

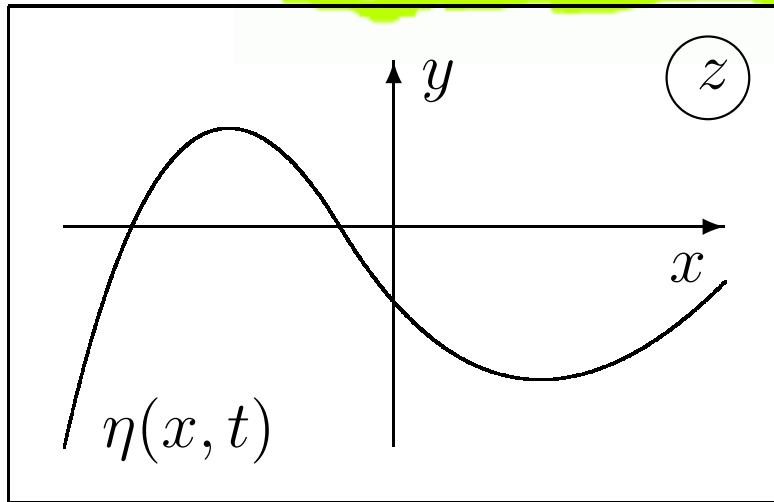


Breaking of Progressive Gravity Surface Wave and Modulation Instability

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Equations



potential irrotational flow

$$\Delta \phi(x, y, t) = 0$$

Boundary conditions:
$$\left[\begin{array}{l} \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g\eta = \frac{P}{\rho}, \\ \frac{\partial \eta}{\partial t} + \eta_x \phi_x = \phi_y \end{array} \right] \text{ at } y = \eta(x, t).$$

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= 0, y \rightarrow -\infty, \\ \frac{\partial \phi}{\partial x} &= 0, |x| \rightarrow \infty. \end{aligned}$$

Hamiltonian and Lagrangian

Hamiltonian H is the total energy of the fluid $H = T + U$

$$\begin{aligned} T &= \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\eta}^{\eta} (\nabla \Phi)^2 dy, & \frac{\partial \eta}{\partial t} &= \frac{\delta H}{\delta \Psi}, \\ U &= \frac{g}{2} \int \eta^2 dx. & \frac{\partial \Psi}{\partial t} &= -\frac{\delta H}{\delta \eta}, \\ & & \Psi(x, t) &= \Phi(x, y, t)|_{y=\eta} \end{aligned}$$

Action $S = \int L dt$

$$\delta S = 0,$$

Lagrangian

$$L = \int \Psi \frac{\partial \eta}{\partial t} dx - H.$$

Conformal Mapping

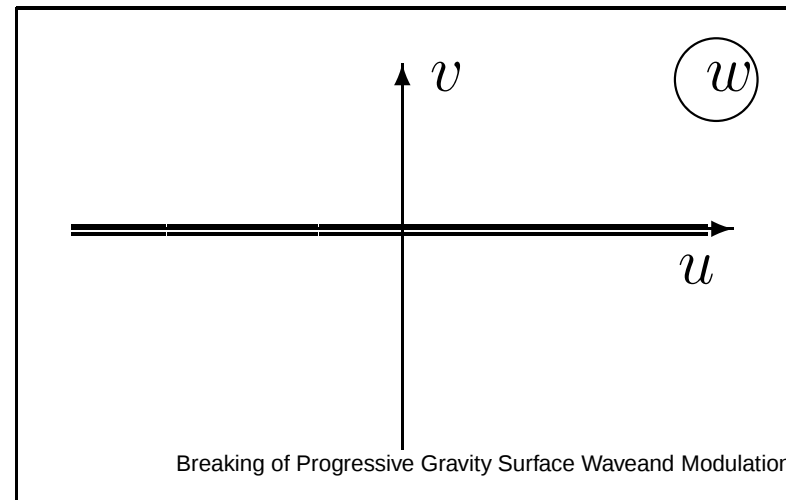
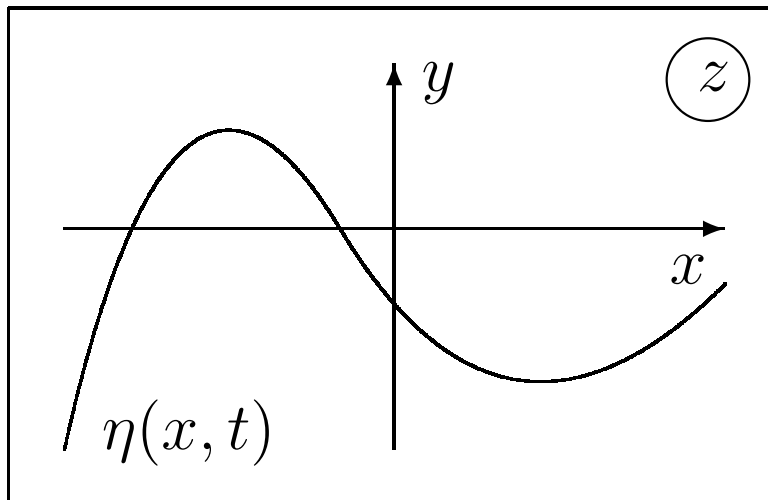
Let us apply the conformal mapping of the domain on the plane $z = x + iy$,

$$-\infty < x < \infty, \quad -\infty < y \leq \eta(x, t),$$

to the lower half-plane,

$$-\infty < u < \infty, \quad -\infty < v \leq 0,$$

on the plane $w = u + iv$.



Conformal Mapping

After this transformation, the shape of the surface is given parametrically by

$$y = y(u, t), \quad x = u + \tilde{x}(u, t).$$

Functions y and $\tilde{x}$ are coupled by the relations:

$$y = \hat{H}\tilde{x} \qquad \tilde{x} = -\hat{H}y.$$

Here $\hat{H}$ is the Hilbert transformation,

$$\hat{H}f(u) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(u')}{(u' - u)} du'.$$

For Fourier harmonics $y_k = i \text{sign}(k) x_k$.

Implicit Equations

Lagrangian can be expressed as follows,

$$L = \int_{-\infty}^{\infty} \Psi(y_t x_u - x_t y_u) du + \frac{1}{2} \int_{-\infty}^{\infty} \left(\Psi \hat{H} \Psi_u du - g y^2 x_u \right) du + \int_{-\infty}^{\infty} f(y - \hat{H} \tilde{x})$$

Here f is the Lagrange multiplier. Hamilton's principle,

$$\frac{\delta S}{\delta \Psi} = 0, \quad \frac{\delta S}{\delta y} = 0, \quad \text{and} \quad \frac{\delta S}{\delta x} = 0.$$

gives the following equations,

$$\begin{aligned} y_t x_u - x_t y_u &= -\hat{H} \Psi_u, \\ \Psi_t x_u - \Psi_u x_t + g y x_u &= \hat{H} (\Psi_t y_u - \Psi_u y_t + g y y_u). \end{aligned}$$

Explicit Equations

If *conformal mapping* has been applied then it is naturally introduce complex analytic functions

$$z = x + iy, \quad \text{and complex velocity potential} \quad \Phi = \Psi + \hat{H}\Psi.$$

$$\begin{aligned} z_t &= iU z_u, \\ \Phi_t &= iU \Phi_u - \hat{P}\left(\frac{|\Phi_u|^2}{|z_u|^2}\right) + ig(z - u). \end{aligned}$$

U is a complex transport velocity:

$$U = \hat{P}\left(\frac{-\hat{H}\Psi_u}{|z_u|^2}\right). \quad u \rightarrow w$$

Projector operator $\hat{P}(f) = \frac{1}{2}(1 + i\hat{H})(f).$

Classical variables Ψ, η

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \Psi \hat{G}(\eta) \Psi dx + \frac{g}{2} \int_{-\infty}^{\infty} \eta^2 dx$$

Normal complex variable a_k :

$$\eta_k = \sqrt{\frac{\omega_k}{2g}} (a_k + a_{-k}^*) \quad \psi_k = -i \sqrt{\frac{2g}{\omega_k}} (a_k - a_{-k}^*) \quad \omega_k = \sqrt{gk}$$

a_k satisfies the equation

$$\frac{\partial a_k}{\partial t} + i \frac{\delta H}{\delta a_k^*} = 0,$$

$$a_k \rightarrow b_k,$$

excluding nonresonant terms.

Classical variables b_k

Hamiltonian without cubic terms:

$$H = \int \omega_k b_k b_k^* dk + \frac{1}{2} \int T_{kk_1}^{k_2 k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \dots$$

$$k + k_1 = k_2 + k_3,$$

$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3},$$

$$T_{kk_1}^{k_2 k_3} = 0!$$

$$T_{k_4 k_5}^{k_1 k_2 k_3} = \frac{2}{g^{1/2} \pi^{3/2}} \sqrt{\frac{\omega_{k_1} \omega_{k_2} \omega_{k_3}}{\omega_{k_4} \omega_{k_5}}} \frac{k_1 k_2 k_3 k_4 k_5}{\max(k_1, k_2, k_3)}$$

Cubic Equations

It turned out, that the equations can be simplified just by changing variables. Introduce instead of $z(w, t)$ and $\Phi(w, t)$ another analytic functions $R(w, t)$ and $V(w, t)$

$$R = \frac{1}{z_w}, \quad \Phi_w = -iV z_w.$$

$$\begin{aligned} R_t &= i [UR' - U'R], \\ V_t &= i \left[UV' - R\hat{P}(V\bar{V})' \right] + g(R - 1). \end{aligned}$$

Complex transport velocity U is defined via $\hat{P}$

$$U = \hat{P}(V\bar{R} + \bar{V}R).$$

Setup the problem

- ⑥ The shape of stationary progressive wave is given by:

$$y = \frac{c^2}{2g} \left(1 - \frac{1}{|z_w|^2} \right),$$

while Φ is related to the surface as

$$\Phi = -c(z - w), \quad V = ic(R - 1).$$

The amplitude of the wave $\frac{h}{L} \simeq 0.088$,

For the sharp peaked limiting wave $\frac{h}{L} \simeq 0.141$.

Setup the problem

- Put 10 such waves in the periodic domain of 2π .
- Add long scale perturbation to this wave train

$$V_k = ic(R_k - 1) + \epsilon\delta_{1,k}.$$

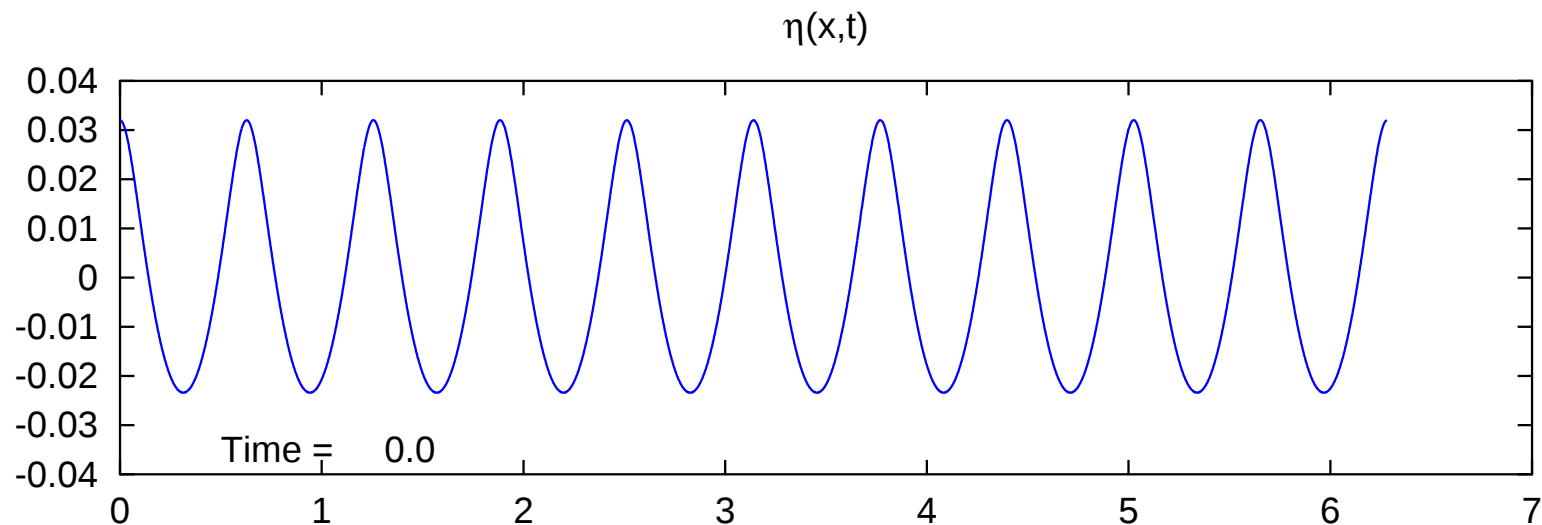


Figure 1: Initial profile of the wave train

Simulation

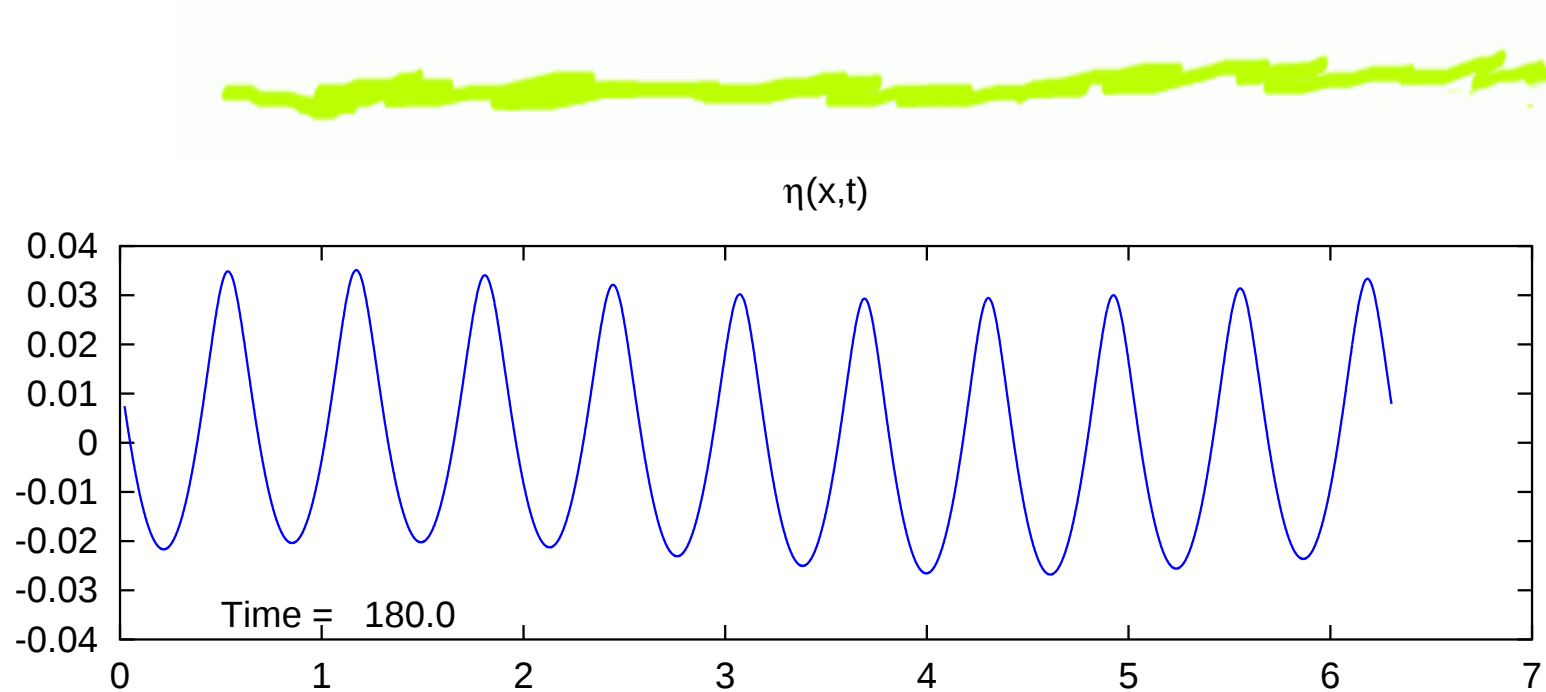
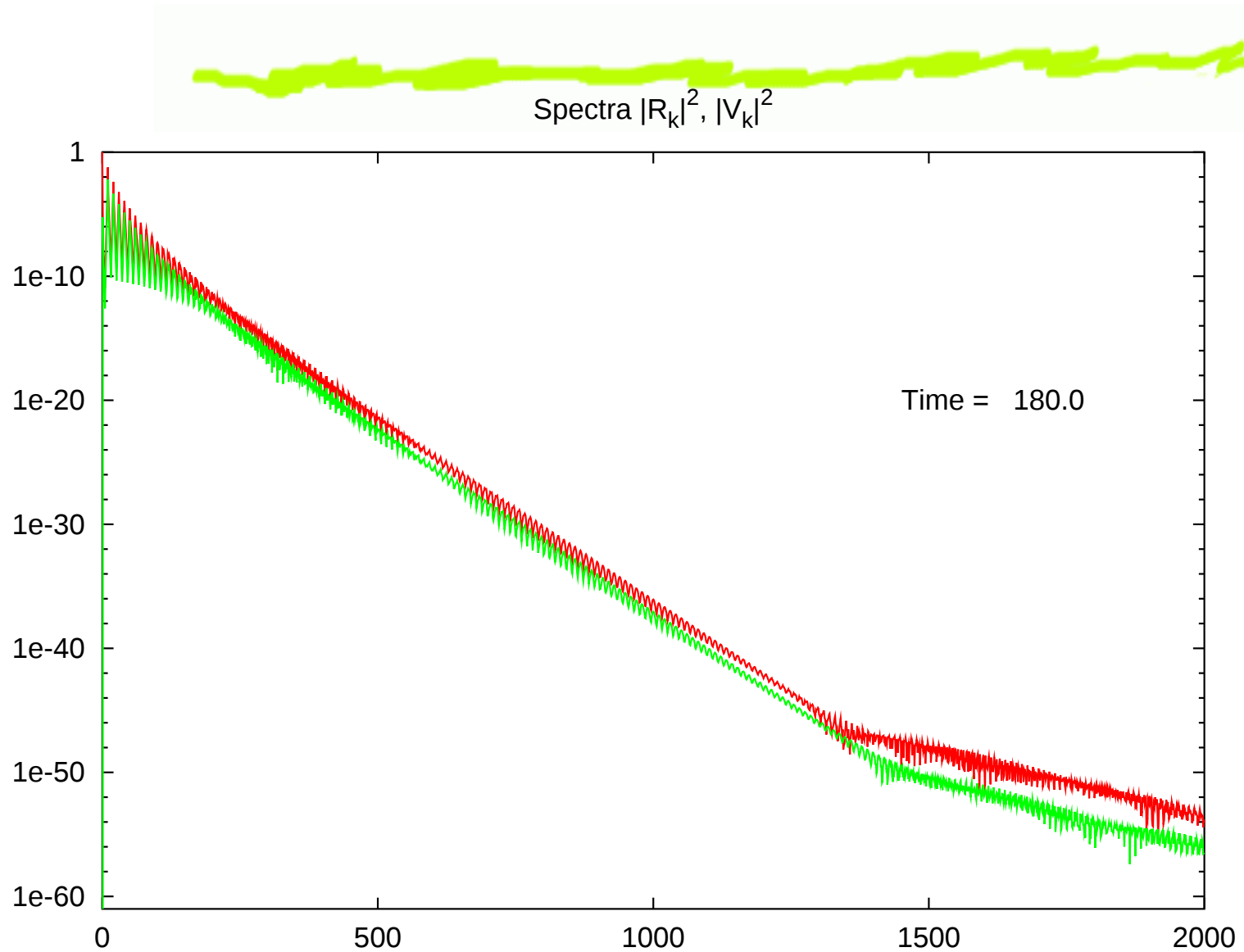


Figure 2: Typical profile of the wave train

Simulation



Wave breaks

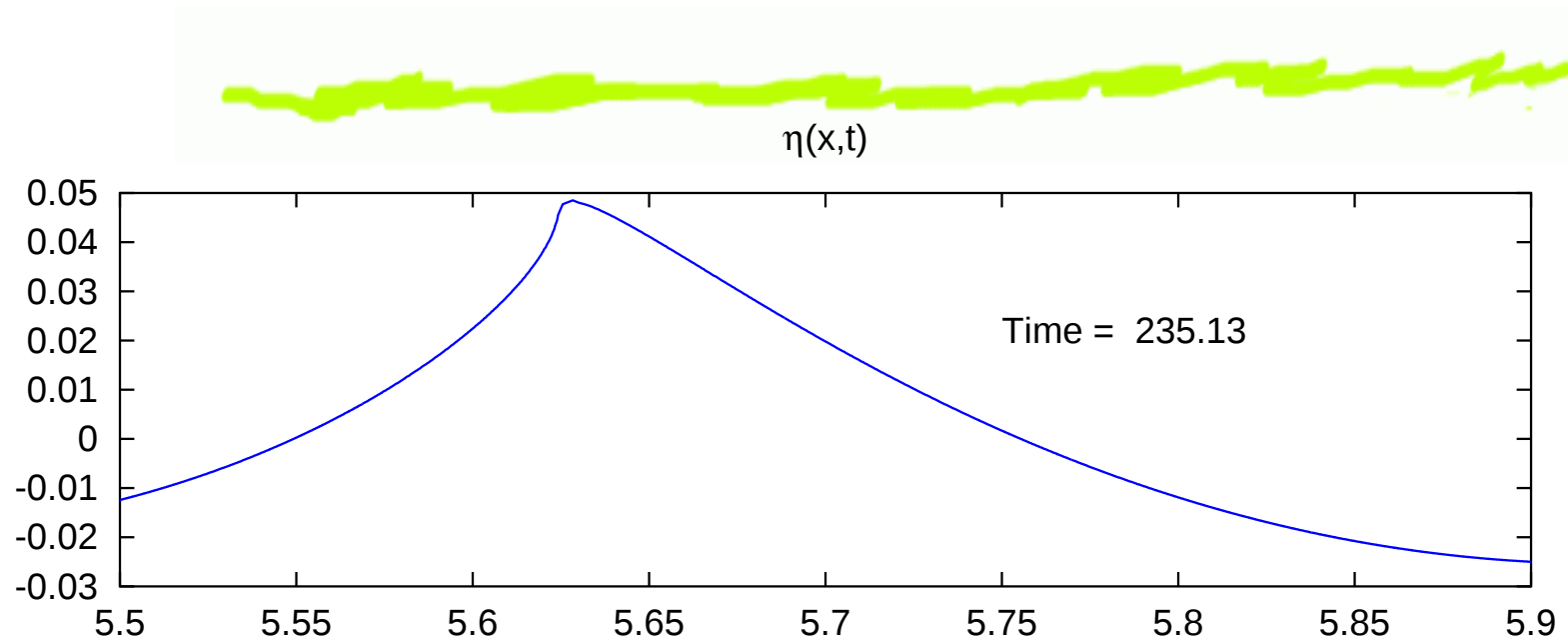


Figure 4: Wave breaks.

Slope of the surface

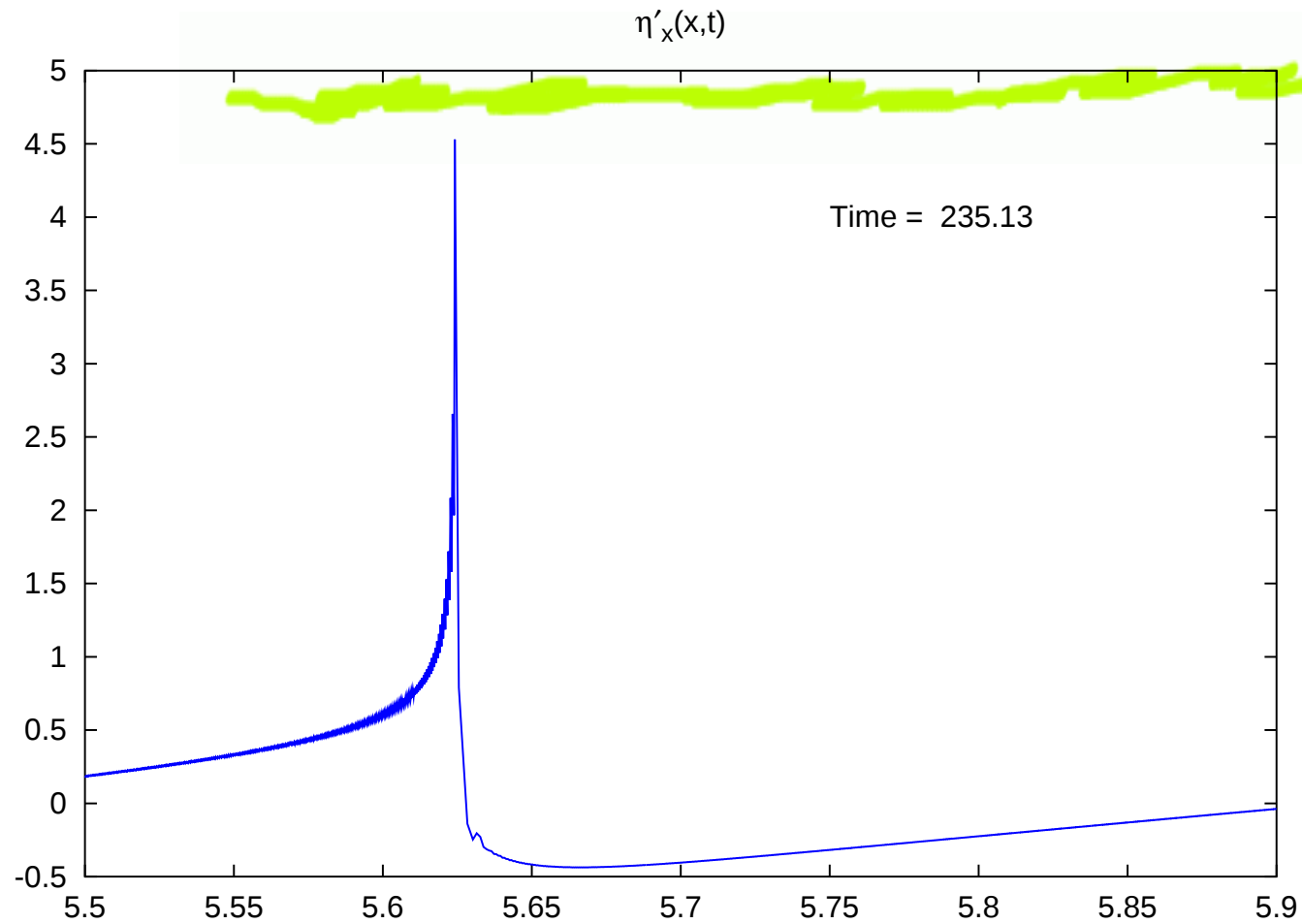


Figure 5: Sharp wedge with the angle $\frac{\pi}{3}$

Power spectrum

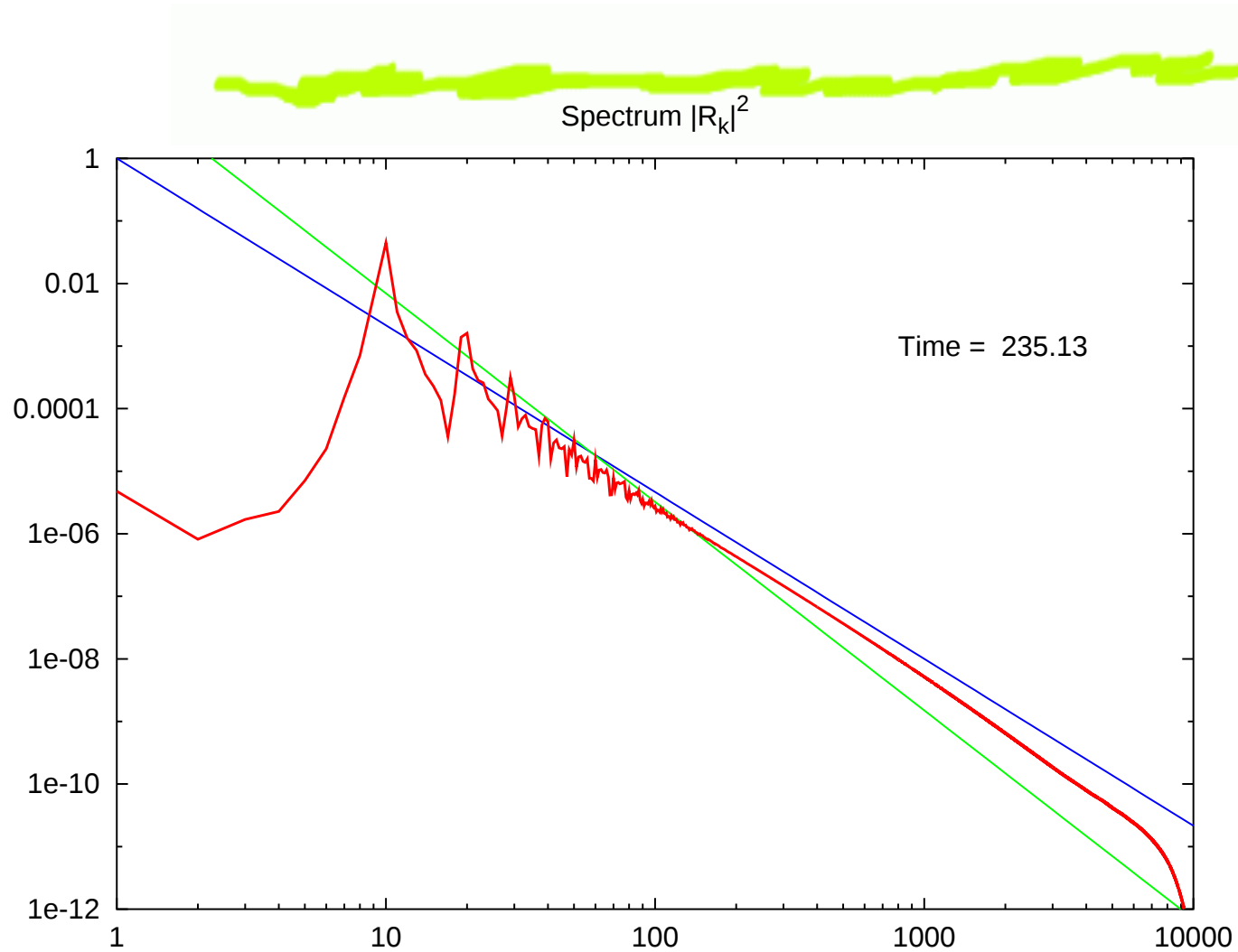


Figure 6: Spectrum of $|R_k|^2$ and $k^{-\frac{8}{3}}$ and $k^{-\frac{10}{3}}$.

???

Sharp peaked stationary gravity wave at the crest

$$z \sim i \frac{c^2}{2g} + \left(\frac{9}{4} \frac{c^2}{g} \right)^{\frac{1}{3}} [-i w^2]^{\frac{1}{3}} + \epsilon w^\alpha + \dots \quad (\sqrt[3]{-i} = e^{-i\frac{\pi}{6}})$$

This is well known result of Stokes with the angle - $\frac{2\pi}{3}$.

For breaking wave numerics shows - $\frac{\pi}{3}$ wedge.

$$z \sim iA(t) + B(t)[-i(w - a(t))(w - b(t))]^{\frac{1}{3}} + \dots$$

Root branch that corresponds to the numerics is

$$z = \begin{cases} -i|u|^{1/3}, & \text{missing real part } u \leq 0; \\ |u|^{1/3}(\frac{\sqrt{3}}{2} - \frac{i}{2}), & u > 0, \end{cases}$$

???

Wave profile in (x, y) coordinates:

$$\begin{cases} y^3 = x, & x \leq 0; \\ y = \frac{1}{\sqrt{3}}x, & x > 0. \end{cases}$$

“Overturning” takes place only near the wave top, main shape of the wave looks like Stokes wave. Distance between branch points a and b is small with respect to wavelength.