating a magnetic field is significantly complicated. This is due to the presence of vertical flows. This result was confirmed by solving the eigenvalue problem in various approximations. In addition, it agrees with numerical simulation data.

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Appendix A

Here we describe the main stages of the derivation of equations for the magnetic field, based on approaches described in [21]. First, we use the Steenbeck–Krause–Raedler equation [3]:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) + \nabla \times (\alpha \boldsymbol{B}) + \eta \Delta \boldsymbol{B} \tag{A1}$$

where $\boldsymbol{V}$ is the regular velocity, $\alpha = -\frac{\tau}{3}\boldsymbol{v} \bullet \nabla \boldsymbol{v}$ is the alpha-effect coefficient, and η is the turbulent velocity coefficient. As for velocity, we can introduce the law $\boldsymbol{V} = r\Omega \boldsymbol{e}_\varphi$, and for the alpha-effect $\alpha = \alpha_0 \frac{z}{h}$, where α_0 is the typical value and h is the half-thickness.

If we introduce the field in Equation (57), the equation will be:

$$\frac{\partial B}{\partial t}\boldsymbol{e}_\varphi + \nabla \times \left(\frac{\partial A}{\partial t}\boldsymbol{e}_\varphi\right) = \nabla \times \left(r\Omega \nabla \times \left(A\boldsymbol{e}_\varphi\right)\right) + \nabla \times \left(\alpha B \boldsymbol{e}_\varphi\right) + \eta \Delta B \boldsymbol{e}_\varphi + \eta\left(\Delta A \boldsymbol{e}_\varphi\right) \tag{A2}$$

We can divide this equation in two parts [21]. The first part corresponds to φ-component, and the second part is included into curl. If we measure the distances in units of radius of the disc and time in units of $\frac{h^2}{\eta}$ we obtain Equations (58) and (59).

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КВАЗИКЛАССИЧЕСКАЯ ДИНАМИКА НЕЛИНЕЙНЫХ ВОЛНОВЫХ СИСТЕМ

Е. А. Кузнецов *

[1] Физический институт им П. Н. Лебедева РАН;
[2] Институт теоретической физики им Л. Д. Ландау РАН;
[3] Сколковский Институт науки и технологии, Сколково, г. Москва, Россия

Представлен краткий обзор по квазиклассической волновой динамике для нелинейного уравнения Шредингера (НУШ) применительно к фокусирующим и дефокусирующим средам. Свойства НУШ существенно различны в зависимости от размерности пространства d. Двумерное НУШ обладает дополнительной симметрией конформного типа относительно преобразований Таланова [1], впервые найденных для стационарной самофокусировки в среде с керровской нелинейностью. Следствием этой симметрии является теорема Власова—Петрищева—Таланова [2], связывающая среднее значение квадрата распределения с гамильтонианом системы. Эта теорема справедлива как для фокусирующих, так и для дефокусирующих сред. В квазиклассическом пределе это позволяет построить анизотропные решения, описывающие как сжатие пучка при самофокусировке, так и разлёт квантовых газов в вакуум в рамках так называемых критических нелинейных уравнений Шредингера, в частности для уравнения Гросса—Питаевского с химическим потенциалом со степенной зависимостью от плотности с показателем $\nu = 2/d$. Для уравнения Гросса—Питаевского случай $d = 2$ отвечает конденсату слабонеидеального бозе-газа, а $d = 3$ описывает конденсат ферми-газа в унитарном пределе. При $d = 3$ уравнения Гросса—Питаевского в квазиклассическом пределе превращается в уравнения газодинамики с показателем адиабаты $\gamma = 5/3$. Автомодельные решения в этом приближении описывают на фоне расширяющегося газа деформации газового облака. Такого типа угловые деформации наблюдаются как при разлёте квантовых газов, так и при воздействии мощного лазерного излучения на вещество. Для сверхкритического фокусирующего трёхмерного НУШ представлены квазиклассические решения коллапсирующего типа, включая точное квазиклассическое решение, описывающее режим сильного коллапса. Выяснено, что все такие квазиклассические коллапсы оказываются неустойчивыми, за исключением самого слабого и одновременно самого быстрого коллапса, соответствующего автомодельному решению НУШ. Рассмотрен вопрос о постколлапсе как продолжении слабого коллапса, в результате которого происходит формирование квазистационарной особенности в виде чёрной дыры, в которую энергия поступает из окружающей коллапсирующей области. Для НУШ при $d \geq 4$ формирование чёрной дыры может быть описано в квазиклассическом приближении. Показано, что анизотропия, обусловленная магнитным полем, существенно изменяет структуру ленгмюровского коллапса, в частности приводит к образованию сильно анизотропных чёрных дыр, описываемых квазиклассически.

ВВЕДЕНИЕ

С тех пор, как был опубликован первый обзор автора по интегральным критериям волновых коллапсов [3] в специальном выпуске журнала «Радиофизика», посвящённом 70-летию Владимира Ильича Таланова, прошло уже 20 лет. Поэтому второй обзор примерно на ту же тему в этом выпуске, посвящённом 90-летию В. И. Таланова, хочется назвать точно так же, как и знаменитый роман Александра Дюма: «Двадцать лет спустя». Этот обзор написан уже после кончины В. И. Таланова, однако его работы по теории самофокусировки света — одного из ярчайших нелинейных явлений — представляют собой фундаментальный вклад в теорию волновых коллапсов. В значительной степени развитию его идей посвящён данный обзор. Основное внимание в нём уделяется квазиклассическому описанию нелинейной волновой динамики в рамках нелинейного

* kuznetso@itp.ac.ru

стационарной самофокусировки света. Другой физический пример критического поведения демонстрирует уравнение Гросса—Питаевского (оно же НУШ) в двумерном случае, которое описывает при положительной длине рассеяния нелинейные колебания конденсата слабонеидеального бозе-газа. При $d = 3$ и $\nu = 2/3$ уравнение Гросса—Питаевского описывает конденсат ферми-газа в так называемом унитарном пределе (см., например, [15]). Для всех этих систем применима теорема Власова—Петрищева—Таланова. Следует отметить, что эта теорема справедлива также при квазиклассическом описании критического НУШ как для фокусирующих, так и для дефокусирующих сред. В квазиклассическом пределе при $d = 3$ и соответсвенно $\nu = 2/3$ уравнение Гросса—Питаевского превращается в уравнения гидродинамики для газа с показателем адиабаты $\gamma = 5/3$, для которых также выполняется теорема Власова—Петрищева—Таланова, связывающая средний размер распределения R с гамильтонианом H.

В данной статье мы рассмотрели, каким образом работает теорема вириала применительно к квазиклассической динамике критических НУШ. Было выяснено, что для дефокусирующего случая, в частности для разлёта квантового и классического газов в вакуум, автомодельные анизотропные решения описывают на угловые деформации газового облака при его расширении. Такого типа деформации наблюдаются как при разлете квантовых газов [25, 26], так и при воздействии мощного лазерного излучения на металл [21], когда применимы уравнения гидродинамики для газа с $\gamma = 5/3$ и соответственно решение Анисимова—Лысикова [18]. В фокусирующем случае автомодельные анизотропные решения, как было показано в [27], приводят к фокусировке на линиях, а не к формированию сильного коллапса, как было позже продемонстрировано Фрайманом [45], а затем в работах [46–48].

Для свехкритического НУШ при $d = 3$ в работе представлено точное квазиклассическое решение, опиывающее режим сильного коллапса, для которого захватываемая в особенность энергия волн конечна. Все другие решения из иерархии квазиклассических решений относятся к слабым коллапсам, для которых формально захваченная в особенность энергия равна нулю. Все такие квазиклассические коллапсы оказывются неустойчивыми, за исключением самого слабого и одновременно самого быстрого коллапса, соответствующего автомодельному решению трёхмерного НУШ [32]. Рассмотрен вопрос о постколлапсе как продолжении слабого коллапса, в результате которого происходит формирование квазистационарной особенности в виде воронки — чёрной дыры, в которую энергия черпается из окружающей коллапс области. Для НУШ высокой размерности $d \geq 4$ формирование воронки может быть описано в квазиклассическом приближении, точность которого улучшается по мере приближения к особенности. Показано, что анизотропия, обусловленная магнитным полем, существенно изменяет структуру ленгмюровского коллапса, в частности приводит к образованию сильно анизотропных чёрных дыр, которые описывются квазиклассически. В этом случае также точность квазиклассического приближения улучшается по мере приближения к особенности.

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NONLINEAR DYNAMICS OF SLIPPING FLOWS

E. A. Kuznetsov,[1,2,3] * **E. A. Mikhailov,**[1,3,4] **and**
M. G. Serdyukov[4]

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We develop a new concept for the formation of behavior features of inviscid incompressible fluids on the rigid boundary due to breaking of slipping flows. The breaking possibility is related to the compressibility of such flows due to the boundary. For two- and three-dimensional inviscid Prandtl equations, we analytically obtain the criteria for a gradient catastrophe for slipping flows. For the two-dimensional Prandtl equations, breaking occurs for both the velocity component parallel to the boundary and the vorticity gradient. The explosive growth of the vorticity gradient correlates with the appearance of a jet in the direction perpendicular to the boundary. For the three-dimensional Prandtl flows, breaking (fold formation) leads to an explosive growth for both the symmetric part of the velocity-gradient tensor and its antisymmetric part, i.e., vorticity. The blow-up generation of vorticity is possible due to the fluid suction from the slipping flow with simultaneous formation of a jet perpendicular to the boundary. These factors can be considered as a tornado-formation mechanism. Within the framework of the two-dimensional Euler equations, we numerically study the problem of the formation of increasing velocity gradients for the flows between two parallel plates. It is revealed that on the rigid boundary, the maximum velocity gradient exponentially increases with time simultaneously with an increase in the vorticity gradient according to the double exponential law. This process is also accompanied by a jet formation in the direction perpendicular to the boundary.

1. INTRODUCTION

Collapse, i.e., the process of the formation of a singularity in a finite time under smooth initial conditions, plays a primary role in understanding the turbulence nature in nonlinear dynamics. This is especially important for a developed hydrodynamic turbulence at large Reynolds numbers $\mathrm{Re} \gg 1$ in the inertial scale interval, in which, according to the Kolmogorov theory [1], the interaction between the fluctuations is considered local and can be described within the framework of the Euler equation. In this scale range, turbulence has a universal character and is determined by the only dimensional quantity, namely, the energy flow ε from large energy-containing scales to small viscous ones. In this case, the velocity fluctuations with scale ℓ are found from the dimension considerations, i.e., $\langle \delta v \rangle \propto \varepsilon^{1/3} \ell^{1/3}$ (hereinafter, the angle brackets denote averaging over the statistical ensemble). Correspondingly, the vorticity fluctuations behave as $\langle \delta\omega \rangle \propto \varepsilon^{1/3} \ell^{-2/3}$, i.e., diverge for $\ell \to 0$. In this case, the transition time turns out to be a finite quantity determined by the characteristic scale of the energy-containing region, which is indicative of the collapse possibility in the three-dimensional Eulerian hydrodynamics. Although many numerical experiments conducted in the late 1990s

* kuznetso@itp.ac.ru

[1] P. N. Lebedev Physics Institute of the Russian Academy of Sciences, Moscow; [2] L. D. Landau Institute of Theoretical Physics of the Russian Academy of Sciences, Chernogolovka; [3] Skolkovo Institute of Science and Technology, Moscow; [4] M. V. Lomonosov Moscow State University, Moscow, Russia. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 66, Nos. 2–3, pp. 145–160, February–March 2023. Russian DOI: 10.52452/00213462_2023_66_02_145 Original article submitted April 10, 2023; accepted April 28, 2023.

dependence between the maximum velocity gradient and its thickness ℓ, i.e., $\max|\partial u/\partial x| \propto \ell^{-2/3}$. It should be noted that we have succeeded in obtaining a much better resolution (by about an order of magnitude) compared with that of our previous numerical experiments [19]. This allowed us not only to refine earlier results related to an exponential increase in the velocity gradient, but also study an increase in the vorticity gradient in detail. It has been demonstrated that it increases according to the double exponential law such that the numerical characteristics of an increase are in full agreement with theoretical prediction given by Eq. (20). It is also worth noting that a superexponential increase leads to an increase in the vorticity gradient for the studied time interval by three orders of magnitude, which would be impossible without an accuracy improvement and use of parallel calculations. It should be mentioned that such regimes with an exponential growth were observed in the three-dimensional Euler equations for the pancake vortex structures and in the two-dimensional Euler equations for the quasi-one-dimensional structures in the form of the vorticity quasi-shocks [12]. For the three-dimensional Prandtl equations, we have shown that the slipping flows demonstrate the blow-up behavior for both the velocity stress tensor and the vorticity. In this case, a jet is formed in the direction transverse to the boundary, which, combined with an explosive increase in the vorticity, can be considered as a tornado-formation mechanism. Thus, it is of fundamental importance to clarify the role of the boundary in the collapse formation in the three-dimensional flows of inviscid fluid within the framework of the Euler equations.

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ЭВОЛЮЦИЯ НЕЛИНЕЙНЫХ ВОЛНОВЫХ ИМПУЛЬСОВ В ТЕОРИИ УРАВНЕНИЯ СИНУС-ГОРДОН

А. М. Камчатнов [a*]

[a] *Институт спектроскопии Российской академии наук*
108840 Москва, Троицк, Россия

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Дано решение модуляционных уравнений Уизема, описывающих эволюцию огибающих однофазных периодических волн, подчиняющихся уравнению синус-Гордон. Методом годографа задача сведена к линейному уравнению в частных производных и описан класс решений этого уравнения с разделяющимися переменными. Теория иллюстрируется примером, в котором получено полное аналитическое решение задачи о самосжатии нелинейного волнового пакета, которое сопровождается уходом волн из области нелинейных колебаний.

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1. ВВЕДЕНИЕ

Явление модуляционной неустойчивости нелинейных волн было открыто независимо в нескольких различных физических контекстах: как самофокусировка интенсивных пучков света, распространяющихся в нелинейной среде [1–3], как разбиение газа ленгмюровских плазмонов на отдельные сгустки [4, 5], как самосжатие волновых пакетов для волн в оптике [6] и на глубокой воде [7, 8]. Если начальное распределение физических параметров волны промодулировано достаточно плавными функциями, то в главном приближении динамика модуляций описывается уравнениями гидродинамического типа, для решения которых могут быть использованы методы газовой динамики. Первые примеры такого подхода были даны в теории самофокусировки, когда эволюция пучка света описывается фокусирующим нелинейным уравнением Шредингера, так что модуляционными параметрами служат интенсивность света и поперечное волновое число световой волны, которые подчиняются уравнениям геометрической оптики, эквивалентным гидродинамическим уравнениям для волн на «опрокинутой мелкой воде». Для этого случая В. И. Таланов нашел решение для пучка с параболическим начальным профилем интенсивности [9], а С. А. Ахманов, А. П. Сухоруков, Р. В. Хохлов [10] — для пучков с начальным профилем интенсивности вида $\mathrm{ch}^{-2}(x)$. В работах [11,12] аналогичный подход был сформулирован независимо от теории НУШ, но также в приближении умеренной амплитуды волны, и в результате модуляционная гидродинамическая система была сведена методом годографа к линейному уравнению эллиптического типа. Примеры решений этого уравнения, описывающих самофокусировку пучков света и самокомпрессию импульсов, были даны в [13] и другие многочисленные примеры можно найти в книге [14].

Естественно, эти решения, предполагающие плавную модуляцию волны, справедливы лишь до момента фокусировки. Кроме того, они неустойчивы относительно малых возмущений, нарушающих плавность профиля волны. Еще в работах [15, 16] было замечено, что в теории фокусирующего НУШ локализованное начальное возмущение однородной плоской волны ведет к образованию расширяющейся со временем промодулированной волновой структуры. Применение модуляционной теории Уизема [17–20] к описанию эволюции этой структуры показало [21–23], что фронт неустойчивости движется с минимальной групповой скоростью действительной ветви закона дисперсии, и этот результат был подтвержден в [24] в рамках метода обратной задачи рассеяния для НУШ. Обобщение

* E-mail: kamch@isan.troitsk.ru

случаями типа «эволюции ступеньки» [21–23] (см. также [47] и приведенные там ссылки). Развитие области неустойчивости в неоднородных системах обсуждалось в работе [25] также только для малоамплитудного края и не затрагивало вопроса об эволюции параметров во всей области нелинейных осцилляций. В настоящей работе дано точное решение модуляционных уравнений Уизема для модуляционно неустойчивой нелинейной волны в модели синус-Гордон. Развитая теория показывает весьма нетривиальную эволюцию нелинейного волнового пакета, которая заключается в его сжатии, сопровождаемом уходом волн из области нелинейных осцилляций, так что за конечное время в этой области остается лишь малое число порядка единицы нелинейных колебаний. На этой стадии теория Уизема теряет свою применимость и вопрос о дальнейшей эволюции волнового импульса должен рассматриваться другими методами.

Можно предполагать, что предложенный здесь подход окажется эффективным и при обсуждении модуляционно неустойчивых для других вариантов нелинейного уравнения Клейна–Гордона, в частности для уравнения, описывающего возбуждение волн ветром (см. [55]) или уравнений, описывающих распространение электромагнитных волн в нелинейных средах (см., например, [56–58]).

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АСИМПТОТИЧЕСКАЯ ТЕОРИЯ СОЛИТОНОВ, ПОРОЖДАЕМЫХ ИЗ ИНТЕНСИВНОГО ВОЛНОВОГО ИМПУЛЬСА

А. М. Камчатнов *

Институт спектроскопии Российской академии наук, 108840, Москва, Троицк, Россия

Поступила в редакцию 26 мая 2023 г.,
после переработки 1 июня 2023 г.
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Развита теория преобразования интенсивного начального волнового импульса в солитоны при асимптотически больших временах эволюции. Наш подход основан на том, что такое преобразование происходит через промежуточную стадию формирования и эволюции дисперсионных ударных волн, так что число нелинейных осцилляций в них оказывается равным числу солитонов в асимптотическом состоянии. С помощью теории интегрального инварианта Пуанкаре–Картана мы показываем, что это число осцилляций, равное классическому действию частицы, ассоциированной с волновым пакетом в окрестности малоамплитудного края дисперсионной ударной волны, остается постоянным при переносе течением, описываемым бездисперсионным пределом рассматриваемых нелинейных волновых уравнений. Это позволяет сформулировать обобщенное правило квантования Бора–Зоммерфельда, которое определяет набор «собственных значений», связанных с физическими параметрами солитонов в асимптотическом состоянии, в частности, с их скоростями. Теория не использует свойств полной интегрируемости нелинейных волновых уравнений, но воспроизводит соответствующие результаты и в этом случае. Аналитические результаты подтверждаются численными решениями нелинейных волновых уравнений.

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1. ВВЕДЕНИЕ

Если в нелинейной волновой системе возможно распространение солитонов, то во многих типичных ситуациях достаточно интенсивный начальный импульс эволюционирует в конечном счете в некоторое число N солитонов и относительно небольшое количество энергии трансформируется в линейные волны. Ввиду универсальности этого физического явления является важным уметь вычислять параметры образующихся солитонов по форме начального импульса. Эта задача легко решается в случае полностью интегрируемых нелинейных волновых уравнений (см., например, книги [1–3] и имеющиеся там ссылки). Каждое такое уравнение связано с некоторой линейной спектральной задачей, в которой распределения волновых переменных играют роль «потенциалов». Хотя эти «потенциалы» изменяются со временем согласно рассматриваемым волновым уравнениям, спектр соответствующей линейной задачи остается постоянным и каждое дискретное собственное значение отвечает солитону в асимптотическом состоянии при $t \to \infty$. Следовательно, если мы найдем спектр линейной задачи для начальных распределений, то дискретные собственные значения дадут нам всю необходимую информацию о параметрах солитонов при асимптотически больших значениях t.

Эта теория существенно упрощается, если число солитонов в конечном состоянии велико, $N \gg 1$, а начальные распределения выражаются достаточно гладкими функциями, так что к линейной спектральной задаче может быть применен квазиклассический метод ВКБ. Например, уравнение Кортевега-де Фриза (КдФ) ассоциировано со стационарным уравнением Шредингера [4], так что хорошо известное правило квантования Бора–Зоммерфельда легко дает приближенные значения

* E-mail: kamch@isan.troitsk.ru

6. ЗАКЛЮЧЕНИЕ

Развитый в этой работе асимптотический метод предлагает простой способ вычисления параметров солитонов, образующихся из исходно гладкого нелинейных уравнений. Применительно к интегрируемым уравнениям он предоставляет простой способ вывода правила квантования Бора–Зоммерфельда для линейной спектральной задачи, ассоциированной с нелинейным уравнением в схеме АКНС. Условие коммутативности производных, определенных формулами (41) или их обобщениями (см. [35]) может использоваться как «тест на интегрируемость»: если они коммутируют, то представляется весьма правдоподобным, что рассматриваемое уравнение полностью интегрируемо в схеме АКНС и формулы (109) предоставляют полезную информацию о квазиклассическом пределе функций, определяющих соответствующую пару Лакса. Наконец, результаты статьи могут быть использованы также при рассмотрении задач, связанных с распространением высокочастотных волновых пакетов по неоднородному и меняющемуся со временем фону, как это было продемонстрировано в работе [35].

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Propagation of generalized Korteweg–de Vries solitons along large-scale waves

A. M. Kamchatnov and D. V. Shaykin

Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow 108840, Russia;
Moscow Institute of Physics and Technology, Institutsky Lane 9, Dolgoprudny, Moscow Region 141700, Russia;
and Skolkovo Institute of Science and Technology, Skolkovo, Moscow 143026, Russia

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We consider propagation of solitons along large-scale background waves in the generalized Korteweg–de Vries (gKdV) equation theory when the width of the soliton is much smaller than the characteristic size of the background wave. Due to this difference in scales, the soliton's motion does not affect the dispersionless evolution of the background wave. We obtained the Hamilton equations for soliton's motion and derived simple relationships which express the soliton's velocity in terms of a local value of the background wave. Solitons' paths obtained by integration of these relationships agree very well with the exact numerical solutions of the gKdV equation.

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I. INTRODUCTION

Perturbation theory for solitons has a long history and a number of publications are devoted to different approaches to it which span from simple variational estimates to rigorous mathematical investigations based on the inverse scattering transform method (see, e.g., review articles [1,2] and references therein). In spite of that, there still exist some specific situations where the methods developed so far are either insufficient or too complicated for practical use, and simpler approaches are needed. One such situation refers to propagation of solitons along a large-scale background wave $u = \overline{u}(x, t)$, where $\overline{u}(x, t)$ obeys in the simplest case of unidirectional propagation the Hopf-like equation

$$\overline{u}_t + V_0(\overline{u})\overline{u}_x = 0. \tag{1}$$

If we denote a characteristic width of a soliton as $\sim\kappa^{-1}$, then it is assumed that $\overline{u}(x, t)$ changes considerably at distances of about l much greater than $\sim\kappa^{-1}$, so that $(\kappa l)^{-1} \ll 1$ is a small parameter of the theory. At the same time, Eq. (1) is just a dispersionless approximation of the nonlinear wave equation under consideration for unidirectional wave propagation. It is supposed that the soliton's propagation does not influence the evolution of the background wave, so this equation does not contain any perturbation terms. This scheme corresponds to the generally accepted qualitative picture according to which the soliton's propagation through a nonuniform and varying with time background can be treated as the motion of a classical particle under the action of an external time-dependent field. Consequently, the first task is to derive equations for the soliton's motion along the evolving background wave $u = \overline{u}(x, t)$. In fact, this problem was solved for Korteweg–de Vries (KdV) solitons in Refs. [3,4] (see also Refs. [5,6]), but this rigorous approach was quite involved mathematically and was not, apparently, widely used in physical literature. Recently this problem was reconsidered in Ref. [7] for propagation of KdV solitons along rarefaction waves by different methods including the Whitham theory of modulations, and this approach was extended to the problem of propagation of KdV solitons along dispersive shock waves (DSWs).

Application of the Whitham modulation theory to this type of problem seems very natural since propagation of the soliton edge of a DSW reduces exactly to the motion of the leading soliton along the background dispersionless wave. For example, this approach easily reproduces equations of motion [8] in Bose-Einstein condensate in the case of the absence of external perturbations (see Ref. [9]). In this paper we combine some ideas developed earlier in the Whitham modulation theory with elementary results of the perturbation theory and reproduce very simply the Hamilton equations of Refs. [3,4] for the soliton's motion. These equations can be integrated to give the useful relationship

$$\kappa = \kappa(\overline{u}) \tag{2}$$

between the soliton's inverse half-width κ and the background wave amplitude $\overline{u}$. Since the soliton's velocity V can be expressed in terms of the dispersion relation $\omega = \omega(k, \overline{u})$ for linear waves with the wave number k propagating along the constant background $\overline{u}$ by the Stokes formula [10] (it was published first in the early edition of Lamb's *Hydrodynamics* [11], Sec. 252; see also Ref. [12] and references therein)

$$V = \frac{\omega(i\kappa, \overline{u})}{i\kappa}, \tag{3}$$

then substitution of Eq. (2) gives the equation

$$\frac{dx}{dt} = V[\overline{u}(x, t)] \tag{4}$$

for the soliton's path $x = x(t)$, which can be easily integrated.

The relationship (2) can be treated as an analytical continuation of the relationship between the carrier wave number k of a short-wavelength wave packet and the background amplitude $\overline{u}$ which follows from the Hamilton theory of propagation of such packets [13] as well as from the Whitham theory for propagation of small-amplitude edges of dispersive and substitution of this expression for V to Eq. (40) gives the dependence of the amplitude along the soliton's path,

$$A(t) = \sqrt{q} - 2\overline{u}[x(t), t]. \tag{46}$$

Of course, this result can also be obtained directly from Eq. (34), which is applicable to the general nonlinearity function $V_0(u)$, with the use of Eqs. (35) and (44).

The Hamilton equations for the soliton's motion in the case of the gKdV equation can be obtained without much difficulty. We denote again $\kappa^2 = f(p)$, and integration of the equation

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = V = V_0(\overline{u}) + \kappa^2 = V_0(\overline{u}) + f(p)$$

gives

$$H = V_0(\overline{u})p + \int f(p)dp. \tag{47}$$

Differentiation of Eq. (43) along the soliton's path yields

$$\frac{d\kappa^2}{dt} = -\frac{2}{3}V_0'(\overline{u})(\overline{u}_t + V\overline{u}_x) = -\frac{2}{3}V_0'(\overline{u})\overline{u}_x\kappa^2.$$

Then the second Hamilton equation $dp/dt = -\partial H/\partial x$ gives

$$\frac{dp}{df}\left(-\frac{2}{3}V_0'(\overline{u})\overline{u}_x f\right) = -V_0'(\overline{u})\overline{u}_x p$$

and, hence, $f(p) = p^{2/3}$, with $p = \kappa^3$. Thus, we obtain the Hamiltonian

$$H = V_0[\overline{u}(x, t)]p + \tfrac{3}{5}p^{5/3} \tag{48}$$

and the Hamilton equations

$$\frac{dp}{dt} = -V_0'(\overline{u})\overline{u}_x p, \quad \frac{dx}{dt} = V_0(\overline{u}) + p^{2/3}. \tag{49}$$

Let us apply the developed theory to the problem of propagation of solitons in the case of the nonlinearity function (37) and for the initial background wave distribution

$$\overline{u}_0(x) = \begin{cases} a(x/5)^{1/\gamma}, & x > 0, \\ 0, & x < 0. \end{cases} \tag{50}$$

Then the solution of the Hopf equation (1) reads

$$\overline{u}(x, t) = a\left(\frac{x}{t/\tau + 5}\right)^{1/\gamma}, \quad \tau = \frac{1}{6a^\gamma}, \tag{51}$$

and $V_0(\overline{u})$ is given by

$$V_0[\overline{u}(x, t)] = \frac{x}{t + 5\tau}. \tag{52}$$

Let the soliton start its motion at the point $x = 0$ at the moment $t = t_0 > 0$ with the initial velocity v_0. Then Eq. (44) takes the form

$$\frac{dx}{dt} = \frac{1}{3}\frac{x}{t + 5\tau} + v_0, \tag{53}$$

and it can be easily solved to give

$$x(t) = \frac{3}{2}v_0(t + 5\tau)\left\{1 - \left(\frac{t_0 + 5\tau}{t + 5\tau}\right)^{2/3}\right\}. \tag{54}$$

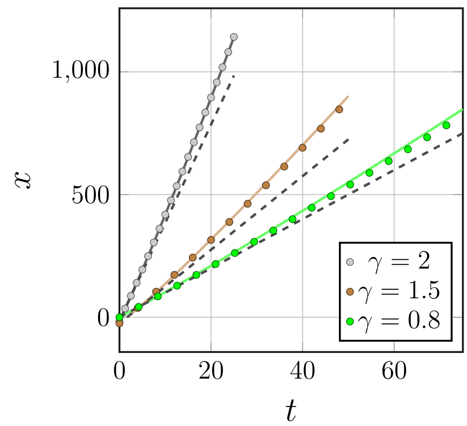

FIG. 2. Paths $x(t)$ of the solitons propagating along the background waves (51) for different data sets (γ, a, v_0, x_0): gray (2, 0.25, 40, −20), brown (1.5, 0.1, 15, −25), and green (0.8, 0.005, 10, 0). The circles correspond to the numerical solution of Eq. (31), the solid lines correspond to Eq. (53), and the dashed lines correspond to the free motion of the soliton along a zero background.

The soliton's amplitude along the path can be found from Eqs. (34) and (35) with

$$\Phi(\overline{u}) = \frac{6}{(\gamma + 1)(\gamma + 2)}\overline{u}^{\gamma+2}, \tag{55}$$

and V is defined by Eq. (53). These analytical predictions are compared with numerical solutions of Eq. (31) in Fig. 2 and very good agreement is observed.

IV. CONCLUSION

We showed that the KdV soliton dynamics along a large-scale background wave can be reduced to Hamilton equations with the use of elementary perturbation theory argumentation. Preservation of the Hamiltonian structure by the dispersionless flow leads to a simple relationship between the inverse half-width of a moving soliton and a local value of the background wave. This relationship can be interpreted as an analytical continuation of the relationship between the carrier wave number of a wave packet propagating along a large-scale background wave which follows from the well-known optical-mechanical analogy where the packet's dynamics is also treated by the Hamilton methods. This type of reasoning first introduced by Stokes allows one to extend the theory to the generalized KdV equation case, and the analytical results are confirmed by comparison with exact numerical solutions.

We believe that our approach based on preservation of Hamiltonian dynamics of both high-frequency wave packets and narrow solitons by dispersionless hydrodynamic flow can be applied to other problems of soliton dynamics.

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ORIGINAL ARTICLE

Vector breathers in the Manakov system

Andrey Gelash[1,2] | **Anton Raskovalov**[1,3,4]

[1]Skolkovo Institute of Science and Technology, Moscow, Russia

[2]Institute of Automation and Electrometry SB RAS, Novosibirsk, Russia

[3]Mikheev Institute of Metal Physics, Ural Branch, RAS, Ekaterinburg, Russia

[4]Institute of Physics and Technology, Ural Federal University, Ekaterinburg, Russia

Correspondence
Andrey Gelash, Skolkovo Institute of Science and Technology, Moscow 121205, Russia.
Email: agelash@gmail.com

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Abstract

We study theoretically the nonlinear interactions of vector breathers propagating on an unstable wavefield background. As a model, we use the two-component extension of the one-dimensional focusing nonlinear Schrödinger equation—the Manakov system. With the dressing method, we generate the multibreather solutions to the Manakov model. As shown previously in [D. Kraus, G. Biondini, and G. Kovačič, Nonlinearity 28(9), 3101, (2015)], the class of vector breathers is presented by three fundamental types I, II, and III. Their interactions produce a broad family of the two-component (polarized) nonlinear wave patterns. First, we demonstrate that the type I and the types II and III correspond to two different branches of the dispersion law of the Manakov system in the presence of the unstable background. Then, we investigate the key interaction scenarios, including collisions of standing and moving breathers and resonance breather transformations. Analysis of the two-breather solution allows us to derive general formulas describing phase and space shifts acquired by breathers in mutual collisions. The found expressions enable us to describe the asymptotic states of the breather interactions and interpret the resonance fusion and decay of breathers as a limiting case of infinite space shift in the case of merging breather eigenvalues. Finally, we demonstrate that only type I breathers participate in the development of modulation

rogue waves has been studied experimentally in a Manakov fiber system.[40,80] At the same time, the experimental observation of vector breathers represents a challenging task for further studies, see also, Ref. 80 where experimental conditions for experimental observation of vector breathers have been discussed.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Nonlinear dynamics of the semi-infinite ferromagnetic samples with an easy-plane anisotropy

V.V. Kiselev [b,c,*], A.A. Raskovalov [a,b,c]

[a] *Skolkovo Institute of Science and Technology (Skoltech), Bolshoy Boulevard 30, bld. 1, Moscow, 121205, Russia*
[b] *M.N. Mikheev Institute of Metal Physics, Ural Branch of the Russian Academy of Sciences, Sofia Kovalevskaya str., 18, Ekaterinburg, 620108, Russia*
[c] *Institute of Physics and Technology, Ural Federal University, Mira str., 19, Ekaterinburg, 620002, Russia*

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ABSTRACT

The modification of the inverse scattering problem is proposed to investigate solitons and dispersive waves in the framework of the Landau–Lifshitz model for semi-infinite ferromagnet with an <<easy-plane>> anisotropy under the mixed boundary conditions, corresponding to different degrees of spin pinning on the edge of the sample. New types of solitons are obtained and their elastic reflection from the edge of the sample is analyzed. Spectral expansions of the integrals of motion for solitons and waves are found. Additional integrals of motion are established that guarantee true boundary conditions for solitons, when they interact with the edge of the sample.

1. Introduction

Ferromagnetic materials attract steady attention and are intensively studied due to the variety of dynamical, thermodynamical, kinetic properties, abundance of structural and phase transitions. Nonlinear properties of ferromagnets are successfully employed in microelectronics and computer engineering, different modulators, multiplicators and elements of logical devices, in diagnostics and control of products. The physics of magnets provides the basis of development and verification for many methods of the modern theory of nonlinear phenomena. It was established that under strong external actions a rich palette of magnetic physical properties is largely determined by the long-lived particle-like states, namely, dynamical and topological solitons. Among the magnetic solitons the most simple are quasi-one-dimensional ones. Their unique properties could be changed and controlled by external fields [1–4] and spin currents [5].

In the last decades, this interest has got a considerable impact, motivated particularly by the increasing availability of new low-dimensional magnetic materials. In the works [6–9] the new methods of autoresonance generation of one-dimensional solitons and driving their properties were found. It was shown that the velocity of solitons motion, as well as their amplitude and energy, could be purposefully controlled by small-amplitude external pumping. This is of a great importance for technological applications. In the works [10,11] a new way to control the quantum bits (qubits) in quantum computers was proposed. It was found that soliton propagating along ferromagnetic chain induces well-localized magnetic field, which can be exploited as a means to manipulate the state of a spin-1/2 localized particle (two-level qubit) that is weakly coupled to the chain.

Investigations of ferromagnetic solitons and their practical applications are largely developed for simplified models, such as scalar sine–Gordon equation [2–4] and generalized nonlinear Shrődinger equation for the envelope of small-amplitude waves [3,6–9,11]. However, the ferromagnetic dynamics is described by the vector Landau–Lifshitz model, which is only approximately could be reduced to the above models. The basis Landau–Lifshitz equations for magnetization are essentially nonlinear and complicated. Therefore, the majority of the works, dedicated to magnetic solitons, gravitate towards the numerical modeling without using the analytical techniques. Micromagnetic simulator mumax3 is widely employed that allows to make corresponding research on the advanced level [12,13]. However, the numerical approach does not show the full picture of the observed phenomena and possibilities of driving solitonic regimes. For this aim the combination of analytical and numerical techniques seems to be more effective.

For ferromagnetic materials the most important are exchange interactions, the energy of crystallographic anisotropy and magnetostatics. Taking these interactions into account, the typical one-soliton states for quasi-one-dimensional ferromagnets were found in [14–16] by direct integration of the Landau–Lifshitz equations. In these works a general concept of the dynamical solitons in the condensed matter physics was

* Corresponding author at: M.N. Mikheev Institute of Metal Physics, Ural Branch of the Russian Academy of Sciences, Sofia Kovalevskaya str., 18, Ekaterinburg, 620108, Russia.

E-mail addresses: kiseliev@imp.uran.ru (V.V. Kiselev), raskovalov@imp.uran.ru (A.A. Raskovalov).

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The only result of reflection of the breather from the boundary of the sample is the shift of the coordinate of the soliton center:

$$z_{+}^{(0)} = l_0 \ln\left|\frac{\kappa}{\tanh\rho \coth\mu}\right|, \qquad z_{-}^{(0)} = l_0 \ln\left|\frac{f(\mu)}{\kappa \tanh\rho \coth\mu}\right|,$$

and change of the initial precession phase:

$$s_{+}^{(0)} = \arg[\tanh\rho \coth\mu\, \kappa^{-1}], \qquad s_{-}^{(0)} = \arg[\kappa \tanh\rho \coth\mu\, f^{-1}(\mu)].$$

The asymptotic formulas (91) coincide with the expressions for the breather of the unbounded medium, which were obtained and analyzed in detail in [17,43,44]. However, in contrast to the case of unbounded medium, the breather on the semiaxis, as well as the rotation wave, is always moving ($\rho \neq 0$, $V \neq 0$).

9. Conclusion

In this work, elastic reflection of solitons from the boundary of an easy-plane ferromagnet is studied by the inverse scattering problem technique in combination with the «image method». The boundary conditions of the problem correspond to the partial spin pinning on the edge of the sample. Let us remind that the exchange interaction and the crystallographic anisotropy of an easy-axis type (the anisotropy axis is parallel to the boundary of the sample) admit the formation of the localized near-boundary solitons with discrete frequencies and specific modulation properties [29,30]. In this work we show that an easy-plane anisotropy (the easy plane is also parallel to the boundary of the sample) excludes the formation of immobile near-boundary solitons.

We predicted the new types of solitons and analyze changes of their dynamical properties depending on the character of spin pinning. We show that the cores of the solitons during their reflection from the surface of the sample undergo strong nonadiabatic deformations, which are accompanied by magnetization changes near the surface of the sample on the value about saturation magnetization. Therefore, such solitons are impossible to describe by the traditional methods of the nonlinear theory for unbounded medium. The obtained peculiarities are typical for the magnetic samples with finite sizes. Their experimental confirmation is of great interest.

It is shown that the magnetization rotation in the center of the particle-like rotation waves after their reflection from the boundary of the sample depends in a threshold way on the amplitude of the surface anisotropy field. It is established that the formation of odd or even number of rotation waves in the sample could be controlled by the character of spin pinning at the boundaries of the sample. The reflection of the breathers from the edge of the sample is accompanied by splash of the specific oscillations and magnetization rotation in the near-surface layer of the sample.

We find the spectral expansions for integrals of motion that allow to treat the strongly excited states of a semi-infinite ferromagnet in terms of an ideal gas of solitons and quasi-particles of the continuous spectrum of spin waves. We obtain additional integrals of motion that guarantee true boundary conditions for solitons under their interaction with the edge of the sample.

The results of the work should be taken into account, when modeling solitonic processes in the samples of finite sizes. Analytical solutions could be useful to verify the numerical calculations.

CRediT authorship contribution statement

V.V. Kiselev: Investigation. **A.A. Raskovalov:** Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Нелинейная динамика полубесконечного ферромагнетика с геликоидальной структурой

© *В.В. Киселев* [1,2], *А.А. Расковалов* [1,2,3,¶]

[1] Институт физики металлов им. М.Н. Михеева УрО РАН,
Екатеринбург, Россия
[2] Уральский федеральный университет им. первого Президента России Б.Н. Ельцина,
Физико-технологический институт (УрФУ),
Екатеринбург, Россия
[3] Сколковский институт науки и технологий,
Москва, Россия
[¶] E-mail: raskovalov@imp.uran.ru

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Для легкоплоскостного ферромагнетика без центра инверсии в рамках модели Ландау–Лифшица найдены и проанализированы новые типы солитонов, встроенных в геликоидальную структуру полубесконечного образца. Учитывались смешанные краевые условия, предельными случаями которых являются свободные и полностью закрепленные спины на границе образца. Все киральные солитоны являются движущимися. Показано, что вблизи поверхности образца их ядра претерпевают сильные деформации, которые сопровождаются перемагничиванием среды. Проанализированы динамические свойства киральных солитонов и особенности их упругого отражения от границы образца в зависимости от характера закрепления краевых спинов.

Ключевые слова: солитоны, уравнение Ландау–Лифшица, волна поворота, легкоплоскостная анизотропия, киральный бризер.

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1. Введение

В последнее десятилетие большое внимание уделяется магнитным материалам, основное состояние которых представляет спиральную структуру. В кристаллах без центра инверсии спиральное упорядочение часто связано с взаимодействием Дзялошинского–Мории, которое теоретически описывается инвариантами Лифшица в разложении свободной энергии [1–4]. Взаимодействие Дзялошинского–Мории конкурирует с обменным взаимодействием, разворачивая спины друг относительно друга на малый угол. Исследованию физических свойств материалов с геликоидальной магнитной структурой посвящено значительное число работ (см., например, [5–11]). Подробный обзор теории одноосных ферромагнетиков с геликоидальным основным состоянием представлен в работе [12].

При включении внешнего магнитного поля перпендикулярно оси геликоидальной структуры магнитная спираль с постоянным шагом превращается в одномерную решетку протяженных доменов. Внутри каждого из них распределение намагниченности почти однородно. Соседние домены разделены узкими доменными стенками — топологическими солитонами, в которых локализуется спиральный поворот намагниченности. Составляющие решетку солитоны ввиду своей мобильности и магниторезистивных свойств перспективны для использования в устройствах спинтроники. Большой интерес представляет исследование движения и устойчивости отдельных доменных стенок и решетки в целом под влиянием электрического тока [13–16].

Спиральное упорядочение реализуется в тяжелых редкоземельных металлах, в большом классе проводящих кубических магнетиков без центра инверсии и ряде других соединений. Среди известных одноосных гелимагнетиков ($CrNb_3S_6$, $CrTaS_6$, CuB_2O_4, $CuCsCl_3$, $Yb(Ni_{1-x}Cu_x)_3Al_9$, $Ba_2CuGe_2O_7$) [17–21] наиболее изучен $CrNb_3S_6$. В нем удалось экспериментально наблюдать решетку киральных солитонов [22].

Киральные мультисолитоны, встроенные в геликоидальную структуру ферромагнетиков, обладают полезными технологическими свойствами [12,23]. Однако их аналитическое описание связано со значительными трудностями из-за нелинейности базовых уравнений теории и по причине неоднородности спирального упорядочения среды. Здесь речь идет об изучении коллективных частицеподобных возбуждений геликоидальной структуры, которая в магнитном поле, ортогональном оси магнитной спирали, сама является существенно нелинейной решеткой из солитонов. В связи с этим имеется мало работ на эту тему. Решение проблемы возможно с привлечением упрощенных моделей, которые корректно учитывают основные взаимодействия и в тоже время допускают точные решения. Одной из таких моделей является популярное квазиодномерное уравнение синус-Гордона. В безграничной среде с однородным основным солитона и энергией геликоидального основного состояния среды без солитона. Корректное вычисление такой энергии — предмет отдельного изучения. Зависимость энергии киральных солитонов от параметров геликоидальной структуры и поверхностной анизотропии следует учитывать, например, при описании термодинамических свойств системы солитонов в полуограниченном образце.

Установлено, что строение киральных волн поворота (7), (18) после отражения от поверхности образца пороговым образом зависит от амплитуды поверхностного поля h. Кроме того, „деформация“ ядра солитона в момент столкновения с поверхностью образца существенно зависит от знака h. Киральные бризеры, в отличие от киральных волн поворота, обладают характерными частотами внутренних пульсаций. Поэтому бризеры можно обнаружить по резонансному поглощению энергии на частотах их колебаний.

Все типы солитонов в геликоидальной структуре являются движущимися частицеподобными объектами. Актуально экспериментальное подтверждение установленных в работе закономерностей их упругого отражения от границы образца.

Столкновения киральных солитонов с поверхностью образца сопровождаются существенным изменением их внутренней структуры и динамических свойств, а также процессами перемагничивания среды на величину порядка намагниченности насыщения. Поэтому киральные солитоны в полуограниченном образце невозможно описать традиционными методами теории возмущений для безграничной среды. Таковая предполагает достаточную „жесткость“ солитонных ядер и малые изменения их свойств под влиянием возмущений.

Результаты работы следует учитывать при моделировании солитонных процессов вблизи поверхностей реальных ферромагнетиков с геликоидальной структурой. Полученные аналитические решения полезны для верификации численных расчетов.

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Undular bore theory for the modified Korteweg–de Vries–Burgers equation

L. F. Calazans de Brito [1] and A. M. Kamchatnov [1,2,3]
[1] *Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia*
[2] *Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow 108840, Russia*
[3] *Skolkovo Institute of Science and Technology, Skolkovo, Moscow 143026, Russia*

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We consider nonlinear wave structures described by the modified Korteweg–de Vries equation, taking into account a small Burgers viscosity for the case of steplike initial conditions. The Whitham modulation equations are derived, which include the small viscosity as a perturbation. It is shown that for a long enough time of evolution, this small perturbation leads to the stabilization of cnoidal bores, and their main characteristics are obtained. The applicability conditions of this approach are discussed. Analytical theory is compared with numerical solutions and good agreement is found.

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I. INTRODUCTION

The modified Korteweg-de Vries (mKdV) equation

$$u_t - 6\alpha u^2 u_x + u_{xxx} = 0 \tag{1}$$

appeared first in the study of the famous KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2}$$

related to Eq. (1) by the Miura transformation [1]. The existence of such a transformation allowed the pioneers of the inverse scattering transform method to discover this method [2–4] for the KdV equation, and it was extended later to many other equations, including the mKdV equation [5,6] (see also, e.g., the books in [7–9] and references therein). The mKdV equation is almost as widely used in physical applications as the KdV equation. Actually, the Gardner equation

$$u_t + 6\beta uu_x - 6\alpha u^2 u_x + u_{xxx} = 0, \tag{3}$$

combining the nonlinear terms of the KdV and mKdV equations, can be transformed into Eq. (1) by a simple change of variables. In addition, in physical applications, it often happens that the coefficient β is very small and can be neglected, so Eq. (3) reduces directly to the equation. The Gardner equation and its simplified mKdV version find applications to the theory of nonlinear waves in stratified fluids, for example, for the description of large-amplitude internal waves [10–12].

One of the most important and universal phenomena in nonlinear physics is the formation and evolution of dispersive shock waves (see, e.g., review articles in [13,14] and references therein). They are called undular bores in water wave physics and they were observed in both surface and internal waves. Their theory was initially developed by Gurevich and Pitaevskii [15], who represented such structures as modulated nonlinear periodic waves governed by the Whitham modulation equations [16,17]. They gave two typical examples of solutions that describe dispersive shock waves: the evolution of an initial discontinuity and the formation of a shock after generic wave breaking for the KdV equation case. The Whitham modulation equations for the mKdV case were derived in Ref. [18], but their application to the theory of dispersive shock waves turned out to be quite a difficult task even in the case of an initial discontinuity problem. The reason for this difficulty is that the mKdV equation is not genuinely nonlinear. (This notion was introduced by Lax in Ref. [19] for hyperbolic systems of first-order partial differential equations and it plays an important role in the classification of wave structures evolving from initial discontinuities in dispersive nonlinear systems; see, e.g., Ref. [20].) This means that in the dispersionless approximation, the nonlinear velocity $6\alpha u^2$ has an extremal (minimal for $\alpha > 0$) value at $u = 0$, whereas in the case of the genuinely nonlinear KdV equation, the nonlinear velocity $6u$ is everywhere a monotonic function of the wave amplitude u. As a result, in the KdV case an initial discontinuity can only evolve into two different structures (rarefaction waves or cnoidal undular bores), whereas in the mKdV case an initial discontinuity evolves into eight different wave structures depending on the parameters of the initial jump of u. Some particular results in this direction were obtained in Ref. [21] and the full solution was given in Ref. [22] in the context of the Gardner equation (3).

In Gurevich-Pitaevskii theory, dispersive shock waves are wave structures that expand with time, so in initial discontinuity-type problems, the change of modulation parameters per unit length decreases with time and can become, at large enough time, smaller than some other physical parameters that were neglected in the derivation of Eq. (1) or (2). For such large values of time, the neglected effects must be taken into account in the modulation theory. For example, small dissipation stops the infinite expansion of undular bores and their length is stabilized at some value inversely proportional to the viscosity coefficient in accordance with the early ideas of Refs. [23,24] about the structure of undular bores in water-wave physics and plasma. The corresponding modified Whitham equations for the KdV theory with weak Burgers dissipation were derived in Refs. [25,26] and were applied in these papers to the description of stationary dispersive shocks whose characteristic length is defined by the small viscosity must be positive. Hence, to realize such a structure the left boundary must satisfy the additional condition

$$u_- > \frac{\gamma}{3\sqrt{\alpha}}. \quad (85)$$

If this condition is not fulfilled, then a combined rarefaction wave matched with a kink is formed (see the discussion of such situations in Ref. [27]).

In region 6 with $u_- < 0$ and $u_+ > 0$ we get a structure with a growing kink, so the intermediate plateau has the amplitude

$$u_* = -u_+ + \frac{\gamma}{3\sqrt{\alpha}} < u_-, \quad (86)$$

and such a structure is realized for

$$u_- < -\frac{\gamma}{3\sqrt{\alpha}}. \quad (87)$$

We compare analytical and numerical solutions for regions 2 and 6 in Figs. 5(c) and 5(d), respectively. Again, quite satisfactory agreement is observed.

It is clear that when u_- reaches the level $u_- = u_*$, the cnoidal bore disappears and the wave structure reduces to a sole kink. After a further increase of u_- we get into region 3 where the left boundary u_- is joined with the plateau u_* by a rarefaction wave (7). Its left edge propagates with velocity $V_{rw}^- = -6\alpha u_-^2$ and its right edge propagates with velocity $V_{rw}^+ = -6\alpha u_*^2$, which must be smaller than the kink's velocity. This gives the condition

$$u_+ < -\frac{2\gamma}{3\sqrt{\alpha}} \quad \text{or} \quad 0 > u_+ > -\frac{\gamma}{6\sqrt{\alpha}} \quad (88)$$

for the realization of such a structure in region 3. A similar structure in the symmetrical region 7 is realized for

$$u_+ > \frac{2\gamma}{3\sqrt{\alpha}} \quad \text{or} \quad 0 < u_+ < \frac{\gamma}{6\sqrt{\alpha}}. \quad (89)$$

As one can see in Figs. 5(e) and 5(f), the analytical theory agrees very well with the numerical solutions for these two regions.

Finally, in regions 4 and 8 the boundary values $u_\pm$ have the same signs, so they are connected by standard rarefaction waves with negligible influence of the Burgers friction [see Figs. 5(g) and 5(h)]. This completes the classification of possible wave structures supported by different boundary conditions in the theory of the mKdVB equation.

VI. WHITHAM EQUATIONS FOR CYLINDRICAL AND SPHERICAL mKdV EQUATIONS

We obtained the Whitham modulation equations in a quite general form (54) where the expression for the perturbation term R in Eq. (33) was not specified. Therefore, this universal form of the Whitham equations can be applied to other problems of the dynamics of mKdV dispersive shock waves. In particular, when we consider cylindrical or spherical dispersive shock waves whose width is much smaller than the radius of the whole wave structure, the curvature of the shock can be treated as a small parameter of the theory and Eq. (54) becomes applicable. Cylindrical or spherical mKdV equations were derived, for example, in Ref. [37] and they can be written in the form

$$u_t - 6\alpha u^2 u + u_{xxx} = -\frac{d}{2(t+t_0)}u, \quad (90)$$

where $d = 1$ or 2 for cylindrical or spherical geometry, respectively. For a large enough time of evolution $t_0 \gg 1$ the perturbative right-hand side term is small, so the dispersive shock wave solutions to this equation can be approximated by periodic solutions of the standard mKdV equation with slowly changing parameters, whose evolution is governed by Eq. (54) with

$$\langle R\rangle = -\frac{d}{2(t+t_0)}k\oint\frac{udu}{\sqrt{f(u)}},$$
$$\langle uR\rangle = -\frac{d}{2(t+t_0)}k\oint\frac{u^2du}{\sqrt{f(u)}}. \quad (91)$$

The integrals here can be expressed in terms of standard Jacobi elliptic integrals of first, second, and third kinds, so we arrive quite easily at the Whitham equations derived earlier by different methods in Ref. [38] for cylindrical cases and in Ref. [39] for spherical cases, respectively. Thus, the Whitham equations (54) can find various applications besides consideration of the effects of the small viscosity.

VII. CONCLUSION

The above theory confirms the general statement that weak dissipative effects stabilize the expanding evolution of dispersive shock waves, so after a long enough time, they converge to stationary structures characterized by some finite length, which is inversely proportional to the viscosity coefficient. The appearance of the new parameter leads to some limitations for the applicability of the Whitham method used in the Gurevich-Pitaevskii approach to description of bores. In particular, the condition that the size of the whole shock is much greater than the typical wavelength inside the shock demands that the jump between the boundary conditions is large enough. Since the mKdV equation is not genuinely nonlinear, we get combined wave structures consisting of a kink and a cnoidal bore or a rarefaction wave. Small viscosity leads to modification of the kink solution found in Ref. [31] and the condition that the two structural elements of a combined structure propagate separately from each other also leads to some limitations for boundary conditions. Although in the case of small viscosity these restrictions are not essential, one should keep in mind their existence in the practical application of the theory.

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Quasiclassical integrability condition in AKNS scheme

A.M. Kamchatnov *, D.V. Shaykin

Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 108840, Russia
Moscow Institute of Physics and Technology, Institutsky lane 9, Dolgoprudny, Moscow region, 141700, Russia
Skolkovo Institute of Science and Technology, Skolkovo, Moscow, 143026, Russia

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ABSTRACT

In this paper, we study the condition of quasiclassical integrability of soliton equations. This condition states that the Hamiltonian structure of equations, which govern propagation of high-frequency wave packets, is preserved by the dispersionless flow independently of initial data. If this condition is fulfilled, then the carrier wave number of any packet is a certain function of the local values of the dispersionless variables pertained to the soliton equation under consideration. We show by several examples that this function together with the dispersion relation for linear harmonic waves determine the quasiclassical limit of the Lax pair functions in the scalar representation of the Ablowitz–Kaup–Newell–Segur scheme.

1. Introduction

The general definition of the "integrability" property of nonlinear wave equations is apparently impossible (see, e.g., discussion in Ref. [1]). In a narrow sense, integrability is often related with existence of the Lax pair of linear systems whose compatibility condition leads to the nonlinear wave equations under consideration. First discovered for the Korteweg–de Vries (KdV) equation [2,3] and nonlinear Schrödinger (NLS) equation [4], Lax pairs were found for a number of wave equations and they were widely used in various physical applications (see, e.g., Refs. [5–7] and references therein). However, the relationship between the physical properties of wave equations and associated with them Lax pairs still remains unclear.

As was noticed in Ref. [8] in the theory of the generalized NLS equation

$$i\psi_t + \frac{1}{2}\psi_{xx} - f(|\psi|^2)\psi = 0, \tag{1}$$

the integrable case with $f(|\psi|^2) = |\psi|^2$ is distinguished by a very special property of propagation of high-frequency wave packets along large scale background waves, and this property can be formulated in purely physical characteristics of Eq. (1). In the problem of packets propagation, we have two very different characteristic length parameters: a small wavelength $\sim k^{-1}$ of harmonics that compose the packet (k is the carrier wave number) and a size $\sim l$ of the background wave, so $(kl)^{-1} \ll 1$ is a small parameter of the theory. Its existence allows one to separate dispersionless evolution of the background wave from propagation of the linear wave packet. The dispersionless evolution is described by the hydrodynamic equations

$$\rho_t + (\rho u)_x = 0, \quad u_t + uu_x + \frac{c^2}{\rho}\rho_x = 0, \tag{2}$$

where ρ and u are defined according to the formula

$$\psi(x,t) = \sqrt{\rho(x,t)}\exp\left(i\int^x u(x',t)dx'\right) \tag{3}$$

and

$$c^2 = \rho f'(\rho). \tag{4}$$

It is worth noticing that substitution (3) allows one to separate fast oscillations of the phase of the ψ-variable for large u from slow changes of u, $u_x \ll 1$, in the hydrodynamic approximation (2). A small-amplitude wave packet propagates along smooth distributions $\rho = \rho(x,t), u = u(x,t)$, given by some specific solution of Eq. (2), according to the Bogoliubov dispersion relation

$$\omega = k\left(u \pm \sqrt{c^2 + \frac{k^2}{4}}\right), \tag{5}$$

where due to the condition $(kl)^{-1} \ll 1$ one can consider u and $c = c(\rho)$ constant within the packet's width. This means that the packet is considered as a point-like particle with coordinate $x = x(t)$ whose motion obeys the Hamilton equations (see, e.g., [9,10])

$$\frac{dx}{dt} = \frac{\partial\omega}{\partial k}, \qquad \frac{dk}{dt} = -\frac{\partial\omega}{\partial x}. \tag{6}$$

As we showed in Ref. [8], the system (2), (6) admits the solution in the form $k = k(\rho, u)$, where the wave number depends on (x,t)-coordinates

* Corresponding author at: Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 108840, Russia.
E-mail addresses: kamch@isan.troitsk.ru (A.M. Kamchatnov), shaykin.dv@phystech.edu (D.V. Shaykin).

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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**ТЕОРЕТИЧЕСКАЯ
И МАТЕМАТИЧЕСКАЯ
ФИЗИКА
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А. М. Камчатнов[*†]

ГАМИЛЬТОНОВА ТЕОРИЯ ДВИЖЕНИЯ ТЕМНЫХ СОЛИТОНОВ В ТЕОРИИ НЕЛИНЕЙНОГО УРАВНЕНИЯ ШРЕДИНГЕРА

Развит метод вывода уравнений Гамильтона, описывающих динамику солитонов при их движении по неоднородному и изменяющемуся со временем крупномасштабному фону для нелинейных волновых уравнений, полностью интегрируемых в схеме Абловица–Каупа–Ньюэлла–Сигура. Метод основан на развитии старых соображений Стокса, позволяющих продолжать аналитически соотношения для линейных волн в солитонную область, и реализован практически на примере дефокусирующего нелинейного уравнения Шредингера. Сформулировано условие, при котором учет внешнего потенциала необходим только при описании эволюции фона, и для этого случая получено уравнение Ньютона для динамики солитона с учетом внешнего потенциала.

Ключевые слова: солитоны, нелинейное уравнение Шредингера, теория возмущений.

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1. ВВЕДЕНИЕ

Солитоны, т. е. распространяющиеся без изменения формы локализованные волны, часто уподобляются частицам, особенно если их динамика описывается полностью интегрируемыми уравнениями, поскольку в этом случае они взаимодействуют друг с другом упруго (см., например, [1]–[3]). Такая аналогия между солитонами и частицами сохраняет смысл и при движении солитонов во внешних полях или по слабо неоднородному и медленно меняющемуся фону, когда с достаточной точностью можно определить зависящую от времени координату солитона $x = x(t)$. Тогда во многих случаях динамика переменной $x = x(t)$ совпадает с динамикой точечной частицы, если пренебречь малыми поправками порядка отношения ширины солитона к характерному расстоянию, на котором изменяется внешнее поле или фон

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[*]Институт спектроскопии Российской академии наук, Москва, Троицк, Россия.
E-mail: kamchatnov@gmail.com

[†]Сколковский институт науки и технологии, Сколково, Московская обл., Россия

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Propagation of dark solitons of DNLS equations along a large-scale background

A.M. Kamchatnov *, D.V. Shaykin

Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 108840, Russia
Moscow Institute of Physics and Technology, Institutsky lane 9, Dolgoprudny, Moscow region, 141700, Russia
Skolkovo Institute of Science and Technology, Skolkovo, Moscow, 143026, Russia

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ABSTRACT

We study dynamics of dark solitons in the theory of the derivative nonlinear Schrödinger equations by the method based on imposing the condition that this dynamics must be Hamiltonian. Combining this condition with Stokes' remark that relationships for harmonic linear waves and small-amplitude soliton tails satisfy the same linearized equations, so the corresponding solutions can be converted one into the other by replacement of the packet's wave number k by $i\kappa$, κ being the soliton's inverse half-width, we find the Hamiltonian and the canonical momentum of the soliton's motion. The Hamilton equations are reduced to the Newton equation whose solutions for some typical situations are compared with exact numerical solutions of the Kaup-Newell DNLS equation.

Dedicated to the memory of Noel Smyth

1. Introduction

In situations when the soliton's width is much smaller than the typical length of the background wave along which the soliton propagates, one can introduce with good enough accuracy the soliton's coordinate $x(t)$ and describe its propagation as a motion of a point-like particle through a non-uniform and varying with time surrounding. In this case, evolution of the background wave is governed by the equations of dispersionless (hydrodynamic) approximation independently of the soliton's motion. However, the soliton's motion cannot be separated from the background wave evolution: this motion causes a counterflow around the soliton and such a counterflow changes drastically the soliton's dynamics. Well-known examples of this back reaction on the soliton's motion are the formation of shelves behind Korteweg-de Vries (KdV) solitons propagating along shallow water with uneven bottom (see, e.g., [1–4]) and change of the frequency of oscillations of a dark soliton in a Bose–Einstein condensate confined in a harmonic trap (see, e.g., [5,6]).

So far, the counterflow effects were studied by different forms of perturbation analysis (see, e.g., the references above). If the wave dynamics is described by a completely integrable equation for which the Whitham modulation equations are already known, then the soliton limit of the Whitham equations provides a convenient tool for investigation of soliton's motion along varying background [7–10]. However, this approach cannot be applied to perturbed integrable equations, and in this case one has to use more general analytical or numerical methods not limited to integrable equations; see, e.g., [11–13] and references therein.

It was recently noticed [14], that if the nonlinear wave equation under consideration can be written in a Hamiltonian form and one assumes that the reduction of the wave evolution to the soliton's motion along the large-scale background wave remains

* Corresponding author at: Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 108840, Russia.
E-mail addresses: kamch@isan.troitsk.ru (A.M. Kamchatnov), shaykin.dv@phystech.edu (D.V. Shaykin).

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These formulas can be obtained from the corresponding formulas for the DNLS-I case by means of the replacements

$$u_I = u_{III} + \rho_{III}, \qquad \rho_I = \frac{1}{2}\rho_{III}, \tag{81}$$

which are accompanied by the change of the time variable

$$t_I = 2t_{III}. \tag{82}$$

The same transformations yield the canonical momentum and the Hamiltonian for the Gerdjikov–Ivanov soliton dynamics.

These results show that at the level of equations for the soliton motion, the gauge equivalence of different DNLS equations reduces to simple linear transformations of the wave variables for the background flows. It is worth noticing that Eqs. (44), (73), (80) can be derived from the soliton limit of the Whitham equations [7–10], however their extension to situations when background flows are affected by external forces needs a more detailed theory for Hamilton dynamics of soliton's motion and transition to Newton equation, so that the influence of the external potentials can be taken into account.

7. Conclusion

The problem of soliton's motion along non-uniform and time-dependent background, especially in presence of external forces, is very difficult because of back reaction of the counterflow caused by a moving soliton in the large-scale background wave. Previously, different versions of the perturbation theory were developed for solving this problem, but they were quite complicated and therefore applied to a very limited number of equations (mainly, KdV and NLS equations). We suggested in Ref. [14] a new approach. Although it is basically also perturbative, many difficulties are removed by imposing the condition that the equations of soliton's motion must be Hamiltonian. Combining this condition with Stokes' remark that some relationships for harmonic linear wave can be converted into the relationships for solitons by replacement of the packet's wave number k by $i\kappa$, κ being the soliton's inverse half-width, we reduce derivation of the Hamiltonian and the canonical momentum of the soliton's motion to a straightforward calculation. The resulting Hamilton equations can be transformed to the Newton equation where the role of the external potential can be taken into account by its inclusion into the equations of the dispersionless (hydrodynamic) flow. The effectiveness of this method was demonstrated in Ref. [18] for the NLS dark soliton, where the results of Ref. [41] were easily reproduced. In the present paper, we applied this method to solitons described by DNLS equations with quite a nontrivial dynamics without reflection symmetry. Validity of our approach is confirmed by its good agreement with exact numerical solutions of the DNLS equation. In this paper, we have considered situations when the equations of asymptotic integrability (39), (70), (77) have exact solutions. If the equation under consideration is not completely integrable, such an exact solution may not exist, nevertheless an approximate solution for large values of the wave number k can still exist, and this is enough for finding an approximate expression for the inverse half-width of narrow solitons. In this case our theory remains applicable, as it was shown for the generalized KdV equation in Ref. [14]. Therefore we believe that this approach can find many other applications.

CRediT authorship contribution statement

A.M. Kamchatnov: Writing – review & editing, Writing – original draft, Supervision, Investigation, Funding acquisition, Formal analysis, Conceptualization. **D.V. Shaykin:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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SPECIAL ISSUE ARTICLE

APPLIED MATHEMATICS WILEY

Advances in the Mathematics and Physics of Solitons in Memory of David Kaup

Hamiltonian mechanics of "magnetic" solitons in two-component Bose–Einstein condensates

A. M. Kamchatnov[1,2]

[1]Russian Academy of Sciences, Institute of Spectroscopy, Troitsk, Moscow, Russia
[2]Skolkovo Institute of Science and Technology, Skolkovo, Moscow, Russia

Correspondence
A. M. Kamchatnov, Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 108840, Russia.
Email: kamchatnov@gmail.com

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Abstract
We consider the motion of a "magnetic" soliton in two-component condensates along a nonuniform and time-dependent background in the framework of Hamiltonian mechanics. Our approach is based on generalization of Stokes' remark that soliton's velocity is related to its inverse half-width by the dispersion law for linear waves continued to the region of complex wave numbers. We obtain expressions for the canonical momentum and the Hamiltonian as functions of soliton's velocity and transform the Hamilton equations to a Newton-like equation. The theory is illustrated by several examples of concrete soliton's dynamics.

KEYWORDS
Bose–Einstein condensates, Hamiltonian mechanics, solitons

1 | INTRODUCTION

The idea that solitons behave under the action of smooth enough external forces similar to point-like particles of classical mechanics is well known. In particular, a number of perturbation methods are based on this qualitative picture[1–4] (see also Refs. [5, 6] and references therein), and these theories are confirmed by numerical and real experiments. The standard perturbation theory is quite involved in case of dark solitons propagating along a nonuniform background[7–12] when separation of the soliton's dynamics from the dynamics of the surrounding background is not obvious: a moving soliton generates a "shelf" in the background, and this leads to an essential back reaction to the soliton's motion. For example, the frequency of oscillations of a dark soliton in a Bose–Einstein condensate (BEC) confined in a harmonic trap with the frequency ω_0 is equal to

One may hope that the suggested method can lead to many other interesting results.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Asymptotic integrability of nonlinear wave equations

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A. M. Kamchatnov[1,2,a)]

AFFILIATIONS

[1]Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow 108840, Russia
[2]Skolkovo Institute of Science and Technology, Skolkovo, Moscow 143026, Russia

[a)]Author to whom correspondence should be addressed: kamch@isan.troitsk.ru

ABSTRACT

We introduce the notion of asymptotic integrability into the theory of nonlinear wave equations. It means that the Hamiltonian structure of equations describing propagation of high-frequency wave packets is preserved by hydrodynamic evolution of the large-scale background wave so that these equations have an additional integral of motion. This condition is expressed mathematically as a system of equations for the carrier wave number as a function of the background variables. We show that a solution of this system for a given dispersion relation of linear waves is related to the quasiclassical limit of the Lax pair for the completely integrable equation having the corresponding dispersionless and linear dispersive behavior. We illustrate the theory with several examples.

At first sight, it seems that the physical explanation of solitonic propagation as a manifestation of a subtle balance of dispersive and nonlinear effects, on one side, and the mathematical idea of a complete integrability of nonlinear wave equations, on the other side, are not related to each other. We show in this paper that a combined consideration of two limiting situations—the dispersionless evolution of large-scale waves and the propagation of high-frequency wave packets according to Hamilton's optical-mechanical analogy—sheds new light on the complete integrability condition. As a result, we formulate a much weaker condition of asymptotic integrability, which means that the Hamiltonian structure of equations for the packet's propagation is preserved by the dispersionless evolution of the background wave, providing, thus, an additional integral of motion. If this criterion is fulfilled, then we arrive at the quasiclassical limit of the Lax pair for some completely integrable equation. This theory is illustrated by several examples.

I. INTRODUCTION

As was explained by classics of physics in the 19th century,[1–4] the phenomenon of solitary wave (or solitonic) propagation is a result of subtle balance of nonlinear and dispersive effects of wave motion. Independently, the mathematicians of the 19th century developed the idea of complete integrability of equations of Newton dynamics (see, e.g., Ref. 5 and references therein), and this idea was based on some specific properties of Hamilton's formulation of classical mechanics. Apparently, these two fundamental ideas had not been related to each other until the discovery of the inverse scattering transform method (IST) of integration of nonlinear wave equations.[6–8] Soon after this discovery, it was realized[9,10] that the nonlinear wave equations solvable by the IST method possess the property of complete integrability generalized to systems with an infinite number of degrees of freedom. Since then, the Hamiltonian approach to nonlinear wave equations has become an important tool for their investigation and, as a result, the soliton physics has become a well-developed part of modern mathematical physics (see, e.g., Refs. 11 and 12 and references therein). At the same time, it seems that the original physical idea of interplay of nonlinear and dispersive effects has not played any essential role in this formal development. The aim of this article is to demonstrate that at least in some situations, elaboration of this classical idea leads to interesting results useful for the nonlinear wave physics.

II. ASYMPTOTIC INTEGRABILITY

We suppose that our physical system is described by two variables, which we will call for definiteness the "density" ρ and the "flow velocity" u. In one-dimensional geometry, they depend on the space

Transition to the Lax pairs according to the rules (10) yields

$$\mathscr{A} = -q + u - \frac{\rho}{q}, \quad \mathscr{B} = -4q - 2u, \tag{93}$$

where q plays the role of the spectral parameter. The resulting completely integrable system is the well-known Zakharov–Ito system[46,47]

$$\begin{aligned} \rho_t + 2u\rho_x + 4\rho u_x &= 0, \\ u_t + 6uu_x - 4\rho_x - u_{xxx} &= 0. \end{aligned} \tag{94}$$

It is worth noticing that although the dispersionless system with the velocities (87) written in (ρ, u)-variables

$$\begin{aligned} \rho_t + 2u\rho_x + 4\rho u_x &= 0, \\ u_t + 6uu_x - 4\rho_x &= 0 \end{aligned} \tag{95}$$

can be transformed to the shallow water system (14) for the variables

$$\overline{\rho} = 4u^2 - 16\rho, \quad \overline{u} = 4u, \tag{96}$$

these substitutions are not compatible with transformation of the dispersion relation (15). In fact, they cast the dispersion relation (88) to the form

$$\omega = k\left(4u + \frac{k^2}{2} \pm \sqrt{\left(2u + \frac{k^2}{2}\right)^2 - 16\rho}\right) \tag{97}$$

different from relation (15) for the Kaup–Boussinesq system.

VIII. CONCLUSION

In this paper, we introduced the physically natural condition of asymptotic integrability of nonlinear wave equations, which means that two asymptotic limits—dispersionless (hydrodynamic) evolution of a smooth background wave and propagation of high-frequency wave packets—are compatible with each other in the sense that the Hamiltonian structure of the packet's propagation is preserved by the evolution of the background wave and, consequently, the Hamilton equations have an integral of motion. In fact, this weaker integrability condition leads to the quasiclassical limit of complete integrability in the framework of the AKNS scheme,[20] and if the Lax pair of the equations under consideration does not depend on the space derivatives of the wave variables, then the asymptotic integrability condition reproduces the exact Lax pair. This observation sheds new light on the origin of the completely integrable nonlinear wave equations.

In addition to that, it should be noted that the relationship for the dependence of the packet's wave number on the background variables can be converted to a similar relationship for the inverse half-width of narrow solitons propagating along a smooth background, and this remark allows one to develop a Hamiltonian theory for the propagation of narrow solitons along a non-uniform and time-dependent background.[48–50]

At last, Eq. (5) of asymptotic integrability can also have approximate solutions correct in the limit of large wave numbers k even for not completely integrable equations. In these quite general situations, one can develop the theory of propagation of small-amplitude wave packets along a non-uniform and time-dependent background[18] and to formulate a generalized Bohr–Sommerfeld quantization rule, which determines the parameters of solitons produced from an intensive initial pulse.[19] Thus, the condition of asymptotic integrability turns out to be a useful tool for investigation of various problems in nonlinear physics.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

A. M. Kamchatnov: Conceptualization (equal); Formal analysis (equal); Writing – original draft (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the author upon reasonable request.

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Perspective

Soliton gas: Theory, numerics, and experiments

Pierre Suret,[1,*] Stephane Randoux,[1] Andrey Gelash,[2,†] Dmitry Agafontsev,[3,4] Benjamin Doyon,[5] and Gennady El[6]
[1]Univ. Lille, CNRS, UMR 8523, PhLAM – Physique des Lasers, Atomes et Molécules, F-59000 Lille, France
[2]Laboratoire Interdisciplinaire Carnot de Bourgogne (ICB), UMR 6303 CNRS-Université Bourgogne Franche-Comté, 21078 Dijon, France
[3]Shirshov Institute of Oceanology of RAS, Nakhimovskiy prosp. 36, Moscow, 117997, Russia
[4]Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, Moscow, 121205, Russia
[5]Department of Mathematics, King's College London, Strand WC2R 2LS, London, United Kingdom
[6]Department of Mathematics, Physics and Electrical Engineering, Northumbria University, Newcastle upon Tyne, NE1 8ST, United Kingdom

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The concept of soliton gas was introduced in 1971 by Zakharov as an infinite collection of weakly interacting solitons in the framework of Korteweg–de Vries (KdV) equation. In this theoretical construction of a diluted (rarefied) soliton gas, solitons with random amplitude and phase parameters are almost nonoverlapping. More recently, the concept has been extended to dense gases in which solitons strongly and continuously interact. The notion of soliton gas is inherently associated with integrable wave systems described by nonlinear partial differential equations like the KdV equation or the one-dimensional nonlinear Schrödinger equation that can be solved using the inverse scattering transform. Over the last few years, the field of soliton gases has received a rapidly growing interest from both the theoretical and experimental points of view. In particular, it has been realized that the soliton gas dynamics underlies some fundamental nonlinear wave phenomena such as spontaneous modulation instability and the formation of rogue waves. The recently discovered deep connections of soliton gas theory with generalized hydrodynamics have broadened the field and opened new fundamental questions related to the soliton gas statistics and thermodynamics. We review the main recent theoretical and experimental results in the field of soliton gas. The key conceptual tools of the field, such as the inverse scattering transform, the thermodynamic limit of finite-gap potentials, and generalized Gibbs ensembles are introduced and various open questions and future challenges are discussed.

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I. INTRODUCTION

Random nonlinear waves in dispersive media have been the subject of intense research in nonlinear physics for more than half a century, most notably in the contexts of water wave dynamics and nonlinear optics. A significant portion of the work in this area has been centered around wave turbulence—the theory of out-of-equilibrium random weakly nonlinear dispersive waves in nonintegrable systems [1,2]. One of the most important results of the wave turbulence theory is the analytical determination in Ref. [3] of the power-law Fourier spectra analogous to the Kolmogorov spectra describing energy flux through scales in dissipative hydrodynamic turbulence.

More recently, a new theme in turbulence theory has emerged in connection with the dynamics of strongly nonlinear random waves described by integrable systems such as the Korteweg–de Vries (KdV) and one-dimensional (1D) nonlinear Schrödinger (NLS) equations. This kind of random wave motion in nonlinear conservative systems, dubbed *integrable turbulence* [4], has attracted significant attention from both the fundamental and applied perspectives. The interest in integrable turbulence is motivated by the inherent randomness of many real-life systems (due to random initial and boundary conditions or to complex interaction mechanisms) even though the underlying physical models may be amenable to the well-established mathematical techniques of integrable systems theory such as the inverse scattering transform or finite-gap theory [5,6].

The integrable turbulence framework is particularly pertinent to the description of modulationally unstable systems which can exhibit highly complex nonlinear behaviors that can be adequately described in terms of the turbulence theory concepts such as probability distribution functions, ensemble averages, Fourier spectra, etc. [7–12]. We stress that the term "turbulence" in this context is understood as complex spatiotemporal dynamics that require a probabilistic description and are not related to the energy cascades through scales, the prime feature of strong hydrodynamic and weak wave turbulence.

The main tool for the analysis of integrable nonlinear dispersive partial differential equations (PDEs) is the inverse scattering transform (IST) [13] which is based on the reformulation of a nonlinear PDE as a compatibility condition of two *linear* problems (the so-called Lax pair): a stationary spectral (scattering) problem and an evolution problem—for the same auxiliary function. Within the classical IST setting formulated for the wave fields decaying sufficiently rapidly as $|x| \to \infty$, the scattering spectrum consists of two components: discrete and continuous, corresponding to two contrasting types of the wave motion: solitary waves (solitons) and dispersive

*Corresponding author: Pierre.Suret@univ-lille.fr
†Present address: Institute of Physics, Swiss Federal Institute of Technology Lausanne (EPFL), CH-1015 Lausanne, Switzerland.

situation was worked out [200], including time dependence, and varying the coupling strength $c \to c(x,t)$ in (78), something which is crucial for comparison with some experiments. External force fields and slowly varying couplings also naturally occur in many situations where soliton gases emerge. The theory from GHD is in principle fully applicable to soliton gases; however, again up to now, the application to soliton gases and the IST perspective on such GHD results are still completely missing.

Hydrodynamics is a derivative expansion, and as such, one may wonder about the higher-derivative corrections. At second derivative, this is the *diffusive correction*, such as the viscosity term in Navier-Stokes equations. Again, an exact expression of the diffusive matrix—or diffusive operator on spectral space—has been evaluated in GHD with convincing comparisons against numerical results, see the review [179]. The form obtained is

$$\frac{\partial}{\partial t}\rho_{\mathrm{p}}(\eta;x,t)+\frac{\partial}{\partial x}[v^{\mathrm{eff}}(\eta;,x,t)\rho_{\mathrm{p}}(\eta;x,t)] = \frac{1}{2}\frac{\partial}{\partial x}\left(\int d\eta'\mathcal{D}_{\eta,\eta'}[\rho_{\mathrm{p}}(\cdot;x,t)]\frac{\partial}{\partial x}\rho_{\mathrm{p}}(\eta';x,t)\right). \quad (93)$$

The diffusion kernel $\mathcal{D}_{\eta,\eta'}[\rho_{\mathrm{p}}]$ is evaluated from the Kubo formula involving space-time integrated current two-point functions, using form factor methods of quantum integrability [184,201]. The general formula, applicable to quantum and classical models alike, is conjectured by comparison with the diffusion kernel obtained in the 1980s for the classical hard-rod gas [202]. Again, the general formula involves the statistical factor $f(\varepsilon)$. The combination of diffusion with external forces has also been evaluated [203]. The third-order, *dispersive* correction was proposed recently [204], although much work is still needed to fully establish it.

Is there diffusion in soliton gases? If so, is it correctly described by the GHD formula? Furthermore, can we evaluate the exact third-order dispersion term? A natural conjecture concerns the condensate limit; in the GHD of quantum integrable models, the condensate limit had been studied earlier, and is known as zero-entropy GHD [205]. The connection between soliton-gas condensate limit and zero-entropy GHD was partially made in Ref. [30]. Do dispersive terms of GHD (soliton gases) reproduce, in the zero-entropy (condensate) limit, dispersive terms of the fundamental dynamical equations (e.g., the KdV equation)?

Finally, the effects of small perturbations that break integrability has been studied. The development is still in its infancy, with various approaches and different physical situations proposed, see the review [206]. The perspective taken in GHD is different from that taken in soliton gases, and it would be fruitful to make a better connection. One important point that has been emphasized [207] generalizes the viewpoint discussed above, whereby the Liouville equation—the kinetic equation for free particles—is seen as a Euler-scale hydrodynamic equation. It is possible to modify the Euler-scale hydrodynamic equation to account for terms that break the conservation laws on which it is based. There are general Kubo-like formulas this modification, and when applied to GHD, these give terms that can be written, at least in quantum models, in a form-factor expansion. Specialized to the GHD of free particles, these terms are nothing else but Boltzmann collision terms from the Boltzmann equation; form factors of interacting integrable models generalize Boltzmann collision terms. Is there a parallel notion of form factors that can be used to evaluate Boltzmann collision terms in soliton gases? Thus, again, we obtain a different viewpoint: the Boltzmann equation, a kinetic equation, is re-interpreted as a hydrodynamic equation, with terms that break the infinitely many conservation laws admitted by free particles. This reinterpretation has, potentially, far-reaching consequences, which still need to be addressed.

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STUDIES IN APPLIED MATHEMATICS WILEY

ORIGINAL ARTICLE

Multisoliton interactions approximating the dynamics of breather solutions

Dmitry Agafontsev[1,2] | **Andrey Gelash**[3] | **Stephane Randoux**[4] | **Pierre Suret**[4]

[1]Shirshov Institute of Oceanology of RAS, Moscow, Russia
[2]Department of Mathematics, Physics and Electrical Engineering, Northumbria University, Newcastle upon Tyne, UK
[3]Laboratoire Interdisciplinaire Carnot de Bourgogne (ICB), Université Bourgogne Franche-Comté, Dijon, France
[4]PhLAM – Physique des Lasers Atomes et Molécules, Université de Lille, Lille, France

Correspondence
Dmitry Agafontsev, Shirshov Institute of Oceanology of RAS, Moscow, Russia.
Email: Dmitry.Agafontsev@gmail.com

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Abstract
Nowadays, breather solutions are generally accepted models of rogue waves. However, breathers exist on a finite background and therefore are not localized, while wavefields in nature can generally be considered as localized due to the limited sizes of physical domain. Hence, the theory of rogue waves needs to be supplemented with localized solutions, which evolve locally as breathers. In this paper, we present a universal method for constructing such solutions from exact multisoliton solutions, which consists in replacing the plane wave in the dressing construction of the breathers with a specific exact N-soliton solution converging asymptotically to the plane wave at large number of solitons N. On the example of the Peregrine, Akhmediev, Kuznetsov–Ma, and Tajiri–Watanabe breathers, we show that constructed with our method multisoliton solutions, being localized in space with characteristic width proportional to N, are practically indistinguishable from the breathers in a wide region of space and time at large N. Our method makes it possible to build solitonic models with the same dynamical properties for the higher

of the plane wave and simply changed its norming constants to $C_n = 1, 1, -1, 1, \dots, -1, 1$ (to the same ones that would be for the 8-soliton model of the Peregrine breather in Section 3), *without the subsequent dressing procedure*. Thus, the only difference between this modified construction and what was discussed earlier is the inaccuracy in the eigenvalues. Despite this inaccuracy, the constructed solitonic model still shows a very good correspondence with the breather. We have repeated this construction for the Akhmediev and higher-order rational breathers using both the semiclassical and exact eigenvalues, and came to the similar results.

Note that according to Equations (13) and (14), the soliton positions and phases evolve linearly with time,

$$x_n(t) = x_{n0} - 2\xi_n t, \tag{36}$$

$$\theta_n(t) = \theta_{n0} + 2\left(\xi_n^2 + \eta_n^2\right)t, \tag{37}$$

where x_{n0} and θ_{n0} are the positions and phases at $t = 0$. In the general case, the frequencies $2(\xi_n^2 + \eta_n^2)$ are incommensurable and may lead to spontaneous synchronization of soliton norming constants and the appearance of rogue wave. For instance, we can let the solitonic model of the plane wave (31)–(32) evolve, and at various moments of time it will turn into a very good approximation of the Akhmediev, Peregrine, and high-order rational breathers.

We believe that the more general synchronization conditions for the soliton norming constants can be found, that will correspond to the emergence of rogue waves surrounded by chaotic perturbations of the wavefield. These rogue waves will appear spontaneously from time to time due to the spontaneous synchronization of soliton norming constants from one synchronization condition to another during the evolution in time. Finding these conditions represents a challenging problem for future studies.

Finally, our solitonic models may prove useful in explaining the process of soliton fission in nonintegrable systems described at leading order with the focusing 1D-NLSE. Indeed, if the initial wavefield can be approximated with an exact multisoliton solution, in which most of the solitons are in a bound state, then the influence of nonintegrable perturbations is expected to gradually destroy this state by changing velocities differently for different solitons, thus leading to the fission. Note that soliton fission plays an important role in the supercontinuum generation[87] and formation of optical rogue waves[88]; very recently it has been observed developing from the Peregrine and higher order rational breathers.[37,38]

AUTHOR CONTRIBUTIONS

Dmitry Agafontsev: Conceptualization; formal analysis; investigation; methodology; project administration; software; supervision; writing—original draft preparation; writing—review and editing. **Andrey Gelash**: Conceptualization; formal analysis; investigation; methodology; software; writing—original draft preparation; writing—review and editing. **Stephane Randoux**: Conceptualization; writing—review and editing. **Pierre Suret**: Conceptualization; writing—review and editing.

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CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Dmitry Agafontsev https://orcid.org/0000-0002-8148-0362
Stephane Randoux https://orcid.org/0000-0001-9309-6539

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Натурные измерения формы морской поверхности и одномерного пространственного спектра волнения

В. В. Стерлядкин[1], **К. В. Куликовский**[1], **С. И. Бадулин**[2,3]

[1] *МИРЭА — Российский технологический университет, Москва, 119454, Россия*
E-mail: sterlyadkin@mail.ru

[2] *Институт океанологии им. П. П. Ширшова РАН, Москва, 117997, Россия*
E-mail: badulin.si@octan.ru

[3] *Сколковский институт науки и технологий, Москва, 121205, Россия*

Представлены результаты натурных измерений профилей морской поверхности, полученных с помощью сканирующего лазерного волнографа вдоль линии сканирования протяжённостью 1700 мм. Отмечается важность регистрации формы поверхности в задачах рассеяния электромагнитного излучения свободной поверхностью раздела. Получены частотные спектры волнения, профилограммы «время – высота», пространственные спектры, включающие капиллярные компоненты волнения. Обсуждается возможность измерения фазовых скоростей различных пространственных компонент волнения, которые можно выделять с помощью частотной фильтрации множества исходных профилей. Отмечается негауссовость распределения возвышений поверхности на основе анализа высших моментов распределений: skewness — коэффициента асимметрии вверх-вниз и коэффициента асимметрии вперёд-назад. Приводятся результаты расчёта пространственного спектра морской поверхности.

Ключевые слова: форма морской поверхности, частотный спектр, пространственный спектр, капиллярные волны, сканирующий лазерный волнограф

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Введение

Форма водной поверхности и характеристики морского волнения являются ключевыми при всех дистанционных измерениях параметров верхнего слоя океана. Форма поверхности определяет отражение радио-, оптического и акустического сигналов, а также отвечает за собственное излучение моря. По этой причине понимание физических механизмов, связывающих форму поверхности и её радиофизические свойства, исключительно важны для дистанционных методов исследования и мониторинга морской поверхности. Спектральное описание широко используется как характеристика водной поверхности (Banner, 1990; Komen et al., 1994). Относительно легко могут быть получены частотные (временные) спектры путём измерения высоты поверхности в единственной точке пространства. Однако для задач дистанционного зондирования обычно недостаточно иметь статистические частотные характеристики волнения, требуется информация о пространственном распределении. Такое распределение может быть получено из частотных спектров только с помощью принятия довольно сильных приближений и часто спорных гипотез.

Фазовые характеристики взволнованной поверхности могут существенным образом определять особенности отражения от морской поверхности. Спектральное описание, оперирующее моделью случайной квазигауссовой поверхности, в частности, не может объяснить заметные различия в отражении сигнала от одной и той же поверхности при зондировании в противоположных направлениях, например по ветру и против ветра (Chen et al., 1993). Требуется учёт амплитудно-фазовых характеристик как функций частоты и направления распространения различных волновых компонент. В этой ситуации прямое измерение пространственных спектров с помощью специальных экспериментальных методов приобретает особое значение. Важной и актуальной задачей является исследование динамических и радиационных процессов на границе морской поверхности и атмосферы, взаимосвязи между тремя фи-

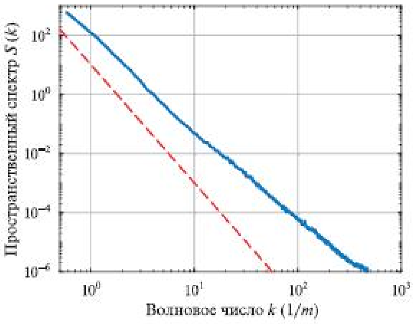

Рис. 10. Пространственный спектр волнения в логарифмическом масштабе. Пунктир — зависимость k^{-4}

В отсутствие углового распределения $D(\kappa, \theta)$ (3) восстановление двумерного пространственного спектра становится неразрешимой задачей. Можно попытаться качественно соотнести измеряемый одномерный пространственный спектр с теоретическими результатами. Следуя анализу размерности, рассмотрим функцию $E(k)/k$, чтобы получить форму двумерного спектра $E(k, l)$. Строго говоря, такое рассуждение справедливо для изотропных распределений, что далеко от нашего случая нескольких анизотропных волновых систем. Результат показан на *рис. 10.*

Зависимость оказывается близкой к закону k^{-4}, т. е. к решению для прямого каскада Колмогорова – Захарова для волн короче полуметра. Проведённая оценка показывает возможные проблемы интерпретации результатов.

Решение всех вышеперечисленных проблем заключается в измерении возвышенностей не по одному, а по двум или более направлениям. Данная экспериментальная задача будет поставлена и реализована в ближайших натурных экспериментах.

Выводы

В работе представлены результаты натурных измерений профилей морской поверхности, полученных с помощью сканирующего лазерного волнографа вдоль линии сканирования протяжённостью 1700 мм. Отмечается важность регистрации формы поверхности в задачах рассеяния электромагнитного излучения свободной поверхностью раздела. Получены частотные спектры волнения, профилограммы «время – высота», пространственные спектры, включающие капиллярные компоненты волнения. Обсуждается возможность измерения фазовых скоростей различных пространственных компонент волнения, которые можно выделять с помощью частотной фильтрации множества исходных профилей. Отмечается негауссовость распределения возвышений поверхности на основе анализа высших моментов распределений: skewness — коэффициента асимметрии вверх-вниз и коэффициента асимметрии вперёд-назад.

Натурные измерения, обработка видеорядов и расчёт спектров проведены за счёт гранта Российского научного фонда (проект № 23-17-00189). Расчёты в разделе «Статистические моменты» выполнены в рамках проекта Российского научного фонда № 19-72-30028.

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Wind-driven sea spectra resilience as statistical attractor

A. N. Pushkarev[+*], V. V. Geogjaev[+−1)], S. I. Badulin[+−]

[+]Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, bld. 1, Moscow 121205, Russia

[*]Lebedev Physical Institute RAS, Leninsky 53, Moscow 119991, Russia

[−]Shirshov Institute of Oceanology RAS, Nahimovskiy 36, Moscow 117997, Russia

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We have observed numerically the resilience phenomenon for ocean wind-driven waves, where the wave spectra return to their original self-similar form after a strong perturbation. This self-similar behaviour is seen as the manifestation of a statistical attractor associated with generalized spectra of Kolmogorov–Zakharov. We have confirmed this interpretation through numerical simulations of random water wave field within the kinetic (Hasselmann) equation. This equation with specific source functions similar to those of conventional wave forecasting models, exhibits families of exact self-similar solutions. These source functions minimize the "non-self-similar" background, allowing us to evaluate the "clean rates" of wave spectra resilience. We use the indices of the exact self-similar solutions as parameters for the attractors of numerical solutions in a two-dimensional phase space.

1. SPECTRAL MODELLING OF RANDOM FIELD OF WIND-DRIVEN WAVES

We study the kinetic equation for surface gravity waves

$$\frac{\partial \varepsilon}{\partial t} = S_{\rm nl} + S_{\rm in} + S_{\rm diss} \qquad (1)$$

This equation is widely used for modeling and forecasting of wind-driven sea. Here $\varepsilon = \varepsilon(\boldsymbol{k}, t)$ is the phase space energy density, $S_{\rm nl}$ — nonlinear four-waves interaction term, $S_{\rm in}$ — wind input, $S_{\rm diss}$ — wave-breaking dissipation. This equation was derived by Klauss Hasselmann [1] while its counterparts were known for decades before [2, 3].

The $S_{\rm nl}$ term in (1) was established in [1] based on the four-wave nonlinear interactions. Some drawbacks of restrictions by four-wave interactions and hypothesis of the wave-field gaussianity were demonstrated in [4, 5]. The effect of quasi-resonant interactions on wave spectra evolution was also studied [6]. Nevertheless we deal with the equation (1) as the conventional model of the modern physics and the basic one of the sea wave forecasting.

The $S_{\rm nl}$ term is derived from the primitive equations of fluid mechanics and has been extensively studied. However, the terms of wind input $S_{\rm in}$ and wave dissipation $S_{\rm diss}$ are only known through empirical parameterizations, which implies an uncertainty in the mathematical model itself. The wind-wave community's efforts are mainly focused on developing new parameterizations for these terms to better align with the available results of experiments and wave monitoring. This will make the corresponding mathematical modeling more suitable for wave forecasting (see e.g. [7]).

The derivation of (1) suggests that the external forcing terms $S_{\rm in}$ and $S_{\rm diss}$ are much smaller than $S_{\rm nl}$. Initially, a basic comparison of the magnitudes of the terms suggests the opposite, indicating that the external forcing terms are rather high [8]. However, this issue has been resolved in [9, 10] by comparing the rates of wave evolution associated with different terms. The dominance of the collision integral $S_{\rm nl}$ in (1) ensures the consistency of the conventional form of the kinetic equation due to its high nonlinearity.

The Hasselmann equation performs well across a wide range of wind-sea modeling and forecasting parameters. In this study we focus on the deep water conditions. In this case, eq. 1 demonstrates the robustness of the wind-wave spectra shaping and growth features. Some of these features, such as spectral shape invariance [11] and typical power-law growth exponents of wave energy and period [12] , have been observed in previous experimental studies. The association of these features with approximate self-similar solutions for the equation has been extensively studied both numerically [13, 14] and through experimental results analysis [15]. In this paper, we analyze the exact self-similar solutions for the Hasselmann equation (1) with a special source function, which reproduces general features of conventional source functions of wave forecasting models [16] and ensures the existence of

1) e-mail: vvg@mail.geogjaev.ru

wind-wave models, deliver exact self-similar solutions for the kinetic equation for deep water waves (the Hasselmann equation). It allows one to refine the problem minimizing the non-self-similar spectra background in our numeric simulations.

Secondly, we propose a finite size (two-dimensional, in fact) parameterization of the evolution of the non-stationary (developing wind sea) and continuous media (wave spectra) to reduce the problem to a classic dynamical attractor. The possibility itself of such a problem reduction has been illustrated for the case of young wind sea, the wave growth regime which is usually realized in the field wave experiments. Extensive numerical studies of the problem are part of our agenda.

We have shown that the classic Kolmogorov-Zakharov (KZ) solutions and their recent generalizations, self-similar spectra localized in wave scales, can be efficiently treated as statistical attractors in the set of already found statistical attractors of nonlinear PDE's, such as soliton in non-integrable NlS.

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7. CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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Article

Magnetic Filaments: Formation, Stability, and Feedback

Evgeny A. Kuznetsov [1,2,3,4,*] and Evgeny A. Mikhailov [1,2,5]

[1] P. N. Lebedev Physical Institute, 119991 Moscow, Russia
[2] Center for Advanced Studies, Skolkovo Institute of Science and Technology, 121205 Moscow, Russia
[3] L. D. Landau Institute for Theoretical Physics, 142432 Chernogolovka, Russia
[4] Space Research Institute, 117997 Moscow, Russia
[5] Department of Physics, M. V. Lomonosov Moscow State University, 119991 Moscow, Russia; ea.mikhajlov@physics.msu.ru
* Correspondence: kuznetso@itp.ac.ru

Abstract: As is well known, magnetic fields in space are distributed very inhomogeneously. Sometimes, field distributions have forms of filaments with high magnetic field values. As many observations show, such a filamentation takes place in convective cells in the Sun and other astrophysical objects. This effect is associated with the frozenness of the magnetic field into a medium with high conductivity that leads to the compression of magnetic field lines and formation of magnetic filaments. We analytically show, based on the general analysis, that the magnetic field intensifies in the regions of downward flows in both two-dimensional and three-dimensional convective cells. These regions of the hyperbolic type in magnetic fields play the role of a specific attractor. This analysis was confirmed by numerical simulations of 2D roll-type convective cells. Without dissipation, the magnetic field grows exponentially in time and does not depend on the aspect ratio between the horizontal and vertical scales of the cell. An increase due to compression in the magnetic field of highly conductive plasma is saturated due to the natural limitation associated with dissipative effects when the maximum magnitude of a magnetic field is of the order of the root of the magnetic Reynolds number $\mathrm{Re_m}$. For the solar convective zone, the mean kinetic energy density exceeds the mean magnetic energy density for at least two orders of magnitude, which allows one to use the kinematic approximation of the MHD induction equation. In this paper, based on the stability analysis, we explain why downward flows influence magnetic filaments, making them flatter with orientation along the interfaces between convective cells.

Keywords: magnetohydrodynamics; convective cells; magnetic field; filaments; feedback

MSC: 76W05

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1. Introduction

The phenomenon of collapse plays a significant role in terms of understanding how turbulence, convection, and other similar phenomena operate in fluids. Collapse is understood as a process of formation of singularities in a finite time for smooth initial conditions. Such processes have been widely studied for quite a long time. According to the classical concepts of the Kolmogorov–Obukhov theory [1,2] in the case of a low-viscosity limit, the vorticity fluctuations in the inertial interval with a scale λ behave proportionally to $\lambda^{-2/3}$. This means that in the limit of small λ, we will have infinite amplitudes of fluctuations, which may indicate that classical turbulence is closely related to the occurrence of collapse. At the same time, when the highly accurate numerical modeling of such problems became possible, it turned out that collapse was in fact not observed in such cases [3] (see also the review paper [4] devoted to this subject). Nevertheless, the tendency for vorticity enhancement remains, but without blow-up behavior. At the same time, for two-dimensional hydrodynamics in the ideal case, solutions associated with collapse are forbidden [5–7]. In hyperbolic region with its center at $y = 0$, $x = 0$. For this reason, the hyperbolic point will be moved towards the counter flow that provides an inverse influence on the growing magnetic field of the convective flow. Because the filament region is small in comparison to the convective cell, such a shift should not significantly influence the convection itself. It is evident that this mechanism shows that the magnetic pressure $B^2/2$ is comparable to the mean kinetic energy density $\langle v^2 \rangle /2$.

7. Conclusions

In this paper, we have analyzed the filamentation of the magnetic field in convective cells in the Sun within the kinematic approximation. This process is associated with the frozenness of the magnetic field into a medium with high conductivity, which leads to the compression of magnetic field lines and the formation magnetic filaments. Based on the general consideration of the convection top flows only, and without knowledge of the cell structure, we demonstrate that the magnetic field intensifies in the regions of downward flows in both two-dimensional and three-dimensional convective cells. These hyperbolic-type regions play the role of a specific attractor of the magnetic field. This theoretical analysis was confirmed by numerical simulations for 2D convective cells of the roll type. Without dissipation, the magnetic field grows exponentially in time and attains its maximal value at the hyperbolic point where the growth rate does not depend on the aspect ratio between the horizontal and vertical scales of the cell. This increase due to the compression of the magnetic filaments is saturated due to the natural limitation associated with finite plasma conductivity when the maximum magnitude of the magnetic field is of the order of the root square of the magnetic Reynolds number. Another effect of saturation of the magnetic field values is connected with feedback of the growing field on the convective flows. Both of these effects on the Sun convective zone give the maximal magnetic field values in filaments the same order of magnitude of about 1 kG. Based on the stability analysis, we have explained why downward flows influence magnetic filaments by making them flatter with orientation along the interfaces between convective cells.

Author Contributions: Conceptualization, E.A.K.; methodology, E.A.K.; numerical results, E.A.M.; writing, E.A.K. and E.A.M. All authors have read and agreed to the published version of the manuscript.

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Three-Dimensional Acoustic Turbulence: Weak Versus Strong

E. A. Kochurin [1,2,*] and E. A. Kuznetsov [2,3,4]

[1] *Institute of Electrophysics, Ural Branch of RAS, 620016, Yekaterinburg, Russia*
[2] *Skolkovo Institute of Science and Technology, 121205, Moscow, Russia*
[3] *Lebedev Physical Institute, RAS, 119991, Moscow, Russia*
[4] *Landau Institute for Theoretical Physics, RAS, Chernogolovka, 142432, Moscow region, Russia*

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Direct numerical simulation of three-dimensional acoustic turbulence has been performed for both weak and strong regimes. Within the weak turbulence, we demonstrate the existence of the Zakharov-Sagdeev spectrum $\propto k^{-3/2}$ not only for weak dispersion but in the nondispersion (ND) case as well. Such spectra in the k space are accompanied by jets in the form of narrow cones. These distributions are realized due to small nonlinearity compared with both dispersion or diffraction. Increasing pumping in the ND case due to dominant nonlinear effects leads to the formation of shocks. As a result, the acoustic turbulence turns into an ensemble of random shocks with the Kadomtsev-Petviashvili spectrum.

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As is known, the developed hydrodynamic turbulence at large Reynolds numbers, $\mathrm{Re} \gg 1$, in the inertial interval represents an example of a system with strong nonlinear interaction, when its energy coincides with the interaction Hamiltonian. Another example relates to acoustic turbulence which demonstrates both strong and weak regimes depending on the ratio between nonlinearity and linear wave characteristics. In this sense, acoustic turbulence is much more diverse and richer than hydrodynamic turbulence. When nonlinear interaction of waves is small compared to linear effects, we have a regime of weak turbulence [1] which can be studied perturbatively by using the random phase approximation. Within the weak turbulence theory (WTT) for arbitrary wave systems, ensembles of waves are described statistically in terms of the corresponding kinetic equations [1,2]. This theory assumes that each wave with its random phase moves long enough time almost freely and undergoes very rarely small changes due to the nonlinear interaction with other waves. To date, the WTT has a lot of applications starting from ocean and plasma waves, waves in solid state physics, in Bose-Einstein condensate, and ending by turbulence both in astrophysics and high energy physics [3–11]. For water waves WTT has been confirmed with high accuracy [12]. The situation changes for acoustic waves without dispersion for which the resonant conditions for three-wave interaction

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2), \qquad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \tag{1}$$

are satisfied only for collinear wave vectors $\mathbf{k}_i$, where $\omega(\mathbf{k}) = kc_s$ is the linear dispersion relation and c_s is the speed of sound. These interacting waves thus form a ray in the k space. Evidently, translating to the system of coordinates moving with c_s along the ray makes this system strongly nonlinear. In one-dimensional (1D) gas dynamics, this nonlinearity leads to the breaking of acoustic waves in accordance with the famous Riemann solution.

However, in a multidimensional situation by means of such transition it is possible to exclude c_s only for one given ray; for all other rays propagating under some angles this exclusion does not work. If we take continuously distributed rays with close propagation angles, we obtain an acoustic beam, which, as is known, is subject to diffraction in the transverse direction. As soon as a ray in the 3D case gets a transverse width, let small, then such a ray will diffract. First time this effect was discussed in the original paper by Zakharov and Sagdeev [13]; practically at the same time the analogous ideas were developed by Newell and Aucoin [14] (see also [15]). From these arguments follows that the weak turbulence regime for acoustic waves can be realized, as we will show in this Letter, not only in the weak wave dispersion case but also in the dispersionless situation due to the diffraction.

For acoustic waves with weak positive wave dispersion,

$$\omega = kc_s(1 + a^2k^2), \qquad (a^2k^2 \ll 1), \tag{2}$$

the resonant conditions (1) are satisfied so that the interacting waves instead of rays form cones with small angles $\sim(ak)^2$ (here a is the dispersion length). The WTT, as known, is applicable for small nonlinearity compared to weak dispersion (WD). In this mode, Zakharov and Sagdeev [13,16] found exact isotropic solution of the kinetic equation for 3D weak acoustic turbulence

*Contact author: kochurin@iep.uran.ru

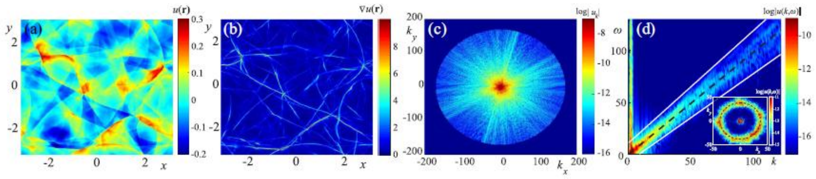

FIG. 3. (a),(b) Dependences $u(\mathbf{r})$ and $|\nabla u(\mathbf{r})|$ for $z = 0$ in shock-wave regime, $t = 200$. (c) Fourier spectrum $|u_k|$ in the $k_z = 0$ plane is shown at $t = 200$. (d) The space-time Fourier transform $|u(\mathbf{k}, \omega)|$; the black dashed line corresponds to the ND wave propagation (2). White solid lines show the frequency broadening δ_ω. The inset shows $|u(\mathbf{k}, \omega_0)|$ with $\omega_0 = 40$ in $k_z = 0$ section.

Figures 3(a) and 3(b) show the spatial distributions of $u(\mathbf{r})$ and its gradient $\nabla u(\mathbf{r})$ at the $z = 0$ plane which show the presence of a set of shocks propagating under various angles. The spectrum shown in Fig. 3(c) clearly indicates the quasi-isotropic behavior of wave energy in Fourier space. Figure 3(d) shows the space-time Fourier spectrum of $u(\mathbf{r}, t)$ that allows us to estimate the frequency broadening $\delta_\omega = 1/\tau_{NL}$, where τ_{NL} is characteristic nonlinear time. We have computed the parameter τ_L/τ_{NL} with linear diffraction time $\tau_L = [\Omega_0^2 k/2]^{-1}$ for both weak and strong WT regimes, see Supplemental Material, Sec. VI [23]. Comparison of these times shows that in the strong turbulence regime, τ_L/τ_{NL} is about 5. The latter means that the wave breaking is more rapid process leading to formation of shocks randomly distributed due to the chaotic pumping. This is a reason for the appearance of the KP spectrum [19].

The main result of the work is that the ZS spectrum of 3D weak acoustic turbulence (3) can indeed be realized. The mechanism for the development of WT is the divergence (diffraction) of acoustic waves preventing their breaking. However, under action of sufficiently large pumping, acoustic turbulence turns into an ensemble of random shocks described by the KP spectrum (4). Thus, the spectra (3) and (4) correspond to different limiting cases of the weakly and strongly nonlinear regimes of 3D acoustic turbulence.

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E. A. Kuznetsov, D. S. Agafontsev, and A. A. Mailybaev

Folding in fluids

Abstract: The formation of the coherent vortical structures in the form of thin pancakes for three-dimensional flows is studied at the high Reynolds regime when, in the leading order, the development of such structures can be described within the Euler equations for ideal incompressible fluids. Numerically and analytically on the base of the vortex line representation, we show that compression of such structures and, respectively, increase of their amplitudes are possible due to the compressibility of the vorticity in the 3D case. It is demonstrated that this growth has an exponential behavior and can be considered as folding (analog of breaking) for the divergence-free fields of vorticity. At high amplitudes, this process in 3D has a self-similar behavior connected the maximal vorticity and the pancake width by the relation of the universal type [1].

Keywords: Folding, vortex line representation, pancake, Kolmogorov spectrum

MSC 2020: 76M99, 76M25

YouTube presentation: https://youtu.be/8ygjVBGXHec

1 Outline

- Motivation: Collapse and the Kolmogorov–Obukhov theory
- Cauchy invariants and the Kelvin theorem
- Vortex line representation (VLR) and its compressibility
- Kolmogorov-type relation for the vortex pancake structures

2 Main question

- The main question of this talk:
  Is it possible to find some compressible entities in incompressible fluid?
- Answer: Yes, such entities do exist!
- These are continuously distributed lines of frozen-in-fluid fields

E. A. Kuznetsov, Lebedev Physical Institute of RAS, Laboratory of Mathematical Physics, 53 Leninsky ave., 119991 Moscow, Russia, e-mail: kuznetso@itp.ac.ru
D. S. Agafontsev, Shirshov Institute of Oceanology of RAS, 36 Nakhimovskii prosp., 117997 Moscow, Russia, e-mail: dmitry.agafontsev@gmail.com
A. A. Mailybaev, Instituto Nacional de Matemática Pura e Aplicada, 110 Estrada Dona Castorina, CEP 22460-320 Rio de Janeiro, Brazil, e-mail: a.mailybaev@gmail.com

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– Increasing with the time number of such structures leads to formation of the Kolmogorov energy spectrum observed numerically in a fully inviscid flow, with no tendency towards finite-time blowup.

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Eigenvalue Problem Describing Magnetorotational Instability in Outer Regions of Galaxies

Evgeny Mikhailov [1,2,3,*] and **Tatiana Khasaeva** [1,4]

[1] Faculty of Physics, M. V. Lomonosov Moscow State University, 119991 Moscow, Russia
[2] P. N. Lebedev Physical Institute, 119991 Moscow, Russia
[3] Center for Advanced Studies, Skolkovo Institute of Science and Technology, 121205 Moscow, Russia
[4] Institute of Earthquake Prediction Theory and Mathematical Geophysics, 117997 Moscow, Russia
* Correspondence: ea.mikhajlov@physics.msu.ru

Abstract: The existence of magnetic fields in spiral galaxies is beyond doubt and is confirmed by both observational data and theoretical models. Their generation occurs due to the dynamo mechanism action associated with the properties of turbulence. Most studies consider magnetic fields at moderate distances to the center of the disk, since the dynamo number is small in the marginal regions, and the field growth should be suppressed. At the same time, the computational results demonstrate the possibility of magnetic field penetration into the marginal regions of galaxies. In addition to the action of the dynamo, magnetorotational instability (MRI) can serve as one of the mechanisms of the field occurrence. This research is devoted to the investigation of MRI impact on galactic magnetic field generation and solving the occurring eigenvalue problems. The problems are formulated assuming that the perturbations may possibly increase. In the present work, we consider the eigenvalue problem, picturing the main field characteristics in the case of MRI occurrence, where the eigenvalues are firmly connected with the average vertical scale of the galaxy, to find out whether MRI takes place in the outer regions of the galaxy. The eigenvalue problem cannot be solved exactly; thus, it is solved using the methods of the perturbation theory for self-adjoint operators, where the eigenvalues are found using the series with elements including parameters characterizing the properties of the interstellar medium. We obtain linear and, as this is not enough, quadratic approximations and compare them with the numerical results. It is shown that they give a proper precision. We have compared the approximation results with those from numerical calculations and they were relatively close for the biggest eigenvalue.

Keywords: eigenvalue problem; magnetorotational instability; perturbation theory; operator

MSC: 76W05; 47A75

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1. Introduction

It is well known that a large variety of astrophysical objects, such as the Sun [1,2], the Earth [3], other planets [4,5] and stars [6–8], accretion discs [9,10], pulsars [11] and some galaxies [12,13], have large-scale magnetic fields. There are different methods of observational study for such fields. For example, for the Sun, we can use the Zeeman effect [14]. As for faraway objects (galaxies and accretion discs), this approach cannot give any proper results, so it is necessary to study the synchrotron emission spectra [15]. Nowadays, for most cases, Faraday rotation measurements are taken [16,17]. This method is based on the fact that the polarized radio wave (passing, for example, from pulsars) changes its polarization plane angle while travelling through a magnetized medium. The angle of rotation is proportional to the integral of magnetic field projection to the line of sight. Also, it depends on the wavelength, being proportional to its square. Thus, comparing polarization angles for different wavelengths, we can rebuild the field structure.

integrals numerically. It would be interesting to also study further approximations, but the comparison with numerical studies shows that they are not likely to make any significant changes. After comparing the numerical results and the analytical approximations, we note that the difference is more significant for large values of C. This may be due to the fact that the approximation results are based on perturbation theory methods, which assume that the perturbations proportional to C are relatively small. The larger C is, the less accurate this approach is.

Two different models for the rotation law were used. It is necessary to emphasize that the Brandt rotational curve is closer to real objects, but the calculations for it are more complex. However, the results do not have substantial differences (see Figures 1 and 2, Tables 3 and 6), so we can use the simpler model.

From a physical point of view, the parameter k_z is of primary importance. It can be shown that $1/k_z$ is the vertical lengthscale for the generated magnetic field, which has the same order as the thickness of the object (in dimensionless units). So, the possibility of such magnetic fields generation seems to be quite real and its lengthscales are comparable with the galaxy thickness. Nevertheless, we do not deny the possibility of other mechanisms producing magnetic fields, such as dynamo [25] or battery mechanisms [41].

Author Contributions: Conceptualization, E.M.; methodology, E.M.; investigation, T.K.; numerical results, T.K.; writing—original draft preparation, E.M. and T.K.; writing—review and editing, E.M. and T.K. All authors have read and agreed to the published version of the manuscript.

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УДК 537.612.2

СТАТИСТИЧЕСКИЕ СВОЙСТВА ГАЛАКТИЧЕСКОГО МАГНИТНОГО ПОЛЯ В МОДЕЛИ ДИНАМО С ФЛУКТУАЦИЯМИ АЛЬФА-ЭФФЕКТА И ТУРБУЛЕНТНОЙ ДИФФУЗИИ

Д. А. Грачев[1], С. А. Елистратов[2,3], Е. А. Михайлов[4,5,6]

Возникновение магнитных полей в галактиках принято ассоциировать с действием динамо. Оно основано на спиральности турбулентных движений (альфа-эффект) и нетвердотельности вращения галактики (дифференциальное вращение). Им противостоит диссипация, которая стремится разрушить регулярные структуры поля. Как правило, уравнения динамо содержат усредненные параметры турбулентности. Подобный подход представляет сложности в случае галактик с сильно неоднородной межзвездной средой, связанной, например, со звездообразованием. Одним из способов решения проблемы является использование случайных коэффициентов в уравнениях для поля. В таком случае решения требуют отдельного статистического анализа. В настоящей работе изучены различные статистические моменты для галактического магнитного поля в рамках модели, предполагающей флуктуации альфа-эффекта и турбулентной диффузии.

Ключевые слова: магнитное поле, галактики, динамо, статистические моменты.

[1] Национальный исследовательский университет "Высшая школа экономики", 109028 Россия, Москва, Покровский бульвар, 11; e-mail: dengrac@mail.ru.

[2] Институт системного программирования имени В. А. Иванникова, 109004 Россия, Москва, ул. А. Солженицына, 25; e-mail: invsbl_mn@mail.ru.

[3] Институт океанологии имени П. П. Ширшова, 117997 Россия, Москва, Нахимовский проспект, 36.

[4] ФИАН, 119991 Россия, Москва, Ленинский пр-т, 53; e-mail: ea.mikhajlov@physics.msu.ru.

[5] МГУ имени М. В. Ломоносова, 119991 Россия, Москва, Ленинские горы, 1, стр. 2.

[6] Сколковский институт науки и технологий, 121205 Россия, Москва, Сколково, Большой бульвар, 30/1.

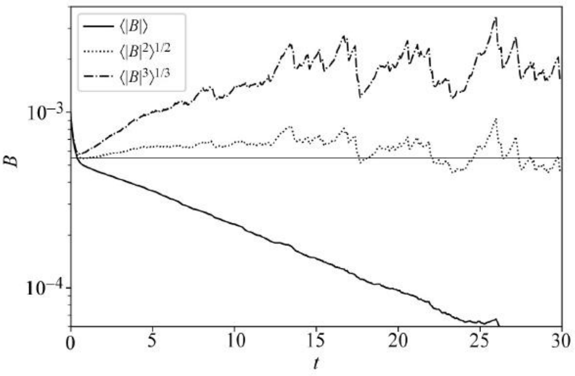

Рис. 3: *Статистические моменты решения при* $p = 0.14$. *Сплошная линия соответствует* $\langle|B(t)|\rangle$, *штриховая* $\langle B^2(t)\rangle^{1/2}$, *штрихпунктирная* – $\langle|B|^3(t)\rangle^{1/3}$. *Результаты приведены для выборки объема* $N = 10^4$.

Кроме того, отмечено, что, начиная с определенного момента, экспоненциальный рост прекращается. Время, в течение которого происходит рост, зависит от объема выборки. Для реальных объектов можно полагать, что объем выборки примерно соответствует количеству турбулентных ячеек, которое соответствует величине порядка 10^5.

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Bi-Solitons on the Surface of a Deep Fluid: An Inverse Scattering Transform Perspective Based on Perturbation Theory

Andrey Gelash,[1,*] Sergey Dremov,[3] Rustam Mullyadzhanov,[2,4] and Dmitry Kachulin[2,3]
[1]*Laboratoire Interdisciplinaire Carnot de Bourgogne (ICB), UMR 6303 CNRS—Université Bourgogne Franche-Comté, 21078 Dijon, France*
[2]*Novosibirsk State University, Novosibirsk 630090, Russia*
[3]*Skolkovo Institute of Science and Technology, Moscow 121205, Russia*
[4]*Institute of Thermophysics SB RAS, Novosibirsk 630090, Russia*

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We investigate theoretically and numerically the dynamics of long-living oscillating coherent structures —bi-solitons—in the exact and approximate models for waves on the free surface of deep water. We generate numerically the bi-solitons of the approximate Dyachenko-Zakharov equation and fully nonlinear equations propagating without significant loss of energy for hundreds of the structure oscillation periods, which is hundreds of thousands of characteristic periods of the surface waves. To elucidate the long-living bi-soliton complex nature we apply an analytical-numerical approach based on the perturbation theory and the inverse scattering transform (IST) for the one-dimensional focusing nonlinear Schrödinger equation model. We observe a periodic energy and momentum exchange between solitons and continuous spectrum radiation resulting in repetitive oscillations of the coherent structure. We find that soliton eigenvalues oscillate on stable trajectories experiencing a slight drift on a scale of hundreds of the structure oscillation periods so that the eigenvalue dynamics is in good agreement with predictions of the IST perturbation theory. Based on the obtained results, we conclude that the IST perturbation theory justifies the existence of the long-living bi-solitons on the surface of deep water that emerge as a result of a balance between their dominant solitonic part and a portion of continuous spectrum radiation.

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Formation of stable localized coherent structures—solitons—is one of the key evolution scenarios of nonlinear wave systems [1]. When such a system is Hamiltonian, solitons emerge due to a balance between nonlinearity and dispersion, while in nonconservative cases, an additional balance between energy gain and loss comes into play [1–3]. Being described by nonlinear partial differential equations (PDEs), systems with solitons can be seen in almost all fields of physics—for example, in hydrodynamics, optics, and plasmas [2,4]. While individual stationary solitons are ubiquitous for nonlinear wave models, long-living multisoliton complexes are not so common and thus draw particular attention and are of great interest for experimental implementation. For example, a bound state of solitons has been observed in mode-locked fiber lasers, Bose-Einstein condensates, and specially designed optical waveguides [5–8].

For a Hamiltonian wave model the presence of recursive multisoliton behavior might be a signature of its integrability or nearly integrable dynamics [9–12]. The inverse scattering transform (IST) theory elucidates the particlelike features of solitons in exactly integrable nonlinear PDEs by proving that solitons correspond to the time-invariant eigenvalue spectrum of an auxiliary scattering problem [9,13]. For example, solitons of the integrable one-dimensional nonlinear Schrödinger equation (NLSE) collide elastically, forming bouncing multisoliton complexes, and preserve their parameters during the whole system evolution [14]. When integrability is broken by adding weak extra terms to the model, solitons can still form long-living but usually inelastic complexes radiating incoherent waves, whose dynamics is described by the IST perturbation theory [10,15,16].

We consider the Hamiltonian models of the 2D hydrodynamics with a free surface: (i) focusing NLSE [17], (ii) Dyachenko-Zakharov envelope equation (DZE) [18], and (iii) fully nonlinear equations for the *R*-*V* variables (RVE) [19–22]. These models are the members of the Hamiltonian hierarchy of equations for the free surface water waves [17,23] in which the NLSE describes only the weakly nonlinear narrow-banded wave trains while the DZE captures many of the nonlinear effects presented in the full model [17,18]. Comparative analysis of the behavior of wave groups in the approximate DZE and the exact RVE models provides insights into how the model objects are expected to be seen in nature [24–26].

Numerical works revealed solitary waves for the DZE and RVE models [27,28] observed later in water wave tank experiments [29,30]. For certain parameters, pairwise collisions of the DZE solitons do not produce any visible

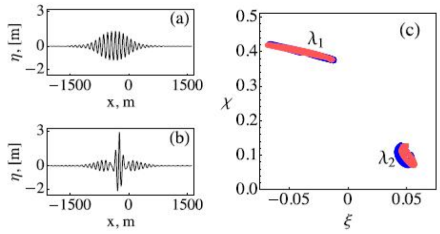

FIG. 4. Nonlinear behavior of the RVE bi-soliton. (a),(b) Surface profiles $\eta(x)$ at minimum and maximum amplitude, respectively. (c) Stable trajectories of soliton eigenvalues with $T \approx 30.0$. Blue and red lines correspond to two complete cycles of the bi-soliton oscillations separated by 200 T.

presented in Supplemental Material [44]. Note that in the case of the RVE bi-solitons, the IST perturbation theory works only quantitatively, which is expected for the fully nonlinear model due to the presence of the complicated structure of its rhs.

The IST analysis of the long-living bi-solitons in the deep water models shows that these oscillatory complexes exist in a stable nearly integrable regime and can be described within the IST perturbation theory. In general, the governing DZE and RVE models are far from being integrable; however, for the bi-solitons all the rhs terms are small throughout the whole oscillation period T. The numerically computed time series of the IST spectrum allows us to accurately reveal the recursive dynamics of the bi-solitons preserving at the scale of hundreds of T. We also show that the bi-soliton complex is stabilized by minor radiation and a nonzero velocity difference between solitons; both of them gradually increase up to the high-amplitude wave field configuration so that the two discrete components and continuous part of the scattering data are in periodic energy and momentum exchange.

In contrast to the approximately solvable models with weakly interacting solitons [65–70], the bi-solitons considered here are fully overlapping and governed by equations of type (7) with such complicated rhs's that cannot be studied analytically with the perturbation framework (12). Here, we propose a perspective of using IST theory in such nonsolvable cases based on the combination of the perturbation approach, exact multisoliton solutions, and numerical IST tools. Our approach provides an IST interpretation of the interaction mechanism for the deep water bi-solitons and opens questions for further studies. One of them is identifying a complete set of initial soliton eigenvalues corresponding to long-living recursive bi-soliton dynamics. Another question concerns the connection of the presented approach with general methods of finding periodic solutions to nonlinear PDEs [71,72]. Besides localized solitonic wave fields, considering a continuous wave background is of fundamental interest for the comparative study of the deep fluid models in the light of the proposed IST approach [32,73,74]. In particular, a combination of the IST with Melnikov's analysis being applied to the DZE and RVE can shed new light on behavior of the so-called breathers—solitons living on the background [75–78]. Our results can be generalized to other physical systems, such as optical waveguides described in the leading order by the NLSE [4,56], and also applied to analyze experimental data.

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*Corresponding author: Andrey.Gelash@u-bourgogne.fr
Present address: Institute of Physics, Swiss Federal Institute of Technology Lausanne (EPFL), CH-1015 Lausanne, Switzerland.

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Ship waves on an elastic floating ice plate

Sergei Badulin [*,†]
Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, Building 1, 121205 Moscow, Russia

Vladimir Gnevyshev [‡]
P.P. Shirshov Institute of Oceanology of the Russian Academy of Sciences, 36 Nakhimovsky Avenue, 117997 Moscow, Russia

Yury Stepanyants [¶,§]
Department of Applied Mathematics, Nizhny Novgorod State Technical University, n.a. R.E. Alekseev, 24 Minin Street, 603950 Nizhny Novgorod, Russia

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We study a wave wake produced by a finite-size source uniformly moving on an ice plate overlying deep water. We describe the kinematic characteristics of source-generated flexural-gravity waves in terms of isophase patterns scaled by the minimum phase speed and show that the wave picture is determined by ad hoc defined Mach and Bond numbers. Then, we show that several bifurcations occur in the wave wake when the source speed increases. In particular, cusps appear in the wake patterns at certain speeds due to the merging at specific points of two or three wave characteristics (wave rays) which correspond to longitudinal (divergent), transverse, and fan waves. To describe the distribution of wave amplitudes in the wake accounting for the source size and shape, we use the reference solution approach. This approach agrees with the stationary phase method in the far-field zone and reproduces also specific wave dynamics at short and intermediate distances from the source. The effect of a source shape on the wave wake pattern is graphically illustrated for the elliptic source as the dependence on the dimensionless Bond number and the source length-to-beam ratio. The illustrations of the source speed and shape effects are given for low-speed regimes when wave patterns are conformed by single-ray solutions. Ways of the generalization of the suggested approach are outlined.

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I. INTRODUCTION

The classic problem of ship waves, i.e., stationary waves generated by a moving source on a water surface (or more generally, in an isotropic dispersive media), was formulated for the first time, apparently, by Thomson [1] (Lord Kelvin after 1892). Since that time, Kelvin's approach was further developed, applied to waves in various media, and presented in many classical textbooks (see, for example, Refs. [2–5]). Nowadays, there is a great interest in the study of ship-type waves

*Also at P.P. Shirshov Institute of Oceanology of the Russian Academy of Sciences, 36 Nakhimovsky Avenue, 117997 Moscow, Russia.
†Contact author: badulin.si@ocean.ru
‡Contact author: avi9783608@gmail.com
¶Contact author: yury.stepanyants@unisq.edu.au
§Also at University of Southern Queensland, West Street, Toowoomba, Queensland 4350, Australia.

speeds and the deflection of ice cover were found numerically for the different shear flow gradients, angles between the source and shear flow velocities, and compression ratio of the ice plate. The wave patterns on the ice plate by a moving source are illustrated numerically for different parameters.

In almost all papers cited above (except Ref. [46]), the authors considered the problem of FGWs generation for the particular sets of dimensional parameters which are quite realistic but do not embrace all possible cases; the results obtained were presented in forms of tables and plots. In our paper, we presented the results in the universal dimensionless variables for infinitely deep ocean without compression or tension forces exerting on a moving source. This allows one to determine a minimal number of governing parameters and manipulate with all other dimensional parameters.

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APPENDIX: REFERENCE SOLUTION METHOD

Following the ideas presented in Ref. [18], consider a linear solution in the dimensional form for the vertical deflection w of an ice plate from the undisturbed position through the Fourier integral (the resultant formulas derived in this Appendix are presented in Sec. IV A in the dimensionless form):

$$w(\mathbf{x}, t) = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\mathbf{k}) e^{i[\mathbf{k}\mathbf{x}-\omega(\mathbf{k})t]}\right) d\mathbf{k}. \tag{A1}$$

Here symbol Re stands for the real part of the corresponding expression, $F(\mathbf{k})$ is the Fourier image of the function $w(\mathbf{x}, 0)$ at initial time moment $t = 0$.

Consider the initial disturbance $w(x, y, 0) \equiv f(x, y)$ in the form of a Gaussian wave packet:

$$f(x, y) = \mathrm{Re}\left(\frac{1}{\Delta x \Delta y}\exp\left[-\frac{x^2}{2(\Delta x)^2} - \frac{y^2}{2(\Delta y)^2}\right] e^{i(k_0 x + l_0 y)}\right). \tag{A2}$$

Note, that in the limit $\Delta x \to 0,\ \Delta y \to 0$ Eq. (A2) reduces to

$$\lim_{\Delta x\to 0,\, \Delta y \to 0} f(x, y) = \tfrac{1}{2}\delta(x - x_0)\delta(y - y_0)\cos(k_0 x + l_0 y). \tag{A3}$$

This property will be used below to construct solutions for a wave of a source with a finite Fourier spectrum. The Fourier transform (A2) of the initial function (A2) is a normalized Gaussian pulse:

$$F(k, l) = \exp\left[-\frac{\kappa_x^2}{2(\Delta k)^2} - \frac{\kappa_y^2}{2(\Delta l)^2}\right], \tag{A4}$$

where $\kappa_x = k - k_0,\ \kappa_y = l - l_0$ and

$$\Delta k = 1/\Delta x; \quad \Delta l = 1/\Delta y. \tag{A5}$$

Assuming the pulse (A4) is narrow-banded, i.e., $\Delta k/k_0 \ll 1,\ \Delta l/l_0 \ll 1$, we approximate the dispersion relation $\omega(\mathbf{k})$ in Eq. (A1) by its Taylor series in the vicinity of the wave packet carrier wave vector $\mathbf{k_0}$:

$$\omega(\mathbf{k}) = \omega_0 + \omega_k' \,|_{\mathbf{k}=}\, \mathbf{k_0}\cdot\kappa_x + \omega_l' \,|_{\mathbf{k}=}\, \mathbf{k_0}\cdot\kappa_y + \tfrac{1}{2}(\mu_x + \mu_y + 2\mu_{xy}) + O(|\kappa|^{\mathbf{3}}), \tag{A6}$$

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Kats–Kontorovich anisotropic solution in simulations of ocean swell☆

Sergei I. Badulin [a,b,*], Vladimir V. Geogjaev [a,b], Andrei N. Pushkarev [a,c]

[a] Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, bld. 1, Moscow, 121205, Russia
[b] P.P. Shirshov Institute of Oceanology of the Russian Academy of Sciences, 36 Nakhimovsky pr., Moscow, 117997, Russia
[c] P.N. Lebedev Physical Institute of the Russian Academy of Sciences, Leninsky pr., 53, Moscow, 119991, Russia

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ABSTRACT

The physical setup of ocean swell is used as a testbed for the results of the weak turbulence theory. The numerical study with the novel Geogjaev-Zakharov approach highlights the importance of isotropic direct and inverse cascade solutions, along with the self-similarity concept of wave spectra, as developed by Vladimir Zakharov and his collaborators. The approximate anisotropic solution proposed by Kats and Kontorovich in 1970-ies is shown to fit wave spectra well at frequencies exceeding three times the spectral peak frequency. This solution can be interpreted as an attractor for a wide variety of initial distributions of a random wave field. In this context, it is a counterpart to the classic isotropic Kolmogorov-Zakharov solutions. The corresponding Kolmogorov constant of the wave momentum transfer is derived analytically. The study also discusses the implications of these results for sea wave modeling.

1. Introduction. Ocean swell within the kinetic equation

The seminal work of Klaus Hasselmann [1] transformed the four-wave kinetic equation from a purely theoretical concept to a powerful tool for modeling and forecasting sea waves (e.g. [2]). While the routine challenges of wave forecasting may have overshadowed the significance of the theoretical results related to this equation [3,4], it is essential to acknowledge that the milestone solutions to the kinetic equation (KE) revealed the fundamental mechanism of the Kolmogorov-Zakharov (KZ) cascading in a random field of weakly nonlinear water waves.

One pivotal solution, presented by Vladimir Zakharov and Natalia Filonenko in 1966 [5], describes the energy transfer to smaller wave scales, predicting a spectral tail of energy spectra $E(\omega) \sim \omega^{-4}$. The physical roots of this power-law behavior have been the subject of extensive debates for decades (e.g. [6]).

The second exact solution, known as the inverse cascade [7], describes the transfer of wave action to lower frequencies. Both the direct and inverse cascade solutions are isotropic and often regarded as theoretical rather than practical achievements. In contrast, the approximate anisotropic solution proposed by Alexander Kats and Victor Kontorovich [8–10] provides a valuable extension which captures the essential feature of sea waves, namely, their anisotropy.

The Kolmogorov-Zakharov solutions have primarily been analyzed through numerical methods [11–15], utilizing various approaches, including direct numerical simulations of the primordial dynamic equations (e.g. [16–18]). These studies have confirmed the power-law dependencies of wave spectra and estimated key physical constants known as Kolmogorov's ones related to the Kolmogorov-Zakharov solutions. However, the simulation setup poses significant challenges in obtaining the KZ solutions, especially, in anisotropic cases. Forcing and dissipation can disrupt the inherent directional features of the wave field. This makes it difficult to satisfy the formal restrictions imposed by KZ solutions, which assume stationarity across a sufficiently broad range of wave scales. This condition is essential to create an inertial interval with quasi-constant spectral fluxes resulting from wave-wave interactions. Simulations of anisotropic, non-stationary wave spectra within the conservative kinetic equation (i.e., without external forcing or wave dissipation) may help resolve these issues [19]. The slow evolution of the emerging spectra likely reflects both the properties of Kolmogorov-Zakharov spectra and observational characteristics of sea waves.

For clarity, we adopt the term *sea swell* for this physical setup, though we recognize that oceanographers may dispute this usage. A primary concern is the kinetic equation's (KE) validity for describing ocean swell spectra, which are typically assumed to be narrow-banded. Yet, KE simulations [19] demonstrate substantially broader spectral distributions. While the formal applicability criteria for the KE (e.g., Eq. 70 in [20]) are difficult to align with experimental data, a rigorous

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* Corresponding author at: Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, bld. 1, Moscow, 121205, Russia.
E-mail address: badulin.si@ocean.ru (S.I. Badulin).

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Declaration of competing interest

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Appendix. Anisotropic Kats–Kontorovich solution

Following Kats & Kontorovich [8] consider a generic form of anisotropic perturbation of the Zakharov–Filonenko spectrum (7)

$$E(\omega,\theta) = 2C_p \frac{(g^4 P)^{1/3}}{\omega^4}\left(1 + u\cdot\omega^{-m}\cos\theta\right) \tag{A.1}$$

with u and m to be arbitrary values to be determined. The parameter u is assumed to be small. The part of $S_{\rm nl}$ that is linear in the first (isotropic) term of (A.1) is related to the Kolmogorov-Filonenko spectrum (7) and, thus, vanishes. The formally small next-order term of the $S_{\rm nl}$, which is linear in the second (anisotropic) term of (A.1), can be expressed in terms of quadruplet representation [31] as follows

$$S_{\rm nl} = 8C_p^3 P\omega^{-1-m}u\cos\theta\, G(m) \tag{A.2}$$

where the dimensionless function $G(m)$ is

$$\begin{aligned} G(m) = {} & \frac{\pi}{2}\,\mathrm{Re}\int T^2\left(\omega_1\omega_2\omega_3\omega_4\right)^{-8} \\ & \times \Big[\left(\omega_2^8-\omega_3^8-\omega_4^8\right)\omega_1^{-m}\exp(i\theta_1)+\left(\omega_1^8-\omega_3^8-\omega_4^8\right)\omega_2^{-m}\exp(i\theta_2) \\ & + \left(\omega_1^8+\omega_2^8-\omega_4^8\right)\omega_3^{-m}\exp(i\theta_3)+\left(\omega_1^8+\omega_2^8-\omega_3^8\right)\omega_4^{-m}\exp(i\theta_4)\Big] \\ & \times \Big[\omega_1^{m+1}\exp(-i\theta_1)+\omega_2^{m+1}\exp(-i\theta_2) \\ & - \omega_3^{m+1}\exp(-i\theta_3)-\omega_4^{m+1}\exp(-i\theta_4)\Big]\,dq. \end{aligned} \tag{A.3}$$

Here ω_i and θ_i are defined by resonant quadruplet q.

The term (A.3) vanishes for $m = 1$ when the last multiplier of the integrand is the condition of four-wave resonance of wavevector components with the dispersion relation (4)

$$\omega_1^2\exp(-i\theta_1)+\omega_2^2\exp(-i\theta_2)-\omega_3^2\exp(-i\theta_3)-\omega_4^2\exp(-i\theta_4)=0 \tag{A.4}$$

Defining the wave momentum spectrum

$$K(\omega,\theta) = E(\omega,\theta)\,g^{-1}\omega\cos\theta \tag{A.5}$$

one can calculate the wave momentum flux

$$M(\omega,\theta) = \int_\omega^\infty d\omega \oint d\theta\,\frac{\partial K}{\partial t} = 8\pi C_p^3\,P\,g^{-1}u\omega^{1-m}\,\frac{G(m)}{m-1} \tag{A.6}$$

The indeterminacy at $m = 1$ in (A.6) can be resolved with the L'Hôpitale rule

$$M = 8\pi C_p^3\,P\,g^{-1}u\,\left.\frac{dG}{dm}\right|_{m=1} \tag{A.7}$$

This allows us to find u in (A.1)

$$u = \frac{C_m}{C_p}\,\frac{gM}{P} \tag{A.8}$$

where C_m is the dimensionless anisotropy Kolmogorov constant

$$C_m = \left(8\pi\,C_p^2 G'(m)|_{m=1}\right)^{-1} \tag{A.9}$$

Calculation of the derivative of G yields

$$\begin{aligned} G'(m)|_{m=1} = {} & \frac{\pi}{2}\int \tilde{T}^2\left(\omega_1\omega_2\omega_3\omega_4\right)^{-8} \\ & \times \Big[\left(\omega_2^8-\omega_3^8-\omega_4^8\right)\omega_1^{-1}e^{i\theta_1}+\left(\omega_1^8-\omega_3^8-\omega_4^8\right)\omega_2^{-1}e^{i\theta_2} \\ & + \left(\omega_1^8+\omega_2^8-\omega_4^8\right)\omega_3^{-1}e^{i\theta_3}+\left(\omega_1^8+\omega_2^8-\omega_3^8\right)\omega_4^{-1}e^{i\theta_4}\Big] \\ & \times \Big[\omega_1^2 e^{-i\theta_1}\ln\omega_1+\omega_2^2 e^{-i\theta_2}\ln\omega_2-\omega_3^2 e^{-i\theta_3}\ln\omega_3-\omega_4^2 e^{-i\theta_4}\ln\omega_4\Big]\,dq. \end{aligned} \tag{A.10}$$

The calculation yields

$$C_m \approx 0.093 \tag{A.11}$$

Thus, the Kats–Kontorovich solution becomes (cf. Eq. (14))

$$E(\omega,\theta) = 2\frac{g^4 P^{1/3}}{\omega^4}\left(C_p + C_m\frac{gM}{\omega P}\cos\theta\right) \tag{A.12}$$

Data availability

No data was used for the research described in the article.

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JFM PAPERS

Properties and asymptotics of water waves nonlinear interaction coefficient

Vladimir Geogjaev[1,2]

[1]Shirshov Institute of Oceanology of Russian Academy of Sciences, Nakhimovsky pr. 36, Moscow 117997, Russia

[2]Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, bld.1, Moscow 121205, Russia

Corresponding author: Vladimir Geogjaev, vvg@mail.geogjaev.ru

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The Hasselmann equation for the nonlinear interactions of deep-water gravity waves differs from other four-wave kinetic equations by the interaction coefficient. The explicit formula for this coefficient (e.g. Krasitskii, *J. Fluid. Mech.*, vol. 272, 1994, pp. 1–20; Zakharov, *Eur. J. Mech. B/Fluids,* vol. 18. issue 3, 1999, pp. 327–344) is of great complexity and leaves its properties obscured. We provide analytical results for the behaviour of the coefficient in different domains. The Phillips curve and discrete interaction approximation-like quadruplets are studied in detail. The coupling coefficient for the long–short wave interactions is calculated and found to be surprisingly small. This smallness greatly reduces the non-locality of the interactions.

Key words: turbulence theory, surface gravity waves, wave-structure interactions

1. Introduction

Gravity water waves are a significant and important example of dynamic systems. The Hamiltonian approach to their evolution (Zakharov 1968) has lead to the construction of the dynamic equation that describes the wave evolution. The wave steepness is used as a small parameter to expand the equation into the series of multi-wave interactions. The exclusion of non-resonant interactions with the use of a canonical transform of the dynamic variables yields the reduced Zakharov equation. The leading-order interactions for water waves are four-wave interactions. The interaction coefficient T for those interactions is the subject of the present paper.

The gravity water waves theory is the basis for the water waves modelling that has significant value in practical areas such as weather prediction and general climate applications. The complexity of four-wave interactions makes the existing wave models computationally expensive. The study of the interaction coefficient not only has a

If $\theta_{1-3} = 0$, then the minimum value of T is achieved. This may be demonstrated by calculating the partial derivatives at this point. Equation (4.15) shows that the derivative by θ_{1-3} is zero. For the other two coordinates, we may choose the $\boldsymbol{k}_{1+2}$ vector. One can see from the symmetry of (2.5) that the derivatives with respect to both its components are zero.

The angular quadruplets are located further apart from the Phillips curve. The quadruplets in between are different both from DIA-like and angular ones.

4.5. *Quadruplets variety*

We have described different classes of quadruplets and the behaviour of T for these classes. Let us summarise what parts of the quadruplet manifold were studied.

The long–short interactions (characterised by $s - \sqrt{2}/2 \ll 1$) directly border the DIA-like quadruplets. In fact, the side part of the Phillips curve may be considered as the long–short interactions as it has smaller vectors of size approximately $1/4$, and larger ones of size approximately $9/4$.

The DIA-like quadruplets occupy the vicinity of the Phiilps curve ($s \approx 1$). This include the large area near the geometric singularity at the centre of the curve.

The quadruplets that are further outside the Phillips curve ($s > 1$) were not covered in the present paper. They occupy an intermediate place between the DIA-like quadruplets and the 'angular' ones, and may have properties similar to both of these kinds. The T coefficient crosses zero in this area; nevertheless, many of these quadruplets having non-zero T are significant.

The outside part of the $\boldsymbol{k}$ plane is occupied by the 'angular' quadruplets ($s \ll 1$). They have a simple asymptotic described above.

5. Conclusions

We have described the properties of the interaction coefficient T in different domains. Taken broadly, these domains cover nearly all of the quadruplet variety, with the notable exception of the quadruplets between the DIA-like and 'angular' ones.

The importance of the interaction quadruplets located at the Phillips curve, and the significance of its central zone, were shown. The behaviour of T in that domain was studied, and an explicit formula (4.2) for DIA-like quadruplets was obtained together with some approximations.

The long–short interaction asymptotics (4.12) were calculated. We have shown that both the first and second order of the T expansion in this case are zeros. Such unusual properties greatly reduce the non-locality of the kinetic equation.

We have studied the 'angular quadruplets', and we have found the minimum value -2 of the normalised T coefficient.

This study may facilitate the development of more efficient numerical models of water waves. The specific role of the singularity at the centre of the Phillips curve cannot be ignored in the models. On the other hand, the unusual smallness of the long–short interaction coefficient justifies the partial omission of such interactions. The estimates of T given above may help in testing the correctness of the models.

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Quadruplet form of water waves kinetic equation ☆

V. Geogjaev *

P.P. Shirshov Institute of Oceanology of the Russian Academy of Sciences, 36 Nakhimovsky pr., Moscow, 117997, Russia
Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30, bld. 1, Moscow, 121205, Russia

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ABSTRACT

We present the quadruplet form of the kinetic equation for deep water surface waves (the Hasselmann equation). This formulation explicitly utilizes quadruplets of interacting waves. We define the integration over the quadruplets and rewrite the interaction integral using this definition. We use our construction to study the kinetic equation, in particular, we calculate the Kolmogorov constants for the Zakharov–Filonenko spectra.

1. Introduction

Gravity water surface waves are a significant and interesting example of dynamic systems. They may be described as a system of wave modes of different wave vectors exchanging their energy through weakly nonlinear interactions. These interactions are resonant and occur between sets of four waves satisfying the resonant conditions. The kinetic equation for the nonlinear interactions was obtained in [1]. This equation is non-local and includes a complicated interaction integral.

The research works of V. E. Zakharov [2–7] generalized and significantly extended previous results on dynamical and statistical descriptions of water waves. The Hamiltonian formulation for potential water waves led to an elegant description of weakly nonlinear waves with the reduced Zakharov equations [8]. These dynamical equations emphasize the role of four-wave interactions in the evolution of water waves and their symmetry features. The derivation of the kinetic equation within the Hamiltonian description helps establish the symmetry properties of the collision integral and its kernel [8]. Within the kinetic equation resonant four-wave interactions emerge as the primary physical mechanism governing the evolution of the wave field.

The key role of the quadruplet interactions is well-known but is not utilized in its full. The conformal mapping of the resonant manifold has been used for derivation of the key results of the weak turbulence theory, e.g. the stationary isotropic Zakharov–Filonenko solution [2] or the anisotropic approximate spectra by Kats and Kontorovich [9]. The Discrete Interaction Approximation (DIA) method of approximate calculation of the collision integral in the numerical wave forecasting models [10] also uses the quadruplet representation of the kinetic equation, though simplified to a sole pair of quadruplets.

In the present paper we develop the quadruplet approach by introducing a quantitative measure to the quadruplet manifold and using it for the construction of the collision integral.

The resonant quadruplets constitute a 3-dimensional manifold. Our approach is to define the integration over this manifold accounting for the essentially different contribution of the manifold domains

The primary application of the quadruplet form of the kinetic equation lies in the field of numerical integration. The details of this field are beyond the scope of the present paper; here we briefly outline the approach. For any reasonable setup one can make the integration grid sufficiently fine to yield the accurate results. A general problem is that the quadruplet manifold is 3-dimensional. Combined with the 2-dimensional spectral grid, this results in a 5-dimensional integral. Such a high dimensionality means that a finer grid would require substantial computational resources. Furthermore, the properties of the quadruplets and their contribution to the collision integral vary significantly; failure to account for this variation can lead to waste of resources for the non-significant parts of the grid. In addition, the nonlocality of interactions further complicates the problem.

Several approaches exist for the problem of the numerical integration of the kinetic equation (see for review [11–14]. The algorithm presented in [15] includes the quadruplet interactions and deals with the singularities resulting from the parameterization of quadruplets. The WRT (Webb–Resio–Tracy) algorithm [16] covers all the quadruplets but does not account for their essentially different contributions and does not optimize the corresponding numerical grids.

The issue of integration over quadruplets is essential in numeric calculation. Numerical integration requires a construction of some

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* Correspondence to: P.P. Shirshov Institute of Oceanology of the Russian Academy of Sciences, 36 Nakhimovsky pr., Moscow, 117997, Russia.
E-mail address: vvg@mail.geogjaev.ru.

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Zeros of F correspond to the stable spectra ($\frac{\partial N_{\mathbf{k}}}{\partial t} = 0$). These zeros are the result of zeroing of the first bracket in (37). For $x = 23/6$ the zeroing is obvious; for $x = 4$ the bracket becomes $\tilde{\omega}_1 + \tilde{\omega}_2 - \tilde{\omega}_3 - \tilde{\omega}_4$ which is zero due to (6).

These zeros are present not only for the whole Eq. (1) but for any given quadruplet or quadruplet set. This means that any approximate model of the kinetic equation, however primitive, has the Zakharov–Filonenko spectra as a solution. The only condition is that the quadruplet form of interaction is kept.

The spectra with $x = 4$ and $x = 23/6$ are stable solutions of (1). First of them has non-zero energy flux P to high frequencies. This flux defines the magnitude of the spectrum:

$$N_k^{(1)} = c_1\, P^{1/3}\, k^{-4} \tag{38}$$

The constant c_1 is

$$c_1 = \left(\frac{2\pi}{3} \left. \frac{dF}{dx} \right|_{x=4} \right)^{-\frac{1}{3}} \tag{39}$$

and may be calculated as the following quadruplet integral (resulting from the differentiation of (37)):

$$c_1 = \left[8\pi^2 \int_q \tilde{T}^2 \left(\tilde{\omega}_1 \ln \tilde{\omega}_1 + \tilde{\omega}_2 \ln \tilde{\omega}_2 - \tilde{\omega}_3 \ln \tilde{\omega}_3 - \tilde{\omega}_4 \ln \tilde{\omega}_4 \right) \times \right.$$
$$\left. \times \left((\tilde{\omega}_2\tilde{\omega}_3\tilde{\omega}_4)^{-8} + (\tilde{\omega}_1\tilde{\omega}_3\tilde{\omega}_4)^{-8} - (\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_4)^{-8} - (\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3)^{-8} \right) \mathrm{d}q \right]^{-\frac{1}{3}} \tag{40}$$

or, in more explicit form

$$c_1 = \left[8\pi^2 \int_{\sqrt{2}/2}^{+\infty} \mathrm{d}s \int_{\theta_m}^{2\pi-\theta_m} \mathrm{d}\tilde{\theta}_1 \int_{\theta_m}^{2\pi-\theta_m} \mathrm{d}\tilde{\theta}_3 \times \right.$$
$$\times J\, \tilde{T}^2 \left(\tilde{\omega}_1 \ln \tilde{\omega}_1 + \tilde{\omega}_2 \ln \tilde{\omega}_2 - \tilde{\omega}_3 \ln \tilde{\omega}_3 - \tilde{\omega}_4 \ln \tilde{\omega}_4 \right) \times$$
$$\left. \times \left((\tilde{\omega}_2\tilde{\omega}_3\tilde{\omega}_4)^{-8} + (\tilde{\omega}_1\tilde{\omega}_3\tilde{\omega}_4)^{-8} - (\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_4)^{-8} - (\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3)^{-8} \right) \right]^{-\frac{1}{3}} \tag{41}$$

In this formula the normed frequencies $\tilde{\omega}_i$ as well as $\tilde{T}$ and J may be derived from the independent variables s, θ_1 and θ_3. The integral does not include δ-functions and can be calculated numerically. Special care should be taken at the singularity at $s = 1$, $\theta_1 = \theta_3 = 0$ and at the long–short interactions zone ($s \to \sqrt{2}/2$).

Numerical calculation yields the value

$$c_1 = \mathbf{0}.194 \tag{42}$$

The second spectrum provides a non-zero flux Q of wavenumber to the small frequencies (which results in the inverse Kolmogorov cascade). It is given by

$$N_k^{(2)} = c_2\, Q^{1/3}\, k^{-23/6} \tag{43}$$

The constant c_2 is

$$c_2 = \left(-\frac{2\pi}{3} \left. \frac{dF}{dx} \right|_{x=23/6} \right)^{-\frac{1}{3}} \tag{44}$$

which is given by the following integral:

$$c_2 = \left[-8\pi^2 \int_q \tilde{T}^2 \left(\ln \tilde{\omega}_1 + \ln \tilde{\omega}_2 - \ln \tilde{\omega}_3 - \ln \tilde{\omega}_4 \right) \times \right.$$
$$\left. \times \left((\tilde{\omega}_2\tilde{\omega}_3\tilde{\omega}_4)^{-\frac{23}{3}} + (\tilde{\omega}_1\tilde{\omega}_3\tilde{\omega}_4)^{-\frac{23}{3}} - (\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_4)^{-\frac{23}{3}} - (\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3)^{-\frac{23}{3}} \right) \mathrm{d}q \right]^{-\frac{1}{3}} \tag{45}$$

Numerical calculation yields the value

$$c_2 = \mathbf{0}.203 \tag{46}$$

5.3. Quadruplet difference

The quadruplet form of the kinetic Eq. (29) allows one to study the properties of different quadruplets and their role in spectral evolution.

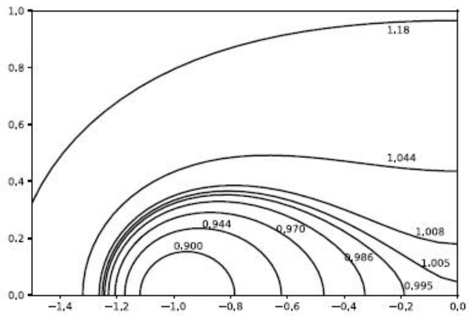

Fig. 3. The lines of constant s chosen so that they divide the quadruplet manifold in ten parts of equal energy transfer rate for the Zakharov–Filonenko spectrum. The corresponding values of s are shown in the figure. (The figure shows a quarter of the **k** plane).

The equation includes a measure on the quadruplet manifold which allows for a quantitative comparison between different domains of the manifold.

As an example we calculate the distribution of the energy transfer rate for the Zakharov–Filonenko spectrum (38). We divide the whole span of the variable s into parts so that the energy transfer rate is equal for each part. Fig. 3 shows the resulting division (calculated numerically). One may see that the most significant energy transfer occurs in the central domain around the Phillips eight curve which accounts for a large part of the transfer.

6. Conclusions

We have constructed the quadruplet form of the kinetic equation for the deep water surface waves. This form explicitly includes the interacting quadruplets of waves and allows one to deal with the separate quadruplets as well as with domains of the quadruplet manifold. The coordinates introduced on this manifold together with the Jacobian J allows the integration over the quadruplets.

This form of the kinetic equation is convenient for both analytic and numerical calculations. It may serve as a basis for the construction of kinetic equation approximations based on the most significant domains of the quadruplets manifold.

The above construction allows the analytic interaction calculation for a number of significant cases such as Zakharov–Filonenko and Zakharov–Zaslavskii spectra. We have obtained the analytic integral formulas for those spectra and calculated numerically the corresponding constants.

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Declaration of competing interest

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Applying canonical transformations to the nonlinear Schrödinger equation for modeling fiber optic communications

Sergey Dremov [a,b,*], Dmitry Kachulin [a,b], Igor Chekhovskoy [b], Oleg Sidelnikov [b], Alexey Redyuk [b], Alexander Dyachenko [c]

[a] Skolkovo Institute of Science and Technology, Moscow, 121205, Russia
[b] Novosibirsk State University, Novosibirsk, 630090, Russia
[c] L.D. Landau Institute for Theoretical Physics RAS, Chernogolovka, 142432, Russia

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ABSTRACT

This paper presents a numerical implementation of a novel algorithm for modeling optical signal propagation under the nonlinear Schrödinger equation. Rooted in the Hamiltonian formalism and canonical transformations, the method is mathematically rigorous and devoid of empirical techniques. We investigate the algorithm's properties and range of applicability both analytically and numerically, highlighting its advantages over existing methods. An extension is also developed to include dissipative effects. The proposed approach is applied to various wave fields, and its performance is compared with standard dispersion and nonlinear phase compensation techniques. These initial results demonstrate the potential of the algorithm for fiber-optic communication applications.

1. Introduction

The nonlinear Schrödinger equation (NLSE) is a fundamental partial differential equation describing the evolution of complex wave envelopes in weakly nonlinear, dispersive media. It is widely used in various physical [1–4], biological [5,6], financial [7,8], and other nonlinear systems. In physics, it occupies a special place in fiber optics and photonics for modeling the propagation of optical signals [9].

Modern bandwidth-intensive services such as online games and video streams, cloud services, and others require ever greater capacity. This rapid growth rate necessitates the development of more efficient data transmission technologies and the optimization of available resources.

The most common method of data transmission is a fiber optic communication system [10]. The current level of research and technology allows for an effective consideration of fiber dispersion [11], noises of various natures [12], and signal power loss [13–15]. However, nonlinearity remains a major limiting factor to increase the overall efficiency of the technology [1,16].

Leaving aside different hardware methods, several widely used approaches take nonlinearity into account. One common method is Digital Back Propagation (DBP) [17,18], which digitally simulates the propagation of a signal through a fiber in the reverse direction. Another common approach is to use the Volterra series [19], which is analytical and universal in the world of nonlinear phenomena [20]. Promising methods also include the Nonlinear Fourier Transform (NFT) [21–25], and the Perturbation-Based Method (PBM) [26,27]. In addition, the Machine Learning (ML) methods are being actively developed and used [28], particularly deep neural networks (DNNs) [29,30]. Research in the field of soliton data transmission technologies is also ongoing [31,32].

However, the DBP, the Volterra Series, and the NFT require significant computational costs in practice. The PBM is applicable only in a certain range of nonlinear parameter values and has limited effectiveness at high signal powers [1]. ML methods, though powerful, often function as "black boxes," lacking interpretability, and are susceptible to overfitting, raising concerns about generalization beyond their training data. Moreover, the majority of these algorithms are united by some empiricism, that is, knowledge of the initial signal is necessary to correctly configure them. This seems to be a convenient approach in practical applications, but in terms of fundamental science, the lack of some mathematical and physical rigor encourages researchers to seek explanations and develop new algorithms.

In this work, we propose another approach, originally suggested earlier in [33], which can be used to model fiber optic communications. It is based on the Hamiltonian structure of the NLSE, and the application of the nonlinear canonical transformation simplifying the Hamiltonian and the corresponding equation of motion. In some sense, the approach represents a different view on the NLSE integrability [34], however, not from the perspective of the scattering problem, but from

* Corresponding author.
E-mail address: S.Dremov@skoltech.ru (S. Dremov).

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revealed. Gaussian pulses were used as test wave fields in numerical simulations, which validated the correctness of the algorithm and confirmed its properties revealed earlier. It was shown that, in contrast to the CDC, the CT can correctly account for nonlinear phase shifts.

- The CT was linearly extended for the dissipative case, the analytical solution with the presence of power losses was found. The solution provides the ability to analytically describe nonlinear effects even taking into account the presence of amplification and signal power loss.
- The CT was applied for the first time for modeling fiber-optic communications. QPSK was used to generate the input symbol sequence, and RRC was used for pulse shaping. It was shown that the CT algorithm can successfully recover such signals in the range of its applicability. The comparison between signal recovery using the CDC with phase minimization and the CT was made. Additional calculation of the L_2-norm demonstrated that the results can be considered comparable.
- Contrary to other known methods for signal recovery, the CT does not use empiricism and has mathematical rigor. That is why it can be used as a tool for approximate analytical description of nonlinear phenomena in fiber optics.

The first step taken opens up a wide range of possibilities for further research. Although for QPSK both algorithms show relatively comparable results, for amplitude coding like QAM16, the nonlinear phase compensation will start to lose accuracy due to the different amplitudes of the constellation points, and the CT can provide more accurate results.

Future plans also include computing the bit error rate (BER) for a full sequence of symbols. A comparison of the BER for both algorithms will certainly be a more rigorous measure of their performance. Moreover, a comparative study with the DBP remains an important and definite objective for future research.

In addition, the extension of the CT for dissipative case showed that it can be applied to model the dynamics of a full fiber optic transmission system with spans in which there are amplifiers and power losses. It may be considered in future.

CRediT authorship contribution statement

Sergey Dremov: Writing – original draft, Investigation, Formal analysis, Software, Visualization; **Dmitry Kachulin:** Investigation, Formal analysis, Software, Writing – review & editing; **Igor Chekhovskoy:** Writing – review & editing, Validation; **Oleg Sidelnikov:** Writing – review & editing, Validation; **Alexey Redyuk:** Writing – review & editing, Project administration; **Alexander Dyachenko:** Conceptualization, Investigation.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. The Extension of the Algorithm to Dissipative Case

Consider again the following equation with dissipative terms:

$$\frac{\partial b_n}{\partial z} = -\frac{\alpha}{2} b_n + i\frac{\beta_2}{2}\omega_n^2 b_n + i\gamma \left[2\sum_{n_1} |b_{n_1}|^2 - |b_n|^2\right] b_n. \tag{38}$$

Due to the presence of dissipation, the square of amplitude modulus b_n is no longer conserved.

$$\frac{\partial |b_n|^2}{\partial z} = -\alpha |b_n|^2, \implies |b_n(z)| = |b_n(0)| e^{-\frac{\alpha}{2} z}. \tag{39}$$

Therefore, the nonlinear phase shift is no longer constant along z.

$$\lambda_n(z) = \gamma \left[2\sum_{n_1} |b_{n_1}(z)|^2 - |b_n(z)|^2\right]. \tag{40}$$

But it depends only on $|b_n|^2$, so the solution can also be obtained in the following form:

$$\lambda_n(z) = \gamma \left[2\sum_{n_1} |b_{n_1}(0)|^2 - |b_n(0)|^2\right] e^{-\alpha z} = \lambda_n(0) e^{-\alpha z}. \tag{41}$$

Consider the general solution of (19) as:

$$b_n(z) = |b_n(0)| e^{-\frac{\alpha}{2} z + i\frac{\beta_2}{2}\omega_n^2 z + i\Phi_n(z)}. \tag{42}$$

Substituting it into (19) yields the equation for $\Phi_n(z)$:

$$\frac{\partial \Phi_n}{\partial z} = \lambda_n(0) e^{-\alpha z}. \tag{43}$$

Then,

$$\Phi_n(z) = \Phi_n(0) + \lambda_n(0)\frac{1 - e^{-\alpha z}}{\alpha}. \tag{44}$$

Here $\Phi_n(0)$ is the phase of each b_n at $z = 0$. A simple limit check of $\alpha \to 0$ gives:

$$\Phi_n(z) = \Phi_n(0) + \lambda_n(0) z \tag{45}$$

Finally, the solution of (19) reads:

$$b_n(z) = b_n(0) e^{-\frac{\alpha}{2} z} e^{i\left[\frac{\beta_2}{2}\omega_n^2 z + \lambda_n(0)\left(\frac{1 - e^{-\alpha z}}{\alpha}\right)\right]}, \tag{46}$$

where

$$\lambda_n(0) = \gamma \left[2\sum_{n_1} |b_{n_1}(0)|^2 - |b_n(0)|^2\right]. \tag{47}$$

An analytical solution is still exists even for different increments $\alpha = \alpha_n$ for each harmonic b_n:

$$|b_n(z)| = |b_n(0)| e^{-\frac{\alpha_n}{2} z}. \tag{48}$$

Nonlinear phase shifts λ_n now are:

$$\lambda_n(z) = \gamma \left[2\sum_{n_1} |b_{n_1}(0)|^2 e^{-\alpha_{n_1} z} - |b_n(0)|^2 e^{-\alpha_n z}\right]. \tag{49}$$

A general solution is the following:

$$b_n(z) = |b_n(0)| e^{-\frac{\alpha_n}{2} z + i\Phi_n(z)}. \tag{50}$$

Again, one can obtain the expression for Φ_n:

$$\frac{\partial \Phi_n}{\partial z} = \lambda_n(z) = \gamma \left[2\sum_{n_1} |b_{n_1}(0)|^2 e^{-\alpha_{n_1} z} - |b_n(0)|^2 e^{-\alpha_n z}\right]. \tag{51}$$

The solution is as follows:

$$\Phi_n(z) = \Phi_n(0) + \gamma \left[2\sum_{n_1} |b_{n_1}(0)|^2 \frac{1 - e^{\alpha_{n_1} z}}{\alpha_{n_1}} - |b_n(0)|^2 \frac{1 - e^{\alpha_n z}}{\alpha_n}\right]. \tag{52}$$

Finally, it yields:

$$b_n(z) = b_n(0)\; e^{\frac{-\alpha_n z}{2}} e^{i\left[\frac{\beta_2}{2}\omega_n^2 z + \gamma\left(2\phi_\Sigma - \phi_n(z)\right)\right]}, \tag{53}$$

where

$$\phi_n(z) = |b_n(0)|^2 \frac{1 - e^{\alpha_n z}}{\alpha_n},$$

$$\phi_\Sigma = \sum_{n_1} \phi_{n_1}(z). \tag{54}$$

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ЗВУКОВАЯ ТУРБУЛЕНТНОСТЬ: ОТ СПЕКТРОВ ЗАХАРОВА—САГДЕЕВА К СПЕКТРУ КАДОМЦЕВА—ПЕТВИАШВИЛИ

Е. А. Кочурин[1,2*], *Е. А. Кузнецов*[2,3,4]

[1] Институт электрофизики Уральского отделения РАН, г. Екатеринбург;
[2] Сколковский институт науки и технологий, г. Москва;
[3] Физический институт им. П. Н. Лебедева РАН, г. Москва;
[4] Институт теоретической физики им. Л. Д. Ландау РАН, г. Черноголовка, Россия

Представлен краткий обзор теоретических и численных работ по трёхмерной акустической турбулентности как в слабонелинейном режиме, когда амплитуды звуковых волн малы, так и в случае сильной нелинейности. Основанием этого обзора стали классические исследования, с одной стороны, В. Е. Захарова и Р. З. Сагдеева [1, 2] по слабой акустической турбулентности, а с другой — Б. Б. Кадомцева и В. И. Петвиашвили [3]. До недавнего времени не было убедительных численых экспериментов, подтверждающих ту или другую точку зрения. В работах [4, 5] авторов данного обзора на основе прямого численного моделирования были найдены веские аргументы в пользу той и другой теории. Показано, что спектр слабой турбулентности Захарова—Сагдеева (с зависимостью от волнового числа k вида $k^{-3/2}$) реализуется не только при малой положительной дисперсии звуковых волн, но и в случае полного отсутствия дисперсии. Рассчитанные спектры турбулентности в слабонелинейном режиме имеют анизотропное распределение: в области малых k формируются узкие конусы (джеты), уширяющиеся в фурье-пространстве. В случае слабой дисперсии джеты сглаживаются, а спектр турбулентности стремится к изотропному в области коротких длин волн. В отсутствие дисперсии спектр турбулентности представляет собой дискретный набор джетов, подверженных дифракционной расходимости. Выяснено, что для каждого отдельного джета нелинейные эффекты намного слабее дифракционных, что препятствуют формированию ударных волн. Таким образом, спектры слабой турбулентности Захарова—Сагдеева реализуются за счёт малости нелинейных эффектов по сравнению с дисперсией или дифракцией. При увеличении уровня накачки в бездисперсионном режиме, когда нелинейные эффекты начинают преобладать, происходит формирование ударных волн — разрывов плотности. В итоге акустическая турбулентность переходит в сильнонелинейное состояние в виде ансамбля случайных ударных волн, который описывается спектром Кадомцева—Петвиашвили, спадающим по закону k^{-2}.

ВВЕДЕНИЕ

Как известно, развитая гидродинамическая турбулентность при больши́х числах Рейнольдса, $\mathrm{Re} \gg 1$, в инерционном интервале представляет собой пример системы с сильной нелинейностью, когда её энергия совпадает с гамильтонианом взаимодействия. Другим классическим примером является акустическая турбулентность, которая демонстрирует как сильные, так и слабые режимы в зависимости от соотношения нелинейных и линейных волновых характеристик. В этом смысле акустическая турбулентность гораздо разнообразнее и богаче гидродинамической турбулентности. Когда нелинейное взаимодействие волн мало по сравнению с линейными эффектами, реализуется режим слабой турбулентности [6], который можно изучать на основе теории возмущений, используя приближение случайных фаз. Теория слабой турбулентности [6, 7] статистически описывает ансамбли взаимодействующих волн в рамках соответствующих кинетических уравнений [6, 7]. Эта теория предполагает, что каждая волна со своей случайной фазой достаточно долго распространяется почти свободно и очень редко претерпевает небольшие деформации из-за нелинейного взаимодействия с другими волнами. На сегодняшний день теория слабой турбулентности

* kochurin@iep.uran.ru

ности быстрых магнитозвуковых волн в плазме с малым β демонстрирует спектр Захарова—Сагдеева, т. к. волны такого типа имеют звуковой закон дисперсии $\omega = kV_{\rm A}$ (здесь $V_{\rm A}$ — альфвеновская скорость) в области частот, меньших циклотронной ионной частоты $\omega_{\rm ci}$. Если рассмотреть медленные магнитозвуковые волны, которые в плазме с малым β можно считать замагниченным ионным звуком, то, как было показано в [42], один из спектров слабой турбулентности, соответствующий постоянному потоку энергии в область коротких волн, несмотря на сильную анизотропию, демонстрирует ту же самую зависимость от модуля k, что и спектр Захарова—Сагдеева. При малых β такую же зависимость от модуля k показывают спектры слабой турбулентности быстрых магнитозвуковых и альфвеновских волн, взаимодействующих с медленными магнитозвуковыми волнами [43]. Недавние эксперименты (см. работы [44, 45] и ссылки в них) по наблюдению спектров флуктуаций магнитного поля в солнечном ветре вблизи Солнца показали зависимости, близкие к спектру $\omega^{-3/2}$, где параметр β мал, $\sim 0{,}1$. Спектр Кадомцева—Петвиашвили, как показывают численные эксперименты [46] с приложениями к астрофизике, часто встречается в магнитогидродинамической турбулентности, равно как и спектр $k^{-3/2}$. Важно, что анизотропия спектров за счёт магнитного поля не влияет на их зависимость от модуля k. Что касается слабой магнитогидродинамической турбулентности, то такое поведение вытекает из простых размерностных оценок.

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Эффекты сильной турбулентности волн на воде

Е. А. Кочурин[+*1)], *Е. А. Кузнецов*[*×○1)]

[+]*Институт электрофизики Уральского отделения РАН, 620016 Екатеринбург, Россия*

[*]*Сколковский институт науки и технологий, 121205 Москва, Россия*

[×]*Физический институт им. П. Н. Лебедева РАН, 119991 Москва, Россия*

[○]*Институт теоретической физики им. Л. Д. Ландау РАН, 142432 Черноголовка, Россия*

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Представлены результаты прямого численного моделирования плоско-симметричной турбулентности волн на воде для потенциальных течений в рамках конформных переменных с учетом низкочастотной накачки и высокочастотного затухания вязкого типа. В данной модели для широкого диапазона амплитуд накачки не обнаружен режим слабой турбулентности. Показано, что для типичных параметров турбулентности главными эффектами являются процессы опрокидывания волн, формирования на их гребнях каспов, которые вносят основной вклад в спектры турбулентности с зависимостью от частоты и волнового числа с одной и той же степенью, равной -4. В этом сильно нелинейном режиме плотность вероятности крутизны волн при больших отклонениях имеет степенные хвосты, ответственные за перемежаемость турбулентности.

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Введение. Хорошо известно, что основным механизмом затухания морского волнения даже при умеренной ветровой накачке является опрокидывание волн (см., например, [1]). Как показывают многочисленные наблюдения, процесс опрокидывания начинается на гребне волны в перпендикулярном направлении к его фронту. Этот процесс носит одномерный характер. На начальном этапе опрокидывания в профиле волны происходит формирование особенности в виде острия. Таким образом, в заданной точке x_0 профиль отклонения поверхности $y = \eta(t)$ и его производная $d\eta(t)/dt$ до момента опрокидывания $t = t_0$ остаются гладкими функциями времени. При $t = t_0$ производная $d\eta(t)/dt$ испытывает скачок, а ее производная будет пропорциональна δ-функции времени. Таким образом, в точке наблюдения $x = x_0$ вторую производную от $\eta(t)$ можно записать в виде

$$\frac{d^2\eta}{dt^2} = \sum_i \Gamma_i \delta(t - t_i) + \text{regular terms}, \qquad (1)$$

где t_i – времена формирования особенностей, Γ_i – скачок величины $d^2\eta/dt^2$ в момент формирования сингулярности. Совершая преобразование Фурье от сингулярной части (1), получаем зависимость η_ω от частоты:

$$\eta_\omega = \frac{1}{\sqrt{2\pi}\omega^2} \sum_i \Gamma_i e^{-i\omega t_i}.$$

Считая величины Γ_i и t_i случайными, отсюда после усреднения находится вклад от сингулярностей в спектр E_ω:

$$E_\omega = \frac{g}{2\pi T}\langle|\eta_\omega|^2\rangle = \frac{g\nu}{2\pi\omega^4}\langle\Gamma^2\rangle, \qquad (2)$$

где g – ускорение поля тяжести, T – время осреднения, ν – средняя частота появления особенностей, угловые скобки $\langle ... \rangle$ означают усреднение по времени. Спектр (2) был получен в работе [2]. Важно отметить, что зависимость (2) от частоты, пропорциональная ω^{-4}, совпадает со спектром слабой турбулентности Захарова–Филоненко [3], который к настоящему времени очень хорошо воспроизведен как экспериментально, так и численно для изотропного распространения волн на воде (см. обзор [4] и ссылки в нем). Это совпадение чисто случайное, поскольку спектр слабой турбулентности предполагает слабо нелинейный режим, когда нелинейность мала по сравнению с дисперсией линейных гравитационных волн на глубокой воде

$$\omega = \sqrt{gk}. \qquad (3)$$

1)e-mail: kochurin@iep.uran.ru; kuznetso@itp.ac.ru

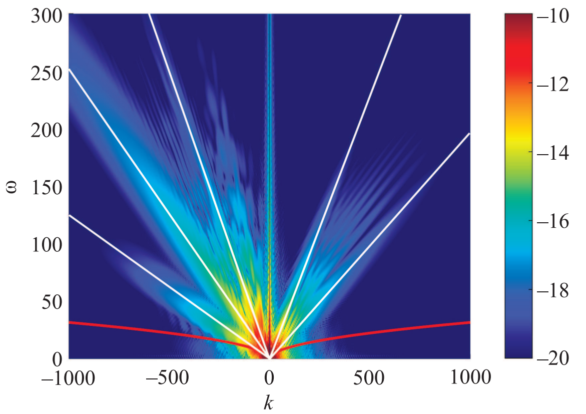

Рис. 7. (Цветной онлайн) Пространственно-временной спектр возмущений поверхности $\log|\eta(\omega,k)|$, красные сплошные линии – линейный закон дисперсии $\omega = k^{1/2}$, белые сплошные линии соответствуют бездисперсионному распространению нелинейных волн $\omega \sim k$

ударно-волновой режим турбулентности реализуется также для слабодисперсионных магнитогидродинамических волн на границах жидкостей [36, 37].

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Magnetic field growth in accretion disc: growth and nonlinear saturation

E.A. Mikhailov[1,2,3], E.N. Zhikhareva[2], M.V. Frolova[2]

[1] *Department of Theoretical Physics, P.N. Lebedev Physical Institute of RAS, Leninskii pr. 53, Moscow, 119991 Russia*
[2] *Faculty of Physics, M.V. Lomonosov Moscow State University, Leninskie gory 1 2, Moscow, 119991 Russia*
[3] *Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30 1, Moscow, 121205 Russia*

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Astrophysical accretion discs associated with massive objects should have some magnetic fields. There are strong arguments that such fields should be, at least partly, connected with dynamo action that is based on properties of small- and large-scale motions in the object. Now there are some works connected with studies of the accretion disc dynamos, and it has been shown that the magnetic field can grow there. However, there is a question connected with field generation and its timescale. The properties of the accretion disc are quite different for various parts of the object, so the growth rate can differ dramatically, too. Another problem is connected with nonlinear saturation connected with the field conservation law. Here we describe different approaches to estimate the field growth and describe its structure.

Keywords: Magnetic fields, growth rate, thin disc model, *RZ* approximation

1 Introduction

Accretion discs play an important role in relativistic astrophysics [1–4]. They surround compact massive objects (black holes, neutron stars, white dwarfs) and contain rapidly rotating medium which can be strongly magnetized. The arguments for the existence of the magnetic field are based on the magnetohydrodynamic processes: they can explain transition of the angular momentum and the medium [1]. Also there are some proofs of the magnetic field at least for such a specific object as an accretion disc surrounding the black hole in the active nuclei of M 87 galaxy [5]. Radioastronomical observations show that there is a remarkable Faraday rotation for electromagnetic waves passing from it, so there should be a magnetic field that produces the rotation

*Email: e.mikhajlov@lebedev.ru

4 Conclusions

We have studied the magnetic field growth in accretion discs using two sufficiently different approaches. First of all, we have studied the field evolution using the thin disc approximation based on the algebraic model for vertical dependence. This model allows us to study the field in the nonaxisymmetric case. However, it has been shown that the field non-uniformities are destroyed by the rotation. After that we used more effective RZ approximation which takes into account that the field depends on the z coordinate according to a differential law. We have studied typical timescales for the magnetic field growth for both cases, and typical field structures.

It can be said that the magnetic field in the nonaxisymmetric case can be modelled using the thin disc approximation. However, vertical structures are described better using the RZ approach.

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Solitons in the semi-infinite ferromagnets with the different types of anisotropy

V.V. Kiselev [b,c,*], A.A. Raskovalov [a,b,c]

[a] Skolkovo Institute of Science and Technology (Skoltech), Bolshoy Boulevard 30, bld. 1, Moscow, 121205, Russia
[b] M.N. Mikheev Institute of Metal Physics, Ural Branch of the Russian Academy of Sciences, Sofia Kovalevskaya str., 18, Ekaterinburg, 620108, Russia
[c] Institute of Physics and Technology, Ural Federal University, Mira str., 19, Ekaterinburg, 620002, Russia

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ABSTRACT

In the framework of the Landau–Lifshitz model the new types of solitons in the semi-infinite ferromagnet are analytically described under the boundary conditions, corresponding to the partial pinning of spins at the boundary of the sample. The problem is successively solved for isotropic, easy-axis, easy-plane and two-axis ferromagnets. Comparative analysis of interaction of solitons with the surface of the sample is performed.

1. Introduction

Under strong external influences on magnets, their dynamical, thermodynamical, transport properties, features of structural and phase transitions are largely determined by long-lived soliton-like states. The basis equations of magnetism – the vector Landau–Lifshitz equations – are considerably nonlinear and complicated. In some special cases they are reducible to more simple equations, which often admit exact solutions. Correspondingly, investigations of ferromagnetic solitons and their practical applications are largely developed for simplified models, such as sine–Gordon equation [1–4] and generalized nonlinear Shrődinger equation for the envelope of small-amplitude waves [5–11]. In the work [12] an interaction of solitons in magnetic bilayer is investigated in the framework of the two-component integrable Manakov model, originally coming from nonlinear optics. As well as in the optical mediums (see, for example, [13] and the literature, cited here), a great importance for magnetic systems has a possibility to control the motion and properties of solitons by driving fields. Considerable results in this field were achieved in the works [7–10], where the new methods of auto-resonance generation of one-dimensional solitons and driving their properties were found. In the works [11,14] a new way to control the quantum bits (qubits) in quantum computers was proposed, meaningful for technological applications.

As for the original Landau–Lifshitz vector model, by now this model is widely used in the most part of investigations, concerning the formation of the domain structure and topological defects, as well as an analysis of the soliton dynamics. As a rule, at such investigation the dissipative term in the Gilbert form is taken into account. Therefore, it is naturally, that studies are mainly performed by means of numerical calculations without using analytical methods [15–17]. That allows to take into account additional terms. They could be responsible, for example, for the spin current, which leads to steady motion of the domain walls in ferromagnets [18,19], the single chiral soliton [20] and chiral soliton lattice in helimagnets [21], the spin inertia and auto-oscillations in a ferromagnet [22]. The work [23] considers the problem of the spin torque oscillator, that is, a uniaxial ferromagnet in an external

* Corresponding author at: M.N. Mikheev Institute of Metal Physics, Ural Branch of the Russian Academy of Sciences, Sofia Kovalevskaya str., 18, Ekaterinburg, 620108, Russia.

E-mail addresses: kiseliev@imp.uran.ru (V.V. Kiselev), raskovalov@imp.uran.ru (A.A. Raskovalov).

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4. Conclusion

In this work, we present the comparative analysis of quasi-one-dimensional solitons in the models of semibounded ferromagnets, taking into account only the main exchange interaction, exchange interaction with different types of crystallographic anisotropy and the magnetostatic interaction. The mixed boundary conditions are considered, the limiting cases of which correspond to free and completely fixed spins at the boundary of the sample. It is established, that exchange interaction and an easy-axis anisotropy (where anisotropy axis is parallel to the surface of the sample) admit two types of precession solitons. Solitons of the first type are localized near the boundary of the sample and are formed when the amplitude of the surface anisotropy field h is more, than some value. The structure and energy of cores of near-boundary solitons depend on the direction of the field h. Solitons of the second type arise under an arbitrary value of h. They move inside the sample, are reflected from the boundary of the sample and elastically collide one with another. The presence of anisotropy constrains the size of the solitons. In the easy-axis ferromagnet under small precession frequencies extended soliton cores are restricted by narrow domain walls. Such states could be treated as immobile edge and moving seeds of magnetization reversal of the material. Near-boundary solitons could be found through characteristic magnetization modulations at the edge of the sample or by the resonance energy absorption on the combinational oscillation frequencies of the multisoliton complexes.

The presence of an easy-plane anisotropy, where the basis plane is parallel to the edge of the sample, as well as the orthorhombic anisotropy and/or the remagnetization field excludes the formation of the near-boundary solitons. Only moving solitons arise, which are divided on two classes. The first class represents solitons without internal degrees of freedom: turning magnetization waves in the easy-plane ferromagnet and the domain walls in the ferromagnet with orthorhombic anisotropy. It is shown, that the magnetization rotation in the centers of such solitons after their reflection from the edge of the sample depends in a threshold way on the amplitude of the surface anisotropy field. The second class of particle-like states contains the precession solitons, which are called the <<breathers>>. In two-axis ferromagnet under small precession frequencies such objects could be treated as moving magnetization reversal nuclei with pulsing sizes. Interaction of breathers with the boundary of the sample leads only to shifts of their positions and initial phases of precession. The measurement of the phase shifts and positions, acquiring by the breathers after their reflection from the surface, could be employed to diagnose the degree of spin pinning at the edge of the sample.

It is established, that the number of the turning waves and the domain walls could be controlled, varying the character of the complete spin pinning at the edge of the sample.

Collisions of the moving solitons with the surface of the sample are accompanied by splash of the magnetization oscillation in the surface layer of the sample. The structure and dynamical properties of solitons are dramatically changed during the collision. Under rotations of spins in the cores of the turning waves and the domain walls during the reflection, the magnetization of the medium varies on the value about the saturation magnetization.

The obtained analytical solutions are impossible to find in the framework of the perturbation theory for solitons in the unbounded medium. Such a theory describes only small changes of structure and properties of the solitons under external influences.

Experimental confirmation of the theoretical predictions, concerning the peculiarities of the elastic reflection of solitons from the surface of the sample, is of importance. The results obtained should be taken into account when modeling solitonic processes near the surfaces of the real ferromagnets of finite sizes.

CRediT authorship contribution statement

V.V. Kiselev: Investigation. **A.A. Raskovalov:** Investigation.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Anton Raskovalov reports financial support was provided by Russian Science Foundation. Vladimir Kiselev declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Spontaneous modulational instability of elliptic periodic waves: The soliton condensate model ☆

D.S. Agafontsev [a,b,c], T. Congy [a], G.A. El [a,*], S. Randoux [d], G. Roberti [a], P. Suret [d]

[a] Department of Mathematics, Physics and Electrical Engineering, Northumbria University, Newcastle upon Tyne, NE1 8ST, United Kingdom
[b] Shirshov Institute of Oceanology of RAS, Moscow, 117997, Russia
[c] Skolkovo Institute of Science and Technology, Moscow, 121205, Russia
[d] Univ. Lille, CNRS, UMR 8523 - PhLAM - Physique des Lasers Atomes et Molecules, Lille, F-59000, France

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ABSTRACT

We use the spectral theory of soliton gas for the one-dimensional focusing nonlinear Schrödinger equation (fNLSE) to describe the statistically stationary and spatially homogeneous integrable turbulence emerging at large times from the evolution of the spontaneous (noise-induced) modulational instability of the elliptic "dn" fNLSE solutions. We show that a special, critically dense, soliton gas, namely the genus one bound-state soliton condensate, represents an accurate model of the asymptotic state of the "elliptic" integrable turbulence. This is done by first analytically evaluating the relevant spectral density of states which is then used for implementing the soliton condensate numerically via a random N-soliton ensemble with N large. A comparison of the statistical parameters, such as the Fourier spectrum, the probability density function of the wave intensity, and the autocorrelation function of the intensity, of the soliton condensate with the results of direct numerical fNLSE simulations with dn initial data augmented by a small statistically uniform random perturbation (a noise) shows a remarkable agreement. Additionally, we analytically compute the kurtosis of the elliptic integrable turbulence, which enables one to estimate the deviation from Gaussianity. The analytical predictions of the kurtosis values, including the frequency of its temporal oscillations at the intermediate stage of the modulational instability development, are also shown to be in excellent agreement with numerical simulations for the entire range of the elliptic parameter m of the initial dn potential.

1. Introduction

Modulational instability (MI) is a fundamental physical phenomenon that has been attracting a major attention from various physics and mathematics communities over the last six decades [1]. Pioneered by Whitham, Lighthill, Benjamin and Feir, Ostrovsky, and Zakharov in 1960-s the theory of modulational instability has developed into a broad area of research with numerous applications in water waves [2], nonlinear optics [3], and condensed matter physics [4]. Typically, MI is manifested as a temporal growth of the amplitude of a small perturbation (modulation) of a weakly nonlinear periodic wave. In many scenarios the dispersion of the wave's envelope plays the dominant role at the initial (linear) stage of the MI development. The initial exponential growth is saturated by nonlinear effects when the modulation amplitude becomes sufficiently large, leading at longer times to the emergence of coherent structures such as solitons and breathers. The eventual fate of the modulationally unstable periodic wave strongly depends on the type of dynamics (integrable vs non-integrable) and the shape of the initial perturbation (localized vs periodic vs random, cf. [5–11]). Some scenarios of the MI development involve the generation of rogue waves—localized coherent structures of unusually large amplitude that emerge unpredictably within otherwise moderate-amplitude wave landscape [12].

It has been well recognized that the one-dimensional cubic focusing NLS equation (fNLSE) represents a paradigmatic model for the description of MI of weakly nonlinear narrow-band short waves (also known as Stokes waves). In the standard normalized form the fNLSE is represented as

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \quad \psi \in \mathbb{C}, \tag{1}$$

where $\psi(x,t)$ is the wave envelope, t is the time-like variable and x – the space-like variable (depending on the application, the t-variable in (1) can have the meaning of a physical spatial variable, e.g. the propagation length in optical fibers, while the x-variable corresponds to the

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* Corresponding author.
E-mail address: gennady.el@northumbria.ac.uk (G.A. El).

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Appendix A. Numerical methods

To study the properties of the asymptotic statistically stationary state developing from the noise-induced MI of a cnoidal wave, we first need to reach this state. For this purpose, following [23], we solve (1) numerically starting from a superposition of cnoidal wave (3) and random, statistically homogeneous in space noise,

$$\psi|_{t=0} = \psi_{\rm dn}(x,0) + \epsilon(x),$$
$$\epsilon(x) = A_0\sqrt{\frac{G_n}{\theta L}}\sum_m e^{-|k_m|^n/\theta^n + i\phi_m + ik_m x}. \tag{A.1}$$

Here A_0 is the noise amplitude, θ is noise spectral width, $k_m = 2\pi m/L$ is the wavenumber, $m \in \mathbb{Z}$ is integer, $\phi_m \in [0, 2\pi)$ are random phases for each k_m and each noise realization, $n \in \mathbb{N}$ is the exponent defining the shape of noise spectrum, $G_n = \pi\, 2^{1/n}/\Gamma_{1+1/n}$ is normalization constant such that the average noise intensity $\langle|\epsilon(x)|^2\rangle$ equals A_0^2, and Γ is Euler's gamma function. We use parameters $n = 32$, $A_0 = 10^{-5}$ and $\theta = 5$, which are slightly different compared to [23] where $n = 2$ has been used; this leads us to slightly different timing when the MI enters its nonlinear stage, but the other results coincide with those reported in that paper.

For the numerical modeling of (1), we use the pseudo-spectral Runge–Kutta fourth-order method in adaptive grid with the grid size Δx set from analysis of the Fourier spectrum of the solution, see [7] for detail. The simulation box $x \in [-L/2, L/2]$ has periodic boundaries. For the cnoidal wave with real and imaginary half-periods $\omega_0 = \pi$ and $\omega_1 = 1.6$, we use $L = 256\pi$ and start simulations on the grid of 16 384 nodes, reaching the final simulation time $t_f = 300$ when the statistical functions are practically stationary, see Fig. 5(a). Note that the number of nodes changes adaptively during the simulations between 16 384 and 262 144, and for other cnoidal waves we sometimes have to use different parameters; see [23] for detail. For each of the studied cnoidal wave, we simulate the time evolution for 1000 random realizations of the initial conditions (A.1) and then average the results over these realizations. To improve the accuracy in the measurement of the stationary values of the statistical functions and exclude influence of the residual temporal oscillations, see e.g. [7,23] and Fig. 5(a), we perform an additional averaging over time interval placed sufficiently far in the nonlinear stage of the MI; for cnoidal wave with $\omega_0 = \pi$ and $\omega_1 = 1.6$ ($m \approx 0.48$), this interval is $t \in [240, 300]$.

We model the SG as a random ensemble containing 200 realizations of N-SS with $N = 128$, and the soliton eigenvalues and norming constants chosen as described in Section 3.2. Computation of the wave fields is performed in the box $x \in [-\tilde{L}/2, \tilde{L}/2]$, $\tilde{L} = 384\pi$, which contains 65 536 nodes, by using the dressing method [18,49] combined with 100-digits precision arithmetics; see [24,38] for detail. As shown in Fig. 3, the constructed 128-SS turn out to be of unity order within a smaller interval $x \in [-L_N/2, L_N/2]$, $L_N \simeq 256\pi$, and decay with increasing $|x|$ outside this interval. The decay is exponential, so that at the edges of the computational box $|x| \simeq \tilde{L}/2$ these wave fields are of 10^{-20} order or smaller. The latter allows us to simulate the time evolution within (1) starting from these 128-SS by using the same numerical scheme with periodic boundary conditions as described above for the MI case. Doing so, we observe that the corresponding statistical functions, averaged over the ensemble of 200 realizations, do not change with time; e.g., see the dashed lines in Fig. 5(a) for the evolution of kinetic and potential energies. Hence, the constructed soliton gas already rests in the statistically stationary state, and to improve the accuracy in the computation of its statistical properties, we perform an additional averaging over the time interval $t \in [0, 300]$.

The fNLSE (1) conserves an infinite series of invariants [18], which can be written in the form

$$\mathcal{I}_j = \frac{1}{L}\int_{-L/2}^{L/2} \psi\, \mathcal{A}_j\, dx, \tag{A.2}$$

$$\mathcal{A}_j = \frac{\partial \mathcal{A}_{j-1}}{\partial x} + \psi \sum_{l+m=j-1} \mathcal{A}_l \mathcal{A}_m, \tag{A.3}$$

where $\mathcal{A}_1 = \psi^*$. The first three invariants are the wave action (the average intensity of fNLSE wave field),

$$\mathcal{N} = \overline{|\psi|^2} = \frac{1}{L}\int_{-L/2}^{L/2} |\psi|^2\, dx = \sum_k |\psi_k|^2, \tag{A.4}$$

the momentum

$$\mathcal{M} = \frac{i}{2L}\int_{-L/2}^{L/2} (\psi_x^*\psi - \psi_x\psi^*)\, dx = \sum_k k|\psi_k|^2, \tag{A.5}$$

and the total energy

$$\mathcal{E} = \mathcal{H}_l + \mathcal{H}_{nl}, \tag{A.6}$$

$$\mathcal{H}_l = \overline{|\psi_x|^2} = \frac{1}{L}\int_{-L/2}^{L/2} |\psi_x|^2\, dx = \sum_k k^2|\psi_k|^2, \tag{A.7}$$

$$\mathcal{H}_{nl} = -\overline{|\psi|^4} = -\frac{1}{L}\int_{-L/2}^{L/2} |\psi|^4\, dx. \tag{A.8}$$

Here $\mathcal{H}_l$ is the kinetic energy (related to dispersion), $\mathcal{H}_{nl}$ is the potential energy (related to nonlinearity), and ψ_k is the Fourier-transformed wave field,

$$\psi_k(t) = \mathcal{F}[\psi] = \frac{1}{L}\int_{-L/2}^{L/2} \psi(x,t)\, e^{-ikx}\, dx.$$

When modeling the time evolution, our numerical scheme conserves the first 10 integrals (A.2)–(A.3) up to the relative errors from 10^{-10} (the first three invariants) to 10^{-6} (the tenth invariant) orders.

We examine the following statistical functions: the ensemble-averaged kinetic $\langle\mathcal{H}_l\rangle$ and potential $\langle\mathcal{H}_{nl}\rangle$ energies, the kurtosis $\kappa_4 = \langle|\psi|^4\rangle/\langle|\psi|^2\rangle^2$, the probability density function (PDF) $\mathcal{P}(I)$ of relative wave intensity $I = |\psi|^2/\langle|\psi|^2\rangle$ where $\langle|\psi|^2\rangle$ is the average intensity, the Fourier spectrum,

$$S_k = \frac{\langle|\psi_k|^2\rangle}{\Delta k}, \tag{A.9}$$

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Asymptotic integrability and its consequences☆

A.M. Kamchatnov *

Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 142190, Russia
Skolkovo Institute of Science and Technology, Skolkovo, Moscow, 143026, Russia

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ABSTRACT

We give a brief review of the concept of asymptotic integrability, which means that the Hamilton equations for the propagation of short-wavelength packets along a smooth, large-scale background wave have an integral independent of the initial conditions. The existence of such an integral leads to a number of important consequences, which include, besides the direct application to the packets propagation problems, Hamiltonian theory of narrow solitons motion and generalized Bohr–Sommerfeld rule for parameters of solitons produced from an intensive initial pulse. We show that in the case of systems with two wave variables and exact fulfillment of the asymptotic integrability condition, the 'quantization' of mechanical systems, associated with the additional integrals, yields the Lax pairs for a number of typical completely integrable equations, and this sheds new light on the origin of the complete integrability in nonlinear wave physics.

Dedicated to the memory of V. E. Zakharov

1. Introduction

The discovery of the inverse scattering transform (IST) method for integration of the Korteweg–de Vries (KdV) equation by M. D. Kruskal and co-authors [1] ushered in a new epoch in nonlinear physics. Its more general formulation given by P. D. Lax [2] suggested possible generality of the IST method, and this possibility was realized in the papers [3,4] by V. E. Zakharov and A. B. Shabat, where the IST method was extended to the nonlinear Schrödinger (NLS) equation for both signs of the nonlinear term. This remarkable achievement resulted in a flood of discoveries of a number of other nonlinear wave equations integrable by the IST method. At last, V. E. Zakharov and L. D. Faddeev [5] and S. C. Gardner [6] showed that the IST method is nothing but the transition from physical variables to the action–angle variables generalized to systems with an infinite number of degrees of freedom, and this Hamiltonian system is completely integrable in the Liouville–Arnold sense [7,8]. This demonstration related the IST method with classical Hamiltonian mechanics and stimulated fast development of modern nonlinear mathematical physics (see, e.g., [9–13] and references therein). However, the relationship between physical properties of wave equations and their complete integrability still seems not clear enough.

As was recently noticed [14,15] (see also Section 3), if we confine ourselves to integrability of nonlinear wave equations in a restricted sense, formulated as the condition of integrability of the Hamilton equations that govern the propagation of short-wavelength packets along a large-scale (hydrodynamic) background wave, then the fulfillment of this *asymptotic integrability* condition means the existence of the integral of the Hamilton equations, that is the carrier wave number k becomes a function of local background wave variables. The very existence of such an integral leads to a number of important consequences.

First of all, the expression for the carrier wave number greatly simplifies solutions of problems related to the propagation of linear wave packets along large-scale background waves [16,17] (see also Section 4). Next, if such a packet corresponds to a small-amplitude edge of a dispersive shock wave in the Gurevich–Pitaevskii theory of these shock waves [18], based on the Whitham theory of modulations [19,20] (see also [21,22]), then this expression allows one to find a path of this edge during the evolution of the shock [23,24]. Then, as was noticed by A. V. Gurevich and L. P. Pitaevskii [25], their theory yields the expression for the speed of entering of oscillations into the dispersive shock region. If the initial wave pulse is localized in space, then these oscillations transform eventually into a train of separate solitons, and, consequently, one can calculate the final number of solitons at asymptotically large time [22,26] for a given initial profile. Natural extension of this theory [27] yields the generalized Bohr–Sommerfeld quantization rule for finding the parameters of these asymptotic solitons, which generalizes earlier theory [28] developed for unidirectional initial pulses (see Section 6).

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* Correspondence to: Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 108840, Russia.
E-mail address: kamch@isan.troitsk.ru.

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and B, C are defined in Eq. (193). It differs only by notation from the original form that was found in Ref. [39].

As we see, quantization of a mechanical system associated with an integral of the asymptotic integrability equations allowed us to derive in a very simple way the Lax pairs of some typical completely integrable nonlinear wave equations.

8. Conclusion

We showed in this paper that the condition of asymptotic integrability of nonlinear wave equations leads to important relationships between the carrier wave number k of high-frequency wave packets propagating along smooth background waves and the local values (r_+, r_-) of the background variables. The existence of these relationships leads to many interesting applications. Besides immediate application to the theory of propagation of the high-frequency packets, it leads to the Hamiltonian theory of motion of narrow solitons along smooth, non-uniform, and time-dependent waves. Specification of this theory on the evolution of the edges of a dispersive shock wave allows one to calculate the number of solitons produced from an intensive wave pulse at asymptotically large time. The formula for the number of solitons turns out to be very general and it admits a general derivation as a consequence of the preservation of the Poincaré–Cartan integral invariant by the hydrodynamic background flow. This observation leads to a natural generalization in the form of the Bohr–Sommerfeld quantization rule for parameters of asymptotic solitons produced from the initial pulse. At last, the quasiclassical quantization according to the Bohr–Sommerfeld rule suggests that it follows from some full quantum theory of the corresponding mechanical system and the simplest Schrödinger and Dirac recipes of finding such an underlying quantum theory yield the Lax pairs of the completely integrable equations. If the asymptotic integrability equations only have an approximate integral, then the Lax pair does not exist, but an approximate Bohr–Sommerfeld rule still can be written, so completely integrable equations share some of their properties with not completely integrable ones. It seems quite plausible that the theory of asymptotic integrability will lead to a number of other important consequences.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

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Dynamics of ring solitons in an expanding cloud of a Bose-Einstein condensate

A. M. Kamchatnov [1,2,3,*] B. I. Suleimanov [4,†] and E. N. Tsoy [5,‡]
[1] Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow 108840, Russia
[2] Skolkovo Institute of Science and Technology, Skolkovo, Moscow 143026, Russia
[3] Higher School of Economics, Physical Department, 20 Myasnitskaya ulica, Moscow 101000, Russia
[4] Institute of Mathematics, Ufa Federal Research Center, RAS Chernyshevsky str. 112, Ufa 450008, Russia
[5] Physical-Technical Institute of the Uzbek Academy of Sciences, Chingiz Aytmatov Str. 2-B, Tashkent 100084, Uzbekistan

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In this paper, we derive equations for the dynamics of ring dark solitons in an expanding cloud of a two-dimensional Bose-Einstein condensate. Assuming that the soliton's width is much smaller than its radius, we obtain the Hamilton equations for its evolution. Then they are transformed into the Newton equation, which is more convenient for applications. The general theory is illustrated by the solution of the Newton equation for the case of the axially symmetric condensate cloud, which expands after switching off a harmonic trap. The validity of our approximate analytical approach is confirmed by comparison with the results of numerical simulations of the Gross-Pitaevskii equation.

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I. INTRODUCTION

Cylindrically symmetric nonlinear wave structures were first predicted and studied theoretically in a two-dimensional generalization of the celebrated Korteweg-de Vries (KdV) equation [1–3]. Thorough investigations of this cylindrical KdV equation showed that it has ring-shaped soliton solutions provided the width of such solitons is much smaller than the radius of the ring (see, e.g., Ref. [4] and references therein). They were also observed in experiments [5–7]. These solitons correspond to the limit of small-amplitude waves propagating in systems with weak dispersion. Since the curvature of ring solitons is a small parameter, they can also be considered as solutions of the Kadomtsev-Petviashvili (KP) equation [8] generalized to the case of dependence of the wave variable on two polar coordinates in Ref. [9]. It is remarkable that all these equations (KdV, cylindrical KdV, KP, and cylindrical KP) can be solved by the inverse scattering transform method [2,10–12] and there are formal links between solutions of different equations (see, e.g., Ref. [13] and references therein). These connections allowed the authors of Refs. [14,15] to demonstrate that a localized pulse of a ring-like soliton is accompanied by a long, small-amplitude tail, but the evolution of the leading pulse is practically indistinguishable from an approximate adiabatic dynamics of the KdV soliton with the account of a small curvature term. Besides that, as is known, soliton solutions of the KP equation in the form of a straight line are unstable with respect to so-called "snake" instability [8,16]. Taking into account the dependence of the wave variable on the azimuthal angle allowed one to study the stability of ring solitons with respect to axial perturbations, and in case of instability, a ring soliton evolves into a chain of localized two-dimensional lumps [15,17].

If we go beyond an approximation of small-amplitude waves, when, for example, their dynamics is described by a two-dimensional nonlinear Schrödinger (NLS) equation in nonlinear optics or a two-dimensional Gross-Pitaevskii (GP) equation in the physics of Bose-Einstein condensates (BECs), then we arrive at similar phenomena. Excitation of a dark soliton leads to the formation of a counterflow shelf around the soliton even in one-dimensional geometry [18]. Such a shelf considerably changes the dynamics of narrow dark solitons under the action of external forces when the GP equation becomes not completely integrable. In particular, the effective mass of a dark soliton quasiparticle is equal to 2 rather than 1 in standard nondimensional units [19–21]. Dark ring solitons were observed in nonlinear optics (see, e.g., Refs. [22,23]) and in BECs (see, e.g., Refs. [24,25]). If the width of a ring-like soliton's profile is much smaller than the ring's radius, then the soliton's curvature can be considered as a small perturbation [26–28]. The accuracy of this approach in describing dark soliton dynamics is confirmed by numerical simulations (see, e.g., Refs. [28,29]). However, so far, this method has been limited to solitons moving through a nonflowing condensate confined in a stationary trap. Another approach based on the Whitham modulation theory (see Refs. [30,31]) describes the interaction of a soliton with the background flow only for situations without the action of external forces and in a one-dimensional geometry. Recently, the Hamiltonian theory for the dynamics of dark solitons was developed in Refs. [32–36], and this method automatically takes into account the effects of counterflow. Besides that, after the transformation of the Hamilton equations to a more convenient Newton-like equation, one can easily take into account the external forces

*Contact author: kamch@isan.troitsk.ru
†Contact author: bisul@mail.ru
‡Contact author: e.n.tsoy@gmail.com

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DATA AVAILABILITY

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Asymptotic integrability and Hamilton theory of soliton motion along large-scale background waves

A. M. Kamchatnov
Institute of Spectroscopy, Russian Academy of Sciences, 108840 Troitsk, Moscow, Russia
and Skolkovo Institute of Science and Technology, 143026 Skolkovo, Moscow, Russia

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We consider the problem of soliton mean-field interaction for the class of asymptotically integrable equations, where the notion of asymptotic integrability means that the Hamilton equations for a high-frequency wave packet's propagation along a large-scale background wave have an integral of motion. Using Stokes' remark, we transform this integral to an integral for the soliton equations of motion and then derive the Hamilton equations for the soliton dynamics in a universal form expressed in terms of the Riemann invariants for the hydrodynamic background wave. The physical properties are specified by the concrete expressions for the Riemann invariants. The theory is illustrated by its application to soliton dynamics, which is described by the Kaup-Boussinesq system.

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I. INTRODUCTION

As is known, the discovery of the inverse scattering transform method [1–3] allowed one to distinguish an important class of completely integrable wave equations [4,5] with remarkable mathematical properties (see, e.g., Refs. [6,7] and references therein). The special properties of these equations found a number of applications to problems of solitons dynamics in different physical situations as, e.g., interaction of solitons, kinetic theory of soliton gases, evolution of soliton's parameters under action of small perturbations. It is remarkable that the property of complete integrability is preserved by some approximate procedures applied to completely integrable equations. For example, asymptotic perturbation theory applied to completely integrable soliton equations yields again completely integrable equations [8] and averaging of periodic solutions leads to the Whitham modulation equations, which also have the property of complete integrability [9]. It is natural to expect that other approximation schemes can also reduce a complicated problem described by a completely integrable system to a much simpler approximate system of completely integrable Hamilton equations.

In this paper, we will consider the problem of propagation of a narrow soliton along a wide and smooth large-scale background wave. In this case, the complete integrability is understood in the restricted sense called "asymptotic integrability" introduced in Ref. [10]. It means that the Hamilton equations, which describe propagation of short-wavelength packets along a smooth large-scale background, have an integral of motion so that existence of such an integral allows one to integrate these equations in a closed form [11]. Using the Stokes remark [12] that the soliton tails and linear harmonic waves obey the same equations, the above-mentioned integral can be cast to the relationship between the soliton's velocity and its inverse half-width. This gives a convenient method of solving problems of interaction of solitons with a background wave (see, e.g., Refs. [13,14]). A different approach based on the Whitham modulation theory was developed in Refs. [15–17]. It was applied to explanation of behavior of solitons propagating along viscous fluid conduits [15] (see also Refs. [18,19]). Perturbation theory of interaction of solitons with nonuniform background in Bose-Einstein condensate was developed in Refs. [20,21] and a Landau quasiparticle approach to the same problem was suggested in Refs. [22–25]. Predictions of these theories agree very well with experimental observations [26,27]. As was shown in Refs. [28–30], Stokes' remark allows one to obtain the Hamilton equations for motion of solitons of the nonlinear Schrödinger (NLS), derivative nonlinear Schrödinger (DNLS), and magnetic soliton equations. We will show here that this approach can be generalized in such a way that the Hamilton equations are obtained for general classes of equations that satisfy the conditions of asymptotic integrability. These general equations are expressed in terms of the Riemann invariants for the dispersionless (hydrodynamic) equations governing the large-scale evolution. The concrete physical realizations of this scheme depend on the expressions for the Riemann invariants in terms of the physical variables of the problem.

In Sec. II we will discuss the concept of asymptotic integrability and in Sec. III we will apply this method to the problem of soliton dynamics in the general formulation applicable to any asymptotically integrable equations. This approach is illustrated in Sec. IV by an example of soliton's motion along a large-scale wave described by the Kaup-Boussinesq system.

II. ASYMPTOTIC INTEGRABILITY

Let the physical system under consideration be described by two wave variables ρ (density) and u (flow velocity). We suppose that dynamics of this system obeys the equations, which can be written in the form

$$\begin{aligned} &\rho_t + F(\rho, u, \rho_x, u_x, \dots) = 0, \\ &u_t + G(\rho, u, \rho_x, u_x, \dots) = 0, \end{aligned} \tag{1}$$

The soliton solution of Eqs. (25) has the form of a dip $\rho(x - Vt)$ in the density ρ_0 and a local distribution $u(x - Vt)$ of the flow velocity caused by the moving soliton. This solution can be written in the form [43]

$$\rho(\xi) = \rho_0 - 2\mu(\xi)[\mu(\xi) - V],$$
$$u(\xi) = 2[V - \mu(\xi)], \tag{78}$$

where

$$\mu(\xi) = V + \frac{\rho_0 - V^2}{\sqrt{\rho_0}\cosh(\kappa\xi) + V} \tag{79}$$

and

$$\xi = x - Vt, \quad V = \sqrt{\rho_0 - \kappa^2/4}. \tag{80}$$

Consequently, we get the approximate solution for the soliton moving along a smooth background $\rho = \rho(x)$ by means of replacements

$$\rho_0 \to \rho(x), \quad V \to \sqrt{\rho(x) - \kappa^2/4},$$
$$\xi \to x - \int_{t_0}^{t} \sqrt{\rho[x(t)] - \kappa^2/4}\, dt - x_0. \tag{81}$$

Although the soliton's inverse half-width remains constant, its amplitude depends on the local density $\rho(x)$ as

$$a = 2(\rho(x) - \sqrt{\rho(x)[\rho(x) - \kappa^2/4]}). \tag{82}$$

In the small-amplitude limit $\kappa^2 \ll \rho$, this solution reduces to the well-known shallow dark KdV soliton. As was shown in Ref. [44], the Kaup-Boussinesq system (25) with $\sigma = +1$ can find applications to dynamics of solitons in two-component Bose-Einstein condensates.

V. CONCLUSION

The concept of complete integrability of nonlinear wave equations plays a crucial role in soliton physics [1–7]. The notion of asymptotic integrability restricts this concept to a particular case of propagation of small-wavelength packets along a large-scale background wave. It turns out that this notion leads directly to the quasiclassical limit of Lax pairs and, in case of systems with two wave variables, this limit admits quite a general description in terms of Riemann invariants of the large-scale hydrodynamic approximation. Such a description is only possible in exact form for special forms of the dispersion relation of linear waves, and we presented here two such universal forms corresponding to a number of completely integrable nonlinear wave equations. In all these cases, there exists an integral of motion of the Hamilton equations that describe the packet's motion. Existence of such an integral allows one to obtain the solution of these Hamilton equations in a closed form [11].

According to Stokes' consideration [12], the soliton's velocity can be expressed via the linear dispersion relation as a function of the soliton's inverse half-width. In a similar way, the mentioned-above integral of motion for the packet's dynamics can be transformed to the integral of motion for the dynamics of a soliton treated as a pointlike particle provided its width is much smaller than the characteristic scale of the background wave. Due to knowledge of such an integral, one can obtain the Hamilton equations for the soliton's dynamics and generalize them to situations when the background profiles are formed under action of external potentials. We obtained the universal expressions for the canonical momentum and Hamiltonian again in terms of the Riemann invariants for the background dynamics. These universal expressions reproduce all particular cases studied earlier in Refs. [28–30].

It is worth noticing that even if the condition of asymptotic integrability is not exactly fulfilled, it may be satisfied approximately in the limit of large wave numbers of the carrier wave. In this case, one can obtain an approximate integral of motion, which can be used in applications to the packet's or soliton's dynamics. Besides that, it allows one to formulate a generalized Bohr-Sommerfeld quantization rule that determines the asymptotic velocities of solitons produced from an intensive initial pulse even for not completely integrable equations [45]. Thus, the concept of asymptotic integrability seems quite fruitful, and one can hope that it will lead to many other interesting results.

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**ТЕОРЕТИЧЕСКАЯ
И МАТЕМАТИЧЕСКАЯ
ФИЗИКА
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 А. М. Камчатнов*

АСИМПТОТИЧЕСКАЯ ИНТЕГРИРУЕМОСТЬ НЕЛИНЕЙНЫХ ВОЛНОВЫХ УРАВНЕНИЙ И КВАЗИКЛАССИЧЕСКИЙ ПРЕДЕЛ ПАР ЛАКСА

Введено понятие асимптотической интегрируемости нелинейных волновых уравнений, означающее интегрируемость уравнений Гамильтона, описывающих распространение высокочастотного волнового пакета по гладкому профилю, динамика которого подчиняется бездисперсионному пределу исходных уравнений. Показано, что этот предельный случай полной интегрируемости позволяет выразить квазиклассический предел пар Лакса через закон дисперсии линейных волн и интеграл уравнений Гамильтона для пакета. Если пара Лакса не зависит от производных волновых переменных, то квазиклассический предел совпадает с точными выражениями. Теория иллюстрируется конкретными примерами.

Ключевые слова: солитоны, пары Лакса, квазиклассическое приближение.

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1. ВВЕДЕНИЕ

Условие полной интегрируемости нелинейных волновых уравнений обычно формулируется как возможность представить рассматриваемое уравнение в виде условия совместности пары Лакса, т. е. двух линейных уравнений со спектральным параметром (см., например, [1]–[3]). Существование такой пары Лакса обычно никак физически не мотивируется и не связывается с какими-либо характеристиками нелинейного волнового уравнения. В настоящей работе мы показываем, что для довольно широкого класса уравнений можно ввести понятие *асимптотической интегрируемости*, которое определяется свойствами закона дисперсии гармонических волн, подчиняющихся линеаризованным уравнениям исходной системы. Согласно оптико-механической аналогии Гамильтона движение высокочастотного пакета по

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*Институт спектроскопии Российской академии наук, Москва, Троицк, Россия.
E-mail: kamchatnov@gmail.com

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АСИМПТОТИЧЕСКАЯ ИНТЕГРИРУЕМОСТЬ НЕЛИНЕЙНЫХ ВОЛНОВЫХ УРАВНЕНИЙ

А. М. Камчатнов *

Институт спектроскопии РАН;
Сколковский институт науки и технологий;
Высшая школа экономики, г. Москва, Россия

Понятие асимптотической интегрируемости нелинейных волновых уравнений, означающее интегрируемость динамики волновых пакетов, распространяющихся по крупномасштабному фону, иллюстрируется примером обобщённого уравнения Кортевега—де Фриза. Показано, что условие асимптотической интегрируемости естественным образом приводит к обобщённому правилу Бора—Зоммерфельда, определяющему параметры асимптотических солитонов, порождаемых из интенсивного начального импульса. Дополнительное предположение, что правило Бора—Зоммерфельда является квазиклассическим пределом для некоторой линейной спектральной задачи, позволяет выделить класс полностью интегрируемых уравнений, найти соответствующую пару Лакса и оценить точность правила Бора—Зоммерфельда для неинтегрируемых уравнений.

ВВЕДЕНИЕ

В истории развития солитонной физики легко различить два этапа. На первом этапе, после первых наблюдений и экспериментальных исследований уединённых волн на мелкой воде (см., например, [1] и имеющиеся там ссылки), первостепенной задачей было дать физическое объяснение этого явления на основе уравнений гидродинамики. Трудность этой задачи заключалась в том, что в то время физики использовали для описания волн на воде два существенно разных приближения. В одном из них амплитуда волны ζ предполагалась исчезающе малой по сравнению с глубиной воды h_0,

$$\zeta \ll h_0, \tag{1}$$

и тогда уравнения гидродинамики в линейном по амплитуде волны приближении привели в исследованиях Лапласа, Пуассона и Коши к закону дисперсии линейных гармонических волн $\zeta \propto \propto \exp[i(kx - \omega t)]$:

$$\omega^2 = gh_0 \mathrm{th}(kh_0), \tag{2}$$

где ω — круговая частота, k — волновое число, g — ускорение свободного падения. В длинноволновом пределе $kh_0 \ll 1$ эта формула сводится к закону дисперсии

$$\omega = \sqrt{gh_0}\; k - \frac{1}{6}\, h_0^2 \sqrt{gh_0}\; k^3. \tag{3}$$

В другом приближённом подходе к теории волн на воде, наоборот, сразу предполагалось, что длина волны много больше глубины воды, но амплитуда волны уже не предполагалась исчезающе малой по сравнению с глубиной. В результате гидродинамические уравнения для волн на поверхности такой «мелкой воды» сводились к уравнениям

$$h_t + (hv)_x = 0, \qquad v_t + vv_x + h_x = 0, \tag{4}$$

* kamch@isan.troitsk.ru, kamchatnov@gmail.com

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Supersonic flow past an obstacle in a quasi-two-dimensional Lee-Huang-Yang quantum fluid

G. H. dos Santos,[1] L. F. Calazans de Brito,[1] A. Gammal,[1] and A. M. Kamchatnov[2,3,4]
[1]*Instituto de Física, Universidade de São Paulo, 05508-090 São Paulo, São Paulo, Brazil*
[2]*Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow Region 108840, Russia*
[3]*Skolkovo Institute of Science and Technology, Skolkovo, Moscow 143026, Russia*
[4]*Higher School of Economics, Physical Department, 20 Myasnitskaya ulica, Moscow 101000, Russia*

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A supersonic flow past an obstacle can generate a rich variety of wave excitations. This paper investigates, both analytically and numerically, two types of excitations generated by the flow of a Lee-Huang-Yang quantum fluid past an obstacle: linear radiation and oblique dark solitons. We show that wave crests of linear radiation can be accurately described by the proper modification of the Kelvin original theory, while the oblique dark soliton solution is obtained analytically by transformation of the one-dimensional soliton solution to the obstacle's reference frame. A comparison between analytical predictions and numerical simulations demonstrates good agreement, validating our theoretical approach.

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I. INTRODUCTION

The study of the Bose-Einstein condensate (BEC) has attracted a significant scientific interest due to its quantum fluid behavior. In the condensate of weakly interacting atoms, most problems can be described within the mean-field approximation by the Gross-Pitaevskii equation (GPE) [1]. However, when the mean-field interspecies attraction in Bose-Bose mixtures exceeds the average repulsive forces, the GPE predicts a collapse of the system. In the case of quantum droplets, this issue can be resolved by incorporating quantum fluctuation effects, leading to a correction of the GPE with a quartic nonlinear term, the Lee-Huang-Yang (LHY) correction, thus opening the study of the beyond-mean-field effects [2,3] (see also Refs. [4,5]).

By manipulating the interaction strength and the number of atoms, it is possible to create a carefully designed mixture of a two-component BEC in which the cubic mean-field term vanishes, allowing quantum fluctuations to be studied through the quartic nonlinear term [6]. This leads to the LHY equation in the form

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + U_{\text{ext}}\Psi + g_{\text{LHY}}|\Psi|^3\Psi, \tag{1}$$

where U_{ext} represents the external potential, which can account for effects such as traps and obstacles, among others. The parameters g_{LHY} and m denote the three-dimensional interparticle interaction constant and the atomic mass, respectively. The quantum fluid governed by the LHY equation exhibits both nonlinear and dispersive effects. These properties enable the propagation of various atomic excitations, including solitons, vortices, dispersive shock waves, and linear radiation. As in the case of the GPE, a supersonic flow past an obstacle in the LHY fluid is expected to generate stationary coherent wave patterns satisfying the Mach-Cherenkov-Landau resonant radiation condition. These wakes are referred to as Kelvin-like wakes due to their similarity to Kelvin's water wave structure generated by a moving ship (commonly known as the Kelvin wake pattern) [7–11]. More specifically, for supersonic flow velocities, the GPE predicts two distinct wave structures: oblique dark solitons inside the Mach cone and linear radiation outside it. In the case of the GPE, these structures have been extensively studied, experimentally [12,13] and through theoretical investigations, particularly regarding linear radiation [14,15] and oblique dark soliton wakes [16,17]. However, for Eq. (1), this problem remains unsolved.

In this work, we develop an analytical framework for the investigation of a supersonic flow past an obstacle in a quasi-two-dimensional (2D) quantum fluid described by Eq. (1). We restrict our analysis to flow velocities that support the propagation of two distinct wake patterns: oblique dark solitons and linear radiation. The paper is structured as follows. Section II presents the analytical theory of linear radiation outside the Mach cone. Section III focuses on oblique solitons inside the Mach cone. In Sec. IV, we compare analytical predictions with numerical simulations. Finally, we conclude with a discussion in Sec. V.

II. DIMENSIONAL REDUCTION AND THE QUASI-2D HYDRODYNAMICAL MODEL

In experimental setups, it is not possible to realize truly low-dimensional systems. Instead, an effective low-dimensional regime can be obtained by "squeezing" a three-dimensional (3D) BEC, tightly confining it in one or two spatial directions using external potentials. Theoretically, beyond-mean-field energy corrections in Bose gases have been investigated within the framework of 3D-one-dimensional (1D)- and 3D-2D-dimensional crossovers in Refs. [18–20].

In our model, we construct a 3D-2D-dimensional crossover by imposing strong confinement along the z axis, such that the system's dynamics are effectively restricted to the x–y plane. This means that the harmonic potential corresponds to a much greater frequency ω_z in z of the atomic motion direc-

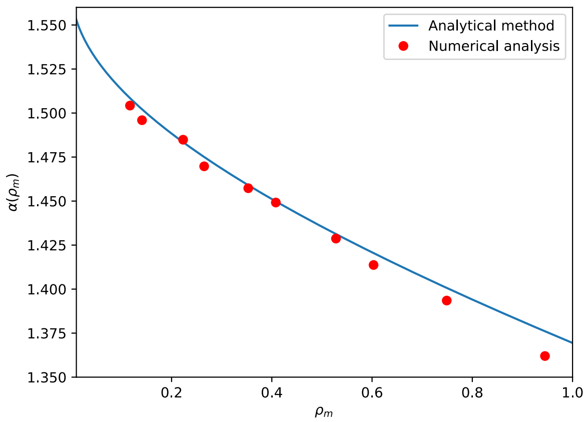

FIG. 6. Analytical and numerical results for the minimum soliton density ρ_m and the oblique soliton angle α. The analytical results are given by Eq. (50). The numerical points are obtained by varying the obstacle radius $0.1 \leqslant R \leqslant 1.5$.

There is very good agreement between the analytical solution for the soliton density profile, given by Eq. (45) for stable oblique solitons, and the numerical profile extracted from Fig. 4 at $x = -80$, as demonstrated in Fig. 5.

Using Eq. (42) and assuming $V = u_0 \cos\alpha$, we can express the relation between the soliton depth ρ_m and the oblique soliton angle α as

$$\alpha(\rho_m) = \arccos\left(\frac{1}{|u_0|}\sqrt{\frac{2\rho_m\left(\frac{3}{5}\rho_0^{5/2} + \frac{2}{5}\rho_m^{5/2} - \rho_0^{3/2}\rho_m\right)}{(\rho_m - \rho_0)^2}}\right). \tag{50}$$

This expression shows that, although both the soliton depth ρ_m and the oblique soliton angle α implicitly depend on the obstacle size, this dependence does not affect the accuracy of the theoretical prediction. This can be observed in Fig. 6, where the pair (ρ_m, α) is obtained varying the value of the obstacle size in the range $0.1 \leqslant R \leqslant 1.5$. However, for sufficiently large obstacles ($R \gg 1$), the number of oblique soliton pairs formed behind the obstacle increases, and the theory should then be applied to each soliton individually.

VI. CONCLUSION

In this paper, we investigated analytically and numerically two types of wave excitations generated by a supersonic flow past an obstacle in a quasi-2D LHY quantum fluid. Numerical simulations confirmed that the wave crests of linear radiation, formed outside the Mach cone, were accurately captured by linear theory (see Fig. 3). The structure of oblique dark solitons are well described by a quasi-one-dimensional analytical model (see Fig. 5), demonstrating excellent agreement with the numerical results.

As proposed in Ref. [29], the framework of a flow past a barrier serves as an experimental tool to measure and analyze critical velocities associated with the emission of collective excitations. The theoretical approach developed here should be useful to understand the behavior of linear waves and solitons generated by the interaction of strong laser beams, approximated by an impenetrable barrier, with quasi-2D supersonic quantum droplets or strongly interacting mixtures, where LHY corrections dominate over the mean-field term.

As a direction for future work, we propose extending this approach to the case of supersonic flow past weak, penetrable obstacles, where the local flow velocity varies along the soliton line. As demonstrated in Ref. [30] for a cigar-shaped BEC, the properties of the obstacle also influence the behavior and stability of the resulting excitations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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Propagation of narrow and fast solitons through dispersive shock waves in hydrodynamics of simple waves

Dmitriy Shaykin *

Russian University of Transport (RUT-MIIT), Obrazcova st.9, Moscow, 127994, Russia
Skolkovo Institute of Science and Technology, Moscow, 127994, 143026, Russia
Moscow Institute of Physics and Technology, Institutsky lane 9, Dolgoprudny, 141700, Moscow region, Russia

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ABSTRACT

We study the propagation of narrow and fast solitons through various profiles of dispersive shock waves (DSWs) in the framework of the generalized Korteweg-de Vries (gKdV) equation. The idea of considering such a motion as a propagation along a smooth effective field is proposed. In the case of KdV and modified KdV this idea is proven rigorously; for other cases, we take this as a natural hypothesis. For cases of self-similar breaking for KdV and mKdV, a specific method for selecting the effective field is proposed, demonstrating high agreement with the numerical solution. For the breaking of a smooth pulse into the resting medium in gKdV case, we propose using the pulse's maximum value as an approximation of the effective field. In the considered special cases, this proposal demonstrates good agreement with the numerical solution only for fast solitons.

1. Introduction

This article is a continuation of a series of works [1–15], devoted to soliton dynamics and related problems in nonlinear dispersive equations.

In the case of simple waves

$$u_t + V(u)u_x + u_{xxx} = 0, \tag{1}$$

the first important work in this direction was Ref. [1], based on an investigation of Whitham's modulation equations [16,17], in which new differential relations were found linking the inverse half-width of solitons k_s and the carrier wave number of small-amplitude wave packets k with the value of the background u, which led to the solution [2,3] of a number of classical problems [18,19]. Based on the optical-mechanical analogy [20] (Hamiltonian formalism) and Stokes' remark [21], the same relations were obtained and related problems were solved [4,5].

The mentioned articles served as the beginning of the study of the dynamics of wave packets [6–8] and solitons [9–11,22] on non-stationary and inhomogeneous smooth waves and related tasks [12]. Strictly speaking, all these works imply the narrowness of the soliton in comparison with the background field ($k_s \ll |\frac{du}{dx}|/u$), because the conclusions are based on approaches treating the soliton as a point particle. The dynamics of such solitons are described by the following equations:

$$k_s^2 = -\frac{2}{3}V(u) + q, \quad \frac{dx}{dt} = \frac{1}{3}V(u) + q, \quad u_t + V(u)u_x = 0, \tag{2}$$

* Corresponding author at: Russian University of Transport (RUT-MIIT), Obrazcova st.9, Moscow, 127994, Russia.
E-mail address: shaykin.dv@phystech.edu.

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Dmitriy Shaykin: Investigation.

Declaration of competing interest

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.chaos.2025.117267.

Data availability

Data will be made available on request.

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Theory of dispersive shock waves induced by the Raman effect in optical fibers

D. V. Shaykin [1,2,3,*] and A. M. Kamchatnov [3,4,5,†]

[1] *Russian University of Transport (RUT-MIIT), Obrazcova st. 9, Moscow 127994, Russia*
[2] *Moscow Institute of Physics and Technology, Institutsky Lane 9, Dolgoprudny, Moscow Region 141700, Russia*
[3] *Skolkovo Institute of Science and Technology, Skolkovo, Moscow 143026, Russia*
[4] *Institute of Spectroscopy Russian Academy of Sciences, Troitsk, Moscow 108840, Russia*
[5] *Higher School of Economics, Physical Department, 20 Myasnitskaya ulica, Moscow 101000, Russia*

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We develop the theory of dispersive shock waves in optical fibers for the case of long-distance propagation of optical pulses, when the small Raman effect stabilizes the profile of the shock. The Whitham modulation equations are derived as the basis for the Gurevich-Pitaevskii approach to the analytical theory of such shocks. We show that the wave variables at both sides of the shock are related by the analog of the Rankine-Hugoniot condition that follows from the conservation laws of the Whitham equations. Solutions of the Whitham equations yield the profiles of the wave variables that agree very well with the exact numerical solution of the generalized nonlinear Schrödinger equation for propagation of optical pulses.

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I. INTRODUCTION

Nonlinear wave structures called dispersive shock waves (DSWs) have been observed in a number of different physical media, from water waves to Bose-Einstein condensates (see, e.g., review articles [1,2] and references therein). Generally speaking, they can be represented as lengthy oscillatory nonlinear wave structures that degenerate at one of their edges to trains of solitons and at the other edge to small-amplitude wavy tails. If such a DSW is formed as a result of a wave breaking of a large-scale wave pulse, so that at the initial stage of evolution dispersion effects dominate over dissipative ones, then this DSW expands with time with an increasing number of oscillations in it. Typically, the wavelength of oscillations in a DSW is much smaller than its whole size. Therefore, this DSW can be represented as a modulated nonlinear periodic wave with parameters (amplitude of oscillations, wavelength, etc.) slowly changing with space and time. However, even small dissipation becomes crucially important at the later stage of evolution of DSWs, when their slow dynamics due to modulation becomes comparable with slow dynamics due to small dissipation. As a result, a DSW stops its expansion and tends to a stationary wave structure whose total length is proportional to the inverse of the small dissipation parameter. In both cases, with or without dissipation, the modulation is small, and this fact was used by Gurevich and Pitaevskii [3] in their approach to the theory of DSWs, which was based on Whitham's theory of modulations of periodic solutions of nonlinear wave equations [4,5], in particular, of the Korteweg-de Vries (KdV) equation. This approach turned out very successful in the theoretical description of DSWs described by the KdV equation both in time-dependent [6–8] and stationary situations with account of small dissipation [9–12].

DSWs in optical fibers were first observed long ago [13,14], but their theoretical description was a difficult problem since its solution needed application of the quite involved inverse scattering transform method, discovered in Refs. [15–18], to the nonlinear Schrödinger (NLS) equation, which describes propagation of light pulses in fibers. Whitham modulation equations for the NLS case without any perturbations were obtained in Refs. [19,20], and their solution for the important problem of evolution of an initial discontinuity was found in Refs. [21,22]. This theory was confirmed in the optical experiment [23] with initial pulses having a specially engineered sharp discontinuity. More general forms of DSWs were studied theoretically, e.g., in Ref. [24], and experimentally in Ref. [25]. However, optical DSWs with dissipation have not been studied much so far, because in the optical case standard forms of dissipation also affect a smooth part of a pulse rather than only the strongly oscillatory region. A quite specific situation of the formation of DSWs by the flow of polariton fluid past an obstacle, when dissipation was compensated by pumping, was discussed in Ref. [26].

As was found in Refs. [27,28], the induced Raman scattering in fibers can play the role of pumping or dissipation. In case of normal group velocity dispersion, formation of dark solitons at sharp edges of a pulse was observed in Ref. [29]. Propagation of pulses in fibers with account of induced Raman scattering is described by the equation [30,31] (see also Ref. [32])

$$i\psi_x + \tfrac{1}{2}\psi_{tt} - |\psi|^2\psi = -\gamma\psi(|\psi|^2)_t, \qquad (1)$$

written here in standard nondimensional form for the case of normal dispersion. Here ψ denotes the strength of an electromagnetic wave in a fiber, x is a coordinate along the fiber, t is the normalized time, and γ is a small parameter

*Contact author: shaykin.dv@phystech.edu
†Contact author: kamch@isan.troitsk.ru

of the full modulated solution, the boundary conditions can be written at once and they lead immediately to the Rankine-Hugoniot-like condition. We can apply this approach to our Eq. (1), which in the hydrodynamiclike form [see Eq. (4)] generalizes to

$$\rho_x + (u\rho)_t = 0,$$
$$u_x + uu_t + \rho_t + \left(\frac{\rho_t^2}{8\rho^2} - \frac{\rho_{tt}}{4\rho} - \gamma\rho_t\right)_t = 0. \quad (43)$$

We look for its solution in the form of a traveling wave $\rho = \rho(\xi), u = u(\xi)$ $(\xi = t - x/V)$. Again we make the replacement $\partial_t \to -V\partial_x$, so elementary integration yields

$$-u/V + \rho u = A,$$
$$-\frac{1}{2V^2} + \frac{A^2}{2\rho^2} + \rho + \frac{\rho_\xi^2}{8\rho^2} - \frac{\rho_{\xi\xi}}{4\rho} - \gamma\rho_\xi = \frac{B}{2}, \quad (44)$$

where A and B are integration constants. These constants are the same at both limits $\xi \to \pm\infty$, so arrive at the boundary conditions

$$\rho^L - V\rho^L u^L = \rho^R - V\rho^R u^R,$$
$$u^L - V(\rho^L + (u^L)^2) = u^R - V(\rho^R + (u^R)^2). \quad (45)$$

Then elimination of V easily yields the conditions (37) and (38) in exact agreement with the Whitham equations. It is important that we can neglect the high-order derivatives directly in the system (43) at both sides of a DSW in the limits $t \to \pm\infty$. Then integration of the shallow-water equations (5) yields at once the boundary conditions (45). The same analog of the Rankine-Hugoniot condition was obtained in Ref. [37] for the Kaup-Boussinesq system, which also reduces to the shallow-water equations in the dispersionless limit. In a similar way, the dispersionless limit of any soliton equations leads to the analog of the Rankine-Hugoniot condition without derivation of the full system of Whitham modulation equations and their conservation laws. However, it is not so easy to find the full description of a DSW without solving the Whitham equations. The system (44) can be studied by the Bogoliubov-Mitropolsky method and then we arrive at the analytical description of the DSW in a way similar to the KdV-Burgers case studied in Ref. [43]. In this sense, our formulas obtained by the Whitham averaging method in Sec. III present the analytical solution of the system (44). Apparently, the Whitham method seems more effective for perturbed completely integrable equations in the AKNS scheme [44], since the most difficult task of averaging the conservation laws has already been done in a universal form in Ref. [45], so it only remains to specify the general formulas to the special case under consideration (see Appendix).

V. CONCLUSION

In recent years, several experiments have been performed in optical or similar systems specially designed for demonstration of DSWs (see, e.g., Refs. [23,25,46,47]). In general, these experimental results agree quite well with earlier theoretical predictions, provided the conditions of their applicability were fulfilled. For example, in Ref. [46], Bose-Einstein condensate was confined in a quasi-one-dimensional trap, but the axial confinement was not strong enough to exclude axial dynamics, so decay of shocks to vortices was effective and the shock had a standard viscous form rather than that of a DSW. In Ref. [25] dissipation was also strong enough, but DSWs were clearly seen, although their profiles could only be calculated numerically. In fibers, the one-dimensional geometry is evident and dissipative effects are negligibly small, so the general structure reproduces theoretical predictions perfectly well [23,47]. In this case, some weaker effects can become crucially important for long-distance propagation of pulses. In the physics of optical fibers, the most important such effects are self-steepening and Raman scattering (see, e.g., Ref. [32]). As was shown in Ref. [48], the steepening effect changes parameters of the DSW, but it preserves the expanding evolution of the shock. On the contrary, the Raman effect can stabilize such an expansion, so the shock acquires a stationary form of a modulated oscillatory profile moving with constant velocity. In the small-amplitude limit, the theory of such shocks is described by the KdV-Burgers equation [31,35], so the results of Refs. [9–12] can be applied. In this paper, we have developed the theory of DSWs induced by the Raman effect for large amplitudes. The Whitham equations are derived and thoroughly studied. It is shown that they have enough number of conservation laws for finding the parameters of the shock for given boundary conditions that have to satisfy the analog of the Rankine-Hugoniot condition. It is important that this Rankine-Hugoniot condition is not universal in the sense that it does not follow from conservation laws for the hydrodynamic equations, as it happens in the classical theory of viscous shocks. However, they can be found from the dispersionless limit of equations under consideration. In our case, these limiting hydrodynamic equations have the form of shallow-water equations (5), the same as in the case of the Kaup-Boussinesq-Burgers equations [37], so the Rankine-Hugoniot conditions are the same. Thus, the theory developed in this paper yields both the method of finding the analogues of the Rankine-Hugoniot conditions for completely integrable equations with dissipative perturbations and the method of calculation of stationary profiles of wave variables. One may hope that this theory can find other interesting applications.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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Magnetic field growth in accretion disc: growth and nonlinear saturation

E.A. Mikhailov[1,2,3], E.N. Zhikhareva[2], M.V. Frolova[2]

[1] *Department of Theoretical Physics, P.N. Lebedev Physical Institute of RAS, Leninskii pr. 53, Moscow, 119991 Russia*
[2] *Faculty of Physics, M.V. Lomonosov Moscow State University, Leninskie gory 1 2, Moscow, 119991 Russia*
[3] *Skolkovo Institute of Science and Technology, Bolshoy Boulevard 30 1, Moscow, 121205 Russia*

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Astrophysical accretion discs associated with massive objects should have some magnetic fields. There are strong arguments that such fields should be, at least partly, connected with dynamo action that is based on properties of small- and large-scale motions in the object. Now there are some works connected with studies of the accretion disc dynamos, and it has been shown that the magnetic field can grow there. However, there is a question connected with field generation and its timescale. The properties of the accretion disc are quite different for various parts of the object, so the growth rate can differ dramatically, too. Another problem is connected with nonlinear saturation connected with the field conservation law. Here we describe different approaches to estimate the field growth and describe its structure.

Keywords: Magnetic fields, growth rate, thin disc model, RZ approximation

1 Introduction

Accretion discs play an important role in relativistic astrophysics [1–4]. They surround compact massive objects (black holes, neutron stars, white dwarfs) and contain rapidly rotating medium which can be strongly magnetized. The arguments for the existence of the magnetic field are based on the magnetohydrodynamic processes: they can explain transition of the angular momentum and the medium [1]. Also there are some proofs of the magnetic field at least for such a specific object as an accretion disc surrounding the black hole in the active nuclei of M 87 galaxy [5]. Radioastronomical observations show that there is a remarkable Faraday rotation for electromagnetic waves passing from it, so there should be a magnetic field that produces the rotation

*Email: e.mikhajlov@lebedev.ru

4 Conclusions

We have studied the magnetic field growth in accretion discs using two sufficiently different approaches. First of all, we have studied the field evolution using the thin disc approximation based on the algebraic model for vertical dependence. This model allows us to study the field in the nonaxisymmetric case. However, it has been shown that the field non-uniformities are destroyed by the rotation. After that we used more effective RZ approximation which takes into account that the field depends on the z coordinate according to a differential law. We have studied typical timescales for the magnetic field growth for both cases, and typical field structures.

It can be said that the magnetic field in the nonaxisymmetric case can be modelled using the thin disc approximation. However, vertical structures are described better using the RZ approach.

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УДК 537.621

ВИХРЕВОЕ ТЕЧЕНИЕ ЭЛЕКТРОЛИТОВ, ВЫЗВАННОЕ ВЗАИМОДЕЙСТВИЕМ ЭЛЕКТРИЧЕСКОГО ТОКА И МАГНИТНОГО ПОЛЯ В ЦИЛИНДРИЧЕСКОЙ ЯЧЕЙКЕ

© 2025 г. Е. А. Михайлов[1,2], А. П. Степанова[1,*], И. О. Тепляков[3]

[1]Федеральное государственное бюджетное учреждение высшего образования «Московский государственный университет имени М.В. Ломоносова», Москва, Россия
[2]Федеральное государственное бюджетное учреждение науки «Физический институт имени П.Н. Лебедева Российской академии наук», Москва, Россия
[3]Федеральное государственное бюджетное учреждение науки «Объединенный институт высоких температур Российской академии наук», Москва, Россия
*E-mail: nastasya_stepanova@mail.ru

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При взаимодействии внешнего магнитного поля с электрическим током в проводящей среде могут возникать вихревые течения. Рассмотрено вращающееся течение во внешнем магнитном поле в цилиндрическом объеме, когда одним из электродов является дно сосуда, а второй электрод имеет малые размеры и погружен в проводящую жидкость сверху. Получены как теоретические, так и экспериментальные результаты: зависимость скорости от радиуса верхнего электрода, аналитическое решение для потенциала и скорости, а также его старшая мода.

Ключевые слова: магнитная гидродинамика, вихревое течение, магнитное поле

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ВВЕДЕНИЕ

При взаимодействии электрического тока, протекающего через проводящую среду, и магнитного поля могут возникать вращающиеся движения жидкости. Это связано с силой Ампера, приводящей к появлению вихревых течений в жидкости, которые встречаются в большом количестве прикладных задач. Так, они возникают при электродуговом и электрошлаковом переплаве в металлургии, могут влиять на процессы в аккумуляторных батареях, играют значимую роль при охлаждении некоторых атомных реакторов.

Результаты для закрученных течений в электромагнитном поле были получены коллективом под руководством Бояревича и Щербинина в Институте физики в Латвии [1]. Их результаты охватывают как теоретические исследования, так и экспериментальные работы. Большой вклад в изучение данного вопроса был внесен пермскими специалистами из Института механики сплошных сред [2]. Течения жидких металлов, а также воспроизводящих ряд их свойств растворов солей, изучались в Магнитогорском государственном техническом университете имени Г. И. Носова [3]. Большой вклад в исследование фундаментальных физических процессов, лежащих в основе изучаемого вопроса, был внесен группой Ю. П. Ивочкина из Объединенного института высоких температур РАН [4]. Схожей тематикой обладают исследования коллег из Екатеринбурга [5].

В работах зарубежных авторов ряд важных разложений решений, интересных как для магнитной гидродинамики, так и для математической физики, были получены Созоу и Пиккерингом (Университет Шеффилда, Великобритания) [6]. Интересные вычислительные результаты, относящиеся к поведению жидкости в электромагнитном поле, принадлежат коллективу Кхариши из Университета Леобена (Австрия) [7] и исследователям из научного центра «Дрезден-Россендорф» (Германия) [8].

В данной работе исследуется вихревое течение, вызванное взаимодействием электрического тока и внешнего магнитного поля в цилиндрическом сосуде. Основной целью работы было получение аналитических результатов для скоростей течений для разных жидкостей, и их сравнение с экспериментом. Боковые стенки являются диэлектрическими, а электроды располагаются снизу и сверху (рис. 1). Нижним электродом является металлическое дно сосуда. Верхний электрод имеет небольшие размеры и погружен в про-

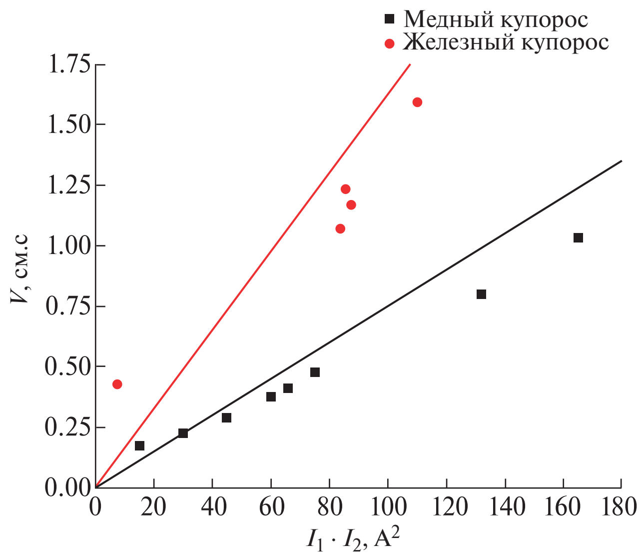

Рис. 3. Экспериментальные результаты для железного купороса и для медного купороса, а также теоретические прямые для них.

теоретической моделью, описанной ранее. Определенную сложность представляло то, что в эксперименте контролировалось значения токов через раствор и через соленоид, а теоретическая модель построена для напряжения и магнитного поля. Тем не менее, с учетом числа витков в катушке и эмпирических оценок для сопротивления между электродами каждой паре токов сопоставлено произведение $U \cdot B$. Теоретические кривые (полученные по формулам (27)—(28)) также приведены на рис. 3. Можно видеть, что результаты в принципе схожи. Несколько завышенная теоретическая скорость может быть связана с действием земного магнитного поля (частично компенсирующее создаваемое соленоидом, а также ухудшение проводимости раствора из-за химических процессов за время проведения эксперимента). Важно отметить, что скорость оказывается пропорциональна произведению силы тока через среду и магнитного поля, создаваемого соленоидом.

ЗАКЛЮЧЕНИЕ

Таким образом, нами было получено как асимптотическое, так и численное решение задачи об вихревом течении под действием электрического тока и внешнего магнитного поля в цилиндрическом сосуде с точечным верхним электродом. Показано, что в таком случае возникает азимутальная скорость, пропорциональная произведению силы тока через среду и магнитного поля соленоида, что хорошо согласуется с теоретическими представлениями о процессе. Численное значение, даваемое теорией, немного завышено; тем не менее, это может быть связано с дополнительными факторами (такими как земное магнитное поле и неоднородность поля соленоида), которые могут быть учтены в дальнейших исследованиях. Данные результаты могут быть полезны в различных случаях, связанных с электромагнитным перемешиванием в различных технических устройствах, а также представлять интерес с точки зрения математической физики. Было получено решение поставленной задачи в общем виде, а также найдена старшая мода, которая в большом количестве подобных задач математической физики достаточно хорошо приближает результат, поэтому интересна как с практической, так и с теоретической точки зрения. Отметим, что магнитные поля могут быть связаны не только с внешним соленоидом, но и с магнитным полем Земли, магнитными полями подводящих проводов и т.д.

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РОЛЬ МАГНИТОРОТАЦИОННОЙ НЕУСТОЙЧИВОСТИ В ПОЯВЛЕНИИ МАГНИТНОГО ПОЛЯ НА ПЕРИФЕРИИ ГАЛАКТИЧЕСКОГО ДИСКА

© 2025 г. Т. Т. Хасаева[1,2,*], Е. А. Михайлов[1,3]

[1]Федеральное государственное бюджетное образовательное учреждение высшего образования «Московский государственный университет имени М.В. Ломоносова», Москва, Россия
[2]Федеральное государственное бюджетное учреждение науки «Институт теории прогноза землетрясений и математической геофизики Российской академии наук», Москва, Россия
[3]Федеральное государственное бюджетное учреждение науки «Физический институт имени П.Н. Лебедева Российской академии наук», Москва, Россия
*E-mail: hasaeva@mitp.ru

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Возникновение магнитных полей галактик принято объяснять действием механизма динамо. Тем не менее, на расстояниях порядка 15–20 кпк действие динамо существенно ослаблено. Магнитное поле в данных областях может объясняться с помощью других механизмов, например, магниторотационной неустойчивости. Найдены решения, отвечающие как механизмам динамо, так и магниторотационной неустойчивости.

Ключевые слова: галактики, динамо, магниторотационная неустойчивость, собственные значения, теория возмущений

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ВВЕДЕНИЕ

В большом количестве спиральных галактик присутствуют крупномасштабные магнитные поля, наличие которых подтверждается с помощью измерений фарадеевского вращения плоскости поляризации радиоволн [1–3]. С теоретической точки зрения их генерация, как правило, описывается с помощью теории динамо, занимающей важное место в космической магнитной гидродинамике. Ключевую роль в работе механизма динамо занимают альфа-эффект и дифференциальное вращения галактического диска. Первый связан со спиральностью турбулентных движений межзвездной среды, второй – с крупномасштабным вращением галактики с меняющейся угловой скоростью. Рост поля ограничивается турбулентной диффузией, размывающей крупномасштабные структуры поля.

Существует ряд работ, посвященных исследованию формирования регулярных структур магнитного поля в галактиках на расстояниях примерно до 10 кпк от ее центра [4–6]. Тем не менее, как показывают исследования [7], в периферийных областях галактических дисков – на расстояниях 15–20 кпк от центра – присутствие магнитных полей несколько меньшей величины также возможно. Важно отметить, что действие динамо там значительно слабее, а поля – примерно на порядок меньше. Тем не менее, в данных областях могут существовать достаточно интенсивные магнитные поля. Косвенным свидетельством этого является характер турбулентных движений, которые могут быть индуцированы с их помощью. Это позволяет предположить, что на больших расстояниях от центра галактики вклад других механизмов в формирование поля может оказаться существенным по сравнению с механизмом динамо.

В магнитной гидродинамике широко известен еще один механизм, который может повлиять на формирование регулярных структур галактического магнитного поля – магниторотационная неустойчивость (МРН). Его суть заключается в возникновении неустойчивости проводящей среды, находящейся в магнитном поле, что приводит к переносу момента количества движения и росту самого поля. В работе [8] было рассмотрено действие МРН в аккреционных дисках. Ввиду того, что многие магнитогидродинамические процессы в аккреционных и галактических дисках схожи, МРН наряду с динамо также может в некоторых случаях, самого старшего, необходимо рассмотреть альтернативные подходы к разрешению системы. Чтобы избавиться от этой проблемы, мы использовали метод немонотонной прогонки, который не требует подобного условия [15].

В рамках данного метода матрица системы (28), как и в классическом случае, приводится к верхнетреугольному виду, однако в данном случае на каждом шаге прямого хода прогонки проверяется выполнение условия диагонального преобладания. В зависимости от его наличия каждая строчка преобразуется при помощи замен, позволяющих схеме быть устойчивой, в отличие от классического метода прогонки, где во время прямого хода ряд коэффициентов может обратиться в ноль или, напротив, существенно возрасти.

Старшие нормированные собственные функции представлены на рис. 2. Отвечающие им собственные значения представлены в табл. 1. Для отвечающих им магнитных полей, которые отличаются в $\sqrt{r}$ раз, собственные значения показаны на рис. 3. Вполне логично, что чем выше номер гармоники, тем большее количество локальных максимумов имеет магнитное поле. Кроме того, «амплитуда колебаний» убывает по мере удаления от центра. Стоит отметить, что полученные результаты соответствуют ранее полученным теоретическим оценкам [14] для спиральных галактик.

ЗАКЛЮЧЕНИЕ

Таким образом, нами были рассмотрены возможные сценарии генерации магнитных полей во внешних областях галактик (превышающих 10 кпк для таких галактик, как Млечный Путь, и 15 кпк для таких объектов, как М 31). Так, действие динамо в этих регионах хотя и возможно, но оказывается сильно подавленным. Магнитные поля могут проникать в данные области за счет нелинейных механизмов – таких, как эффект Колмогорова–Петровского–Пискунова, который связан с формированием нелинейной волны, распространяющейся после того, как поле во внутренних областях достигает значений, сопоставимых с равнораспределением [7]. Тем не менее, магнитные поля, сгенерированные такими способами, оказываются достаточно слабыми.

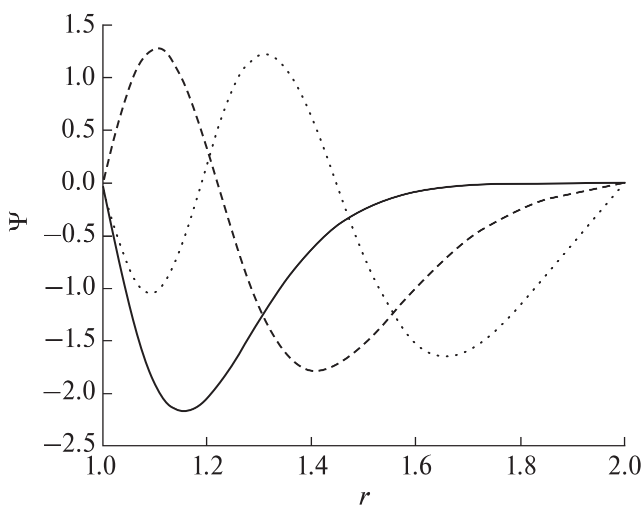

Рис. 2. Первая (сплошная), вторая (штриховая) и третья (пунктирная) собственные функции в безразмерных единицах.

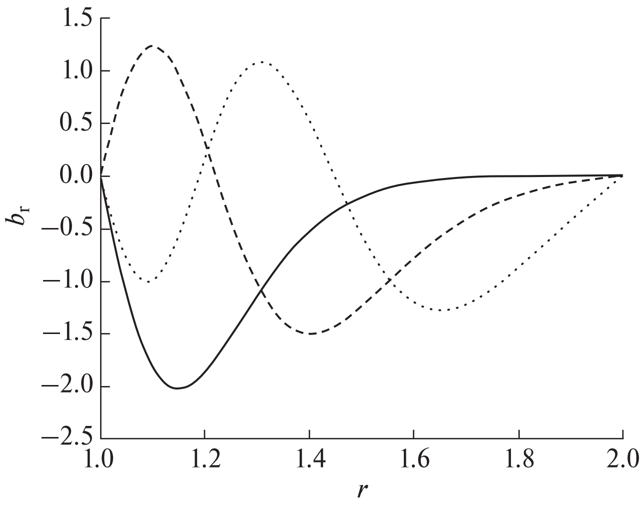

Рис. 3. Радиальная компонента магнитного поля, соответствующая первой (сплошная), второй (штриховая) и третьей (пунктирная) собственным функциям в безразмерных единицах.

Более эффективной оказывается магниторотационная неустойчивость, которая возникает за счет градиента угловой скорости вращения галактики. Она приводит к генерации различных мод магнитного поля, которые могут объяснять те или иные структуры поля. Ключевую роль играет волновое число k_z, которое характеризует вертикальный масштаб возникающих полей $\frac{1}{k_z}$. Нами были получены соответствующие собственные значения и можно видеть, что эти масштабы соответствуют соотношению между полутолщиной диска и радиусом его основной части (порядка 10^{-2}). Это говорит о том, что данный механизм может играть существенную роль в генерации магнитных полей во внешних областях галактик.

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МАГНИТНЫЕ ПОЛЯ, ГЕНЕРИРУЕМЫЕ В АККРЕЦИОННЫХ ДИСКАХ С ПОМОЩЬЮ МЕХАНИЗМА ДИНАМО

© 2025 г. Е. А. Михайлов[1,2,*], М. В. Фролова[2], Е. Н. Жихарева[2]

[1]Федеральное государственное бюджетное учреждение науки
Физический институт имени П.Н. Лебедева Российской академии наук, Москва, Россия
[2]Федеральное государственное бюджетное учреждение высшего образования
«Московский государственный университет имени М.В. Ломоносова», Москва, Россия
*E-mail: ea.mikhajlov@physics.msu.ru

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Магнитные поля играют важную роль в структуре и эволюции аккреционных дисков. Существуют серьезные аргументы в пользу того, что они связаны с действием механизма динамо. Обсуждается структура магнитного поля, которую можно получить с использованием различных приближений. Отдельное внимание уделяется возможности возникновения существенно неосесимметричных решений.

Ключевые слова: аккреционные диски, динамо, магнитное поле, приближение

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ВВЕДЕНИЕ

Аккреционные диски, окружающие массивные астрофизические объекты — черные дыры, нейтронные звезды и белые карлики — играют большую роль в релятивистской астрофизике [1]. Они тесно связаны с гамма-всплесками [2, 3], возникновением джетов [4] и другими важными процессами. Аккреционный диск состоит из диффузной среды, которая падает на центральный объект под действием гравитационных сил, но при этом обладает значимым моментом количества движения. Еще в классических работах показано, что многие процессы, важные для существования диска, невозможно объяснить без наличия магнитного поля [1]. Так, даже достаточно большие градиенты угловой скорости не позволили бы различным частям диска обмениваться друг с другом моментом импульса [1, 5—7].

Несмотря на серьезные теоретические аргументы, наблюдательные подтверждения наличия магнитных полей в аккреционных дисках долгое время отсутствовали. В первую очередь это связано с достаточно малыми размерами аккреционных дисков и недостаточным разрешением наблюдательной техники. Лишь относительно недавно были получены данные о фарадеевском вращении плоскости поляризации радиоволн, измеренном при исследовании аккреционного диска, окружающего черную дыру в центре галактики M87 [8, 9].

Если говорить о причинах возникновения магнитных полей аккреционных дисков, то можно отметить несколько подходов. Наиболее простой из их связан с переносом магнитного поля с падающей средой [10]. Широко известно, что поле вморожено в ионизованный газ, и можно ожидать, что это станет источником магнетизма в диске. Очевидной проблемой такого подхода является то, что за счет турбулентности потоков в среде крупномасштабные структуры поля разрушаются, оно будет иметь в различных точках случайные направления и небольшую величину. Поэтому данный механизм может обуславливать лишь начальные поля, которые станут «затравкой» для других, более эффективных механизмов. Другой подход относится к влиянию поля центрального объекта. Известно, что ряд черных дыр, нейтронных звезд и белых карликов имеют значительный для их размеров магнитный момент; следовательно, можно было бы ожидать, что он будет индуцировать магнитные поля в окружающих их аккреционных дисках. Тем не менее, численное моделирование для таких систем показало, что создаваемые таким образом магнитные поля не будут достаточно интенсивными [11].

По указанным причинам, наиболее вероятным представляется возникновение полей в аккреционных дисках за счет механизма динамо. Он обуславливает магнетизм целого ряда космических объектов, таких как Солнце, другие

УЧЕТ ВЕРТИКАЛЬНОЙ СТРУКТУРЫ ПОЛЯ

Между тем, с учетом того, что аккреционный диск расширяется по мере удаления от центра, возникает необходимость рассмотрения более сложной зависимости поля от вертикальной координаты, нежели простой косинусоидальный закон. С учетом того, что выше была показана оправданность осесиммметричного приближения, будем рассматривать поле в виде $B = \vec{\nabla} \times \left(A\vec{e}_{\phi}\right) + B\vec{e}_{\phi}$. В таком случае уравнение Штеенбека–Краузе–Рэдлера сведется к системе из таких двух уравнений (так называемое RZ-приближение) [19]:

$$\frac{\partial A}{\partial t} = \frac{\Omega l^2}{h^2} zB + \nu\left(\frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial \phi} - \frac{A}{r^2}\right); \quad (12)$$

$$\frac{\partial B}{\partial t} = \Omega \frac{\partial A}{\partial z} + \nu\left(\frac{\partial^2 B}{\partial z^2} + \frac{\partial^2 B}{\partial r^2} + \frac{1}{r}\frac{\partial B}{\partial \phi} - \frac{B}{r^2}\right). \quad (13)$$

Для азимутальной компоненты магнитного поля B на границах области выбиралось нулевой значение, а для азимутальной составляющей его векторного потенциала A – условие нулевой нормальной производной. Рассматривались значения координат $r_{\min} < r < R$, $-h(r) < z < h(r)$, где $h(r) = h_0\left(\frac{r}{r_{\min}}\right)^{9/8}$.

Данные уравнения решались численно, результат для структуры азимутального магнитного поля представлен на рис. 4. Можно отметить, что хотя магнитное поле и демонстрирует более сложную структуру, приближенно можно представлять его в виде $\cos\left(\frac{\pi z}{2h(r)}\right)$.

ЗАКЛЮЧЕНИЕ

Таким образом, нами рассмотрен процесс роста магнитного поля в аккреционных дисках. Показано, что оно может быть создано с помощью механизма динамо. С помощью моделирования подтверждена допустимость использования осесимметричных моделей: показано, что «пестрая» структура магнитного поля, созданная в начальный момент времени, постепенно размывается, и в течение долгого времени сохраняются лишь радиальные неоднородности. В свою очередь, сами эти неоднородности могут являться предметом самостоятельного исследования в рамках теории контрастных структур.

Также изучена возможность использования планарного приближения, которое было разработано для тонких галактических дисков. С этой целью уравнения магнитной гидродинамики были решены с использованием RZ-приближения, которое более аккуратно учитывает вертикальную структуру поля. Показано, что отличия поля от косинусоидального закона, используемого в рамках планарного приближения, являются не очень существенными. Отметим, что в других работах, относящихся к применению данного приближения к галактическим объектам, показана возможность генерации дипольных структур поля. Впрочем, это требует крайне интенсивных течений среды, которые могут реализовываться лишь в довольно специфических объектах.

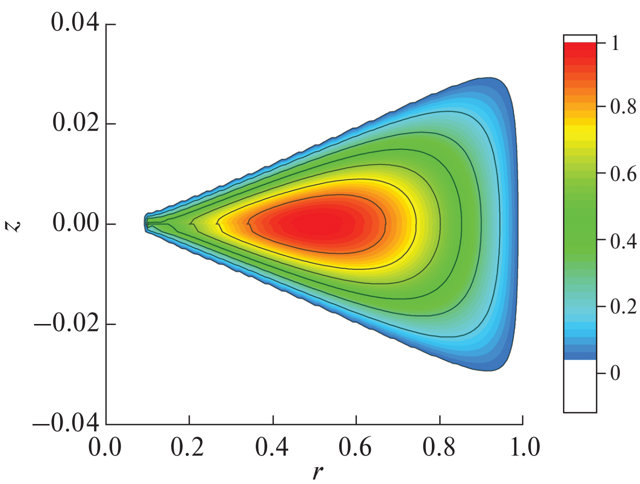

Рис. 4. Структура магнитного поля в рамках RZ-модели

Работа Е. А. Михайлова по формулировке задачи о времени существования неоднородностей в рамках планарного приближения выполнена при поддержке Российского научного фонда (проект № 19-72-30028). Работа М. В. Фроловой по исследованию магнитных полей в рамках RZ-модели выполнена при поддержке Фонда развития теоретической физики и математики «Базис» (проект № 24-2-2-36-1). Работа Е. Н. Жихаревой и Е. А. Михайлова по численной реализации модели с учетом случайных начальных условий выполнена в рамках государственного задания МГУ имени М.В. Ломоносова.

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