Nonlinear Waves, Solitons and Collapses

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CHAPTER 1

Finite Dimensional Hamiltonian Systems

We start by recalling basic facts from classical mechanics. More details can be found in many books on classical mechanics, see e.g. [**LL89b, Arn89**]. The readers with a prior knowledge of classical mechanics can jump directly to Section (1.12) or even to Chapter (2.1) while looking at previous sections when helpful.

1.1. Canonical Hamiltonian mechanics

Canonical Hamiltonian equations

$$\begin{aligned}\frac{dq_j}{dt} &= \frac{\partial H}{\partial p_j},\\ \frac{dp_j}{dt} &= -\frac{\partial H}{\partial q_j}, \quad j = 1, \ldots, N\end{aligned} \tag{1.1}$$

are the system of the even number $2N$ (N is the positive integer) of ordinary differential equations (ODE). That system is fully defined by the Hamiltonian H which is the given smooth function $H = H(p_1, \ldots, p_N, q_1, \ldots q_N, t)$ of independent variables $\mathbf{p} := (p_1 \ldots, p_N)$, $\mathbf{q} := (q_1 \ldots, q_N)$ and time t. Here all components of $\mathbf{p}$, $\mathbf{q}$ and as well as H and t are real numbers $\mathbb{R}$. The variables $\mathbf{p}$ and $\mathbf{q}$ are usually called by the generalized momenta and coordinates, respectively. The set of all possible values of $\mathbf{p}$ and $\mathbf{q}$ is called by a *phase space* P. The canonical Hamilton equations together with the independent variables $\mathbf{q}$ and $\mathbf{p}$ and the phase space P form the equations of the Hamiltonian mechanics. For the beginning we can assume that P is the set in $\mathbb{R}^{2N}$, the vector space of $2N$ real numbers. Generally, the phase space P is a *smooth manifold* of dimension $2N$ (see Appendix A.3 for the definition of smooth manifold) and $(\mathbf{q}, \mathbf{p})$ are local coordinates in P.

A time dependence of H is given by

$$\frac{dH}{dt} = \sum_{j=1}^{N}\left[\dot p_j \frac{\partial H}{\partial p_j} + \dot q_j \frac{\partial H}{\partial q_j}\right] + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}, \tag{1.2}$$

where we used the dynamic equations (1.1) for $\dot p_j$ and $\dot q_j$. Here here and below we use the notation $\dot f := \frac{df}{dt}$ for any function $f(t)$. In most cases below we assume that the Hamiltonian H does not depends on t explicitly (unless we explicitly specify the opposite), i.e. $\frac{\partial H}{\partial t} = 0$. Then the Hamiltonian is the constant of motion as follows from (1.2):

$$\frac{dH}{dt} = 0. \tag{1.3}$$

For the following particular form of the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{{p_i}^2}{2m_i} + U(\mathbf{q}) \tag{1.4}$$

we recover from (1.1) the Newton equations for the motion of the particles with the coordinates $\mathbf{q}$, the momenta $\mathbf{p}$ and the masses m_i in the potential $U(\mathbf{q})$: $p_i = m_i\dot{q}_i,\ i = 1,\dots N,\ \dot{\mathbf{p}} = -\frac{\partial U}{\partial \mathbf{q}}$, where here and below we use the notation $\frac{\partial}{\partial \mathbf{q}} := \left(\frac{\partial}{\partial q_1},\dots,\frac{\partial}{\partial q_N}\right)$. For N/D particles in $D-$dimensional space with coordinates $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{N/D} \in \mathbb{R}^D$ we set $\mathbf{q} = (\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{N/D}) \in \mathbb{R}^N$ (masses m_i are split in that case in N/D groups). Each component of $\mathbf{q}$ and $\mathbf{p}$ can take the arbitrary real values so the phase space is $\mathbb{R}^N \times \mathbb{R}^N = \mathbb{R}^{2N}$. In the equation (1.4), the terms $\sum\limits_{i=1}^{N} \frac{{p_i}^2}{2m_i}$ and $U(\mathbf{q})$ are called by the kinetic energy and the potential energy, respectively. Thus the Newton equations of the potential motion of particles is the particular case of the Hamiltonian mechanics with the Hamiltonian (1.4) and the phase space is $\mathbb{R}^{2N}$ provided $U(\mathbf{q})$ is the smooth function of $\mathbf{q}$.

The system of harmonic oscillators is one of the simplest nontrivial example of the Hamiltonian (1.4) with the quadratic potential

$$U(\mathbf{q}) = \sum_{j=1}^{N} \frac{m_j\omega_j^2}{2}{q_j}^2. \tag{1.5}$$

Together with (1.4) it gives the harmonic oscillator Hamiltonian

$$H = \sum_{i=1}^{N} \left(\frac{{p_j}^2}{2m_j} + \frac{m_j\omega_j^2}{2}{q_j}^2\right), \tag{1.6}$$

which is quadratic both in $\mathbf{q}$ and $\mathbf{p}$. Then the dynamic equations (1.1) turns into the system of linear ODEs with constant coefficients which are decoupled for each pair of canonical variables $(p_j, q_j),\ j = 1,\dots,N$. The general solution of that system is

$$\begin{aligned} p_j &= m_j\omega_j a_j \cos(\omega_j t + \varphi_j), \\ q_j &= a_j \sin(\omega_j t + \varphi_j), \quad j = 1,\dots,N, \end{aligned} \tag{1.7}$$

where $\mathbf{a} = (a_1,\dots,a_N)$ and $\varphi = (\varphi_1,\dots,\varphi_N)$ are $2N$ arbitrary real constants. These constants are uniquely determined by the initial value problem $\mathbf{p}(t_0) = \mathbf{p}_0$ and $\mathbf{q}(t_0) = \mathbf{q}_0$ for the system (1.1) and (1.6).

One of the simplest example of the Hamiltonian system with the nontrivial phase space is an ideal planar pendulum which is the points mass m in the gravitational field g attached by a rigid rod of zero mass and length L to the point O such that it can rotate freely at any angle around O as shown in Figure 1. It is convenient to define the generalized coordinate θ as the angle of rotation in counterclockwise direction with respect to the negative direction of the vertical axis as shown in Figure 1. The Hamiltonian is the sum of the kinetic energy $\frac{m}{2}L^2\dot{\theta}^2$ and the potential energy $-mgL\cos\theta$ giving

$$H = \frac{p^2}{2mL^2} - mgL\cos\theta, \tag{1.8}$$

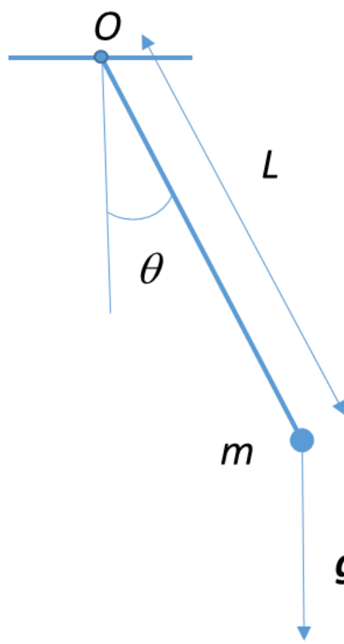

FIGURE 1. A schematic of planar pendulum of the length L and mass m. g is the acceleration of gravity and θ is the angle with respect to the negative (downward) direction of the vertical axis.

where we defined the canonical momentum as $p = mL^2\dot{\theta}$, where $\dot{\theta} := \frac{d\theta}{dt}$. Equations (1.1) and (1.8) can be solved in elliptic functions. The qualitative behaviour of the system can be however extracted from the conservation of the Hamiltonian H: if $H < mgL$ then even for $p = 0$ the angle cannot reach the value $\theta = \pm\pi$ and the pendulum oscillates nether passing through the vertical position $\theta = \pm\pi$; if $H > mgL$ then the pendulum rotates nonstop either clockwise of counterclockwise depending on the sign the initial angular velocity $\dot{\theta}|_{t=0}$. Values of $|p(t)|$ can be arbitrary large (depending on initial conditions) while θ spans between $-\pi$ and π because of the periodicity in θ. Thus the phase space of the planar pendulum is the cylinder $[-\pi, \pi) \times \mathbb{R}$.

1.2. Poisson bracket and action-angle variables

It is often convenient to formulate the Hamiltonian equations (1.1) through a Poisson bracket. Consider the functions $F(\mathbf{p}, \mathbf{q})$ and $G(\mathbf{p}, \mathbf{q})$ assumed to be infinitely differentiable in both arguments. The Poisson bracket $\{.,.\}$ in canonical variables $\mathbf{p}$ and $\mathbf{q}$ is defined in the phase space P as

$$\{F, G\} = \sum_{j=1}^{N} \left(\frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right). \tag{1.9}$$

The Hamiltonian equations (1.1) can be equivalently expressed through the Poisson brackets between $\mathbf{p}$, $\mathbf{q}$ and the Hamiltonian H as follows

$$\begin{aligned} \frac{d\mathbf{q}}{dt} &= \{\mathbf{q}, H\}, \\ \frac{d\mathbf{p}}{dt} &= \{\mathbf{p}, H\}. \end{aligned} \tag{1.10}$$

Using the definition (1.9) we immediately obtain that $\{\mathbf{q}, H\} = \frac{\partial H}{\partial \mathbf{p}}$ and $\{\mathbf{p}, H\} = -\frac{\partial H}{\partial \mathbf{q}}$ thus recovering (1.1) from (4.85).

It follows from the definition (1.9) that the Poisson bracket is antisymmetric

$$\{F, G\} = -\{G, F\} \tag{1.11}$$

and satisfies a Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \tag{1.12}$$

for any infinitely differentiable functions $F(\mathbf{p}, \mathbf{q})$, $G(\mathbf{p}, \mathbf{q})$ and $H(\mathbf{p}, \mathbf{q})$.

Time-derivative of any function $F(\mathbf{p}, \mathbf{q})$ with $\mathbf{p}(t)$ and $\mathbf{q}(t)$ being the solution of (1.1) is conveniently represented through the Poisson bracket as follows

$$\frac{dF}{dt} = \dot{\mathbf{p}} \cdot \frac{\partial F}{\partial \mathbf{p}} + \dot{\mathbf{q}} \cdot \frac{\partial F}{\partial \mathbf{q}} = -\frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial F}{\partial \mathbf{p}} + \frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial F}{\partial \mathbf{q}} = \{F, H\}. \tag{1.13}$$

Here and below $\dot{\mathbf{f}} := \frac{d\mathbf{f}}{dt}$ for any $\mathbf{f}$. A function $F(\mathbf{p}, \mathbf{q})$ is called a *constant of motion* (also called by *integral of motion*) if F is constant along the solution of (1.1), i.e. $\frac{dF}{dt} = 0$, which implies by equation (1.13) that

$$\{F, H\} = 0. \tag{1.14}$$

Two functions F and G are called to be in *involution* if $\{F, G\} = 0$. Thus (1.14) implies that any constant of motion is in involution with the Hamiltonian. If there are N functionally independent integrals of motion $F_j,\ j = 1, \ldots, N$, which are all involution, i.e. $\{F_i, F_j\} = 0$ for any $i, j = 1, \ldots, N$ then the Hamiltonian system is called *Liouville integrable* (also called by completely integrable system in Liouville sense). Here by functional independence of F_j we mean that a set of $2N$-dimensional gradients $(\nabla_{\mathbf{p}}, \nabla_{\mathbf{q}})F_j,\ j = 1, \ldots, N$ spans an N-dimensional subspace of phase space almost everywhere except on sets of zero measure. In other words, vectors $(\nabla_{\mathbf{p}}, \nabla_{\mathbf{q}})F_j,\ j = 1, \ldots, N$ are linearly independent almost everywhere in the phase space. Liouville-Arnold theorem [**Arn89**] proves for any Liouville integrable system that if the level set $M_{\mathbf{f}} = \{(\mathbf{p}, \mathbf{q}) :\ F_j = f_j,\, f_j \in \mathbb{R}, j = 1, \ldots, N, \}$ is compact then it is diffeomorphic (can be transformed into by a smooth invertible transformation) to N-dimensional torus $\mathbb{T} = \{\phi_1, \ldots, \phi_N\}$, where $\phi_j \in \mathbb{R}$ modulo 2π.

Liouville-Arnold theorem also proves that exists a neighborhood of $M_{\mathbf{f}} \in P$ where it is possible to find action-angle variables $(\mathbf{I}, \phi)$ such that

$$\begin{aligned} \frac{d\mathbf{I}}{dt} &= 0, \\ \frac{d\phi}{dt} &= \nabla_{\mathbf{I}} H(\mathbf{I}) := \omega(\mathbf{I}). \end{aligned} \tag{1.15}$$

Here we define the action coordinate $\mathbf{I} \in \mathbb{R}^N$ and the angle coordinate $\phi \in \mathbb{T}^N$ in such a way that the Hamiltonian depends only on the action

$$H = H(\mathbf{I}) \tag{1.16}$$

but not on ϕ. The action is respectively the function of $\mathbf{F} = (F_1, \ldots, F_j)$: $\mathbf{I} = \mathbf{I}(\mathbf{F})$.

Equations (1.15) are simple to integrate which gives

$$\begin{aligned} \mathbf{I}(t) &= \mathbf{I}(0) = const, \\ \phi(t) &= \phi(0) + \omega(\mathbf{I})t. \end{aligned} \tag{1.17}$$

Thus the solution of Liouville integrable system is quasiperiodic with the rotation of each angular coordinate ϕ_j at the angular frequency ω_j, $j = 1, \ldots, N$.

For $N = 1$ any Hamiltonian system (1.1) is Lioville integrable because we use the Hamiltonian as the integral of motion $H = F_1$. E.g., for the harmonic oscillator $I = H$, where $H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$.

Problems 1.2

1.2.1 Show that the Jacobi identity (4.86) follows from the definition (1.9) of the Poisson bracket.

1.3. Canonical transformations and generating functions

The differential equations (1.15) in action-angle variables $(\mathbf{I}, \phi)$ is the particular case of the Hamiltonian equations (1.1) with $\mathbf{I}$ being the generalized momentum and ϕ being the generalized coordinate. Thus the change of variables from the original variables $(\mathbf{p}, \mathbf{q})$ into action-angle variables save the general form (1.1) of the Hamiltonian equations. In this Section we study the general transformation of the Hamiltonian equations from the original variables $(\mathbf{p}, \mathbf{q})$ into new variables variables $(\mathbf{P}, \mathbf{Q})$ such that

$$\begin{aligned} \frac{dQ_j}{dt} &= \frac{\partial \tilde{H}}{\partial P_j}, \\ \frac{dP_j}{dt} &= -\frac{\partial \tilde{H}}{\partial Q_j}, \quad j = 1 \ldots N \end{aligned} \tag{1.18}$$

with $\mathbf{P} \in \mathbb{R}^N$ and $\mathbf{Q} \in \mathbb{R}^N$ being the new generalized momentum and generalized coordinate, respectively. Here $\tilde{H}(\mathbf{P}, \mathbf{Q}, t)$ is the Hamiltonian in new variables. For generality in this section we allow the explicit time dependence of the Hamiltonian: $H(\mathbf{p}, \mathbf{q}, t)$.

We start by deriving the Hamiltonian equations (1.1) from the principle of least action (or more accurately is sometimes called by the principle of stationary action as well as it is known as the Hamilton's principle) for the action S defined as the integral

$$S = \int_{t_1}^{t_2} \left[\mathbf{p}(t) \cdot \dot{\mathbf{q}}(t) - H\left(\mathbf{p}(t), \mathbf{q}(t), t\right)\right] dt, \tag{1.19}$$

between two fixed times t_1 and t_2. The functional S here is understood as the line integral along a curve in $2N+1$-dimensional real vector space $(\mathbf{p}, \mathbf{q}, t) \in \mathbb{R}^{2N+1}$. The principle of the least action states that the motion of the system between the specified points $\mathbf{q}_1 := \mathbf{q}(t_1)$ and $\mathbf{q}_2 := \mathbf{q}(t_2)$ is determined by the the stationary value (extremum) of the action S. Assume that the functions $\mathbf{p}(t)$ and $\mathbf{q}(t)$ realize such stationary value. Then it means that the replacement of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ in (1.46) by their independent infinitesimal variations in the form $\mathbf{p}(t) + \delta\mathbf{p}(t)$ and $\mathbf{q}(t) + \delta\mathbf{q}(t)$ with fixed generalized coordinates at initial and finite times,

$$\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = 0, \tag{1.20}$$

do not change the value of S in the linear order of both $\delta\mathbf{p}(t)$ and $\delta\mathbf{q}(t)$. It is assumed here that $\mathbf{p}(t)$, $\mathbf{q}(t)$, $\delta\mathbf{p}(t)$ and $\delta\mathbf{q}(t)$ are smooth functions at $t_1 \le t \le t_2$ to ensure the differentiability of S. One can assume for example, that these functions are infinitely differentiable. The variation of S is given by

$$\begin{aligned}\delta S = &\int_{t_1}^{t_2} [(\mathbf{p}+\delta\mathbf{p})\cdot(\dot{\mathbf{q}}+\delta\dot{\mathbf{q}}) - H(\mathbf{p}+\delta\mathbf{p},\mathbf{q}+\delta\mathbf{q},t)]\,dt \\ &- \int_{t_1}^{t_2} [\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)]\,dt = \int_{t_1}^{t_2} \left(\delta\mathbf{p}\cdot\dot{\mathbf{q}} + \mathbf{p}\cdot\delta\dot{\mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}}\cdot\delta\mathbf{q} - \frac{\partial H}{\partial \mathbf{p}}\cdot\delta\mathbf{p}\right)dt \\ &+ h.o.t.,\end{aligned} \tag{1.21}$$

where $h.o.t.$ means the higher order terms (quadratic and above) in $\delta\mathbf{q}$ and $\delta\mathbf{p}$. Below unless otherwise specified, we neglect $h.o.t.$ in δS, i.e. by δS we assume the first variation (only linear in $\delta\mathbf{q}$ and $\delta\mathbf{p}$ contributions). Also the scalar products $\frac{\partial H}{\partial \mathbf{q}}\cdot\delta\mathbf{q} = \sum_{j=1}^{N}\frac{\partial H}{\partial q_j}\delta q_j$ and similar expressions are implied here and below. The assumption of the stationary value of S on $\mathbf{p}(t)$ and $\mathbf{q}(t)$ implies that $\delta S = 0$. Then integrating (1.48) by part for the term with $\delta\dot{\mathbf{q}} = \frac{d\delta\mathbf{q}}{dt}$ we obtain that

$$\delta S = \mathbf{p}\cdot\delta\mathbf{q}\big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2}\left(\dot{\mathbf{q}} - \frac{\partial H}{\partial \mathbf{p}}\right)\cdot\delta\mathbf{p}\,dt + \int_{t_1}^{t_2}\left(-\dot{\mathbf{p}} - \frac{\partial H}{\partial \mathbf{q}}\right)\cdot\delta\mathbf{q}\,dt = 0. \tag{1.22}$$

Taking into account the condition (1.20) we conclude that the integral in (1.49) vanishes for any independent variations $\delta\mathbf{p}$ and $\delta\mathbf{q}$. It means that the expressions in brackets at each integrand in (1.22) also vanish resulting in the Hamiltonian equations (1.1). Thus the Hamiltonian mechanics can be considered as the formulation of the classical mechanics using the principle of least action.

We note that the above derivation of the Hamiltonian equations (1.1) from the variation of S needs that the motion of the mechanical system to be the stationary value of S but does not actually require S to be the minimum. In most cases however, for short pieces of the mechanical system trajectories, S is the minimum. It justifies the historical use of the principle of least action terminology. Although more accurately it should be called by the principle of stationary action. Later in the book we additionally discuss the variational principle by introducing the Lagrangian in Section 1.7.

Now we return to the transformation between variables $(\mathbf{p},\mathbf{q})$ which satisfy (1.1) and $(\mathbf{P},\mathbf{Q})$ which follows (1.18). Because of (1.18), new variables must also satisfy the Hamilton's principle $\delta\tilde{S} = 0$ with

$$\tilde{S} = \int_{t_1}^{t_2} \left[\mathbf{P}\cdot\dot{\mathbf{Q}} - \tilde{H}\left(\mathbf{P},\mathbf{Q},t\right)\right] dt, \tag{1.23}$$

Both $\delta S = 0$ with (1.19) and $\delta\tilde{S} = 0$ (1.23) are simultaneously satisfied if the integrands of (1.19) and $\delta\tilde{S} = 0$ (1.23) are different by the total time derivative of any smooth function F as follows

$$\mathbf{p}\cdot\dot{\mathbf{q}} - H = \mathbf{P}\cdot\dot{\mathbf{Q}} - \tilde{H} + \dot{F} \tag{1.24}$$

because the difference in S and $\tilde{S}$ is reduced to the constant term $F|_{t=t_1}^{t=t_2}$ which vanishes under variation[1].

We consider the transformation

$$\mathbf{P} = \mathbf{P}(\mathbf{p},\mathbf{q},t) \quad \text{and} \quad \mathbf{Q} = \mathbf{Q}(\mathbf{p},\mathbf{q},t) \tag{1.25}$$

from the old variables $(\mathbf{p},\mathbf{q})$ to the new variables $(\mathbf{P},\mathbf{Q})$, were $\mathbf{p},\mathbf{q},\mathbf{P}$ and $\mathbf{Q}\in\mathbb{R}^N$. The transformation (1.25) is called *the canonical transformation* if both (1.1) and (1.18) are simultaneously satisfied and there exists a function F such that (1.24) is valid. We define F in one of four possible forms as follows:

$$F = F_1(\mathbf{q},\mathbf{Q},t), \tag{1.26a}$$

$$F = F_2(\mathbf{P},\mathbf{q},t) - \mathbf{P}\cdot\mathbf{Q}, \tag{1.26b}$$

$$F = F_3(\mathbf{p},\mathbf{Q},t) + \mathbf{p}\cdot\mathbf{q}, \tag{1.26c}$$

$$F = F_4(\mathbf{p},\mathbf{P},t) + \mathbf{p}\cdot\mathbf{q} - \mathbf{P}\cdot\mathbf{Q}. \tag{1.26d}$$

Each function $F_j,\ j = 1,\ldots,4$ is called *the generating function* of the canonical transformation. We call these generating functions as type 1, 2, 3 and 4, respectively. Thus the equations (1.26a)-(1.26d) allow the generating function to be the function of any pair of the old and new canonical variables and time. Four different types of canonical transformations are obtained from these four forms of the generating function. Taking the first form $F = F_1$ from (1.26a) and substituting the total time derivative of F_1 into (1.24) by F_1 result in

$$\mathbf{p}\cdot\dot{\mathbf{q}} - H = \mathbf{P}\cdot\dot{\mathbf{Q}} - \tilde{H} + \dot{F}_1 = \mathbf{P}\cdot\dot{\mathbf{Q}} - \tilde{H} + \frac{\partial F_1}{\partial\mathbf{q}}\cdot\dot{\mathbf{q}} + \frac{\partial F_1}{\partial\mathbf{Q}}\cdot\dot{\mathbf{Q}} + \frac{\partial F_1}{\partial t}. \tag{1.27}$$

We assume the the old coordinate $\mathbf{q}$ and the new coordinate $\mathbf{Q}$ are separately independent. Then the equation (1.27) is identically valid if the expressions multiplying both $\dot{\mathbf{q}}$ and $\dot{\mathbf{Q}}$ are both identically zero as well as the sum of the remaining terms is zero which give the expressions for the old and new generalized momenta as well as the relation between the old and new Hamiltonians as

$$\mathbf{p} = \frac{\partial F_1}{\partial\mathbf{q}}, \quad \mathbf{P} = -\frac{\partial F_1}{\partial\mathbf{Q}}, \quad \tilde{H} = H + \frac{\partial F_1}{\partial t}. \tag{1.28}$$

We can define the transformation $(\mathbf{p},\mathbf{q}) \to (\mathbf{P},\mathbf{Q})$ from (1.28) as follows. The first equation in (1.28) defines $\mathbf{p}$ as the function of $\mathbf{q}$, $\mathbf{Q}$ and t. We assume that this function can be inverted to give $\mathbf{Q}$ as the function of $\mathbf{p}$, $\mathbf{q}$ and t, thus giving the second part of the transformation (1.25). Then $\mathbf{Q}(\mathbf{p},\mathbf{q},t)$ can be substituted

[1]If we replace (1.24) with a more general expression $\mathbf{p}\cdot\dot{\mathbf{q}} - H = \lambda(\mathbf{P}\cdot\dot{\mathbf{Q}} - \tilde{H} + \dot{F})$, where λ is the arbitrary real constant, then we obtain the *extended canonical transformation* , see e.g. Refs. ??? for more discussion of such transformations.

into the second equation in (1.28) resulting in $\mathbf{P}(\mathbf{p}, \mathbf{q}, t)$, i.e. the first part of the transformation (1.25). After that using the inverse of the transformation (1.25) in the third equation in (1.28) we immediately the new Hamiltonian $\tilde{H}$ in terms of the the new canonical variables $(\mathbf{P}, \mathbf{Q})$.

In a similar way, taking F from (1.26b),(1.26c) and (1.26d), respectively, and substituting the total time derivative of F of into (1.24) result in

$$\mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}}, \quad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}}, \quad \tilde{H} = H + \frac{\partial F_2}{\partial t}, \tag{1.29a}$$

$$\mathbf{q} = -\frac{\partial F_3}{\partial \mathbf{p}}, \quad \mathbf{P} = -\frac{\partial F_3}{\partial \mathbf{Q}}, \quad \tilde{H} = H + \frac{\partial F_3}{\partial t}, \tag{1.29b}$$

$$\mathbf{q} = -\frac{\partial F_4}{\partial \mathbf{p}}, \quad \mathbf{Q} = \frac{\partial F_4}{\partial \mathbf{P}}, \quad \tilde{H} = H + \frac{\partial F_4}{\partial t}. \tag{1.29c}$$

The last expression in each of the equations (1.28), (1.29a), (1.29b) and (1.29c) give the formula for the new Hamiltonian $\tilde{H}$ in terms of the new canonical variables $(\mathbf{P}, \mathbf{Q})$. We also note from these equations that if the generating function F does not depend on t explicitly then $H = \tilde{H}$. In that case to obtain the new Hamiltonian $\tilde{H}$ it is sufficient to express $\mathbf{p}$ and $\mathbf{q}$ in the old Hamiltonian H through the new variables $\mathbf{P}$ and $\mathbf{Q}$.

The canonical transform does not have to be precisely one of four types (1.26a). Instead it could be a mixture of all four types. As a simple example consider the generating function of the mixed type

$$\tilde{F}(\mathbf{p}^{(1)}, \mathbf{Q}^{(1)}, \mathbf{P}^{(2)}, \mathbf{q}^{(2)}), \tag{1.30}$$

where $\mathbf{p}^{(1)} = (p_1, \dots p_{N_1})$, $\mathbf{Q}^{(1)} = (Q_1, \dots Q_{N_1})$, $\mathbf{P}^{(2)} = (P_{N_1+1}, \dots P_N)$ and $\mathbf{q}^{(2)} = (q_{N_1+1}, \dots p_N)$, $1 \leq N_1 \leq N - 1$, i.e. it is of the type 3 for first N_1 components of the canonical momenta and coordinates and it is of the type 2 for the other $N - N_1$ components with F in the equation (1.24) given by

$$F = \tilde{F}(\mathbf{p}^{(1)}, \mathbf{Q}^{(1)}, \mathbf{P}^{(2)}, \mathbf{q}^{(2)}) + \mathbf{p}^{(1)} \cdot \mathbf{q}^{(1)} - \mathbf{P}^{(2)} \cdot \mathbf{Q}^{(2)}. \tag{1.31}$$

The distinction between canonical coordinates and canonical momenta is essentially lost under the canonical transformation. This can be seen if we perform a simple canonical transformation $\mathbf{P} = -\mathbf{q}$ and $\mathbf{Q} = \mathbf{p}$ which interchanges the canonical coordinates and canonical momenta. That canonical transformation is of the type 1 with $F = F_1 = \mathbf{q} \cdot \mathbf{Q}$. It suggest to call canonical coordinates and canonical momenta as canonically conjugated quantities.

A generating function can be found for any canonical transformation. Thus it provides a full description of all canonical transformations.

1.4. Canonical transformations and symplectic matrix

Another way to look at the canonical transformation (1.25) is to directly make sure that (1.18) holds provided (1.1) is true without using of the generating function. We define vectors

$$\mathbf{x} := (\mathbf{q}, \mathbf{p}) = (q_1, \dots, q_N, p_1, \dots, p_N) \tag{1.32}$$

and

$$\tilde{\mathbf{x}} := (\mathbf{Q}, \mathbf{P}) = (Q_1, \dots, Q_N, P_1, \dots, P_N) \tag{1.33}$$

for old and new canonical variables, respectively. We also introduce an antisymmetric matrix in block matrix form

$$\mathbf{J} := \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \tag{1.34}$$

where I_N is $N \times N$ identity matrix. Then the Hamiltonian equations (1.1) can be written in the matrix form

$$\dot{\mathbf{x}} = \mathbf{J}\frac{\partial H}{\partial \mathbf{x}}, \tag{1.35}$$

where $\mathbf{x}$ is viewed as the column vector.

Assume for now that the transformation (1.25) does not depend explicitly on t, i.e. $\mathbf{P} = \mathbf{P}(\mathbf{p}, \mathbf{q})$ and $\mathbf{Q} = \mathbf{Q}(\mathbf{p}, \mathbf{q})$. Inverting that transformation we obtain that $\mathbf{p} = \mathbf{p}(\mathbf{P}, \mathbf{Q})$ and $\mathbf{q} = \mathbf{q}(\mathbf{P}, \mathbf{Q})$. Plugging that inverted transformation into (1.35) and using chain rule of the differentiation results in

$$\dot{\tilde{\mathbf{x}}} = \mathbf{M}\mathbf{J}\mathbf{M}^T\frac{\partial H}{\partial \tilde{\mathbf{x}}}, \tag{1.36}$$

where

$$M_{ij} = \frac{\partial \tilde{x}_i}{\partial x_j} \tag{1.37}$$

is the Jacobian matrix for the transformation (1.25) and $\tilde{H} = H$.

Comparing (1.36) with the canonical Hamiltonian equations (1.18) allows to conclude that the canonicity of transformation (1.25) requires that

$$\mathbf{M}\mathbf{J}\mathbf{M}^T = \mathbf{J} \tag{1.38}$$

which is called a *symplectic condition* for the canonical transformation. The matrix $\mathbf{M}$ satisfying the condition (1.25) is called a *symplectic matrix.* The same symplectic condition is valid for the general canonical transformation (1.25) with the explicit dependence on time.

If the condition (1.38) is not satisfied then the Hamiltonian equations (1.36) takes the general noncanonical form

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{J}}\frac{\partial H}{\partial \tilde{\mathbf{x}}}, \tag{1.39}$$

where $\tilde{\mathbf{J}}(\tilde{\mathbf{x}})$ is the skew-symmetic nonsingular ($det(\tilde{\mathbf{J}}) \neq 0$) matrix,

$$\tilde{\mathbf{J}}^T = -\tilde{\mathbf{J}} \tag{1.40}$$

which is still called the symplectic matrix.

A theorem attributed to Darboux Ref??? states that locally (in the small enough neigbourhood of each point of the phase space $\tilde{\mathbf{x}}$) the general skew-symmeric matrix $\tilde{\mathbf{J}}(\tilde{\mathbf{x}})$ can be transformed into the canonical form (1.34).

1.5. Canonical transformation through Poisson bracket

The Poisson bracket is invariant with respect to the canonical transformation, i.e. the Poisson bracket is *canonical invariant.* To show that we define the Poisson bracket with respect to the canonical variables $\mathbf{Q}$ and $\mathbf{P}$ as follows

$$\{F, G\}_{\mathbf{Q},\mathbf{P}} = \sum_{j=1}^{N}\left(\frac{\partial F}{\partial Q_j}\frac{\partial G}{\partial P_j} - \frac{\partial F}{\partial P_j}\frac{\partial G}{\partial Q_j}\right). \tag{1.41}$$

Then (1.9) defines $\{F, G\}_{\mathbf{q},\mathbf{p}}$. Substitution (1.25) together with the condition of canonicity of transformation (1.1) and (1.18) result in

$$\{F, G\}_{\mathbf{Q},\mathbf{P}} = \{F, G\}_{\mathbf{q},\mathbf{p}}. \tag{1.42}$$

Thus the Poisson bracket does not change under the canonical transformation and below we use $\{\cdot, \cdot\}$ for Poisson bracket over any set of canonical Hamiltonian variables omitting subscripts unless necessary to specify over which the canonical variables the Poisson bracket is considered.

Using (1.9) and for $\{\cdot, \cdot\}_{\mathbf{q},\mathbf{p}}$ we obtain that

$$\begin{aligned} &\{Q_j, Q_{j'}\} = \{P_j, P_{j'}\} = 0, \\ &\{Q_j, P_{j'}\} = \delta_{j,j'}, \quad j, j' = 1, \dots, N, \end{aligned} \tag{1.43}$$

where $\delta_{j,j'}$ is the Kronecker delta function: $\delta_{j,j'} = 0$ for $j \neq j'$ and $\delta_{j,j} = 1$. Taking into account (1.42) we conclude that (1.43) is valid for any canonically transformed variables. Stronger statement is also valid that (1.43) are the necessary and sufficient condition for the transformation (1.25) to be canonical.

Problems 1.5

1.5.1 Show by explicit computation that the expression (1.42) is valid for the transformation (1.25) if it satisfies the conditions of canonicity of transformation (1.1) and (1.18).

1.6. Canonical transformation as the motion of the Hamiltonian system

Definition of the transformation through a symplectic condition (1.38) shows that the canonical transformation can be defined independent of the particular form of the Hamiltonian. We can choose another Hamiltonian $X(\mathbf{P}, \mathbf{Q}, t)$ with the Hamiltonian equations

$$\begin{aligned} \frac{dQ_j}{d\tau} &= \frac{\partial X}{\partial P_j}, \\ \frac{dP_j}{d\tau} &= -\frac{\partial X}{\partial Q_j}, \quad j = 1 \dots N \end{aligned} \tag{1.44}$$

for the new time τ. Solving (1.44) for the initial conditions $\mathbf{Q}|_{\tau=0} = \mathbf{q}$ and $\mathbf{P}|_{\tau=0} = \mathbf{p}$ results in the canonical transformations (1.25) for each value of τ. This follows immediately from the Hamilton's principle that the variation of the action (1.19) does not change under a translation (shift) of a time variable. A particular choice $H = X$ allows to conclude that the motion of the Hamiltonian system (1.1) itself can be viewed as the canonical transformation.

1.7. Lagrangian mechanics

The canonical Hamiltonian system (1.1) has the equivalent form called by the Lagrange equations (also sometimes called by the Euler-Lagrange equations or the Lagrange equations of the second kind)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{1.45}$$

which is the system of N second order (in time) ODEs. That system is fully defined by the Lagrangian L which is the given function $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ of independent variables t, $\mathbf{q} \in \mathbb{R}^N$ and $\dot{\mathbf{q}} \in \mathbb{R}^N$ ($\dot{\mathbf{q}}$ is used as the second independent variable in contrast to the Hamiltonian system (1.1), where $\mathbf{p}$ is the second independent variable). The Lagrange equations together with the independent variables $\mathbf{q}$ and $\dot{\mathbf{q}}$ form the Lagrangian mechanics.

Similar to the Hamiltonian equations, the Lagrange equations follow from the principle of least action (or more accurately as the principle of stationary action as well as it is also called by the Hamilton's principle) for the action S defined as

$$S = \int_{t_1}^{t_2} L\left(\mathbf{q}(t), \dot{\mathbf{q}}(t), t\right) dt \tag{1.46}$$

between two fixed times t_1 and t_2. The functional S here is understood as the line integral along a curve in $N+1$-dimensional real vector space $(\mathbf{q}, t) \in \mathbb{R}^{N+1}$. The principle of the least action states that the motion of the system between the specified points $\mathbf{q}_1 := \mathbf{q}(t_1)$ and $\mathbf{q}_2 := \mathbf{q}(t_2)$ is determined by the the stationary value (extremal) of the action S. Assume that the function $\mathbf{q}(t)$ realizes such stationary value. Then it means that the replacement $\mathbf{q}(t)$ in (1.46) by its infinitesimal variation in the form $\mathbf{q}(t) + \delta\mathbf{q}(t)$ satisfying

$$\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = 0 \tag{1.47}$$

does not change the value of S in the linear order of $\delta\mathbf{q}(t)$. It is assumed here that both $\mathbf{q}(t)$ and $\delta\mathbf{q}(t)$ are smooth functions at $t_1 \le t \le t_2$ to ensure the differentiability of S. One can assume for example, that these functions are infinitely differentiable. The variation of S is given by

$$\delta S = \int_{t_1}^{t_2} L\left(\mathbf{q} + \delta\mathbf{q}, \dot{\mathbf{q}} + \delta\dot{\mathbf{q}}, t\right) dt - \int_{t_1}^{t_2} L\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}}\delta\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}}\delta\dot{\mathbf{q}}\right) dt + h.o.t., \tag{1.48}$$

where $h.o.t.$ means the higher order terms (quadratic and above) in $\delta\mathbf{q}$ and $\delta\dot{\mathbf{q}}$. Below unless otherwise specified, we neglect $h.o.t.$ in δS, i.e. by δS we assume the first variation (only linear in $\delta\mathbf{q}$ and $\delta\dot{\mathbf{q}}$ contribution). Also the scalar products $\frac{\partial L}{\partial \mathbf{q}}\delta\mathbf{q} = \sum_{j=1}^{N} \frac{\partial L}{\partial q_j}\delta q_j$ and similar expressions are implied here and below. The assumption of the stationary value of S on $\mathbf{q}(t)$ implies that $\delta S = 0$. Then integrating (1.48) by part for the term with $\delta\dot{\mathbf{q}} = \frac{d\delta\mathbf{q}}{dt}$ we obtain that

$$\delta S = \frac{\partial L}{\partial \dot{\mathbf{q}}}\delta\mathbf{q}\Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\delta\mathbf{q}\, dt = 0. \tag{1.49}$$

Taking into account the condition (1.47) we conclude that the integral in (1.49) vanishes for any $\delta\mathbf{q}$. It means that the integrand in (1.49) also vanishes resulting in the Lagrange equations (1.45). Thus the Lagrangian mechanics can be considered as the reformulation of the classical mechanics using the principle of least action.

We note that the above derivation of the Lagrange equations (1.45) needs that the motion of the mechanical system to be the stationary value of S but does not actually require S to be the minimum. In most cases however, for short pieces of the mechanical system trajectories, S is the minimum. It justifies the historical use of the principle of least action terminology. Although more accurately it should be called by the principle of stationary action.

1.8. Equivalence of the Lagrangian and Hamiltonian mechanics

We introduce the generalized momentum $\mathbf{p}$ in the Lagrangian mechanics as follows

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}. \tag{1.50}$$

Then the Lagrange equations (1.45) can be rewritten as the system of two sets of equations

$$\begin{aligned} \dot{\mathbf{p}} &= \frac{\partial L}{\partial \mathbf{q}}, \\ \mathbf{p} &= \frac{\partial L}{\partial \dot{\mathbf{q}}}. \end{aligned} \tag{1.51}$$

We define the relation between the Hamiltonian $H(\mathbf{p}, \mathbf{q}, t)$ and the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ describing the same mechanical system as follows

$$H(\mathbf{p}, \mathbf{q}, t) = \mathbf{p}\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t). \tag{1.52}$$

The equation (1.52) together with (1.52) define the Légendre transform from the independent variables $\mathbf{p}$, $\mathbf{q}$ and t to the independent variables $\mathbf{q}$, $\dot{\mathbf{q}}$ and t. The exact differential dH of H is given by

$$dH = \frac{\partial H}{\partial \mathbf{p}} d\mathbf{p} + \frac{\partial H}{\partial \mathbf{q}} d\mathbf{q} + \frac{\partial H}{\partial t} dt \tag{1.53}$$

while the exact differential of the right-hand side (rhs) of (1.52) is given by

$$\begin{aligned} d\left(\mathbf{p}\dot{\mathbf{q}} - L\right) &= \mathbf{p}\, d\dot{\mathbf{q}} + \dot{\mathbf{q}}\, d\mathbf{p} - \frac{\partial L}{\partial \mathbf{q}} d\mathbf{q} - \frac{\partial L}{\partial \dot{\mathbf{q}}} d\dot{\mathbf{q}} - \frac{\partial L}{\partial t} dt \\ &= \dot{\mathbf{q}}\, d\mathbf{p} - \dot{\mathbf{p}} d\mathbf{q} - \frac{\partial L}{\partial t} dt. \end{aligned} \tag{1.54}$$

Here we used (1.52). Because rhs of (1.53) equals to rhs of (1.54), we immediately recover the Hamilton equations (1.1) from the common terms of $d\mathbf{p}$ and $d\mathbf{q}$. Thus we started from the Lagrange equations (1.51) together with (1.52) and derived the canonical Hamilton equations (1.1). It is straightforward to show that (1.1) with (1.1) results in the Lagrange equations (1.51). It proves the equivalence of the Lagrangian and the Hamiltonian mechanics. Also the terms with dt results in the following relation

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{1.55}$$

In most cases below both H and L do not explicitly depend on t meaning that (1.55) is zero. Such systems are called *autonomous*.

For the motion of particles in the potential (1.4) we obtain from the Légendre transform (1.52) expressing $\mathbf{p}$ through $\dot{\mathbf{q}}$ that

$$L = \sum_{i=1}^{N} \frac{m_i \dot{q}_i{}^2}{2} - U(\mathbf{q}). \tag{1.56}$$

Notice that the potential energy U enters with the opposite sign in comparison with the Hamiltonian (1.4).

1.9. Holonomic constraints and the Lagrangian mechanics on manifolds

The analysis of numerous mechanical systems can be greatly simplified if we consider ideal objects like rigid body. The planar pendulum considered in Section 1.1 is the example of such simplification with the ideal rigid massless rod attached to the point mass. That system can be considered as the limit of the Newton equations of the potential motion where the mass of the rod approaches zero while its stiffness (represented by the potentials for its stretching, bending and twisting) goes to infinity. That limit can be taken rigorously (see e.g. [**Arn89**]). Such type of constrains are called by holonomic constraints. These constraints can also depend on time t.

Consider the general system of N_p point masses in the D-dimensional real vector space $\mathbb{R}^D$ (usually $D = 1, 2, 3$) as in equation (1.4). The positions of these masses are given by vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_K \in \mathbb{R}^D$, i.e. by a point in the real vector space $\mathbb{R}^{N_pD}$. Applying N_h independent holonomic constraints we restrict the allowed values of these coordinates to $N_pD - N_h$-dimensional surface M in $\mathbb{R}^{N_pD}$. From point of view of equation (1.4) it means that we take the limit of the potential U to be infinitely large if the holonomic constraints are not satisfied. Assume that $N = N_pD - N_h$ and $\mathbf{q} := (q_1 \ldots, q_N)$ is the set of generalized coordinates on M. The dynamics of N_p point masses with N_h constraints is equivalent to the Lagrangian mechanics for the generalized coordinates $\mathbf{q}$ on the surface M[**Arn89**].

A planar motion of a point mass ($N_p = 1$ and $D = 2$) without constraint is characterized by two coordinates (e.g. vertical and horizontal coordinates). The planar pendulum considered in Section 1.1 has one holonomic constraint because the rigid rod allows for the mass to move on the circle S^1 only. The position of the point mass on that circle is fully characterized by a single generalized coordinate $q = \theta$.

The surface M is called the configuration space of the Lagrange equations (1.45). M is the N-dimensional differential manifold. It means that the small neighborhood of each point of M is homeomorphic to the N-dimensional Eucledian space $\mathbb{E}^N$ as well as that M has a globally defined differential structure (see Appendix A.3 for the detailed definition). *The Eucledian space* $\mathbb{E}^N$ is the vector space $\mathbb{R}^N$ together with the standard scalar product $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^N x_i y_i$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. The dimension N of the configuration space is called *the number of degrees of freedom.*

Below we often consider the dynamical systems with constraints which do not have any simple mechanical analogies and do not allow easy derivation of the constraints from the unconstrained systems. However the notion of constraints remains extremely helpful for many Hamiltonian systems.

Consider a smooth curve $\mathbf{q}(t)$ on the N-dimensional configuration space M parameterized by a scalar $t \in \mathbb{R}$. Then $\dot{\mathbf{q}}(t)$ is the tangent vector to M. The set of all tangent vectors at point $\mathbf{q}$ (can be obtained by parameterization of all curves on M which pass through $\mathbf{q}$) forms the N-dimensional vector space $TM_{\mathbf{q}}$ which is called *the tangent space* to M at the point $\mathbf{q}$. If M is embedded into the Eucledian space $\mathbb{E}^{N_e}$ (as in the case of N_p point masses described above with

$N_e = N_pD$) then $TM_{\mathbf{q}}$ can be equivalently defined as the orthogonal complement to the set of vectors $\{\nabla f_1, \nabla f_2, \ldots, \nabla f_{N_e-N}\}$. Here $N_e - N$ scalar functions $f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_{N_e-N}(\mathbf{x})$, $\mathbf{x} \in \mathbb{E}^{N_e}$ define the configuration space M by $N_e - N$ holonomic constraints $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, \ldots, f_{N_e-N}(\mathbf{x}) = 0$. These constraints are assumed to be independent at M meaning that the set of vectors $\{\nabla f_1, \nabla f_2, \ldots, \nabla f_{N_e-N}\}$ is linearly independent at each $\mathbf{q} \in M$.

The union of all tangent spaces $TM_{\mathbf{q}}$ for all $\mathbf{q} \in M$ is called *the tangent bundle* of M and is denoted by TM. Each point of TM consist of the point $\mathbf{q} \in M$ and the tangent vector ξ to M at $\mathbf{q}$. TM is $2N$-dimensional differential manifold. Assume that $(\xi_1, \xi_2, \ldots, \xi_N)$ are the components of ξ in the local coordinates $(q_1, q_2, \ldots, q_N)$ of M. Then $(q_1, q_2, \ldots, q_N, \xi_1, \xi_2, \ldots, \xi_N)$ forms a local coordinate system in TM.

For autonomous systems the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ on the manifold can be considered as the mapping from TM into $\mathbb{R}$. Similarly, for nonautonomous systems the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is the mapping $TM \times \mathbb{R} \to \mathbb{R}$. The least action principle and all equations of Section 1.45 are valid assuming that $\mathbf{q}(t) \in M$ and $(\mathbf{q}(t), \dot{\mathbf{q}}(t)) \in TM$ thus defining the Lagrangian mechanics on manifolds.

Assume that $\mathbf{x} \in \mathbb{E}^{N_e}$ are the coordinates for the Newtonian equations of the potential motion of point masses given by the equations (1.56) and (1.45). Consider the coordinate transform

$$\mathbf{x} = \mathbf{g}(\mathbf{q}) \tag{1.57}$$

from the generalized coordinates $\mathbf{q} \in \mathbb{R}^N$ to $\mathbf{x}$, where N is the dimension of the configuration space M. We choose the rank of the Jacobian matrix $\frac{\partial g_i}{\partial q_j}$ to be N for any $\mathbf{q} \in M$. By the time derivative of (1.57) we obtain the relation between the velocities $\dot{\mathbf{q}}$ and $\dot{\mathbf{x}}$ in both coordinate systems as follows

$$\dot{x_i} = \sum_{j=1} \dot{\mathbf{q}} \cdot \frac{\partial g_i}{\partial \mathbf{q}}. \tag{1.58}$$

Then the kinetic energy $T = \sum\limits_{i=1}^{N_e} \frac{m_i \dot{x_i}^2}{2}$ is transformed into the quadratic form for $\dot{\mathbf{q}}$ and the Lagrangian (1.56) takes the following form

$$L = \sum_{i,j=1}^{N} \frac{A_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j}{2} - U(\mathbf{q}), \tag{1.59}$$

where

$$A_{ij} = \sum_{k=1}^{N_e} m_k \frac{\partial g_k(\mathbf{q})}{\partial q_i} \frac{\partial g_k(\mathbf{q})}{\partial q_j} \tag{1.60}$$

are the elements of the symmetric $N \times N$ matrix $\mathbf{A} = \mathbf{A}^T$ ($\mathbf{A}^T$ means the transposed matrix of $\mathbf{A}$). It follows from (1.60) and positivity of masses $m_i > 0$, $i = 1, 2, \ldots, N_e$ that $\mathbf{A}$ positive-definite matrix (i.e. $\xi^T \mathbf{A} \xi > 0$ for any nonzero column vector $\xi \in \mathbb{R}^N$). We also abused a notation by setting $U(\mathbf{q}) = U(\mathbf{x})$.

We use the example (1.59) to define the kinetic energy of in the Lagrangian mechanics as the general symmetric positive-definite quadratic form in $\mathbf{q}$:

$$T = \sum_{i,j=1}^{N} \frac{A_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j}{2}. \tag{1.61}$$

The potential energy U is defined as the general function of $\mathbf{q}$. The Lagrangian equations have the *natural form* if $L = T - U$. Then the Lagrange equations take the following form

$$\sum_{j=1}^{N} A_{ij}(\mathbf{q})\ddot{q}_j + \text{other terms} = 0, \quad i = 1, 2, , \dots, N, \tag{1.62}$$

where "other terms" stands for the terms which depend of $\mathbf{q}$ and $\dot{\mathbf{q}}$ only but not on higher derivatives of $\mathbf{q}$ over time. The symmetry the positive-definiteness of $\mathbf{A}$ ensures its invertibility. It means that (1.62) is solvable for $\ddot{\mathbf{q}}$ which provides the existence and the uniqueness of the solutions of the Cauchy problem for the natural Lagrange equations.

More general non-natural Lagrange systems have a Lagrangian L which is a general general function of $\mathbf{q}$ and $\dot{\mathbf{q}}$ (and t for nonautonomous systems). Then it is generally impossible to separate the Lagrangian into the kinetic and potential energy. The existence and the uniqueness of the solutions of the Cauchy problem for such systems require that the matrix $\frac{\partial^2 L}{\partial q_i \partial q_j}$ to be nonsingular on M.

1.10. Oscillations

Consider the equilibrium point $\mathbf{q}_0 \in M$ of the Lagrangian system (1.45), i.e. $\dot{\mathbf{q}}|_{\mathbf{q}=\mathbf{q}_0} = 0$. For the natural Lagrangian system (1.59),(1.45) it implies that $\mathbf{q}_0$ is the stationary point of $U(\mathbf{q})$,

$$\left.\frac{\partial U(\mathbf{q})}{\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}_0} = 0. \tag{1.63}$$

Consider the motion in the neighborhood of the equilibrium point $\mathbf{q}_0$. Assume that both $\mathbf{q}-\mathbf{q}_0$ and $\dot{\mathbf{q}}$ are small and of the same order and expand the Lagrangian (1.59) in the Taylor serious near $\mathbf{q} = \mathbf{q}_0$ and $\dot{\mathbf{q}} = 0$. Without loss of generality we set $\mathbf{q}_0 = 0$ (can be done by the translation of the generalized coordinates $\mathbf{q}$) and $U(0) = 0$ (addition of the arbitrary time-independent real number to $U(\mathbf{q})$ does not change the Lagrangian equations (1.45)). A first nontrivial contribution to the Lagrangian (1.59) is quadratic in $\mathbf{q}$ and $\dot{\mathbf{q}}$ as follows

$$L_2 = \sum_{i,j=1}^{N} \frac{A_{ij}\dot{q}_i\dot{q}_j}{2} - \sum_{i,j=1}^{N} \frac{B_{ij}q_iq_j}{2}, \tag{1.64}$$

where $L = L_2 + O(\dot{\mathbf{q}}^2\mathbf{q}) + O(\mathbf{q}^3)$,

$$A_{ij} := A_{ij}(0) \tag{1.65}$$

and

$$B_{ij} := \left.\frac{\partial^2 U(\mathbf{q})}{\partial q_i \partial q_j}\right|_{\mathbf{q}=0} \tag{1.66}$$

are the constant symmetric $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. We also used (1.63) to remove linear in $\mathbf{q}$ terms.

The Lagrange equations (1.45) for the Lagrangian L_2 results in the system of N linear ODEs of the second order

$$\mathbf{A}\ddot{\mathbf{q}} = -\mathbf{B}\mathbf{q}. \tag{1.67}$$

The kinetic energy $T = \frac{1}{2}\dot{\mathbf{q}} \cdot \mathbf{A}\dot{\mathbf{q}}$ and the potential energy $U = \frac{1}{2}\mathbf{q} \cdot \mathbf{B}\mathbf{q}$ in the Lagrangian L_2 are the quadratic forms in independent variables $\dot{\mathbf{q}}$ and $\mathbf{q}$. The kinetic energy is the positive-definite quadratic form because $\mathbf{A}$ is the positive-definite matrix, as we found in Section 1.9.

Consider the quadratic forms $\mathbf{q} \cdot \mathbf{A}\mathbf{q}$ and $\mathbf{q} \cdot \mathbf{B}\mathbf{q}$. According to the standard theorem of linear algebra (see e.g. [**Gan90**]), these two quadratic forms, one of which is positive-definite, can be simultaneously converted to the sum of squares by a linear nonsingular transformation

$$\mathbf{Q} = \mathbf{C}\mathbf{q}, \tag{1.68}$$

where $\mathbf{C}$ is the $N \times N$ matrix of the transformation and $\mathbf{Q} \in \mathbb{R}^N$. For the quadratic form $\dot{\mathbf{q}} \cdot \mathbf{A}\dot{\mathbf{q}}$ we can use the same transformation

$$\dot{\mathbf{Q}} = \mathbf{C}\dot{\mathbf{q}}, \tag{1.69}$$

which results in the expression for L_2 in new variables $\mathbf{Q}$ and $\dot{\mathbf{Q}}$ as follows

$$L_2 = \frac{1}{2}\sum_{i=1}^{N}\left(\dot{Q_i}^2 - \lambda_i Q_i^2\right) \tag{1.70}$$

and the Lagrangian equations (1.67) turn into the the following decoupled system of ODEs:

$$\ddot{Q}_i = -\lambda_i Q_i. \tag{1.71}$$

Here $\lambda_1, \lambda_2, \ldots \lambda_N$ are the eigenvalues of $\mathbf{B}$ with respect to $\mathbf{A}$ determined by the characteristic equation

$$\det(\mathbf{B} - \lambda\mathbf{A}) = 0. \tag{1.72}$$

Notice that the positive-definiteness of $\mathbf{A}$ ensures its the invertibility and allows to rewrite (1.72) as the usual characteristic equation $\det(\mathbf{A}^{-1}\mathbf{B} - \lambda\mathbf{I}) = 0$ for the matrix $\mathbf{A}^{-1}\mathbf{B}$, where $\mathbf{I}$ is the identity matrix.

The quadratic Lagrangian (1.70) is in the form (1.56) which implies that the Hamiltonian of that system is

$$H_2 = \frac{1}{2}\sum_{i=1}^{N}\left(P_i{}^2 + \lambda_i Q_i^2\right), \tag{1.73}$$

where the canonical momenta are

$$P_i = \dot{Q}_i,\ i = 1, \ldots, N. \tag{1.74}$$

Problems 1.10

1.10.0.1 Show that the transformation from $\mathbf{q}$ and $\mathbf{p}$ to $\mathbf{Q}$ and $\mathbf{P}$ defined by (1.64), (1.69), (1.73) and (1.74) is the canonical transformation for the Hamiltonian equations (1.1) with the quadratic Hamiltonian (1.73).

The Hamiltonian (1.6) for the harmonic oscillator in physical variables is transformed into (1.112) by the following trivial change of variables

$$Q_j = m^{1/2}q_j,\ P_j = m^{-1/2}p_j,\ j = 1, \ldots, N, \tag{1.75}$$

which results in the Hamiltonian (1.73) with $\lambda_j = \omega_j^2,\ j = 1, \ldots, N$.

For positive eigenvalues $\lambda_j = \omega_j^2 > 0$ we recover from (1.71) a set of decoupled harmonic oscillators as in the equation (1.5) with the general solution $Q_j = a_j \cos(\omega_j t + \varphi_j)$, similar to (1.7). Zero eigenvalues $\lambda_j = 0$ correspond to the general solution $Q_j = c_{j,1}t + c_{j,2}$, where c_1 and c_2 are the arbitrary real constants. The general solution for the negative eigenvalues $\lambda_j = -\nu_j^2 < 0$ is given by $Q_j = c_{j,1}e^{-\nu_j t} + c_{2,j}e^{\nu_j t}$. Using the transformation (1.69) we write the general solution of the Lagrange equations (1.67) as follows

$$\begin{aligned} \mathbf{q} = \sum_{j=1}^{N_1} a_j \cos(\omega_j t + \varphi_j)\mathbf{q}_j + \sum_{j=N_1+1}^{N_1+N_2} \left(c_{j,1}e^{-\nu_j t} + c_{2,j}e^{\nu_j t}\right)\mathbf{q}_j \\ + \sum_{j=N_1+N_2+1}^{N} \left(c_{j,1}t + c_{j,2}\right)\mathbf{q}_j, \end{aligned} \tag{1.76}$$

where $\mathbf{q}_j$ is jth eigenvector of $\mathbf{A}^{-1}\mathbf{B}$, N_1 is the number of positive eigenvalues, N_2 is the number of negative eigenvalues and $N - N_1 - N_2$ is the number of zero eigenvalues for $\mathbf{A}^{-1}\mathbf{B}$ with all eigenvalues counted according to their algebraic multiplicity. Each term in (1.76) is the particular solution of (1.67) called by a *normal mode* of (1.67). Notice that although some of λ_j can have algebraic multiplicity more than one, their algebraic multiplicity is always equal to their geometric multiplicity, i.e. eigenspace of all eigenvalues spans $\mathbb{R}^N$. It immediately follows from the simultaneous transformation by (1.69) of both quadratic forms $\mathbf{q}\cdot\mathbf{A}\mathbf{q}$ and $\mathbf{q}\cdot\mathbf{B}\mathbf{q}$ to the sum of squares. It also implies that no extra power of t appears in (1.76) for Lagrange equations (contrary to the case of general linear ODE system with repeated eigenvalues).

If $\mathbf{q} = 0$ is the strict minimum of the potential energy then the matrix $\mathbf{B}$ is positive-definite. It implies that all eigenvalues λ_j are positive and the equation (1.76) reduces to the sum of oscillating terms only

$$\mathbf{q} = \sum_{j=1}^{N_1} a_j \cos(\omega_j t + \varphi_j)\mathbf{q}_j. \tag{1.77}$$

Any solution of the Lagrange equations is a superposition of oscillations in that case.

1.10.0.1. *Example.* Consider the Newton dynamics of a molecule which consists of N_p atoms with different masses $m_k > 0$ located at points $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_p} \in \mathbb{R}^3$. All atoms can move with respect to each other (i.e. there are no holonomic constraints). The configuration space is the $N = 3N_p$-dimensional Eucledian space $\mathbb{E}^N$. Assume that $\mathbf{r}_{1,0}, \mathbf{r}_{2,0}, \ldots, \mathbf{r}_{N/3,0}$ are equilibrium positions of the atoms in the molecule. Choose a vector $\mathbf{q} = (\mathbf{r}_1 - \mathbf{r}_{1,0}, \mathbf{r}_2 - \mathbf{r}_{2,0}, \ldots, \mathbf{r}_{N/3} - \mathbf{r}_{N/3,0}) \in \mathbb{E}^N$ as the vector of the configuration space so that $\mathbf{q} = 0$ is the equilibrium. Consider small deviations from that equilibrium which allows to use the quadratic Lagrangian (1.64). The absence of the holonomic constraints implies that the kinetic energy has the diagonal form $T = \sum\limits_{k=1}^{N/3} \frac{m_k}{2}(q_{3k-2} + q_{3k-1} + q_{3k}) = \sum\limits_{j=1}^{N} \frac{\tilde{m}_j \dot{q}_j{}^2}{2}$. Here $\tilde{m}_{3k-2} = \tilde{m}_{3k-1} = \tilde{m}_{3k} := m_k$ is the mass the kth atom. The potential energy has the general form $U = \sum\limits_{i,j=1}^{N} \frac{B_{ij} q_i q_j}{2}$

as in (1.64) and the Lagrangian is given by

$$L = \frac{\dot{\mathbf{q}} \cdot \mathbf{M}\dot{\mathbf{q}}}{2} - \frac{\mathbf{q} \cdot \mathbf{B}\mathbf{q}}{2}, \tag{1.78}$$

where $\mathbf{M} = \mathbf{A}$ is the diagonal matrix with the main diagonal $(\tilde{m}_1, \dots, \tilde{m}_N)$. The Hamiltonian is given by

$$H = \sum_{j=1}^{N} \frac{{p_j}^2}{2\tilde{m}_j} + \frac{1}{2} \sum_{i,j=1}^{N} B_{ij} q_i q_j, \tag{1.79}$$

where the momentum $\mathbf{p} = \mathbf{M}\dot{\mathbf{q}}$.

The Hamilton equations for (1.79) (or equivalently the Lagrange equations for (1.78)) result in the linear system

$$\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{B}\mathbf{q}. \tag{1.80}$$

Assume additionally that all atoms interacting only pairwise with each other by the harmonic potentials with the positive interaction constants $\alpha_{ij} = \alpha_{ji} > 0$. The Hamiltonian of that system takes the following form

$$H = \sum_{j=1}^{N} \frac{{p_j}^2}{2\tilde{m}_j} + \frac{1}{4} \sum_{i,j=1}^{N} \alpha_{ij}(q_i - q_j)^2. \tag{1.81}$$

In this case $B_{ii} = \sum\limits_{j=1,\ j\neq i}^{N} \alpha_{ij}$ and $B_{ij} = -\alpha_{ij}$, for $i \neq j$ with $i, j = 1 \dots, N$. It also immediate follows from (1.81) that the matrix $\mathbf{B}$ is non-negative (i.e. $\mathbf{x} \cdot \mathbf{B}\mathbf{x} \geq 0$ for any $\mathbf{x} \in \mathbb{R}^N$) and respectively all λ_j are nonnegative. Zero values of λ_j have a simple physical meaning representing the translation of the molecule in any direction. There are three independent directions for such translation (i.e. for the motion of the molecule's center of mass). The general solution (1.76) is then reduced to

$$\mathbf{q} = \sum_{j=1}^{N-3} a_j \cos(\omega_j t + \varphi_j)\mathbf{q}_j + \sum_{j=N-2}^{N} (c_{j,1}t + c_{j,2})\, \mathbf{q}_j. \tag{1.82}$$

1.10.0.2. *Example.* Consider a variation of Example 1.10.0.1 when the motion of $N+1$ particles (atoms) of equal mass m is restricted to the single dimension $D = 1$ and the interaction is only between the neighboring particles with the leftmost and rightmost particles fixed as schematically shown in Fig. 2 by the springs connecting the particles. These springs are assumed to be the same so that the Hamiltonian from (1.81) is then reduced to

$$H_2 = \sum_{j=1}^{N-1} \left[\frac{{p_j}^2}{2m} + \frac{U_2}{2}(q_{j+1} - q_j)^2 \right] = \sum_{j=1}^{N-1} \left[\frac{m{\dot{q}_j}^2}{2} + \frac{U_2}{2}(q_{j+1} - q_j)^2 \right], \tag{1.83}$$

where $U_2 > 0$ is the constant and we set

$$q_0 = q_N = p_0 = p_N \equiv 0 \tag{1.84}$$

to reflect the conditions that 0th and Nth particles have fixed positions. Thus we have to consider the Hamiltonian variables $\mathbf{q} = (q_1, \dots, q_{N-1})$ and $\mathbf{p} = (p_1, \dots, p_{N-1})$. Also the Hamilton equations (1.1) imply that $\mathbf{p} = m\dot{\mathbf{q}}$ and

$$m\ddot{q}_j = U_2(q_{j+1} - 2q_j + q_{j-1}),\ j = 1, \dots, N-1. \tag{1.85}$$

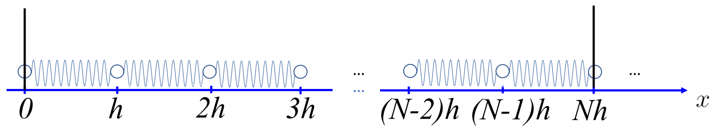

FIGURE 2. A chain of $N+1$ particles connected by springs aligned along the direction x. Here $x = jh$, $h > 0$ is the equilibrium position of jth particle and q_j is the displacement of jth particle from the equilibrium position. Vertical lines represent that 0th and Nth particles are fixed (e.g. attached to walls).

The normal variables $\mathbf{Q} = (Q_1, \dots, Q_{N-1})$ and $\mathbf{P} = (P_1, \dots, P_{N-1})$ are immediately obtained by the discrete sine transform as follows

$$
\begin{aligned}
Q_k &= \frac{2^{1/2}}{N^{1/2}} \sum_{j=1}^{N-1} q_j \sin\left[\frac{\pi j k}{N}\right], \quad k = 1, 2, \dots, N-1, \\
P_k &= \frac{2^{1/2}}{N^{1/2}} \sum_{j=1}^{N-1} p_j \sin\left[\frac{\pi j k}{N}\right], \quad k = 1, 2, \dots, N-1,
\end{aligned} \tag{1.86}
$$

and its inverse

$$
\begin{aligned}
q_j &= \frac{2^{1/2}}{N^{1/2}} \sum_{k=1}^{N-1} Q_k \sin\left[\frac{\pi j k}{N}\right], \quad j = 1, 2, \dots, N-1, \\
p_j &= \frac{2^{1/2}}{N^{1/2}} \sum_{k=1}^{N-1} P_k \sin\left[\frac{\pi j k}{N}\right], \quad j = 1, 2, \dots, N-1.
\end{aligned} \tag{1.87}
$$

We notice that in Eq. (1.87) one can also use both indexes $j = 0$ and $j = N$ which recovers the zero boundary conditions (1.84).

Plugging (1.87) into (1.83) and using the orthogonality relation

$$
\sum_{n=1}^{N-1} \sin\left[\frac{\pi k n}{N}\right] \sin\left[\frac{\pi k' n}{N}\right] = \frac{N}{2} \delta_{k,k'}, \quad k, k', N \in \mathbb{N}. \tag{1.88}
$$

we obtain that in the normal variables the Hamiltonian (1.83) has the form of the sum of decoupled oscillators as follows

$$
H_2 = \frac{1}{2} \sum_{k=1}^{N-1} \left({P_k}^2 + \omega_k^2 Q_k^2 \right), \tag{1.89}
$$

with the frequencies

$$\omega_k = 2\left(\frac{U_2}{m}\right)^{1/2}\sin\left(\frac{\pi k}{2N}\right),\ k = 1,\ldots,N-1. \tag{1.90}$$

The corresponding Hamiltonian equations (1.1) are given by

$$m\ddot{Q}_k = -4U_2\sin^2\left(\frac{\pi k}{2N}\right)Q_k,\ k = 1,\ldots,N-1. \tag{1.91}$$

Each such decoupled oscillator represents the collective motion of all $N-1$ particles of the lattice. If e.g. only one mode Q_k is nonzero, generally all components of $\mathbf{q}$ are nonzero because they are obtained from the the discrete sine transform (1.86). In physics these collective motions (also referred as excitations) are called by *phonons* [**LL80, LP81**].

Remark. The mapping from $(\mathbf{q},\mathbf{p})$ to $(\mathbf{Q},\mathbf{P})$ is the canonical transformation. To show that we write the discrete sine transform (1.86) in the matrix form as

$$\mathbf{Q} = \mathbf{U}\mathbf{q}, \quad \mathbf{P} = \mathbf{U}\mathbf{p}, \tag{1.92}$$

where $\mathbf{U}$ is $N\times N$ real matrix with the components,

$$U_{kj} = \left(\frac{2}{N}\right)^{1/2}\sin\left[\frac{\pi jk}{N}\right]. \tag{1.93}$$

The orthogonality relation (1.88) implies that $\mathbf{U}$ is the orthogonal matrix

$$\mathbf{U}^{-1} = \mathbf{U}^T. \tag{1.94}$$

A Jacobian matrix (1.37) of the transformation

$$\mathbf{X} = \mathbf{X}(\mathbf{x}) \tag{1.95}$$

from $\mathbf{x} = (\mathbf{q},\mathbf{p}) \in \mathbb{R}^{2N}$ to $\mathbf{X} = (\mathbf{Q},\mathbf{P}) \in \mathbb{R}^{2N}$ is given by a block matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{U} & 0 \\ 0 & \mathbf{U} \end{pmatrix}. \tag{1.96}$$

Equations (1.93), (1.94) and (1.96) imply that (1.38) is satisfied thus (1.95). That canonical transformation does not change the Hamiltonian because it does not have an explicit dependence on time.

1.10.0.3. *Example.* Another variation of Examples 1.10.0.1 and 1.10.0.2 is N particles connected by springs with the periodic boundary conditions

$$q_0 = q_N,\ p_0 = p_N \tag{1.97}$$

representing e.g. particles connected with springs in the annular region with the assumed negligible contribution from the curvature of the annulus. In that case the Hamiltonian (1.83) is replaced by adding the terms with $j=0$ as follows

$$H_2 = \sum_{j=0}^{N-1}\left[\frac{p_j^2}{2m} + \frac{U_2}{2}\left(q_{j+1}-q_j\right)^2\right] = \sum_{j=0}^{N-1}\left[\frac{m\dot{q}_j^2}{2} + \frac{U_2}{2}\left(q_{j+1}-q_j\right)^2\right], \tag{1.98}$$

as well as the dynamical equations (1.85) are replaced by a similar system of linear second order ODEs

$$m\ddot{q}_j = U_2(q_{j+1} - 2q_j + q_{j-1}),\ j = 0,\ldots,N-1. \tag{1.99}$$

Here we assume that $q_{-1} = q_{N-1}$ by periodicity (1.97) over N particles.

The periodic boundary conditions (1.97) imply that we cannot expand in sines only (as we did in previous examples in equations (1.86)-(1.87)). Instead we solve equation (1.99) using Discrete Fourier Transform (DFT)

$$\begin{aligned} Q_k &= \frac{1}{N^{1/2}} \sum_{j=0}^{N-1} q_j \exp\left[-\frac{2\pi \mathrm{i} jk}{N}\right], \quad k \in \mathbb{N}, \\ P_k &= \frac{1}{N^{1/2}} \sum_{j=0}^{N-1} p_j \exp\left[-\frac{2\pi \mathrm{i} jk}{N}\right], \quad k \in \mathbb{N} \end{aligned} \tag{1.100}$$

and Inverse DFT

$$\begin{aligned} q_j &= \frac{1}{N^{1/2}} \sum_{k=0}^{N-1} Q_k \exp\left[\frac{2\pi \mathrm{i} jk}{N}\right], \quad j = 0, 1, 2, \ldots, N-1, \\ p_j &= \frac{1}{N^{1/2}} \sum_{k=0}^{N-1} P_k \exp\left[\frac{2\pi \mathrm{i} jk}{N}\right], \quad j = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{1.101}$$

where Q_k and P_k are the kth Fourier harmonics of $\mathbf{q} := (q_0, q_1, \ldots, q_{N-1})$ and $\mathbf{p} := (p_0, p_1, \ldots, p_{N-1})$, respectively.

Equation (1.100) and $q_j \in \mathbb{R}$ imply the symmetry relation with respect to the complex conjugation,

$$Q_k = \bar{Q}_{N-k}, \; P_k = \bar{P}_{N-k}, \text{ for any } k, m \in \mathbb{N} \tag{1.102}$$

and ad with respect to the shift of the subscript by any integer multiple of N,

$$Q_{k+mN} = Q_k \;\; P_{k+mN} = P_k \text{ for any } k, m \in \mathbb{N}. \tag{1.103}$$

defining the subscripts of Q_k and P_k modulo N.

Applying DFT to both l.h.s and r.h.s. of (1.99) we obtain decoupled harmonic oscillators

$$m\ddot{Q}_k = -4U_2 \sin^2\left(\frac{\pi k}{N}\right) Q_k, \; k \in \mathbb{N} \tag{1.104}$$

thus indicating that $Q_k, \; k \in \mathbb{N}$ represents the normal modes of oscillations. These decoupled oscillators have the frequencies

$$\omega_k = 2\left(\frac{U_2}{m}\right)^{1/2} \sin\left(\frac{\pi k}{N}\right), \; k \in \mathbb{N}, \tag{1.105}$$

which can be compared with the frequencies of Example 1.10.0.2 (equation (1.90)) to conclude that the factor $\sin\left(\frac{\pi k}{2N}\right)$ is now replaced by $\sin\left(\frac{\pi k}{N}\right)$. Another difference between equations (1.90) and (1.105) is the zero frequency in (1.105) for $k = 0$. That zero frequency corresponds to the constant velocity motion of the center of mass of all particles (in previous Example such mode was not considered because of the zero boundary conditions (1.84) which fixed the reference frame). Without loss of generality one can assume in the current Example that the center of mass is at the rest.

Plugging (1.101) into (1.98) and using the orthogonality relation of DFT

$$\sum_{n=0}^{N-1} \exp\left[\frac{2\pi \mathrm{i} kn}{N}\right] \exp\left[\frac{-2\pi \mathrm{i} k' n}{N}\right] = N\delta_{k,k'}, \quad k, k', N \in \mathbb{N}, \tag{1.106}$$

together with the symmetries (1.102) and (1.103), we obtain that the Hamiltonian (1.83) in the variables Q_k, P_k has the form of a sum of decoupled oscillators as follows

$$H_2 = \frac{1}{2}\sum_{k=0}^{N-1}\left(|P_k|^2 + \omega_k^2|Q_k|^2\right), \tag{1.107}$$

where ω_k is given by (1.105).

Equation (1.107) suggests that one can use a subset $k = 0, \ldots, N-1$ out of all possible subscripts $k \in \mathbb{N}$ of both Q_k and P_k to fully characterize the system of periodically coupled harmonic springs. However, that subset is still excessive if one notices that N complex values of $Q_k,\ k = 0, \ldots, N-1$, generally correspond to $2N$ real values while there are only N real degrees of freedom in $\mathbf{q}$ (and similar for $\mathbf{p}$). To reduce that subset to the minimally required set of values of k, one uses that

$$Q_k = \bar{Q}_{N-k},\ P_k = \bar{P}_{N-k} \tag{1.108}$$

as follows from equations (1.102) and (1.103). Equation (1.108) means that out of N complex harmonics Q_k (and similar for P_k) only $\lfloor N/2 \rfloor + 1$ harmonics are independent, where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$ (the floor function) such that

$$\lfloor k \rfloor = k \quad \text{and} \quad \lfloor k + 1/2 \rfloor = k \quad \text{for any} \quad k \in \mathbb{N}. \tag{1.109}$$

In particular, if N is even then there are $\lfloor N/2 \rfloor + 1 = N/2 + 1$ independent Fourier harmonics $(Q_0, Q_1, \ldots, Q_{N/2})$. Two of them, Q_0 and $Q_{N/2}$, are real ones (as follows from equation (1.100)) and remaining $N/2-1$ are complex so together they have the equivalent of N real degrees of freedom. This equals to N real values determined by $\mathbf{q}$. If N is odd then there are $\lfloor N/2 \rfloor + 1 = (N+1)/2$ independent Fourier harmonics $(Q_0, Q_1, \ldots, Q_{(N-1)/2})$. One of them, Q_0, is real and remaining $(N-1)/2$ are complex. Thus together they also form the equivalent of N real degrees of freedom. These N real degrees of freedom can be chosen as normal modes (both the real and imaginary parts of Eq. (1.104) ensure that both $Re(Q_k)$ and $Im(Q_k)$ are the normal modes). Then these N components and the corresponding N components of P_k can form the new Hamiltonian variables $\mathbf{Q} \in \mathbb{R}^N$ and $\mathbf{P} \in \mathbb{R}^N$. Similar to Example 1.10.0.2, one can define a canonical transformation from $(\mathbf{q}, \mathbf{p})$ to $(\mathbf{Q}, \mathbf{P})$ which we leave as the exercise for the readers.

1.11. Harmonic oscillator and normal complex variables

Consider the quadratic Hamiltonian (1.73) and assume that all λ_i are positive, $\lambda_i = \omega_i^2$, i.e. all normal modes are oscillations. It is often convenient to introduce in (1.73) complex variables a_j and their complex conjugate $\bar{a}_j$, $j = 1, \ldots, N$ as follows

$$\begin{aligned} a_j &= \frac{1}{(2\omega_j)^{1/2}}\left(\omega_j Q_j + iP_j\right), \\ \bar{a}_j &= \frac{1}{(2\omega_j)^{1/2}}\left(\omega_j Q_j - iP_j\right), \end{aligned} \tag{1.110}$$

where we choose the positive sign $\omega_j > 0$ for all j. Solving (1.110) for Q_j and P_j we obtain that

$$
\begin{aligned}
Q_j &= \frac{1}{(2\omega_j)^{1/2}} \left(a_j + \bar{a}_j\right), \\
P_j &= -i \left(\frac{\omega_j}{2}\right)^{1/2} \left(a_j - \bar{a}_j\right).
\end{aligned}
\tag{1.111}
$$

The Hamiltonian (1.73) simplifies by (1.111) into

$$
H_2 = \sum_{j=1}^{N} \omega_j |a_j|^2, \tag{1.112}
$$

where the positive value of H_2 is ensured by our choice $\omega_j > 0$.

In a similar way, the Hamiltonian (1.6) of the harmonic oscillator is transformed into (1.112) by the following change of variables

$$
\begin{aligned}
q_j &= \frac{1}{(2m\omega_j)^{1/2}} \left(a_j + \bar{a}_j\right), \\
p_j &= -i \left(\frac{m\omega_j}{2}\right)^{1/2} \left(a_j - \bar{a}_j\right)
\end{aligned}
\tag{1.113}
$$

with the inverse transform given by

$$
\begin{aligned}
a_j &= \frac{1}{(2m_j\omega_j)^{1/2}} \left(m_j\omega_j q_j + i p_j\right), \\
\bar{a}_j &= \frac{1}{(2m_j\omega_j)^{1/2}} \left(m_j\omega_j q_j - i p_j\right).
\end{aligned}
\tag{1.114}
$$

Consider now complex variables (1.110) for a general time-independent Hamiltonian $H(\mathbf{P}, \mathbf{Q})$, where $\omega_1, \ldots, \omega_N$ are N arbitrary positive constants. Using the Hamiltonian equations (1.1) together with (1.110) we obtain that

$$
\begin{aligned}
\frac{da_j}{dt} &= \frac{1}{(2\omega_j)^{1/2}} \left(\omega_j \frac{dQ_j}{dt} + i\frac{dP_j}{dt}\right) = \frac{1}{(2\omega_j)^{1/2}} \left(\omega_j \frac{\partial H}{\partial P_j} - i\frac{\partial H}{\partial Q_j}\right), \\
\frac{d\bar{a}_j}{dt} &= \frac{1}{(2\omega_j)^{1/2}} \left(\omega_j \frac{dQ_j}{dt} - i\frac{dP_j}{dt}\right) = \frac{1}{(2\omega_j)^{1/2}} \left(\omega_j \frac{\partial H}{\partial P_j} + i\frac{\partial H}{\partial Q_j}\right).
\end{aligned}
\tag{1.115}
$$

Expressing P_j and Q_j in H through a_j and $\bar{a}_j$ by (1.111) we obtain the complex form of the Hamiltonian equation for the pairs of independent variables a_j and $\bar{a}_j$ as follows

$$
\frac{da_j}{dt} = -i\frac{\partial H}{\partial \bar{a}_j}. \tag{1.116}
$$

The equation for $\frac{da_j}{dt}$ is obtained the complex conjugation if we recall that H is the real function:

$$
\frac{d\bar{a}_j}{dt} = i\frac{\partial H}{\partial a_j}. \tag{1.117}
$$

Calculating partial derivatives in (1.116) and (1.117) we imply that $\frac{\partial \bar{a}_j}{\partial a_{j'}} = 0$ and $\frac{\partial \bar{a}_{j'}}{\partial a_j} = 0$ for any $j, j' = 1, \ldots, N$ because we consider a_j and $\bar{a}_j$ as independent variables.

For the particular case of the quadratic Hamiltonian (1.112) we obtain from (1.116) the system of first order complex ODEs

$$\frac{da_j}{dt} = -i\omega_j a_j, \ j = 1, \ldots, N, \tag{1.118}$$

which has a solution $a_j(t) = a_j(0)e^{-i\omega_j t}$ representing the complex form of the solution of N decoupled harmonic oscillators.

Poisson brackets for a and $\bar{a}$ are immediately obtained from (1.114) and (1.43)

$$\begin{aligned} &\{a_j, a_{j'}\} = \{\bar{a}_j, \bar{a}_{j'}\} = 0, \\ &\{a_j, \bar{a}_{j'}\} = -i\,\delta_{j,j'}, \quad j, j' = 1, \ldots, N. \end{aligned} \tag{1.119}$$

Similar to (1.43), the conditions (1.119) are necessary and sufficient conditions for the transformation to be canonical. Last equation of (1.119) has $-i$ factor. To avoid that factor when dealing with complex variables a and $\bar{a}$ we redefine the Poisson bracket as follows

$$\{F, G\} := \sum_{j=1}^{N} \left(\frac{\partial F}{\partial a_j} \frac{\partial G}{\partial \bar{a}_j} - \frac{\partial F}{\partial \bar{a}_j} \frac{\partial G}{\partial a_j} \right), \tag{1.120}$$

which is different from the definition (1.9) by multiplication on i. Then equations (1.119) are replaced by

$$\begin{aligned} &\{a_j, a_{j'}\} = \{\bar{a}_j, \bar{a}_{j'}\} = 0, \\ &\{a_j, \bar{a}_{j'}\} = \ \delta_{j,j'}, \quad j, j' = 1, \ldots, N. \end{aligned} \tag{1.121}$$

Complex variables a_j and $\bar{a}_j$ are the classical analogs of the Bose creation and annihilation operators of quantum mechanics (see e.g. [**LL76**]). The transformation (1.110) is the classical analog of the quantum mechanical transformation from the coordinate-momentum representation to the representation by the Bose creation and annihilation operators.

1.12. Complex variables in weakly nonlinear case

Complex variables (1.111) are especially useful if the Hamiltonian can be expanded in a power series of the canonical variables P_j and Q_j. E.g. it occurs if we expand the Hamiltonian in the power series near the stationary point (1.63) and take into account next terms beyond quadratic terms considered in Section 1.10. The linearity of the transformation (1.111) implies a power series in variables a_j and $\bar{a}_j$ for the same Hamiltonian expanded in the power series.

Consider a nonlinear oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + U(q), \tag{1.122}$$

such that the potential $U(q)$ has a minimum at $q = 0$. Without loss of generality we set $U(0) = 0$ because the Hamiltonian equations (1.1) do not change if shift $U(q)$ by an arbitrary real constant.

Assume that q is small and expand $U(x)$ in a Taylor series about $q = 0$:

$$U(q) = \frac{m\omega^2}{2} q^2 + \alpha q^3 + \beta q^4 + \ldots, \tag{1.123}$$

where we define ω in the same way as in (1.5) so it would have a meaning of frequency of small oscillations.

The Hamiltonian has now a form of power series in canonical variables p and q as follows

$$\begin{aligned} H_2 &= \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2, \\ H_3 &= \alpha q^3, \\ H_4 &= \beta q^4, \\ &\dots, \end{aligned} \tag{1.124}$$

where subscripts in H represent the order of terms in powers of both q an p.

Limiting to $H = H_2$ we recover the harmonic oscillator considered in Section 1.11 with the dynamical equation (1.11). Using equations (1.114) to express q through a and $\bar{a}$ in the second equation of (1.124), we obtain that

(1.125)

$$H_3 = \alpha \left(\frac{1}{2m\omega}\right)^{3/2} (a^3 + \bar{a}^3 + 3a\bar{a}^2 + 3\bar{a}a^2) = \frac{U}{3}\left[a^3 + \bar{a}^3\right] + V\left[|a|^2\bar{a} + |a|^2 a\right],$$

which results by (1.116) in the following Hamiltonian equation for $H = H_2 + H_3$:

$$\frac{da}{dt} + i\omega a = -iU\bar{a}^2 - iVa^2 - 2iVa\bar{a}, \tag{1.126}$$

where

$$U = V := 3\alpha \left(\frac{1}{2m\omega}\right)^{3/2}. \tag{1.127}$$

Similar, at the next order $H = H_2 + H_3 + H_4$ we obtain from (1.111) and (1.124) that

$$H_4 = \beta q^4 = \beta \left(\frac{1}{2m\omega}\right)^2 [a^4 + \bar{a}^4 + 6|a|^4 + 4|a|^2(\bar{a}^2 + a^2)] \tag{1.128}$$

and the corresponding dynamic equation takes the following form

(1.129)

$$\frac{da}{dt} + i\omega a = -iU\bar{a}^2 - iVa^2 - 2iVa\bar{a} - i\beta\left(\frac{1}{2m\omega}\right)^2 (4\bar{a}^3 + 12|a|^2 a + 12|a|^2\bar{a} + 4a^3).$$

1.13. Nonlinear oscillator and the generation of multiple harmonics

To analyze the equation (1.129) we note that its leading order solution (more precise meaning the leading order solution will be found below in this section) in the limit of small a is reduced to the harmonic oscillator equation $\frac{da}{dt} + i\omega a = 0$ with the exact solution $a = C_1 e^{-i\omega t}$. We call that solution by a *first* (or *fundamental*) *harmonic*. We assume that nonlinear terms in r.h.s of (1.129) result in slow time dependence of C_1 at times $\gg 1/\omega$ together with the generation of frequncies which are integer multiple of ω, which motivates looking for the solution of (1.129) in the following form

$$a = C_1 e^{-i\omega t} + C_{-1} e^{i\omega t} + C_0 + C_2 e^{-2\omega t} + C_{-2} e^{2i\omega t} + \dots\ , \tag{1.130}$$

where $C_1(t),\ C_{-1}(t), C_0(t),\ C_2(t),\ C_{-2}(t), \dots$ are the slow functions of time. We refer to $C_0(t)$ and $C_2(t)$ as the amplitudes of the zeroth and second harmonics, respectively. C_{-1} and $C_{-2}(t)$ can by called by the "minus" first and the "minus" second harmonics, respectively. All these harmonics are generated by the quadratic nonlinear terms in r.h.s of (1.129). To see that we first substitute a in r.h.s. (1.129) by the leading order harmonic solution $a \simeq a_1 := C_1(t)e^{-i\omega t}$ and neglect for now cubic terms which gives

$$\frac{da}{dt} + i\omega a = -iU\bar{C}_1^2 e^{2i\omega t} - iV{C_1}^2 e^{-2i\omega t} - 2iV|C_1|^2 \tag{1.131}$$

In other words, our "linear" oscillator (represented by l.h.s of (1.131) is forced by the terms with frequencies 2ω, -2ω and 0 at r.h.s of (1.131) producing nonzero values of $C_{\pm 2}$ and C_0 in arbitrary small time. Note that C_{-1} harmonic is not generated by r.h.s. of equation (1.131) in that approximation. We approximately solve equation (1.131) by replacing a in l.h.s. of (1.131) through its expansion (1.130) in harmonics and collect all terms with frequencies $0,\ \pm 2\omega$. At the leading order we neglect the time derivatives of $C_0(t),\ C_2(t),\ C_{-2}(t)$ in (1.131) which results in the closed expressions for these amplitudes as follows

$$C_0 = -\frac{2V}{\omega}|C_1|^2;\ C_2 = \frac{V}{\omega}C_1^2;\ C_{-2} = -\frac{U}{3\omega}\bar{C}_1^2. \tag{1.132}$$

Thus second and zero harmonics have the quadratic dependence $\propto |C_1|^2$. It allows to specify the assumed above smallness of a as $|C_0|,\ |C_{\pm 2}| \sim |V/\omega||C_1|^2 \ll |C_1|$, i.e. $|C_1| \ll |\omega/V|$. Then notion of the leading order takes the meaning of the series expansion of the solution $a(t)$ in the powers of the small parameter

$$\epsilon := |C_1||V/\omega| \ll 1. \tag{1.133}$$

These higher harmonics C_0 and $C_{\pm 2}$ also modify the dynamics of the first harmonic C_1. To understand that modification we return to the equation (1.126), collect all terms which $\propto e^{-i\omega t}$ corresponding to the fundamental harmonic and use (1.132) to express $C_0(t),\ C_2(t),\ C_{-2}(t)$ through $C_1(t)$. That procedure can be qualitatively interpreted as the projection of a into the "state" $e^{-i\omega t}$ and gives that

$$\begin{aligned} \frac{dC_1}{dt} &= -2iU\bar{C}_{-2}\bar{C}_1 - i2VC_1C_0 - 2iVC_1\bar{C}_0 - 2iVC_2\bar{C}_1 \\ &= i\frac{20}{3\omega}V^2|C_1|^2C_1, \end{aligned} \tag{1.134}$$

where we used that $U = V$.

Returning back from C_1 to the full form of the fundamental harmonic $a_1 \equiv C_1(t)e^{-i\omega t}$ results in the equation

$$\frac{da_1}{dt} + i(\omega + \Delta\tilde{\omega})a_1 = 0 \tag{1.135}$$

where

$$\Delta\tilde{\omega} = -\frac{20}{3}\frac{V^2}{\omega}|a_1|^2. \tag{1.136}$$

Using equations (1.133) and (1.135) we obtain that

$$\Delta\tilde{\omega}/\omega \sim \epsilon^2, \tag{1.137}$$

i.e. $\Delta\tilde{\omega}$ corresponds to *second-order perturbation theory* (meaning $O(\epsilon^2)$) while values of $C_0,\ C_{\pm 2}$ correspond to *first-order perturbation theory* because $|C_0/C_1|,\ |C_{\pm 2}/C_1| = O(\epsilon)$ according to equation (1.132).

The expressions (1.135) and (1.136) are however not sufficient because we also need to find a similar contribution to $\Delta\omega$ from H_4. For that we take into account the term with β in (1.129) and again project that term on the state $e^{-i\omega t}$ which results in the equation similar to (1.134) but with the added term $-i\beta\left(\frac{1}{2m\omega}\right)^2 12|C_1|^2 C_1$ in r.h.s. Thus instead of equations (1.135) and (1.136) we obtain a full expression

$$\frac{da_1}{dt} + i(\omega + \Delta\omega)a_1 = 0 \tag{1.138}$$

which includes all terms of the order $O(|a_1|^2 a_1)$. Here $\Delta\omega$ is called by the *nonlinear frequency shift* and is given by

$$\Delta\omega = \left[3\beta\left(\frac{1}{m\omega}\right)^2 - \frac{20}{3}\frac{V^2}{\omega}\right]|a_1|^2. \tag{1.139}$$

Typically two terms in square brackets of equation (1.139) are different by the numerical factor ~ 1 and we replace equation (1.137) by

$$\Delta\omega/\omega \sim \epsilon^2. \tag{1.140}$$

The equations (1.138) and (1.139) imply that $\Delta\omega \in \mathbb{R}$ and $\frac{d|a_1|^2}{dt} = 0$. It allows to solve (1.138) and (1.139) explicitly and obtain that

$$a_1(t) = a_1(0)e^{-i(\omega+\Delta\omega)t}, \tag{1.141}$$

which has the same form as the solution $a_1(t) = a_1(0)e^{-i\omega t}$ for the harmonic oscillator (1.118) except that the frequency ω is replaced by $\omega + \Delta\omega$.

Equation (1.138) has the complex Hamiltonian form (1.116) with the Hamiltonian

$$H = \omega|a_1|^2 + \frac{T}{2}|a_1|^4 := H_2 + H_{4,eff}, \tag{1.142}$$

where $H_2 = \omega|a_1|^2$ is the quadratic part of the Hamiltonian and

$$T := 3\beta\left(\frac{1}{m\omega}\right)^2 - \frac{20}{3}\frac{V^2}{\omega} \tag{1.143}$$

is the *interaction constant* (also sometimes called by coupling constant) for the effective 4th-order Hamiltonian $H_{4,eff}$. There are two contributions to T. The first one is given by $3\beta\left(\frac{1}{m\omega}\right)^2$ and can be immediately obtained from the averaging of the original Hamiltonian H_4: $\langle H_4\rangle = \frac{3}{2}\beta\left(\frac{1}{m\omega}\right)^2|a_1|^4$, where $\langle\ldots\rangle$ means averaging over fast oscillations at time scale $1/\omega$. The second contribution, $-\frac{10}{3}\frac{V^2}{\omega}$, comes from taking into account of the zero and second harmonics in the cubic terms H_3 (1.125) as explained above. That contribution corresponds to the second order $O(\epsilon^2)$ of the perturbation theory (at the first order $O(\epsilon)$ we calculated C_0, C_2, C_{-2} in (1.132) and at the second order we found the influence of these terms on the dynamics of a_1). In other words, the interaction constant $3\beta\left(\frac{1}{m\omega}\right)^2$ of $\langle H_4\rangle$ is renormalized by the cubic terms H_3 of the Hamiltonian in the second order of the perturbation theory. That renormalization has the negative sign which is consistent with the well-known fact of the quantum mechanics that the shift of the ground state level from the second-order perturbation theory is negative one [**LL76**]. The nonlinear frequency shift (1.139) can be rewritten using equation (1.143) as

$$\Delta\omega = T|a_1|^2. \tag{1.144}$$

We conclude that there are two effects of nonlinearity of the oscillator:

1. Nonlinearity results in the formation of multiple harmonics.
2. Nonlinearity produces the nonlinear frequency of the fundamental harmonic.

We now return to the approximation made in equation (1.131) by neglecting time derivatives to obtain the expressions (1.132). Retaining these time derivatives because of a slow time dependence of $C_1(t)$, $C_0(t)$, $C_2(t)$, $C_{-2}(t)$ would result in the addition of small terms $\sim \left|\frac{dC_1^2}{dt}\right|\omega^{-1} \ll |C_1|^2$ into r.h.s, of each expression in (1.132). The nonlinear frequency shift (1.139) together with (1.132) immediately implies that $\left|\frac{dC_2}{dt}\omega^{-1}\right| \sim \left|\frac{dC_1^2}{dt}\right||V/\omega^2| \sim \epsilon^2|C_2|$, i.e. the relative contribution from time derivative to C_2 is $O(\epsilon^2)$. In addition, all multiple harmonics $C_{\pm n},\ n=0,1,2,\ldots$ multiple frequencies $0,\ \pm\omega,\ \pm 2\omega,\ \pm 3\omega,\ \pm 4\omega,\ldots.$ are produced in arbitrary small time if instead of equation (1.131) one uses the full expansion (1.130) in r.h.s. of equation (1.129) instead of the approximation $a \simeq a_1$. One then obtains that

(1.145)
$$|C_{-1}| \sim \epsilon^2|C_1|,\ |C_{\pm 3}| \sim \epsilon^2|C_1|,\ |C_{\pm 4}| \sim \epsilon^3|C_1|,\ldots,|C_{\pm n}| \sim \epsilon^{n-1}|C_1|,\ n>1.$$

The technique of obtaining equation (1.141) represents the *averaging method* (averaging over fast oscillations $e^{-i\omega t}$ by assuming weak nonlinearity results in the calculation of the nonlinear frequency shift (1.139)). The averaging ensures that the solution (1.141) is valid at times well exceeding the linear time $1/\omega$. To estimate the time of the validity of the solution (1.141) one can calculate the next order correction $\Delta\omega_4$ to the nonlinear frequency shift $\Delta\omega$ (which replaces solution (1.141) by $a_1(t) = a_1(0)e^{-i(\omega+\Delta\omega+\Delta\omega_4)t}$) from higher order harmonics using the scalings (1.145) which gives $\Delta\omega_4 \sim \epsilon^4\omega$. Then Taylor series expansion of $e^{-i\Delta\omega_4 t}$ provides the estimate $t \lesssim 1/(\epsilon^4\omega)$ of the validity of the leading-order solution (1.139) and (1.141).

Problems 1.13

1.13.0.1 Find the nonlinear frequency shift for the pendulum (1.8).

1.13.0.2 Show the validity of the estimates (1.145).

1.13.0.3 Find the explicit expression for the next order correction for nonlinear frequency shift $\Delta\omega$ for equation beyond the leading-order expression (1.139).

1.13.1. Resonance in nonlinear oscillator. Consider a dynamics/excitation of the nonlinear oscillator with an external periodic forcing. In the linear approximation (i.e. neglecting the nonlinear frequency shift $\Delta\omega$), the equation (1.138) with the added forcing $-if_0e^{-i\Omega t}$ takes the following form

$$\frac{\partial a_1}{\partial t} + i(\omega - i\gamma)a_1 = -if_0e^{-i\Omega t}, \tag{1.146}$$

where $f_0 \in \mathbb{C}$ and $\Omega \in \mathbb{R}$ are constants (i.e. the forcing is monochromatic). Also we added a small damping coefficient $\gamma > 0$ ($\gamma \ll \omega$) into l.h.s. of equation (1.146).

A solution of (1.146) is the sum of the transient solution $\propto e^{-i\omega t-\gamma t}$ (corresponds to the solution of the homogeneous part with forcing excluded of (1.146)) and the particular solution of the full nonhomogeneous equation (1.146) which is $\propto e^{-i\Omega t}$. Assuming $t\to\infty$, the transient solution can be neglected and the remaining solution takes the following form

$$a_1 = \frac{f_0}{\Omega - \omega + i\gamma}\, e^{-i\Omega t},$$

which gives a time-independent resonance solution in the form of Lorentzian function for $|a_1|^2$ as follows

$$|a_1|^2 = \frac{|f_0|^2}{(\Omega - \omega)^2 + \gamma^2}. \tag{1.147}$$

It follows from here that $|a_1|^2 \to \infty$ at $\gamma \to 0$ and $\Omega \to \omega$, i.e. usual linear resonance.

As we obtained in Section 1.13, taking into account nonlinearity amounts to the addition of the nonlinear frequency shift (1.144), which is achieved by the replacement of equation (1.146) by

$$\frac{\partial a_1}{\partial t} + i(\omega - i\gamma)a_1 = -iT|a_1|^2 a_1 - if_0 e^{-i\Omega t}. \tag{1.148}$$

Then equation (1.147) with the same assumption $a_1 \propto e^{-i\Omega t}$ modifies into

$$|a_1|^2 = \frac{|f_0|^2}{(\nu - T|a_1|^2)^2 + \gamma^2}, \quad \nu := \Omega - \omega, \tag{1.149}$$

where ν is the *frequency detuning* between the linear oscillator frequency ω and the pumping frequency Ω.

The matrix element T (1.143) used in equations (1.148) and (1.149) determines the nonlinear frequency shift (1.144), see Section 1.13. T is real-valued but can be both positive and negative depending to the values of the parameters in equation (1.143). We notice that a change of sign in ν is equivalent to the change of sign of T in equation (1.149). A complex conjugation of equation (1.148) also allows to change the sign of T with the additional replacement of the signs of frequencies and both the change of the sign and complex conjugation of the amplitude f_0. It is found below in this section that the contribution to the results from f_0 depends on $|f_0|$ only. Thus below in this section we assume $T > 0$ without loss of generality.

Equation (1.149) represents the cubic equation for the unknown amplitude $|a_1|^2$. Contrary to the linear resonance, that amplitude is bounded for all values of ν, γ and f_0 including $\gamma = \nu = 0$. To show that boundness we assume by contradiction that $|a_1|^2 \to \infty$ for some values of ν, γ and f_0. That assumption is inconsistent with equation (1.149) (l.h.s $\to \infty$ while r.h.s would $\to 0$ in that case). Thus the nonlinear frequency shift regularizes the linear resonance which is sometimes called by "frozenness" of the linear resonance by the nonlinearity.

At the conditions $\nu \to 0$ and $\gamma \to 0$ of the linear resonance, the frozen nonlinear resonance implies from equation (1.149) that

$$|a_1|^2 \to |a_1|^2_{sat} := \left(\frac{|f_0|}{|T|}\right)^{2/3}.$$

For nonzero damping $\gamma > 0$, the value of $|a_1|^2$ remains close to $|a_1|^2_{sat}$ at $\nu = 0$ provided $|T||a_1|^2_{sat} = |f_0|^{2/3}|T|^{1/3} \gg \gamma$.

Taking the limit $f_0 \to 0$ for each fixed values of $\gamma > 0$ and ν, we recover the linear solution (1.147). In that limit there is only one real solution of (1.149) for $|a_1|^2$. Figure 3a shows a typical dependence $|a_1|^2$ on ν for small $|f_0|$. If we increase $|f_0|$ at $|f_0| = |f_{0cr}|$, the slope of $|a_1|^2(\nu)$ turns infinite at one point as shown in

Figure 3b, while for $|f_0| > |f_{0cr}|$ the dependent $|a_1|^2(\nu)$ becomes multivalued as seen in Figure 3c. Here

$$|f_{0cr}| = \left(\frac{8\gamma^3}{3^{3/2}|T|}\right)^{1/2} \tag{1.150}$$

is the critical value of the forcing $|f_0|$ corresponding to the appearance of two additional real roots. To find $|f_{0cr}|$ we multiply equation (1.149) by the denominator of its r.h.s. which gives that

$$|a_1|^2 \left[\left(\nu - T|a_1|^2\right)^2 + \gamma^2\right] = |f_0|^2 \tag{1.151}$$

and then differentiate equation (1.151) over ν assuming that $|a_1|^2$ is the function of μ. It results in

$$\frac{d|a_1|^2}{d\nu}\left[\left(\nu - T|a_1|^2\right)^2 + \gamma^2 - 2|a_1|^2 T\left(\nu - T|a_1|^2\right)\right] + 2|a_1|^2\left(\nu - T|a_1|^2\right) = 0. \tag{1.152}$$

The slope $\frac{d|a_1|^2}{d\nu}$ turns infinite,

$$\frac{d|a_1|^2}{d\nu} = \infty, \tag{1.153}$$

at two values $\nu = \nu_{1,2}$ as seen in Figure 3c. Equation (1.152) results in

$$\left(\nu - T|a_1|^2\right)^2 + \gamma^2 - 2|a_1|^2 T\left(\nu - T|a_1|^2\right) = -\frac{2|a_1|^2\left(\nu - T|a_1|^2\right)}{d|a_1|^2/d\nu},$$

which together with equation (1.153) implies that

$$\left(\nu - T|a_1|^2\right)^2 + \gamma^2 - 2|a_1|^2 T\left(\nu - T|a_1|^2\right) = 0. \tag{1.154}$$

Solving equation (1.154) for $|a_1|^2$ results in

$$|a_1|^2 = \frac{2\nu \pm \sqrt{\nu^2 - 3\gamma^2}}{3T}. \tag{1.155}$$

Both roots coincide provided

$$\nu = \sqrt{3}\gamma \tag{1.156}$$

which implies that

$$|a_1|^2 = \frac{2\gamma}{\sqrt{3}T}. \tag{1.157}$$

Equations (1.151), (1.156) and (1.157) result in the threshold (1.150).

Above the threshold, $|f_0| > |f_{0cr}|$, there are three real solutions of equation (1.151) for $|a_1|^2$ in the finite range $\nu_1 \le \nu \le \nu_2$. Values of ν_1 and ν_2 are determined by the condition that two of these three solutions merge together roots, i.e. equation (1.153) is satisfied. The explicit expressions for ν_1 and ν_2 can be obtained by plugging (1.155) into equation (1.151) and solving for ν. These expressions are however somewhat bulky so we do not provide them here. Figure 3c shows a typical dependence of $|a_1|^2$ on ν for $|f_0| > |f_{0cr}|$.

It is sees from Figures 3a, 3b and 3c that $|a_1|^2(\nu)$ has a single maximum for all values of $|f_0|$. At the maximum $\frac{d|a_1|^2}{d\nu} = 0$ which implied through equation (1.152) that

$$\nu := \nu_{max} = T|a_1|^2. \tag{1.158}$$

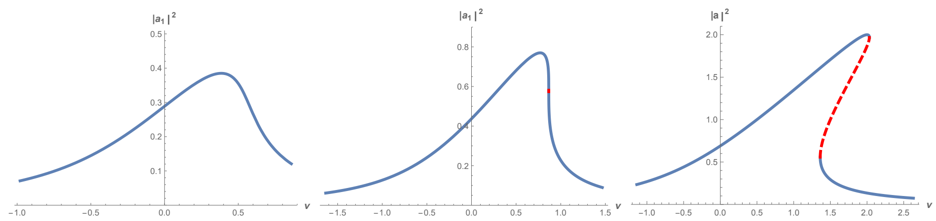

FIGURE 3. The dependence of the square amplitude $|a_1|^2$ of the nonlinear oscillator (1.148) on the frequency detuning ν for $T = 1$, $\gamma = 0.5$ as given by the real-valued solution of (1.149) for different values of $|f_0|^2$. (a) $|f_0|^2 = |f_{0cr}|^2/2 < |f_{0cr}|^2$, i.e. below the threshold (1.150). (b) $|f_0|^2 = |f_{0cr}|^2$, i.e. exactly at the threshold. Red dot corresponds to equations (1.156) and (1.157) with $\frac{d|a_1|^2}{d\nu} = \infty$. (c) $|f_0|^2 = 4\gamma^3/T > |f_{0cr}|^2$, i.e. above the threshold. Solid and dashed lines in (c) represent stable and unstable solutions, respectively. $\nu_1 = 1.36178\ldots$ and $\nu_2 = 2.03176\ldots$ are the points of infinite slope $\frac{d|a_1|^2}{d\nu} = \infty$ where solid lines join different ends of the dashed line.

Plugging (1.158) into equation (1.151) results in

$$|a_1|^2_{max} := |a_1|^2 = \frac{|f_0|^2}{\gamma^2} \tag{1.159}$$

and

$$\nu_{max} = \frac{|f_0|^2 T}{\gamma^2}. \tag{1.160}$$

We now analyze the linear stability of the solution (1.149) by replacing in equation (1.148) of $a_1 \to a_1 + \delta a_1$, where $a_1 \propto e^{-i\Omega t}$ is the solution of (1.149) while δa_1 is the small perturbation such that $|\delta a_1/a_1| \ll 1$. The terms of the zero order of perturbation $(\delta a_1)^0$, $(\delta a_1^*)^0$ in equation (1.148) are identically satisfied because of equation (1.149). For the linearization we keep only linear terms $\propto \delta a_1$, δa_1^* which results in

$$\begin{aligned} &\frac{d\delta a_1}{dt} - i(\nu - 2T|a_1|^2 + i\gamma)\delta a_1 + iTC\delta a_1^* = 0, \\ &\frac{d\delta a_1^*}{dt} + i(\nu - 2T|a_1|^2 - i\gamma)\delta a_1^* - iTC^{*2}\delta a_1 = 0. \end{aligned} \tag{1.161}$$

Assuming that δa_1, $\delta a_1^* \propto e^{\lambda t}$, we obtain from equation (1.161) the eigenvalue problem for λ which has two solutions

$$\lambda = \lambda_\pm = -\gamma \pm \sqrt{T^2|a_1|^4 - (\nu - 2T|a_1|^2)^2}. \tag{1.162}$$

The solution $\lambda = \lambda_-$ is always stable (i.e. $Re(\lambda) < 0$ for any $\gamma > 0$, $|a_1|$, ν and T). The solution $\lambda = \lambda_+$ is unstable provided the expression under the square root in (3.62) is large enough to ensure that $Re(\lambda) > 0$ which results in the inequality

$$T^2|a_1|^4 - (\nu - 2T|a_1|^2)^2 - \gamma^2 > 0. \tag{1.163}$$

We notice that l.h.s of equation (1.163) coincides with l.h.s. of equation (1.154) multiplied by -1. It ensures the neutral stability,

$$\lambda_+ = 0 \text{ at } \nu = \nu_{1,2}. \tag{1.164}$$

Using equations (1.159) and (1.160) we also obtain at $\nu = \nu_{max}$ that

$$\lambda_+|_{\nu=\nu_{max}} = -\gamma < 0, \tag{1.165}$$

i.e. the solution (1.149) is stable at the maximum $|a_1|^2 = |a_1|^2_{max}$.

Assume that $|f_0| < |f_{0cr}|$. Then there are no real values of $\nu_{1,2}$ as follows from the above analysis of this section. Thus λ_+ (3.62) never crosses the zero value and by continuity over ν from the negative value (1.165), $Re(\lambda_+)$ remains negative for all $\nu \in \mathbb{R}$. It implies that the solution (1.149) is stable at all ν for $|f_0| < |f_{0cr}|$.

Assume that $|f_0| > |f_{0cr}|$. Then there are two real values of $\nu_{1,2}$ as discussed above in this section. It is easy to show (see the problem at the end of this section) that the derivative of the expression under the square root in l.h.s. of (1.163) is nonzero at $\nu = \nu_{1,2}$,

$$\frac{d}{d|a_1|^2}\left(T^2|a_1|^4 - [\nu(|a_1|^2) - 2T|a_1|^2]^2 - \gamma^2\right)\Big|_{\nu=\nu_{1,2}} \neq 0, \tag{1.166}$$

where $\nu = \nu(|a_1|^2)$ is considered to be the function of $|a_1|^2$ to follow the solution curve given by equation and shown on Figure 3c. The condition (1.166) implies together with (1.163) and (1.164) that λ_+ (3.62) crosses from the negative to the positive value at $\nu = \nu_2$ if one moves downwards along the solution curve. Then we continue moving along the solution curve from $\nu = \nu_2$ downwards and to the left represented by dashed line on Figure 3c until reaching $\nu = \nu_1$. Equation (1.154) ensures λ_+ is positive all way until reaching the zero at $\nu = \nu_1$. Thus the solution branch represented by the dashed line of Figure 3c is unstable. After crossing $\nu = \nu_1$ to the lowest branch of solution, λ_+ again changes sign to $\lambda_+ < 0$ and keeps that sign all way as move along the solution curve to $\nu \to \infty$. We summarize that for $|f_0| > |f_{0cr}|$ two branches of the solution (1.149) of equation (1.148) represented by solid lines in Figure 3c are linearly stable while the branch shown by the dashed is linearly unstable. As $|f_0| \to |f_{0cr}|+$, the dashed line shrinks to the single point shown in Figure 3b, so the region of instability disappears.

Such stability/instability represents a dependence of the state of the system (1.148) on its history which is called *hysteresis.* To understand that dependence we fix the value of forcing with $|f_0| > |f_{0cr}|$ and adiabatically slowly in time increase ν from $\nu < \nu_1$ up to the range $\nu_1 \le \nu < \nu_2$. The stability of the stationary solution $|a_1|^2$ of equation (1.149), corresponding to the upper solid line of Figure 3, implies that the system slowly moves over that upper stationary solution. Further increase of ν results in the jump of the solution into the lower stable branch (lower solid line of Figure 3) at $\nu = \nu_2$. In contrast, a slow decrease of ν from $\nu > \nu_2$ down to $\nu_2 \ge \nu > \nu_1$ results in a slow motion over the lower stable stationary solution with the jump to the upper one at $\nu = \nu_1$. Thus in the range $\nu_1 < \nu < \nu_2$ the system settles at either the lower or the upper branches depending on the history. The hysteresis is closely related to *bi-stability*, i.e. a coexistence of two stable stationary solutions of equation (1.149) for the same values of ν.

Problems 1.13.1

1.13.1.1 Show that (1.166) is valid for $|f_0| > |f_{0cr}|$.

1.13.2. Parametric resonance. Another type or resonance occurs if the oscillator frequency ω has a periodic time dependence. Simplest example of the periodic time dependence is given by

$$\omega^2(t) = \omega_0^2(1 + \varepsilon \cos \Omega t), \tag{1.167}$$

where $\varepsilon \ll 1$ is the small parameter. The dynamics of oscillator without damping is given by the Mathieu's differential equation

$$\frac{d^2q}{dt^2} = -\omega^2(t)q, \tag{1.168}$$

where q is the coordinate. To bring that equation to the complex form we introduce the momentum as $p = \dot{q}$ (for simplicity we set the mass $m = 1$) an define a and $\bar{a}$ similar to (1.114) as

$$\begin{aligned} a &= \frac{1}{(2\omega_0)^{1/2}} \left(\omega_0 q + ip\right), \\ \bar{a} &= \frac{1}{(2\omega_0)^{1/2}} \left(\omega_0 q - ip\right), \end{aligned} \tag{1.169}$$

except that now time-independent part ω_0 of the frequency $\omega(t)$ is used for that transformation.

Consider the following Hamiltonian

$$H = \omega_0|a|^2 + \frac{\varepsilon\omega_0}{4}(a + \bar{a})^2 \cos(\Omega t), \tag{1.170}$$

then the Hamiltonian equations (1.116) and (1.117) result in

$$\begin{aligned} \frac{da}{dt} &= -i\omega_0 a - i\frac{\epsilon\omega_0}{2}(a + \bar{a})\cos(\Omega t), \\ \frac{d\bar{a}}{dt} &= i\omega_0 \bar{a} + i\frac{\epsilon\omega_0}{2}(a + \bar{a})\cos(\Omega t). \end{aligned} \tag{1.171}$$

Solving that systems of ODEs for q results in equation (1.169). Thus the Mathieu's differential equation(1.169) is equivalent to the Hamiltonian equations (1.116) and (1.170).

We now solve (1.116) and (1.170) assuming that the leading order solution is given by

$$a(t) = C(t)e^{-i\omega_0 t} \text{ and } \bar{a}(t) = \bar{C}(t)e^{i\omega_0 t},$$

where $C(t)$ is the slow function of time in comparison with $e^{-i\omega_0 t}$. We separate the Hamiltonian (1.170) into the harmonic part $H_2 = \omega_0|a|^2$ and the interaction part

$$H_{int} = \frac{\varepsilon\omega_0}{4}(a + \bar{a})^2 \cos(\Omega t).$$

We again use averaging of H_{int} over the fast time scale $1/\omega_0$. Only nonoscillating terms survive the averaging. Recalling that $\cos \Omega t = (e^{i\Omega t} + e^{-i\Omega t})/2$ as well as that at leading order $a \propto e^{-i\omega_0 t}$ and $\bar{a} \propto e^{i\omega_0 t}$, we conclude that many terms in H_{int} oscillate and average out to zero, such as $|a|^2 \cos \Omega t$. Nonzero average is possible for $a^2 e^{i\Omega t}$ provided $2\omega_0 \simeq \Omega$ which is near to the condition

$$2\omega_0 = \Omega \tag{1.172}$$

of the parametric resonance. In that case we obtain that

$$\langle H_{int}\rangle = \frac{\varepsilon\omega_0}{8}\left[a^2e^{i\Omega t} + \bar{a}^2e^{-i\Omega t}\right]. \tag{1.173}$$

Also $\langle H_2\rangle = H_2 = \omega_0|a|^2$, i.e. the harmonic part of the Hamiltonian is not affected by the averaging.

The parametric resonance condition (1.172) has an analog in quantum mechanics. We rewrite equation (1.173) as $\langle H_{int}\rangle = \frac{\omega_0}{2}\left(\bar{h}a^2 + h\bar{a}^2\right)$, where $h(t) := \frac{\epsilon}{4}e^{i\Omega t}$ is the pumping amplitude. Recall that a and $\bar{a}$ are the classical analogs of the annihilation and creation operators, respectively. Then the term $h\bar{a}^2$ in $\langle H_{int}\rangle$ describes the annihilation of a quantum of pumping and the creation of two quanta of the oscillator. That process requires the energy conservation which is insured by the condition (1.172).

Consider the dynamical equations for the averaged Hamiltonian $\langle H\rangle = H_2 + \langle H_{int}\rangle$ (1.171) and (1.173):

$$\begin{aligned}\frac{da}{dt} + \gamma a &= -i\frac{\partial\langle H\rangle}{\partial\bar{a}} = -i\omega_0 a - i\frac{\epsilon\omega_0}{4}\bar{a}e^{-i\Omega t},\\ \frac{d\bar{a}}{dt} + \gamma\bar{a} &= i\frac{\partial\langle H\rangle}{\partial a} = i\omega_0\bar{a} + i\frac{\epsilon\omega_0}{4}ae^{i\Omega t},\end{aligned} \tag{1.174}$$

where we added a linear damping term γa into l.h.s., similar to Section (1.13.1). Comparison between first equations in (1.171) and (1.174) shows that (1.174) does not include terms which oscillate with a different frequency than $e^{-i\omega_0 t}$. Taking into account such non-resonant terms in (1.171) can be done by the expansion in multiple harmonics, qualitatively similar to the expansion (1.130). These terms however would modify (1.174) by the inclusion of $O(\epsilon^2)$ terms which we neglect here. Thus the averaging of the Hamiltonian is equivalent to the neglect of non-resonant terms in the dynamic equations (1.171).

We introduce new variables c and $\bar{c}$ as follows

$$a = c(t)e^{-i\frac{\Omega t}{2}},\ \bar{a} = \bar{c}(t)e^{+i\frac{\Omega t}{2}},$$

which transform (1.174) into ODE system with constant coefficients

$$\begin{aligned}&\frac{\partial c}{\partial t} + \left[i\left(\omega_0 - \frac{\Omega}{2}\right) + \gamma\right]c + i\frac{\epsilon\omega_0}{4}\bar{c} = 0,\\ &\frac{\partial\bar{c}}{\partial t} + \left[-i\left(\omega_0 - \frac{\Omega}{2}\right) + \gamma\right]\bar{c} - i\frac{\epsilon\omega_0}{4}c = 0.\end{aligned} \tag{1.175}$$

Assuming $c,\ \bar{c} \sim e^{\Gamma t}$ we obtain from (1.175) a homogeneous system of linear equations

$$\begin{aligned}&\left[\Gamma + \gamma + i\left(\omega_0 - \frac{\Omega}{2}\right)\right]c = -i\frac{\epsilon\omega_0}{4}\bar{c},\\ &\left[\Gamma + \gamma - i\left(\omega_0 - \frac{\Omega}{2}\right)\right]\bar{c} = i\frac{\epsilon\omega_0}{4}c\end{aligned} \tag{1.176}$$

for the unknowns c and $\bar{c}$. That system is solvable provided 2×2 matrix of its coefficients has a zero determinant which gives

$$\Gamma = -\gamma \pm \sqrt{\frac{(\epsilon\omega_0)^2}{16} - \left(\omega_0 - \frac{\Omega}{2}\right)^2}. \tag{1.177}$$

It follows from (1.177) that the instability ($\Gamma > 0$) is possible for $\epsilon > \epsilon_{cr} = 4\gamma/\omega_0$ provided ω_0 is close enough to $\Omega/2$ to ensure that the expression under the square root in (1.177) is positive as well as to overcome dissipation rate γ. This instability is called the *parametric instability.* The growth rate Γ of the parametric instability reaches maximum if the condition (1.172) is exactly satisfied.

A difference between the forced oscillator of Section (1.13.1) and the parametric amplification of Section (1.13.2) can be seen from the everyday experience in playing on a children's swing. Rocking back and forth pumps the swing as a forced harmonic oscillator. Once moving, the swing can also be parametrically amplified by alternately standing and squatting at key points in the swing arc, i.e. by periodically changing the moment of inertia of the swing and hence the resonance frequency. The rate of that change is twice of the natural frequency of the swing satisfying the parametric resonance condition (1.172).

Problems 1.13.2

1.13.2.1 Find a frequency width of the parametric resonance $\Delta\Omega$, i.e. the range of values of Ω at which a parametric excitation ($\Gamma > 0$) occurs according to equation (1.177) assuming that $\epsilon > \epsilon_{cr}$.

1.14. Fermi-Pasta-Ulam-Tsingou system

Enrico Fermi, John Pasta, Stanislaw Ulam and Mary Tsingou (FPUT) [**FPU55**] considered a chain of N particles connected by nonlinear springs (anharmonic one dimensional lattice) as schematically shown in Figure 2. It means that only nearest neighbor interaction between particles is considered. The position of jth particle ($j = 0, 1, \ldots, N-1$) along the spatial direction x is given by the displacement q_j along the chain from the equilibrium position $x = jh$, where h is the distance between nearest neighbors in equilibrium and x is the coordinate along the chain. The Hamiltonian of the FPUT system is given by

$$H = \sum_{j=0}^{N-1} \frac{p_j^2}{2m} + U = \sum_{j=0}^{N-1} \frac{m\dot{q}_j^2}{2} + U, \tag{1.178}$$

where the first term in r.h.s. is the kinetic energy, $\dot{q}_j := \frac{dq}{dt}$, the momentum p_j is given by

$$p_j = m\frac{dq}{dt} := m\dot{q}_j. \tag{1.179}$$

as follows from equation (1.1) and U is the potential energy of the nonlinear springs given by

$$U = \sum_{j=0}^{N-1} \left[\frac{U_2}{2} \left(q_{j+1} - q_j\right)^2 + \frac{U_3}{3} \left(q_{j+1} - q_j\right)^3 + \frac{U_4}{4} \left(q_{j+1} - q_j\right)^4 \right] \tag{1.180}$$

Here U_2, U_3 and U_4 are the coefficient of the quadratic, cubic and fourth order terms which can be obtained e.g. from the Taylor series expansion of the general interparticle pairwise interaction potential $U_{interparticle}(q_{j+1} - q_j)$. It is assumed that

$U_2 > 0$, i.e. $U_{interparticle}(q_{j+1}-q_j)$ has the minimum at $q_{j+1}-q_j = 0$ which also implies that the first derivative $U'_{interparticle}(0) = 0$ so there is no linear term in equation (1.180). Also without loss of generality we assume that $U_{interparticle}(0) = 0$ (any nonzero value of $U_{interparticle}(0)$ does not enter into the Hamiltonian equations (1.1)). The coefficients U_3 and U_4 are responsible in equation (1.180) for the anharmonicity of the lattice. We assume either fixed boundary conditions

$$q_0 = q_N = p_0 = p_N \equiv 0 \tag{1.181}$$

or the periodic boundary conditions

$$q_N = q_0, \ p_N = p_0. \tag{1.182}$$

We note that historically FPUT system was sometimes called by the names of Fermi, Pasta and Ulam as FPU while more recently the historically accurate addition of the name of Mary Tsingou was made [**T.08**]. Another historical remark is that Ref. [**FPU55**] used the notations $U_3 \equiv \alpha$ and $U_4 \equiv \beta$. Since then it is often customary to call the particular case $U \neq 0, \ U_4 = 0$ by "α-model" and the particular $U_4 \neq 0, \ U_4 \neq 0$ by "β-model", respectively.

Equation (1.1), (1.178) and (1.180) result in the dynamical equations

$$\begin{aligned} m\frac{d^2 q_j}{dt^2} := m\ddot{q}_j = U_2(q_{j+1} - 2q_j + q_{j-1}) + U_3\left[(q_{j+1} - q_j)^2 - (q_j - q_{j-1})^2\right] \\ +U_4\left[(q_{j+1} - q_j)^3 - (q_j - q_{j-1})^3\right], \ j = 0, \ldots, N-1. \end{aligned} \tag{1.183}$$

The simplest limiting case $U_2 \neq 0$ and $U_3 = U_4 = 0$ was considered in Examples (1.10.0.2) and (1.10.0.3) turning the FPUT Hamiltonian (1.178), (1.180) into the quadratic Hamiltonian for N coupled harmonic oscillators in normal variables. If a specific kth Fourier harmonics is excited by the external force (e.g. a laser beam a given k) then it will not exchange energy between different harmonics in such limiting case. Nonlinearity is required to observe such exchange. Enrico Fermi, John Pasta, Stanislaw Ulam [**FPU55**] together with Mary Tsingou [**T.08**] performed in 1953 simulations of the nonlinear chain (1.183) with zero boundary conditions (1.181) in two cases: $U_3 \neq 0, \ U_4 = 0$ and $U_3 = 0, \ U_4 \neq 0$. The idea of their simulations was to observe a thermalization of energy, i.e. the equipartition of energy between different Fourier harmonics (normal modes) because of the nonlinear interactions. They provided the initial conditions with the single initially excited $k = 1$ Fourier harmonic and zero initial velocity, i.e. in equation (1.87) it corresponds at $t = 0$ to $Q_1 \neq 0, \ Q_2 = \ldots = Q_{N-1} = Q_N = 0, \ P_1 = \ldots = P_{N-1} = P_N = 0$. To their big surprise, they do not observed equipartition of energy. At the intermediate times the energy was indeed transferred to modes with $k = 2, 3, \ldots$. However, for the case $U_3 \neq 0, \ U_4 = 0$, after 157 periods of the mode $k = 1$, about 97% of the total energy is transferred back to the mode $k = 1$. This phenomenon is called *FPUT recurrence* and it repeats again and again at later times. Such nearly periodic in time transfer of energy between different Fourier modes is shown in Figure 4. The first explanation of that highly unexpected result was provided in Ref. [**ZK65**] in a *long wavelength limit* through the elastic nonlinear interaction of localized excitations of FPUT system later called by solitons.

The long wavelength limit (also called by continuous limit) assumes that that the displacement q_j of the jth particle is a slow function of j. That slow dependence

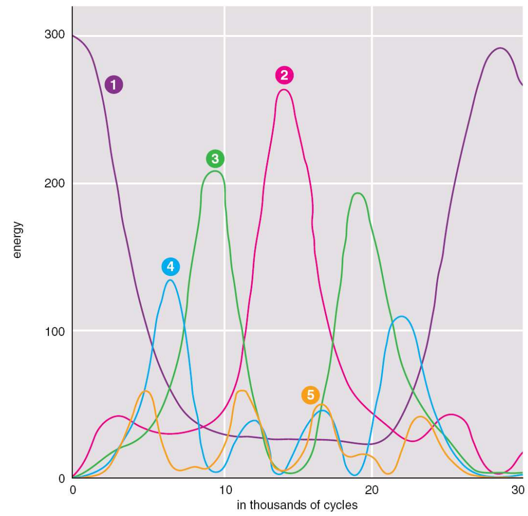

FIGURE 4. A dependence of different Fourier harmonics of FPUT system on time for $N = 32$. ???

allows to define the continuous approximation $q(x)$ for q_j as follows

$$q_j := q(jh), \tag{1.184}$$

where x is the distance along the lattice, h is the distance between equilibrium positions of neighboring particles. A Taylor series expansion

$$q_{j\pm 1} = q(x \pm h) = q(x) \pm hq'(x) + \frac{h^2}{2!}q''(x) + \frac{h^3}{3!}q'''(x) + \dots, \; x := jh \tag{1.185}$$

in equation (1.183) with $U_3 \neq 0$, $U_4 = 0$ results in the following partial differential equation (PDE)

$$\begin{aligned} mq_{tt} - U_2 h^2 q_{xx} &= U_2 \frac{h^4}{12} q_{xxxx} + 2h^3 U_3 q_{xx} q_x \\ &+ O(U_2 h^6 q_{xxxxxx}) + O(U_3 h^5 q_{xxxx} q_x) + O(U_3 h^5 q_{xxx} q_{xx}). \end{aligned} \tag{1.186}$$

L.h.s. of (1.186) is the linear wave equation with the wave velocity

$$c := \left(\frac{U_2 h^2}{m}\right)^{1/2}. \tag{1.187}$$

We introduce scaled dimensionless variables

$$\begin{aligned} \tilde{t} &= \frac{ct}{L}, \\ \tilde{x} &= \frac{x}{L}, \\ \tilde{q} &= \frac{q}{h}, \end{aligned} \tag{1.188}$$

where the length L e.g. can be chosen to be be equal to the total length of the system $L = Nh$. Another natural choice is to take L to be of the order of the typical width of at which q changes along x, i.e. $L \sim |q/q_x|$. Equations (1.190) transform (1.187) into the dimensionless form

$$\begin{aligned} \tilde{q}_{\tilde{t}\tilde{t}} - \tilde{q}_{\tilde{x}\tilde{x}} = \frac{h^2}{12L^2}\tilde{q}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \frac{2h^4U_3}{Lmc^2}\tilde{q}_{\tilde{x}\tilde{x}}\tilde{q}_{\tilde{x}} + O(\frac{h^4}{L^4}q_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}}) \\ +O\left(\frac{h^6U_3}{L^3mc^2}\tilde{q}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}\tilde{q}_{\tilde{x}}\right) + O\left(\frac{h^6U_3}{L^3mc^2}\tilde{q}_{\tilde{x}\tilde{x}\tilde{x}}\tilde{q}_{\tilde{x}\tilde{x}}\right). \end{aligned} \tag{1.189}$$

Here $\epsilon := h/L$ is the small parameter as follows from long wavelength approximation. We look for weakly nonlinear solution of (1.189) which is close to the solution of the linear wave equation meaning that the nonlinear term in r.h.s of (1.189) as well as the linear term $|\frac{h^2}{12L^2}\tilde{q}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}|$ are small in comparison with $|q_{\tilde{t}\tilde{t}}|$ and $|\tilde{q}_{\tilde{x}\tilde{x}}|$. Solution of l.h.s. of (1.189) set to zero is a linear combination of the waves propagating in the right and left directions, $\tilde{q} \simeq f(\tilde{x}-\tilde{t}) + g(\tilde{x}+\tilde{t})$, respectively. Assume that the right-propagating wave dominates and neglect left-propagating wave. It occurs if we have well-enough localized initial condition $|\tilde{q}(\tilde{x},\tilde{t}=0)| \to 0$ as $|\tilde{x}| \to \infty$ (e.g. assuming that exists a positive constant α such that $|\tilde{q}(\tilde{x},\tilde{t}=0)| < e^{-\alpha|\tilde{x}|}$) and wait long enough so that $\tilde{t}$ well exceeds a typical width $\tilde{x}_0 < 1/\alpha$ of $|\tilde{q}(\tilde{x},\tilde{t}=0)|$. Then in the moving frame $\tilde{x}-\tilde{t}$ one can neglect $g(\tilde{x}+\tilde{t})$ and focus on $f(\tilde{x}-\tilde{t})$. R.h.s. of (1.189) results in more complicated solution compare with $f(\tilde{x}-\tilde{t})$. To account for that we introduce new time τ and new coordinate y its as follows

$$\begin{aligned} &\tau = \epsilon\tilde{t}, \quad \epsilon := \frac{h^2}{24L^2} \ll 1, \\ &y = \tilde{x} - \tilde{t}. \end{aligned} \tag{1.190}$$

Then

$$\frac{\partial^2}{\partial\tilde{t}^2} - \frac{\partial^2}{\partial\tilde{x}^2} = -2\epsilon\frac{\partial^2}{\partial\tau\partial y} + \epsilon^2\frac{\partial^2}{\partial\tau^2}$$

and equation (1.189) is transformed into

$$\tilde{q}_{\tau y} + \tilde{q}_{yyyy} + \delta^2\tilde{q}_{yy}\tilde{q}_y + O(\epsilon) = 0, \tag{1.191}$$

where

$$\delta^2 := \frac{24h^2LU_3}{mc^2} \tag{1.192}$$

and we abused notation by setting $\tilde{q}(\tilde{x},\tilde{t}) = \tilde{q}(y,\tau)$. We assume that $\delta^2 \sim 1$ and define a new unknown

$$u(y,\tau) = \frac{\delta^2}{6}q_y(y,\tau), \tag{1.193}$$

then (1.191) takes the following form

$$u_\tau + u_{yyy} + 6uu_y = 0, \tag{1.194}$$

where $O(\epsilon)$ term was neglected. Equation (1.194) is called by a *Korteweg–de Vries equation* (KdV). The KdV equation was first introduced by Boussinesq in 1877 for water waves problems and later rediscovered by Diederik Korteweg and Gustav de Vries in 1895.

Simulations of equation (1.183) with $U_3 \neq 0$, $U_4 = 0$ (α-model of FPUT) in Ref. [**ZK65**] were performed with the periodic boundary conditions (1.182) and $N = 32$??? The initial formation of nearly shock wave decaying into multiple solitons of KdV was observed (1.194) ????

Problems 1.14

1.14.0.1 Assume that $U_3 = 0,\ U_4 \neq 0$ in equation (1.183) (β-model of FPUT) and derive the modified Korteweg-de Vries equation (mKdV)

$$u_\tau + u_{yyy} + 6u^2 u_y = 0, \tag{1.195}$$

in the same long wavelength limit as was used in derivation of KdV equation (1.194).

1.14.0.2 Prove the orthogonality condition (1.106) of DFT (1.100).

1.14.0.3 Find the nonlinear frequency shift of α-model of FPUT.

1.14.0.4 Find the nonlinear frequency shift of β-model of FPUT.

1.15. Poisson Mechanics

In this section we consider Hamiltonian systems of more general form than in (1.1), in which it is impossible to make a unique separation of variables into coordinates and momenta. Such systems are conveniently described in terms of generalized coordinates $\mathbf{x} \in \mathbb{R}^N$, that are generally speaking not canonical. Let G, the phase space of the system, be a manifold covered by some system of maps. We assume that on the manifold G there is given a symplectic structure which is a nondegenerate closed two-form Ω. This means that at each point a twice covariant anti-symmetric tensor $\Omega_{ij} = -\Omega_{ji},\ i, j = 1, \ldots, N$, is defined. Suppose that x_i are the local coordinates of some point. The closure condition implies that Ω_{ij} satisfies the system of equations

$$\frac{\partial \Omega_{ij}}{\partial x_k} + \frac{\partial \Omega_{jk}}{\partial x_i} + \frac{\partial \Omega_{ki}}{\partial x_j} = 0 \tag{1.196}$$

with $\det||\Omega_{ij}|| \neq 0$.

A system of differential equations defined on G is said to be Hamiltonian if there exists a function H on G, so that, in the neighborhood of each point identified by x_i one has

$$\sum_{j=1}^{N} \Omega_{ij} \dot{x}_j = \frac{\partial H}{\partial x_i}. \tag{1.197}$$

It is easy to see that under changes of coordinates $x_i = x_i(\tilde{x}_1, ..., \tilde{x}_N)$, for which Jacobian $\partial(x_1, ..., x_N)/\partial(\tilde{x}_1, ..., \tilde{x}_N) \neq 0$, then the equation (1.197) remains invariant. In this case the matrix Ω transforms as follows,

$$\tilde{\Omega}_{lm} = \sum_{i,j=1}^{N} \frac{\partial x_i}{\partial \tilde{x}_l} \Omega_{ij} \frac{\partial x_j}{\partial \tilde{x}_m}.$$

A manifold on which one can assign symplectic structure is said to be symplectic. It necessarily has even dimension (otherwise $\det\ \Omega_{ij} = 0$).

Within each simply connected region Eq.(1.196) can be integrated to read:

$$\Omega_{ij} = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}, \tag{1.198}$$

where $A_i(x)$ are the "potential" of the form Ω_{ij}. If the solutions of the system (1.197) do not extend beyond the limits of this region, the variational principle

$\delta S = 0$ is valid, where

$$S = \int \left(A_i \, \dot{x}_i + H\right) dt. \tag{1.199}$$

It has been noted that the variational principle (1.199) exists globally only if the form Ω_{ij} is exact, i.e., if the relation (1.198) can be extended over the whole manifold G. Generally speaking, A_i are multivalued functions on G, that acquire a nonzero addition in going around any cycle not homologous to zero. Locally, in each simply connected region one can, by a suitable change of variables, bring the system to canonical coordinates, i.e., to the form (1.1) (Darboux's theorem). However, globally (over all G) doing this is generally not possible, even if the form (1.198) is exact. Due to the assumption of the nondegeneracy of the form Ω_{ij}, Eq. (1.197) can be written in the form

$$\dot{x}_i = \sum_{j=1}^{N} R_{ij} \frac{\partial H}{\partial x_j}. \tag{1.200}$$

Here $R_{ij} = -R_{ji}$ is the inverse matrix of Ω_{ij}, i.e. $R^{-1} = \Omega$. It is then easily verified that the relations (1.196) are equivalent to the relations

$$\sum_{m=1}^{N} \left(R_{im} \frac{\partial}{\partial x_m} R_{jk} + R_{km} \frac{\partial}{\partial x_m} R_{ij} + R_{jm} \frac{\partial}{\partial x_m} R_{ki}\right) = 0 \tag{1.201}$$

for any $i, j, k = 1, \ldots, N$.

Next, by means of the matrix R one defines the general Poisson bracket between any functions A and B given on G:

$$\{A, B\} = \sum_{i,j=1}^{N} R_{ij} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j}. \tag{1.202}$$

The definition (1.202) generalizes the canonical Poisson bracket defined in (1.9).

It it follows from the antisymmetry of R_{ij} that

$$\{A, B\} = -\{B, A\},$$

while the relations (1.201) guarantee that the Jacobi identity

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0 \tag{1.203}$$

is satisfied. Because of the nondegeneracy of the matrix Ω_{ij}, in each coordinate system the matrix R_{ij} is also nondegenerate. The matrix R is called the *cosymplectic operator*, and it plays the same role as the metric tensor g_{ij} in Eucledian geometry. The condition (1.201) is the analog of the vanishing of the curvature tensor for Eucledian space, and, respectively, the canonical form

$$\Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

has the same meaning as

$$g = I$$

in Eucledian space.

The next step for generalizing the Hamiltonian system is to drop the requirement of nonsingularity of R. This variant of the Hamiltonian mechanics is called by the Poisson mechanics. If $\det||R_{ik}|| = 0$, then a return to the form (1.197) is impossible. Suppose that $\xi_i^\alpha(\alpha = 1, ..., k)$ is based in the co-kernel of the operator R_{ij} (i.e., $\sum_{i=1}^N \xi_i R_{ij} = 0$). Then equation (1.200) implies that

$$\sum_{i=1}^{N} \xi_i^\alpha \dot{x}_i = 0, \ \alpha = 1, ..., k. \tag{1.204}$$

In a simply connected domain in which rank of the matrix R is constant, due to the Frobenious theorem, Eqs.(1.204) can be integrated:

$$f^\alpha(x_1, ..., x_n) = \text{const} \ , \alpha = 1, ..., k \ .$$

In their turn, these relations are connected with vectors ξ_i^α by the evident formulas:

$$\xi_i^\alpha = \frac{\partial f^\alpha}{\partial x_i}.$$

The constants f^α are called the *Casimirs invariants*. Moreover, the Frobenious theorem and the relations (1.201) guarantee that all these k invariants are functionally independent. They are evidently integrals of motion for our Hamiltonian system. These integrals separate G into manifolds invariant under the system (1.200) (symplectic leaves). On each of them one can introduce the usual Hamiltonian mechanics. From our remarks it is clear that the possibility of introducing Poisson brackets implies the system under consideration to be Hamiltonian in the weakest sense.

Of special interest is the case where the metric elements R_{ij} are linearly dependent on the coordinates as follows:

$$R_{ij} = \sum_{m=1}^{N} e_{ij,m} x_m. \tag{1.205}$$

From condition (1.201) it now follows that the $e_{ij,m}$ are subject to the relations

$$\sum_{i=m}^{N} [e_{ik,m} e_{jm,l} + e_{ji,m} e_{km,l} + e_{kj,m} e_{im,l}] = 0$$

i.e., they are the structure constants of some Lie algebra L. Calculating the bracket between quantities x_i, x_j, it can be checked that

$$\{x_i, x_j\} = R_{ij} = \sum_{m=1}^{N} e_{ij,m} x_m. \tag{1.206}$$

Thus, the space G itself is now a Lie algebra L. The constants f^α in this case are just the Casimir invariants of this Lie algebra, and they commute with any elements from the algebra,

$$\{f^\alpha, \cdot\} = 0.$$

The matrix R_{ij} is in general degenerate. However, the relations (1.204) are always integrable. Consider the algebra L^*, dual to L, and the corresponding Lie group l^*. Here the algebra L forms the co-adjoint representation of the group l^*. Relations (1.204) are invariant under the action of the group l^*, and so conditions (1.206) hold, and define the orbits of the action of the group l^* in L. On these orbits (cf. Kirillov [**?**], Kostant [**?**]) there exists a fully valid Hamiltonian mechanics.

If the Hamiltonian is polynomial in its variables, then the equations are also polynomials in the canonical coordinates, and they have a nonlinearity that is one degree lower. If the degree of nonlinearity of the investigated system coincides with the degree of nonlinearity of the Hamiltonian, then the matrix is linear in the coordinates, and the symplectic manifold G is the orbit of some Lie group in its co-adjoint representation. This currently happens for equations of hydrodynamic type.

Another interesting case is the situation when the Poisson structure R depends on coordinates x_i quadratically. In this case it can be considered as the classical R-matrix which plays the important role in theory of Hamiltonian systems integrable by the inverse scatterring transform. This theory, however, is far from a scope of this book.

Problems 1.15

1.15.0.1 Show that the Jacobi identity (1.203) is satisfied for the Poisson bracket (1.202) provided the condition (1.201) is valid.

1.16. Symplectic leaves. Example of free motion of rigid body

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CHAPTER 2

Hamiltonian Systems in continuous media

2.1. Linear waves

Small amplitude waves in unbounded linear media are typically described by the system the first linear homogeneous PDEs with constant coefficients of the general type

$$\frac{\partial \mathbf{u}}{\partial t} = L(\nabla)\mathbf{u}, \tag{2.1}$$

where $\mathbf{u}(\mathbf{r}, t) \in \mathbb{C}^N$ is the vector function of time t and the spatial variable $\mathbf{r} \in \mathbb{R}^D$. The components of $\mathbf{u}$ can be either complex-valued or real-valued and consist of physical variables (unknowns) for each given physical system. $L(\nabla)$ is the $N \times N$ matrix linear operator which depends only on the different powers of components of the spatial gradients ∇. The multiple particular examples of $L(\nabla)$ are provided below in this chapter with $L(\nabla)$ being the linearization operator of nonlinear physical systems. The assumption of the system (2.1) does not exclude higher order in time PDEs from the consideration because they can be always reduced to the first order system by addition of separable variable for each second and higher order time derivative.

2.1.0.1. *Example.* The linear wave equation with the constant velocity c is given by

$$q_{tt} = c^2 q_{xx}, \tag{2.2}$$

with $q \in \mathbb{R}$, see e.g. the derivation of such equation in Section 1.14 as the leading order linear approximation of the nonlinear string given by l.h.s. of equation (1.186). Defining $q := u_1$ and $q_t := u_2$, we transform the second order PDE (2.2) into the system of two first order in time PDEs

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= u_2, \\ \frac{\partial u_2}{\partial t} &= c^2 \frac{\partial^2 u_1}{\partial x^2}, \end{aligned} \tag{2.3}$$

which corresponds to the system (2.1) with $\mathbf{u} = (u_1, u_2)^T$ and

$$L(\nabla) = \begin{pmatrix} 0 & 1 \\ c^2 \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}. \tag{2.4}$$

2.1.0.2. *Example.* We now extend the Example 2.1.0.1 by adding the next order linear term from r.h.s. of equation (1.186) which results in

$$q_{tt} = c^2 q_{xx} + \gamma q_{xxxx}, \tag{2.5}$$

where $\gamma > 0$ is the constant. Then equation (2.6) is replaced by

$$L(\nabla) = \begin{pmatrix} 0 & 1 \\ c^2\frac{\partial^2}{\partial x^2} + \gamma\frac{\partial^4}{\partial x^4} & 0 \end{pmatrix} \tag{2.6}$$

with the same definition of $\mathbf{u}$ through q and q_t as in Example 2.1.0.1.

2.1.0.3. *Example.* The time-dependent Schrödinger equation of a single particle of mass m in the free space is given by

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi, \tag{2.7}$$

where $\Psi \in \mathbb{C}$ is the wave function and $\hbar$ is the reduced Planck's constant. Equation (2.7) corresponds to $\mathbf{u} = \Psi$ and

$$L(\nabla) = \mathrm{i}\frac{\hbar}{2m}\nabla^2. \tag{2.8}$$

The system (2.1) can be efficiently solved by the spatial Fourier transform (FT)

$$f_{\mathbf{k}} = \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} f(\mathbf{r})e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}, \tag{2.9}$$

of the function $f(\mathbf{r})$ over the spatial variable $\mathbf{r}$ together with inverse FT (IFT)

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} f_{\mathbf{k}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}, \tag{2.10}$$

see Appendix A.4 for the overview of FT. In this book we always assume (if not explicitly specified otherwise) that both FT and inverse FT (IFT) exists as well as IFT of $f_{\mathbf{k}}$ recovers $f(\mathbf{r})$ pointwise.

FT (2.9) of equation (2.1) results in

$$\frac{d\mathbf{u}_{\mathbf{k}}}{dt} = L(\mathrm{i}\mathbf{k})\mathbf{u}_{\mathbf{k}}, \tag{2.11}$$

which is the linear ODE system with the constant coefficients. We look for the particular solutions of that system in the exponential form $\mathbf{u}_{\mathbf{k}} \propto e^{-\mathrm{i}\omega t}$ with the frequency $\omega \in \mathbb{R}$ resulting in the eigenvalue problem

$$-\mathrm{i}\omega\mathbf{u}_{\mathbf{k}} = L(\mathrm{i}\mathbf{k})\mathbf{u}_{\mathbf{k}} \tag{2.12}$$

for $-\mathrm{i}\omega$.

Remark 1: A little faster way to obtain the system (2.12) is by looking at a particular solution of the system (2.1) in the exponential form $\mathbf{u}_{\mathbf{k}} \propto e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}-\mathrm{i}\omega t}$ which immediately results in (2.12).

Remark 2: One can work directly with the higher order PDEs, such as equations (2.2) and (2.5) in examples 2.1.0.1 and 2.1.0.2, without reducing them to the first order system (2.1). Then FT converts these PDEs to the higher order linear ODEs instead of the first order system (2.11). These higher order ODEs can be solved by the standard ODE methods including looking for the particular solutions $\propto e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}-\mathrm{i}\omega t}$. Below we however work only with the more general system (2.1).

The solvability condition of the homogeneous linear system (2.12) is given by characteristic equation $\det|-\mathrm{i}\omega I_N - L(\mathrm{i}\mathbf{k})| = 0$ for the matrix $L(\mathrm{i}\mathbf{k})$, where I_N is $N \times N$ identity matrix. Solutions of that characteristic equation provide the relations

$$\omega = \omega_j(\mathbf{k}) := \omega_{\mathbf{k},j}, \ j = 1, \dots, N \tag{2.13}$$

between the frequency ω and the wave vector $\mathbf{k}$ which is called by the *dispersion relation.*

There are generally N branches $j = 1, \ldots, N$ of the dispersion relation. Although some of $\omega_{\mathbf{k},j}$ can potentially have algebraic multiplicity more than one (i.e. $\omega_{\mathbf{k},j}$ are the same for several distinct values of j), their algebraic multiplicity for linear waves is always equal to their geometric multiplicity, i.e. the eigenspace $\{\mathbf{u}_{\mathbf{k},1}, \ldots, \mathbf{u}_{\mathbf{k},N}\}$ of all eigenvalues of $L(\mathrm{i}\mathbf{k})$ spans $\mathbb{C}^N$, where the eigenvector $\mathbf{u}_{\mathbf{k},j}$ corresponds to the eigenvalue $-\mathrm{i}\omega_{\mathbf{k},j}$ for $j = 1, \ldots, N$. In other words, $L(\mathrm{i}\mathbf{k})$ is the diagonalizable matrix with the diagonalization ensured by the similarity transformation $D = P^{-1}L(\mathrm{i}\mathbf{k})P$, where D is the diagonal matrix with the eigenvalues on the main diagonal, $D_{j,l} = -\mathrm{i}\omega_{\mathbf{k},j}\delta_{j,l}$, and the matrix P is formed by N column vectors $\{\mathbf{u}_{\mathbf{k},1}, \ldots, \mathbf{u}_{\mathbf{k},N}\}$ of the eigenspace. It implies that any particular solution of the system (2.12) is purely exponential, $\mathbf{u}_{\mathbf{k}} \propto e^{-\mathrm{i}\omega t}$, without any nonzero powers of t multiplying these exponents.

Particular examples of such matrices are skew-Hermitian matrices (also called by antihermitian matrices), $L(\mathrm{i}\mathbf{k})^\dagger = -L(\mathrm{i}\mathbf{k})$, where $\dagger$ means the conjugate transpose (also called by Hermitian conjugate) of the matrix, i.e. $L(\mathrm{i}\mathbf{k})^\dagger := \overline{L(\mathrm{i}\mathbf{k})}^T$. All eigenvalues of skew-Hermitian matrices all purely imaginary which ensures that $\omega_{\mathbf{k},j}$ are real-valued for all $j = 1, \ldots, N$. For example, FT of equation (2.8) gives 1×1 skew-Hermitian matrix (purely imaginary scalar). However, FT of equations (2.2) and (2.5) are not skew-Hermitian matrices although diagonalizable. Thus for linear waves we cannot restrict $L(\mathrm{i}\mathbf{k})$ to the class of skew-Hermitian matrices but have to assume more general class of diagonalizable matrices $L(\mathrm{i}\mathbf{k})$ with all purely imaginary eigenvalues.

The general solution of ODE system (2.11) is given by the linear superposition

$$\mathbf{u}_{\mathbf{k}}(t) = \sum_{j=1}^{N} c_{\mathbf{k},j}\mathbf{u}_{\mathbf{k},j}e^{-\mathrm{i}\omega_{\mathbf{k},j}t} = \Phi_{\mathbf{k}}(t)\mathbf{c}_{\mathbf{k}} \tag{2.14}$$

of the arbitrary functions $c_{\mathbf{k},j}$ of $\mathbf{k}$. Here $\mathbf{c}_{\mathbf{k}} := (c_{\mathbf{k},1}, \ldots, c_{\mathbf{k},N})^T$ is the column vector and

$$\Phi_{\mathbf{k}}(t) := (\mathbf{u}_{\mathbf{k},1}e^{-\mathrm{i}\omega_{\mathbf{k},1}t}, \ldots, \mathbf{u}_{\mathbf{k},N}e^{-\mathrm{i}\omega_{\mathbf{k},N}t}) \tag{2.15}$$

is the fundamental matrix of ODE system (2.11) which consists of the column eigenvectors $\{\mathbf{u}_{\mathbf{k},1}e^{-\mathrm{i}\omega_{\mathbf{k},1}t}, \ldots, \mathbf{u}_{\mathbf{k},N}e^{-\mathrm{i}\omega_{\mathbf{k},N}t}\}$.

The arbitrary functions $c_{\mathbf{k},j}$ can be chosen to satisfy the initial condition $\mathbf{u}_{\mathbf{k}}(0)$ at $t = 0$ which results in

$$\mathbf{u}_{\mathbf{k}}(t) = \sum_{j=1}^{N} c_{\mathbf{k},j}\mathbf{u}_{\mathbf{k},j}e^{-\mathrm{i}\omega_{\mathbf{k},j}t} = \Phi_{\mathbf{k}}(t)\Phi_{\mathbf{k}}^{-1}(0)\mathbf{u}_{\mathbf{k}}(0). \tag{2.16}$$

IFT (2.10) of equation (2.16) together with the definition (2.15) provides the solution of the initial value problem $\mathbf{u}(\mathbf{r}, t)|_{t=0} := \mathbf{u}_0(\mathbf{r})$ in $\mathbf{r}$ space as follows

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{(2\pi)^{D/2}} \int\limits_{\mathbb{R}^D} \Phi_{\mathbf{k}}(t)\Phi_{\mathbf{k}}^{-1}(0)\mathbf{u}_{\mathbf{k}}(0)d\mathbf{k}, \tag{2.17}$$

where $\mathbf{u}_{\mathbf{k}}(0)$ is FT of $\mathbf{u}_0(\mathbf{r})$.

Instead of working with the full solution (2.17), below for clarity of presentation we address each branch $\omega_{\mathbf{k},j}$ of the dispersion relation separately, i.e. assume that the initial condition is $\mathbf{u}_{\mathbf{k}}(0) = c_{\mathbf{k},j}\mathbf{u}_{\mathbf{k},j}$ for some given j with the corresponding solution $\mathbf{u}_{\mathbf{k}}(t) = c_{\mathbf{k},j}\mathbf{u}_{\mathbf{k},j}e^{-\mathrm{i}\omega_{\mathbf{k},j}t}$. We are not loosing any generality because if needed we can recover the general solution by the superposition of separate solutions for all branches. Below we omit the index j, i.e $\omega_{\mathbf{k},j} \equiv \omega_{\mathbf{k}}$ as well as we work with a single component of $\mathbf{u}_{\mathbf{k}}(t)$ which we call by $u_{\mathbf{k}}(t) \in \mathbb{C}$ and the same for $\mathbf{u}(\mathbf{r},t)$ vs. $u(\mathbf{r},t)$. Again, we are not loosing any generality because we can recover $\mathbf{u}_{\mathbf{k}}(t)$ by combining all N components at any desired time.

Thus (2.17) is reduced to

$$u(\mathbf{r},t) = \frac{1}{(2\pi)^{D/2}} \int\limits_{\mathbb{R}^D} u_{\mathbf{k}}(0)e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}-\mathrm{i}\omega_{\mathbf{k}}t}d\mathbf{k}, \tag{2.18}$$

Equation (2.18) can be viewed as the expansion of the solution of equation (2.1) into the plane waves $e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}-\mathrm{i}\omega_{\mathbf{k}}t}$. Respectively, equation (2.17) is the superposition of such expansion taking into account all branches of the dispersion relation (2.13).

2.2. Propagation of wavepackets in linear continuous media

In this section without loss of generality we consider the solutions of equation (2.1) in the particular form (2.18). Depending on $u_{\mathbf{k}}(0)$ that integral can have quite complicated behaviour at different times. Reductions to *quasi-monochromatic wave* and *wavepackets* often drastically simplify the analysis of wave dynamics as well as such reductions provide a convenient framework to extend into nonlinear waves later in this chapter.

Quasi-monochromatic wave means a well localized distribution of waves in $\mathbf{k}$ space. Wavepacket is quasi-monochromatic wave which is also well localized $\mathbf{r}$ space (and sometimes in time).

We start from solutions which are localized only in $\mathbf{k}$ space. The simplest solution of equation (2.18) with extreme localization in $\mathbf{r}$ space is the *monochromatic wave* $u_{\mathbf{k}}(t) = (2\pi)^{D/2}u_0e^{-\mathrm{i}\omega(\mathbf{k}_0)t}\delta(\mathbf{k}-\mathbf{k}_0)$, which gives after IFT that

$$u(\mathbf{r},t) = u_0e^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}-\mathrm{i}\omega(\mathbf{k}_0)t}, \tag{2.19}$$

where $u_0 \in \mathbb{C}$ is the arbitrary constant. That solution is also called by the *plane wave* which propagates in the direction of the vector $\mathbf{k}_0$. The phase of (2.19) changes only along the direction of $\mathbf{k}_0$ while it remains constant in the plane perpendicular to $\mathbf{k}_0$ for each fixed t. A *phase speed*

$$v_{ph} := \frac{\omega(\mathbf{k}_0)}{|\mathbf{k}_0|} \tag{2.20}$$

is the velocity at which the phase $\mathbf{k}_0\cdot\mathbf{r}-\omega(\mathbf{k}_0)t$ propagates in the direction $\mathbf{k}_0$. In other words, if we choose the (perpendicular to $\mathbf{k}_0$) plane of the phase $\theta := \mathbf{k}_0\cdot\mathbf{r}-\omega(\mathbf{k}_0)t$, then θ remains constant as time t progresses if we move that plane with the velocity v_{ph} in the direction $\mathbf{k}_0$. We can also define the phase velocity as the vector $\mathbf{v}_{ph} := \frac{\omega(\mathbf{k}_0)}{|\mathbf{k}_0|^2}\mathbf{k}_0$.

We note that the solution (2.19) is intrinsically complex one. It means that it directly applies to the complex-valued field $u(\mathbf{r},t)$ such as in Example 2.1.0.3. If instead $u(\mathbf{r},t)$ is the real-valued as in Examples 2.1.0.1 and 2.1.0.2, then the monochromatic wave must include the wavevector $-\mathbf{k}_0$ so that equation (2.19) is replaced by

$$u(\mathbf{r},t) = u_0e^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}-\mathrm{i}\omega(\mathbf{k}_0)t} + \bar{u}_0e^{-\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}+\mathrm{i}\omega(\mathbf{k}_0)t}, \tag{2.21}$$

with the Fourier harmonic taking the form $u_{\mathbf{k}}(t) = (2\pi)^{D/2}[u_0 e^{-i\omega(\mathbf{k}_0)t}\delta(\mathbf{k}-\mathbf{k}_0) + \bar{u}_0 e^{i\omega(\mathbf{k}_0)t}\delta(\mathbf{k}+\mathbf{k}_0)]$. Thus $u(\mathbf{r},t) \in \mathbb{R}$ implies that all wave vectors show up in pairs $\mathbf{k}$ and $-\mathbf{k}$. The superposition principle indicates that for the linear systems (2.1) we can still use the complex form (2.19) for intermediate calculations but assume that for the final physical form we replace (2.19) by $2Re(u(\mathbf{r},t))$. However, the inclusion of nonlinear terms in subsequent chapters generally imply interaction between Fourier modes with $\mathbf{k}$ and $-\mathbf{k}$ with the lack of the superposition principle. It means that for the general nonlinear waves with real-valued $u(\mathbf{r},t)$ we have to use the representation (2.21). Below in this section we assume that $u(\mathbf{r},t)$ is complex valued with all results easily generalized to real-valued case by adding the complex conjugated terms as in equation (2.21).

Beyond the monochromatic wave (2.19), the another important particular case is the propagation of quasi-monochromatic wave centered in Fourier space around a wave vector $\mathbf{k}=\mathbf{k}_0$ such that the spectrum $|f_{\mathbf{k}}|$ is narrow being localized in the domain $|\mathbf{k}-\mathbf{k}_0| \lesssim \Delta k$ with $\Delta k/|\mathbf{k}_0| \ll 1$. Then one can consider the Taylor series expansion

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_0) + (\mathbf{k}-\mathbf{k}_0)\cdot\frac{\partial\omega(\mathbf{k}_0)}{\partial\mathbf{k}_0} + O(\Delta k^2), \tag{2.22}$$

where

$$\mathbf{v}_g := \frac{\partial\omega(\mathbf{k}_0)}{\partial\mathbf{k}_0} \tag{2.23}$$

is called the *group velocity.*

Taking into account only the linear term in equation (2.22), we obtain from equations (2.18) and (2.22) that

$$\begin{aligned} u(\mathbf{r},t) &\simeq \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} u_{\mathbf{k}}(0)\exp\left\{i\mathbf{k}\cdot\mathbf{r} - i\left[\omega(\mathbf{k}_0) + (\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{v}_g\right]t\right\}d\mathbf{k} \\ &= \frac{e^{i\mathbf{k}_0\cdot\mathbf{v}_g t - i\omega(\mathbf{k}_0)t}}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} u_{\mathbf{k}}(0)e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}_g t)}d\mathbf{k} = e^{i[\mathbf{k}_0\cdot\mathbf{v}_g - \omega(\mathbf{k}_0)]t}u_0(\mathbf{r}-\mathbf{v}_g t), \end{aligned} \tag{2.24}$$

which describes the propagation (translation) of the initial condition $u(\mathbf{r},t=0) = u_0(\mathbf{r})$,

$$u_0(\mathbf{r}) = \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} u_{\mathbf{k}}(0)e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}, \tag{2.25}$$

with the group velocity $\mathbf{v}_g$ without the change of the shape of the solution except of its multiplication by the complex time-dependent exponent $e^{i[\mathbf{k}_0\cdot\mathbf{v}_g - \omega(\mathbf{k}_0)]t}$.

The approximate solution (2.24) is valid only for moderate times $t \lesssim$??? while the term $O(\Delta k^2)$ can be neglected in the expansion (2.22). To find these times we add the next order term in the Taylor series expansion

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_0) + (\mathbf{k}-\mathbf{k}_0)\cdot\frac{\partial\omega(\mathbf{k}_0)}{\partial\mathbf{k}_0} + \sum_{j.l=1}^{D}\frac{1}{2}(k_j-k_{0,j})(k_l-k_{0,l})\omega_{jl} + O(\Delta k^3), \tag{2.26}$$

where

$$\omega_{jl} := \left.\frac{\partial^2\omega(\mathbf{k})}{\partial k_j\partial k_l}\right|_{\mathbf{k}=\mathbf{k}_0} \tag{2.27}$$

is the tensor of *group velocity dispersion* (GVD) which determines how group velocity $\mathbf{v}_g$ changes with the variation of the carrier wavevector $\mathbf{k}_0$.

Instead of working directly with the integral (2.18) and the expansion (2.26), it is often more convenient to replace (2.18) by the equivalent ODE

$$\frac{du_{\mathbf{k}}}{dt} = -\mathrm{i}\omega_{\mathbf{k}} u_{\mathbf{k}} \tag{2.28}$$

and use the expansion (2.26) there. Then one can use IFT to obtain PDE for $u(\mathbf{r}, t)$. For example, if we neglect GVD (2.26), i.e. replace $\omega(\mathbf{k})$ by equation (2.22), then IFT results in

$$\frac{\partial u}{\partial t} + \mathrm{i}[\omega(\mathbf{k}_0)u - \mathbf{v}_g \cdot \mathbf{k}_0]u + \mathbf{v}_g \cdot \nabla u = 0 \tag{2.29}$$

which has the exact solution given by r.h.s of equation (2.24).

The Taylor series (2.26) implies that the dependence on the wavevector $\mathbf{k}$ enters into all terms as $\mathbf{k} - \mathbf{k}_0$. Then that one can define

$$\boldsymbol{\kappa} := \mathbf{k} - \mathbf{k}_0 \tag{2.30}$$

and use $\boldsymbol{\kappa}$ as the independent variable. Then equation (2.26) transforms into

$$\omega(\mathbf{k}_0 + \boldsymbol{\kappa}) = \omega(\mathbf{k}_0) + \boldsymbol{\kappa} \cdot \mathbf{v}_g + \sum_{j.l=1}^{D} \frac{1}{2} \kappa_j \kappa_l \, \omega_{jl} + O(\Delta k^3). \tag{2.31}$$

We also define a new unknown

$$\psi_{\boldsymbol{\kappa}}(t) := u_{\mathbf{k}_0 + \boldsymbol{\kappa}}(t) \tag{2.32}$$

which is centered around $\boldsymbol{\kappa} = 0$ instead of centering at $\mathbf{k} = \mathbf{k}_0$ for $u_{\mathbf{k}}$.

IFT of $u_{\mathbf{k}}$ is

$$\begin{aligned} u(\mathbf{r}, t) = \frac{e^{i\mathbf{k_0}\mathbf{r}}}{(2\pi)^{D/2}} \int u_{\mathbf{k}}(t) e^{i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}} d\mathbf{k} =& \frac{e^{i\mathbf{k_0}\cdot\mathbf{r}}}{(2\pi)^{D/2}} \int u_{\mathbf{k}_0+\boldsymbol{\kappa}}(t) e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} d\boldsymbol{\kappa} \\ =& \frac{e^{i\mathbf{k_0}\cdot\mathbf{r}}}{(2\pi)^{D/2}} \int \psi_{\boldsymbol{\kappa}}(t) e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} d\boldsymbol{\kappa} \end{aligned} \tag{2.33}$$

which suggests to define $\psi(\mathbf{r}, t)$ in $\mathbf{r}$ space as IFT of $\psi_{\boldsymbol{\kappa}}(t)$, i.e.

$$\psi(\mathbf{r}, t) := \frac{1}{(2\pi)^{D/2}} \int \psi_{\boldsymbol{\kappa}}(t) e^{\mathrm{i}\boldsymbol{\kappa}\cdot\mathbf{r}} d\boldsymbol{\kappa}. \tag{2.34}$$

Equations (2.33) and (2.34) result in the relation

$$u(\mathbf{r}, t) = e^{i\mathbf{k_0}\cdot\mathbf{r}} \psi(\mathbf{r}, t) \tag{2.35}$$

implying that the fast spatial dependence of $u(\mathbf{r}, t)$ is included into the factor $e^{i\mathbf{k_0}\cdot\mathbf{r}}$ while $\psi(\mathbf{r}, t)$ is the slow varying function in $\mathbf{r}$ called by the *spatial envelope* of $u(\mathbf{r}, t)$ of simply the envelope, see Figure ??? for the schematic representation of the envelope for each given time t.

Equation (2.28) turns into

$$\frac{d\psi_{\boldsymbol{\kappa}}}{dt} = -\mathrm{i}\omega_{\mathbf{k}_0+\boldsymbol{\kappa}} \psi_{\boldsymbol{\kappa}}. \tag{2.36}$$

Using equation (2.31) and applying IFT (2.34) to equation (2.36) result in PDE for the envelope $\psi(\mathbf{r}, t)$ as follows

$$\frac{\partial \psi}{\partial t} + \mathrm{i}\omega_{\mathbf{k}_0}\psi + \mathbf{v}_g \cdot \nabla\psi - \frac{i}{2}\sum_{j,l=1}^{D} \omega_{jl}\frac{\partial^2 \psi}{\partial x_j \partial x_l} + \ldots = 0, \tag{2.37}$$

where $\ldots$ means higher order terms with three and more derivatives of $\mathbf{r}$ from $O(\Delta k^3)$ terms in equation (2.31). Here we used that IFT results in replacing $\kappa \to -\mathrm{i}\nabla$.

One can define the *spatial-temporal envelope* $\psi_0(\mathbf{r}, t)$ as

$$u(\mathbf{r}, t) = e^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{r} - \mathrm{i}\omega_{\mathbf{k}_0} t}\psi_0(\mathbf{r}, t) \tag{2.38}$$

which removes the leading order term $\mathrm{i}\omega_{\mathbf{k}_0}\psi$ from equation (2.39) resulting in

$$\frac{\partial \psi_0}{\partial t} + \mathbf{v}_g \cdot \nabla\psi_0 - \frac{\mathrm{i}}{2}\sum_{j,l=1}^{D} \omega_{jl}\frac{\partial^2 \psi_0}{\partial x_j \partial x_l} + \ldots = 0. \tag{2.39}$$

Thus $\psi_0(\mathbf{r}, t)$ is the slow function of both space and time.

The term with $\mathbf{v}_g$ is generally dominant over GVD term in equation in accordance with the Taylor series (2.31) resulting in the translation of the solution with the group velocity $\mathbf{v}_g$ as in equation (2.24). However, a Galilean transformation

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_g t$$
$$t' = t,$$

removes such translation by switching to the frame of reference moving with the velocity $\mathbf{v}_g$. Then the partial derivatives of equation (2.39) are transformed to

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \mathbf{v}_g\frac{\partial}{\partial \mathbf{r}'}, \quad \frac{\partial}{\partial \mathbf{r}} = \frac{\partial}{\partial \mathbf{r}'},$$

resulting in

$$\mathrm{i}\frac{\partial \psi_0}{\partial t} + \frac{1}{2}\sum_{j,l=1}^{D} \omega_{jl}\frac{\partial^2 \psi_0}{\partial x'_j \partial x'_l} = 0, \tag{2.40}$$

where we neglected the smaller $\ldots$ terms.

The definition (2.27) implies that $\boldsymbol{\omega} := (\omega_{jl})$ is the symmetric matrix, $\omega_{jl} = \omega_{lj},\ j, l = 1, \ldots, D$. Then $\boldsymbol{\omega}$ is diagonalizable with all real eigenvalues, i.e. it exists the orthogonal matrix $U = (U^T)^{-1}$ (column vectors of U are eigenvectors of $\boldsymbol{\omega}$ which form the orthonormal basis in $\mathbb{R}^D$) such that

$$U^{-1}\boldsymbol{\omega}U = U^T\boldsymbol{\omega}U = \begin{pmatrix} \lambda_1 & 0 & \ldots \\ & \ddots & \\ 0 & \ldots & \lambda_D \end{pmatrix} \tag{2.41}$$

is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ being the real eigenvalues of $\boldsymbol{\omega}$.

Choosing a new spatial variable

$$\mathbf{y} := U^T\mathbf{x}' \tag{2.42}$$

and using equation (2.41) we obtain that

$$\sum_{j,l=1}^{D}\omega_{jl}\frac{\partial^2}{\partial x'_j\partial x'_l}=\sum_{j,j=1}^{D}\sum_{m,n=1}^{D}\omega_{jl}\frac{\partial y_n}{\partial x'_l}\frac{\partial y_m}{\partial x'_j}\frac{\partial^2}{\partial y_n\partial y_m}=\sum_{j=1}^{D}\lambda_j\frac{\partial^2}{\partial y_j^2}. \tag{2.43}$$

Equations (2.40), (2.42) and (2.43) result in

$$\mathrm{i}\frac{\partial\psi_0}{\partial t}+\sum_{j=1}^{D}\lambda_j\frac{\partial^2\psi_0}{\partial y_j^2}=0. \tag{2.44}$$

If all λ_j have the same sign, then rescaling y_j (and complex conjugation of equation (2.44) for all negative λ_j) shows that equation (2.44) is equivalent to the linear Schrödinger equation (2.7).

The dispersion law $\omega(\mathbf{k})$ in the isotropic media can only depend on $k:=|\mathbf{k}|$, i.e. $\omega(\mathbf{k})=\omega(k)$, where we abused notation and used the same letter ω for functions of both $\mathbf{k}$ and its absolute value k. Using the relation $\frac{\partial k}{\partial\mathbf{k}}=\frac{\mathbf{k}}{k}$, we obtain that the GVD tensor (2.27) is transformed into

$$\omega_{jl}=\frac{k_jk_l}{k^2}\omega''(k)+\frac{v_g}{k}\left(\delta_{jl}-\frac{k_jk_l}{k^2}\right)\Bigg|_{\mathbf{k}=\mathbf{k}_0}, \tag{2.45}$$

where $\omega''(k)=\frac{d^2\omega(k)}{dk^2}$ and $v_g=\omega'(k)=\frac{d\omega(k)}{dk}=|\mathbf{v}_g|$.

Without loss of generality we can assume that $\mathbf{k}_0$ is pointed in the direction of x_D axis. Then equations (2.40) and (2.45) result

$$\mathrm{i}\frac{\partial\psi_0}{\partial t}+\frac{v_g}{2k_0}\nabla_\perp^2\psi_0+\frac{\omega''(k_0)}{2}\frac{\partial^2\psi_0}{\partial x_D^2}=0, \tag{2.46}$$

where

$$\nabla_\perp^2=\frac{\partial^2}{x_1^2}+\ldots+\frac{\partial^2}{x_{D-1}^2} \tag{2.47}$$

is the transverse Laplacian (i.e. the part of the full Laplacian ∇^2 which is transverse to the direction of x_D axis). Thus the equation (2.40) in the isotropic media, i.e. equation (2.46), has the diagonal form (2.44) without the need of any additional coordinate transform (2.42). If $v_g\omega''(k_0)>0$, then then a rescaling the coordinates x_j (and complex conjugation of equation (2.46) for $v_g<0$) shows that equation (2.46) is equivalent to the linear Schrödinger equation (2.7).

The diagonal form (2.44) is convenient to solve the Cauchy problem $\psi_0(\mathbf{y},t)|_{t=0}=\psi_{0,ini}(\mathbf{y})$ for the initial condition $\psi_{0,ini}(\mathbf{y})$ by FT (2.9) which gives that

$$\mathrm{i}\frac{d\psi_{0,\boldsymbol{\kappa}}}{dt}-\psi_{0,\boldsymbol{\kappa}}\sum_{j=1}^{D}\lambda_j\kappa_j^2=0, \tag{2.48}$$

i.e.

$$\psi_{0,\boldsymbol{\kappa}}(t)=\psi_{0,\boldsymbol{\kappa}}(0)\exp\left[-\mathrm{i}t\sum_{j=1}^{D}\lambda_j\kappa_j^2\right]. \tag{2.49}$$

IFT of equation (2.49) provides the solution for $\psi_0(\mathbf{y},t)$ as follows

$$\psi_0(\mathbf{y},t) = \frac{1}{(2\pi)^{D/2}} \int \psi_{0,\boldsymbol{\kappa}}(0) \exp\left[-\mathrm{i}t\sum_{j=1}^{D}\lambda_j\kappa_j^2\right] e^{i\boldsymbol{\kappa}\cdot\mathbf{y}}d\boldsymbol{\kappa}. \tag{2.50}$$

2.2.1. Gaussian linear waves. Assume that IC $\psi_0(\mathbf{y},0)$ has the Gaussian form

$$\psi_0(\mathbf{y},0) = A\exp\left[\sum_{j=1}^{D}\left(-\frac{1}{2}a_jy_j^2 + b_jy_j\right)\right], \tag{2.51}$$

where $A, a_j, b_j, j = 1,\ldots,D$ are the complex constants with the condition $Re(a_j) > 0$ to ensure a decay of $\psi_0(\mathbf{y},0)$ at $|\mathbf{y}| \to 0$. Using equations (2.34), (2.51) and (A.502) we obtain that

$$\psi_{0,\boldsymbol{\kappa}}(0) = \frac{A}{(a_1\cdot\ldots\cdot a_D)^{1/2}}\exp\left[-\sum_{j=1}^{D}\frac{(\kappa_j+\mathrm{i}b_j)^2}{2a_j}\right], \tag{2.52}$$

which has the Gaussian form in $\boldsymbol{\kappa}$. Equations (2.50), (2.52) and (A.502) result in

$$\begin{aligned}\psi_0(\mathbf{y},t) &= \frac{1}{(2\pi)^{D/2}} \int \psi_{0,\boldsymbol{\kappa}}(0) \exp\left[-\mathrm{i}t\sum_{j=1}^{D}\lambda_j\kappa_j^2\right] e^{i\boldsymbol{\kappa}\cdot\mathbf{y}}d\boldsymbol{\kappa}\\ &= \frac{A}{[(1+2\mathrm{i}a_1\lambda_1t)\cdot\ldots\cdot(1+2\mathrm{i}a_D\lambda_Dt)]^{1/2}}\\ &\quad\times\exp\left[\sum_{j=1}^{D}\left(-\frac{a_jy^2}{2(1+2\mathrm{i}a_j\lambda_jt)}+\frac{b_jy}{1+2\mathrm{i}a_j\lambda_jt}+\frac{\mathrm{i}b_j^2\lambda_jt}{1+2\mathrm{i}a_j\lambda_jt}\right)\right]\\ &= \frac{A}{[(1+2\mathrm{i}a_1\lambda_1t)\cdot\ldots\cdot(1+2\mathrm{i}a_D\lambda_Dt)]^{1/2}}\exp\left[\sum_{j=1}^{D}\left(-\frac{a_j(y-b_j/a_j)^2}{2(1+2\mathrm{i}a_j\lambda_jt)}+\frac{b_j^2}{2a_j}\right)\right].\end{aligned} \tag{2.53}$$

It is seen from Eq. (2.53) that the Gaussian initial condition (2.51) results in the Gaussian solution with time-dependent coefficients which is the *Gaussian linear wave.*

Problems 2.2

2.2.1.1 Show the validity of equation (2.45) for $\omega(\mathbf{k}) = \omega(k)$, i.e. when the dispersion depends on k only which corresponds to isotropic media.

2.3. Schrödinger equation in linear optics

The propagation of the monochromatic laser beam in the paraxial approximation in the linear homogenous and transparent media is described by the linear Schrodinger equation,

$$\frac{\partial}{\partial z}\mathcal{E} = \frac{\mathrm{i}}{2k_0}\nabla_\perp^2\mathcal{E}, \tag{2.54}$$

where $\mathcal{E}(\mathbf{r}_\perp, z)$ is the complex envelope of the laser beam, z is the direction of the laser beam propagation and $\nabla_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is the transverse gradient which corresponds to the transverse coordinates $\mathbf{r}_\perp := (x, y)$. Here k_0 is the wavenumber of the laser. The electric field $\mathbf{E}$ of the laser beam is recovered from $\mathcal{E}(\mathbf{r}_\perp, z)$ as follows

$$\mathbf{E} = \frac{\mathbf{n}}{2}\left[\mathcal{E}e^{ik_0 z - i\omega t} + c.c.,\right] \tag{2.55}$$

where we assume the linear polarization of the wave[1] in the direction determined by the unit vector $\mathbf{n}$ which is perpendicular to the direction of z. See Section 2.5 for the detailed derivation of Eq. (2.54) from the Maxwell's equations.

Eq. (2.54) is corresponds to Eq. (2.44) with $D = 3$ and t replaced by z as well as $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2k_0}$. Then the Gaussian linear wave (2.53) provides the exact solution of Eq. (2.54). This solution in optics is called by the *Gaussian beam* and is traditionally written in the following form

$$\mathcal{E}(\mathbf{r}_\perp, z) = \mathcal{E}_0 \frac{w_0}{w(z)} \exp\left(-\frac{x^2+y^2}{w^2(z)} - \mathrm{i}\zeta(z) + \mathrm{i}\frac{k_0(x^2+y^2)}{2R(z)}\right), \tag{2.56}$$

where w_0 is the Gaussian beam waist, $w(z)$ is the spot size, $R(z)$ is the radius of curvature, and $\zeta(z)$ is the Guoy phase shift and $\mathcal{E}_0$ is amplitude of the wave with

$$\begin{aligned}
w^2(z) &= w_0^2\left(1 + \frac{z^2}{z_R^2}\right), \\
R(z) &= z\left(1 + \frac{z_R^2}{z^2}\right), \\
\zeta(z) &= \arctan\left(\frac{z}{z_R}\right), \\
z_R &= \frac{k_0 w_0^2}{2}.
\end{aligned} \tag{2.57}$$

Here z_R is the Rayleigh length. It is assumed in Eq. (2.56) without loss of generality that the laser beam has the zero curvature of the wave front, $R(0) = \infty$, at $z = 0$. The the Rayleigh length determines the distance at which the spot size $w(z)$ (also called by the beam width) is increased by a factor $\sqrt{2}$ as follows from Eq. (2.57). See also Fig. ??? for the schematic of the Gaussian beam propagation.

A general non-Gaussian beam can be expanded into *Hermite-Gaussian modes* $u_{n,m}(x, y, z)$ as follows

$$\begin{aligned}
\mathcal{E}(x,y,z) = \sum_{n=0,\, m=0}^{\infty} a_{n,m} u_{n,m}(x,y,z) \equiv \sum_{n=0,\, m=0}^{\infty} a_{n,m} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{1}{2^{n+m} n! m! w_x(z) w_y(z)}\right)^{1/2} \\
\times H_n\left(\frac{2^{1/2}x}{w_x(z)}\right) H_m\left(\frac{2^{1/2}y}{w_y(z)}\right) \\
\times \exp\left(-\frac{x^2}{w_x^2(z)} - \frac{y^2}{w_y^2(z)} - \mathrm{i}\frac{(2n+1)\zeta_x(z) + (2m+1)\zeta_y(z)}{2}\right. \\
\left. + \mathrm{i}\frac{k_0 x^2}{2R_x(z)} + \mathrm{i}\frac{k_0 y^2}{2R_y(z)}\right),
\end{aligned} \tag{2.58}$$

[1] See e.g. Ref. [**LL84**] for more discussion on polarizations of electromagnetic waves.

where $a_{n,m}$ are the expansion coefficients and

$$
\begin{aligned}
w^2_{x(y)}(z) &= w^2_{0,x(y)}\left(1+\frac{z^2}{z^2_{R,x(y)}}\right),\\
R_{x(y)}(z) &= z\left(1+\frac{z^2_{R,x(y)}}{z^2}\right),\\
\zeta_{x(y)}(z) &= \arctan\left(\frac{z}{z_{R,x(y)}}\right),\\
z_{R,x(y)} &= \frac{k_0 w^2_{0,x(y)}}{2}
\end{aligned}
\tag{2.59}
$$

which are the generalizations of Eqs. (2.57) to have separate expressions in x and y directions designated by the corresponding subscripts. It is seen from Eq. (2.58) that the functions $w_x(z)$ and $w_y(z)$ defined in Eq. (2.59) play the role of the parameters of the expansion (2.58). Also $H_n(x),\ n=0,1,2\ldots$ in Eq. (2.58) are the Hermite polynomials [**AS72**] which are the orthogonal polynomial with the following recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{2.60}$$

and $H_0(x)=1$. Using this recurrent relation we immediately obtain that $H_1(x) = 2x$, $H_2(x)=4x^2-2$, $H_3(x)=8x^3-12x$, ...

The validity of equation (2.58) as the exact solution of equation (2.54) is immediately verified by the direction substitution of equation (2.58) into equation (2.54) if we take into account that the Hermite polynomials satisfy the Hermite differential equation

$$\frac{d^2H_n(x)}{dx^2} - 2x\frac{dH_n(x)}{dx} + 2nH_n(x) = 0,\ n=0,1,2,\ldots \tag{2.61}$$

The orthogonality condition

$$\int\limits_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = \pi^{1/2}2^n n!\delta_{n,m} \tag{2.62}$$

of the Hermite polynomials ensures that

$$\iint_{-\infty}^{\infty} u^*_{n,m}(x,y,z)u_{n',m'}(x,y,z)|_{z=0}dxdy = \delta_{n,n'}\delta_{m,m'} \tag{2.63}$$

which implies that the expansion coefficients $a_{n,m}$ can be explicitly calculated from the $\mathcal{E}(x,y,z)$ at any fixed z as follows

$$a_{n,m} = \iint_{-\infty}^{\infty}\mathcal{E}(x,y,z)u^*_{n,m}(x,y,z)dxdy. \tag{2.64}$$

It follows from equations (2.58) and (2.63) that the power P of the laser beam is given by

$$P = \iint_{-\infty}^{\infty} |\mathcal{E}(x,y,x)|^2 dxdy$$
$$= \sum_{n=0,\, m=0}^{\infty} |a_{n,m}|^2 \iint_{-\infty}^{\infty} |u_{n,m}(x,y,z)|^2 dxdy$$
$$= \sum_{n=0,\, m=0}^{\infty} |a_{n,m}|^2. \tag{2.65}$$

The Hermite-Gaussian expansion (2.58) represent an alternative to the expansion of the solution in Fourier modes in Section 2.2. The Hermite-Gaussian expansion is especially useful in optics because the laser cavities often produce the laser beams with only few such modes, see e.g. Refs. ??? Also ideal lenses and mirrors typically convert one Gaussian beam to another Gaussian beam. There are many other basis used in expansion of laser beams such as Laguerre-Gaussian modes, Ince-Gaussian modes and Hypergeometric-Gaussian modes, see e.g. Refs??? The usefulness of all these expansions in nonlinear optics is somewhat diminished because usually they do not represent exact solutions if nonlinearity is taken into account beyond the linear equation (2.54).

2.4. Nonlinear sound waves in compressible hydrodynamics and canonical variables

Consider the motion of ideal (inviscid) compressible fluid (gas) with the density ρ, the velocity $\mathbf{v}$ and the pressure p. These variables are defined at each spatial point in $\mathbf{r} \in \mathbb{R}^D$ and time t. Assume that fluid is *barotropic* meaning that its pressure p has an explicit dependence on the density only,

$$p = p(\rho). \tag{2.66}$$

The conservation of the mass of fluid implies the first Euler equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{2.67}$$

and the dynamics of velocity is given by the second Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p. \tag{2.68}$$

Here $\nabla := \partial_{\mathbf{r}} = (\partial_{r_1}, \ldots, \partial_{r_D})$ is the spatial gradient and $\operatorname{div}\mathbf{f} := \nabla\cdot\mathbf{f}$ is the divergence of the vector field $\mathbf{f}$.

A particular case of Euler equations (2.67) and (2.68) is the *potential flow* when one can fully describe the velocity as the gradient of the *velocity potential* ϕ,

$$\mathbf{v} = \nabla\phi. \tag{2.69}$$

More general non-potential flows are considered in Chapter 3.

Using a vector identity

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = \nabla\frac{\mathbf{v}^2}{2} - [\mathbf{v}\times \mathtt{curl}\,\mathbf{v}],$$

where $\texttt{curl}\,\mathbf{f} := \nabla \times \mathbf{f}$ is the curl operator, we obtain that

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = \nabla\frac{\mathbf{v}^2}{2} \quad \text{for} \quad \mathbf{v} = \nabla\phi \tag{2.70}$$

because $\texttt{curl}\ \mathbf{v} = \texttt{curl}\,\nabla\phi = 0$.

Using (2.73) we obtain from equations (2.68) and (2.69) that

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{(\nabla\phi)^2}{2} + w(\rho)\right] = 0, \tag{2.71}$$

where $w(\rho)$ is called the *enthalpy* and defined by $\nabla w(\rho) := \rho^{-1}\nabla p(\rho)$, i.e.

$$w(\rho) = \int \rho^{-1}p'(\rho)d\rho. \tag{2.72}$$

Integrating equation (2.71) over $\mathbf{r}$ we obtain that

$$\frac{\partial\phi}{\partial t} + \frac{(\nabla\phi)^2}{2} + w(\rho) = f(t), \tag{2.73}$$

where $f(t)$ is the arbitrary function of velocity. One can shift ϕ by an arbitrary function of time, $\phi \to \phi + g(t)$ without changing the equation of motion because of definition of the velocity through ϕ in (2.69). Thus one can always set $f(t) = 0$ in (2.71) which results in the *unsteady Bernoulli equation*

$$\frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + w(\rho) = 0. \tag{2.74}$$

Coupling (2.74) with the continuity equation (2.67) for the potential flow,

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\nabla\phi) = 0, \tag{2.75}$$

results in the closed system of equations for the density ρ the velocity potential ϕ.

Generally there is no systematic way to find the canonical Hamiltonian variables for the continuous media.

Consider a total energy of the fluid,

$$H = \int\left[\frac{\rho\,(\nabla\phi)^2}{2} + \varepsilon(\rho)\right]d\mathbf{r}, \tag{2.76}$$

where integral is taken over $\mathbb{R}^D$, the first term gives the kinetic energy of the fluid, and the second represents the internal fluid energy. Also $\varepsilon(\rho)$ is the internal energy density per unit volume which is related to the enthalpy w as follows

$$\frac{\partial\varepsilon(\rho)}{\partial\rho} = w(\rho). \tag{2.77}$$

We aim to represent equations (2.74) and (3.228) in the canonical Hamiltonian form for continuous media

$$\begin{aligned} \frac{\partial q}{\partial t} &= \frac{\delta H}{\delta p}, \\ \frac{\partial p}{\partial t} &= -\frac{\delta H}{\delta q}, \end{aligned} \tag{2.78}$$

where the canonical coordinate q and the canonical momentum p are continuous function of $\mathbf{r}$: $q = q(\mathbf{r}, t)$ and $p = p(\mathbf{r}, t)$.

Consider a variation of the Hamiltonian H (2.76) under a variation of ϕ (i.e. assuming $\delta\rho = 0$ and varying ϕ only) which gives

$$\delta H|_{\delta\rho=0} = \int \nabla\phi \cdot \nabla\delta\phi \, d\mathbf{r}.$$

Using Green's identity to integrate by parts one obtains that

$$\delta H = -\int \operatorname{div}(\rho\nabla\phi)\,\delta\phi(r)d\mathbf{r}, \tag{2.79}$$

where we assumed that $\phi \to 0$ as $|\mathbf{r}| \to 0$ (if $\phi \to \phi_0 = const$ as $|\mathbf{r}| \to 0$, one can replace $\phi \to \phi + \phi_0$ which ensure $\phi \to 0$ as $|\mathbf{r}| \to 0$ without changing the equations of motion because of equation (2.69)).

It is discussed in Appendix ??? that a variational derivative for a continuous function $f(\mathbf{r})$ is given by

$$\frac{\delta f(\mathbf{r}_1)}{\delta f(\mathbf{r}_2)} = \delta\left(\mathbf{r}_1 - \mathbf{r}_2\right). \tag{2.80}$$

Together with equation (2.79) it results in

$$\frac{\delta H}{\delta\phi} = -\operatorname{div}(\rho\nabla\phi). \tag{2.81}$$

A variation of the Hamiltonian H (2.76) under a variation of $\delta\rho$ (assuming $\delta\phi = 0$) gives

$$\delta H|_{\delta\phi=0} = \int \left[\frac{(\nabla\phi)^2}{2} + w(\rho)\right]\delta\rho \, d\mathbf{r}.$$

Together with equation (2.80) it implies that

$$\frac{\delta H}{\delta\phi} = \frac{(\nabla\Phi)^2}{2} + w(\rho). \tag{2.82}$$

Equations (2.81) and (2.82) allow to write the Euler equations (2.74) and (3.228) in the canonical Hamiltonian form (2.78) as follows

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= \frac{\delta H}{\delta\phi}, \\ \frac{\partial\phi}{\partial t} &= -\frac{\delta H}{\delta\rho}, \end{aligned} \tag{2.83}$$

where ρ is the generalized coordinate and ϕ is the generalized momentum.

This result can also be obtained from a constrained Lagrangian. To do that we generalize the Lagrangian (1.56) of finite-dimensional mechanical system to the continuous medium taking it as the difference between the kinetic energy $\int \frac{\rho\mathbf{v}^2}{2}d\mathbf{r}$ and the potential energy $\int \varepsilon(\rho)d\mathbf{r}$ with an added constraint to satisfy the continuity equation (3.228). Assume that ϕ is the Lagrangian multiplier of the constraint (3.228). Then the action S is given by

$$S = \int L dt = \int \left\{\frac{\rho\mathbf{v}^2}{2} - \varepsilon(\rho) + \phi\left[\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v})\right]\right\} d\mathbf{r}dt, \tag{2.84}$$

where we omitted limits of integration in time between t_1 and t_2. A variation of the action (2.85) with respect to the variable $\mathbf{v}$ results in the condition for the potential flow, $\mathbf{v} = \nabla\phi$, and variations with respect to the variables ρ and ϕ lead to Eqs. (2.74), (3.228), respectively. Here we used equation (2.77) and while integrating by parts in t for the variation in ρ we applied the constraints $\delta\rho(\mathbf{r}, t_1) = \delta\rho(\mathbf{r}, t_2) = 0$, which are similar to the constraints (1.20). A transition to the Hamiltonian is accomplished by the Légendre transformation (1.52) generalized to the continuous medium which gives

$$H = \int \phi \frac{\partial \rho}{\partial t} d\mathbf{r} - L \tag{2.85}$$

and results in the Hamiltonian (2.76).

Assume that $\rho(\mathbf{r}, t) := \rho_0 + \rho_1(\mathbf{r}, t)$, where ρ_1 is the fluctuation of density above the average (background) value ρ_0 such that $\rho \to \rho_0$ and $\mathbf{v} \to 0$ as $|\mathbf{r}| \to \infty$. We perform a series expansion of the Hamiltonian (2.76) in powers of the canonical variables ϕ and ρ_1. A Taylor series expansion of the internal energy density $\varepsilon(\rho)$ gives

$$\varepsilon(\rho) = \varepsilon(\rho_0) + \rho_1 \varepsilon_\rho(\rho_0) + \frac{\rho_1^2}{2!}\varepsilon_{\rho\rho}(\rho_0) + \frac{\rho_1^3}{3!}\varepsilon_{\rho\rho\rho}(\rho_0) + \ldots \tag{2.86}$$

The first term in r.h.s. of (2.86) is set to zero to make sure that the Hamiltonian (2.76) takes a finite value (i.e. we take into account only fluctuations of the internal energy about solution with $\rho \equiv \rho_0$. The second term in r.h.s. of (2.86) gives zero contribution to the Hamiltonian because of the conservation of mass $\int \rho_1 d\mathbf{r} \equiv 0$. Thus the first nontrivial contribution to the power series expansion of the Hamiltonian appears in the quadratic terms

$$H_2 = \int \left[\frac{1}{2}\rho_0 \left(\nabla\phi\right)^2 + c_s^2 \frac{\rho_1^2}{2\rho_0} \right] d\mathbf{r}, \tag{2.87}$$

where

$$c_s := [\rho_0 \varepsilon_{\rho\rho}(\rho_0)]^{1/2} = [p_\rho(\rho_0)]^{1/2} \tag{2.88}$$

is the *speed of sound* in compressible fluid. To understand that c_s is indeed the speed of sound, one uses the canonical Hamiltonian equations (2.83) with $H = H_2$ which gives

$$\begin{aligned} &\frac{\partial}{\partial t}\rho_1 + \rho_0 \nabla^2 \phi = 0, \\ &\frac{\partial \phi}{\partial t} + \frac{c_s^2}{\rho_0}\rho_1 = 0. \end{aligned} \tag{2.89}$$

Excluding ϕ one obtains the linear wave equation

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \nabla^2 \rho_1. \tag{2.90}$$

We define a spatial Fourier transform (FT) of the canonical variables,

$$\rho_{\mathbf{k}}(t) = \frac{1}{(2\pi)^{D/2}} \int \rho_1(\mathbf{r}, t) e^{-i\mathbf{k}\mathbf{r}} d\mathbf{k}, \tag{2.91}$$

$$\phi_{\mathbf{k}}(t) = \frac{1}{(2\pi)^{D/2}} \int \phi(\mathbf{r}, t) e^{-i\mathbf{k}\mathbf{r}} d\mathbf{k}, \tag{2.92}$$

which satisfies the equations

$$\rho_{\mathbf{k}} = \bar{\rho}_{-\mathbf{k}}, \quad \phi_{\mathbf{k}} = \bar{\phi}_{-\mathbf{k}}, \tag{2.93}$$

which result from the condition that both $\rho(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$ are real-valued functions. Plugging Inverse FT,

$$\begin{aligned}\rho_1(\mathbf{r},t) &= \frac{1}{(2\pi)^{D/2}}\int \rho_{\mathbf{k}}(t)e^{i\mathbf{k}\mathbf{r}}d\mathbf{k},\\ \phi(\mathbf{r},t) &= \frac{1}{(2\pi)^{D/2}}\int \phi_{\mathbf{k}}(t)e^{i\mathbf{k}\mathbf{r}}d\mathbf{k},\end{aligned} \tag{2.94}$$

into the quadratic Hamiltonian H_2 (2.87) and using Parseval's identity result in

$$H_2 = \int \left\{\frac{\rho_0 k^2}{2}|\phi_{\mathbf{k}}|^2 + \frac{c_s^2}{2\rho_0}|\rho_{\mathbf{k}}|^2\right\} d\mathbf{k}. \tag{2.95}$$

Note that this result is immediately obtained if one plugs in the expressions (A.482) into H_2 (2.87) and perform integration over $\mathbf{r}$. That integration results in Dirac delta function (distribution) $\delta(\mathbf{k})$ according to the identity

$$\int e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r} = (2\pi)^D\delta(\mathbf{k}), \quad \mathbf{r},\ \mathbf{k}\in\mathbb{R}^D, \tag{2.96}$$

see Appendix 2.116 for more details on Dirac delta function of Fourier transforms.

The canonical Hamiltonian equations (2.83) after FT take the canonical form

$$\begin{aligned}\frac{\partial\rho_{\mathbf{k}}}{\partial t} &= \frac{\delta H}{\delta\bar{\phi}_{\mathbf{k}}},\\ \frac{\partial\phi_{\mathbf{k}}}{\partial t} &= -\frac{\delta H}{\delta\bar{\rho}_{\mathbf{k}}},\end{aligned} \tag{2.97}$$

for the generalized coordinates $\rho_{\mathbf{k}}$ and the generalized momenta $\phi_{\mathbf{k}}$. FT (2.91) is the continuous analog of DFT of Section 1.10.0.2. It is also necessary to remember that the conditions (2.91) ensure that only half of the total number of Fourier modes are independent.

The Hamiltonian equations (2.97) with the Hamiltonian $H = H_2$ (2.95) result in the dynamical equations

$$\begin{aligned}&\frac{\partial\rho_{\mathbf{k}}}{\partial t} - k^2\rho_0\phi_{\mathbf{k}} = 0,\\ &\frac{\partial\phi_{\mathbf{k}}}{\partial t} + \frac{c_s^2}{\rho_0}\rho_{\mathbf{k}} = 0,\end{aligned} \tag{2.98}$$

where in calculating variational derivative of (2.97) we took into account that both $\bar{\rho}_{\mathbf{k}} = \rho_{-\mathbf{k}}$ and $\bar{\phi}_{\mathbf{k}} = \phi_{-\mathbf{k}}$ appear twice in the integral for each $\mathbf{k}$ because $|\rho_{\mathbf{k}}|^2 = |\rho_{-\mathbf{k}}|^2$ and $|\phi_{\mathbf{k}}|^2 = |\phi_{-\mathbf{k}}|^2$.

The dynamics of each pair of the canonical variables $\rho_{\mathbf{k}}$ and $\phi_{\mathbf{k}}$ in equations (2.98) represents a harmonic oscillator with a frequency

$$\omega_k = kc_s, \quad k := |\mathbf{k}|, \tag{2.99}$$

which is called by a *dispersion law* of the sound waves. That dispersion law is convenient to obtain from the system (2.98) by looking at the solution in the form $\rho_{\mathbf{k}}, \phi_{\mathbf{k}} \propto e^{-i\omega_k t}$ which results in a linear homogeneous system with constant coefficients

$$\begin{aligned}&-i\omega_k\rho_{\mathbf{k}} - \rho_0 k^2\phi_{\mathbf{k}} = 0,\\ &-i\omega_k\phi_{\mathbf{k}} + \frac{c_s^2}{\rho_0}\rho_{\mathbf{k}} = 0,\end{aligned}$$

for unknowns $\rho_{\mathbf{k}}$ and $\phi_{\mathbf{k}}$. That system, rewritten in a matrix form

$$\mathbf{A}\begin{pmatrix}\rho_{\mathbf{k}}\\ \phi_{\mathbf{k}}\end{pmatrix}=0, \quad \mathbf{A}=\begin{pmatrix}-i\omega_k & -\rho_0 k^2\\ \frac{c_s^2}{\rho_0} & -i\omega_k\end{pmatrix},$$

has a nontrivial solution (also called by a compatibility condition) provided the determinant $\det \mathbf{A}=0$, which results in the dispersion law (2.99). *Remark 1:* The same result (2.99) is immediately obtained by calculating ω_k^2 as the product of the coefficients multiplying $\frac{|\phi_{\mathbf{k}}|^2}{2}$ and $\frac{|\rho_{\mathbf{k}}|^2}{2}$ in the integrand of the quadratic Hamiltonian (2.95).

The oscillators in (2.98) with different $\mathbf{k}$ are decoupled thus representing the continuous analog of normal modes of Section 1.10. The interaction of different Fourier modes occurs if we take into account a next order of nonlinearity (cubic nonlinearity) from the Hamiltonian (2.76) beyond the quadratic Hamiltonian H_2 (2.87) which gives the interaction Hamiltonian H_{int} as follows

$$H_{int}=H_3=\int\left[\frac{1}{2}\rho_1\left(\nabla\phi\right)^2+c_s^2 q\frac{\rho_1^3}{2\rho_0^2}\right]d\mathbf{r}, \tag{2.100}$$

where we used the Taylor series expansion (2.86) truncated at cubic term $O(\rho_1^3)$ to approximate the potential energy. Here $q:=\frac{\rho_0^2}{3c_s^2}\varepsilon_{\rho\rho\rho}(\rho_0)=\frac{\rho_0}{3}\frac{\varepsilon_{\rho\rho\rho}(\rho_0)}{\varepsilon_{\rho\rho}(\rho_0)}$ is the dimensionless constant of the order of one. In contrast, the kinetic energy does not have terms of higher order than cubic so that the Hamiltonian H_2+H_3 in (2.87) and (2.100) take into account terms of all order for the kinetic energy $K=\frac{1}{2}\int(\rho_0+\rho_1)(\nabla\phi)^2 d\mathbf{r}$ of the full Hamiltonian (2.76).

Similar to the case of finite number of oscillators considered in Section 1.11, we introduce the complex variables $a_{\mathbf{k}}$ and $\bar{a}_{\mathbf{k}}$ for the continuous media as follows

$$\begin{aligned}\rho_{\mathbf{k}}&=\left(\frac{\rho_0 k^2}{2\omega_k}\right)^{1/2}(a_{\mathbf{k}}+\bar{a}_{-\mathbf{k}}),\\ \phi_{\mathbf{k}}&=-i\left(\frac{\omega_k}{2\rho_0 k^2}\right)^{1/2}(a_{\mathbf{k}}-\bar{a}_{-\mathbf{k}}),\end{aligned} \tag{2.101}$$

where ω_k is given by the dispersion law (2.99).

Using equations (A.482) and (2.101) we rewrite H_3 (2.100) in the following form

$$\begin{aligned}H_3=&\int(V_{k_1k_2k_3}\bar{a}_{k_1}a_{k_2}a_{k_3}+c.c.)\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\\ &+\frac{1}{3}\int(U_{k_1k_2k_3}a_{k_1}a_{k_2}a_{k_3}+c.c.)\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,\end{aligned} \tag{2.102}$$

where the matrix elements of U and V have the following symmetry properties:

$$U_{k_1k_2k_3}=U_{k_1k_3k_2}=U_{k_3k_2k_1},\ V_{k_1k_2k_3}=V_{k_1k_3k_2}$$

and are given by

$$
\begin{aligned}
U_{k_1k_2k_3} = V_{k_1k_2k_3} &= \frac{1}{4(2\pi)^{D/2}}\left(\frac{1}{2\rho_0}\right)^{1/2} \\
&\times\left\{3qc_s^2\frac{k_1k_2k_3}{(\omega_{k_1}\omega_{k_2}\omega_{k_3})^{1/2}} + \left(\frac{\omega_{k_1}\omega_{k_2}}{\omega_{k_3}}\right)^{1/2} k_3\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{k_1k_2}\right. \\
&\left.+\left(\frac{\omega_{k_1}\omega_{k_3}}{\omega_{k_2}}\right)^{1/2} k_2\frac{(\mathbf{k}_1\cdot\mathbf{k}_3)}{k_1k_3} + \left(\frac{\omega_{k_2}\omega_{k_3}}{\omega_{k_1}}\right)^{1/2} k_1\frac{(\mathbf{k}_2\cdot\mathbf{k}_3)}{k_2k_3}\right\}.
\end{aligned} \tag{2.103}
$$

We note that $U_{k_1k_2k_3} = V_{k_1k_2k_3}$ in equation (2.103). This is however a pure coincidence because the matrix elements $U_{k_1k_2k_3}$ and $V_{k_1k_2k_3}$ are defined on the different resonant manifolds determined according to two Dirac δ functions in equation (2.102) by the conditions

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3 \tag{2.104}$$

and

$$\mathbf{k}_1 = -\mathbf{k}_2 - \mathbf{k}_3, \tag{2.105}$$

respectively.

Equations (2.102), (2.104) and (2.105) have a simple interpretation in terms of quantum mechanics if we recall that a_k and $\bar{a}_k$ are the classical analog of the annihilation and creation operators, respectively. Also $\mathbf{k}$ is the analog of the momentum. Then the term $\propto \bar{a}_{k_1}a_{k_2}a_{k_3}\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$ in equation (2.102) corresponds to the process of the annihilation of two quanta with momenta $\mathbf{k}_2$ and $\mathbf{k}_3$ with the creation of the quantum with the momentum $\mathbf{k}_1$. The term $\propto a_{k_1}\bar{a}_{k_2}\bar{a}_{k_3}\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$ in equation (2.102) corresponds to the process of the creation of two quanta with momenta $\mathbf{k}_2$ and $\mathbf{k}_3$ with the annihilation of the quantum with the momentum $\mathbf{k}_1$. In a similar way, the terms $\propto a_{k_1}a_{k_2}a_{k_3}\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$ and $\bar{a}_{k_1}\bar{a}_{k_2}\bar{a}_{k_3}\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$ in equation (2.102) correspond to the processes of the annihilation and the creation of three quanta with momenta $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$.

2.5. Electromagnetic waves in nonlinear dielectric and nonlinear Schrödinger equation

Propagation of electromagnetic waves in nonlinear dielectric is described by the Maxwell's equations [**LL84**]

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{2.106a}$$

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}, \tag{2.106b}$$

$$\nabla\cdot\mathbf{D} = 0, \tag{2.106c}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{2.106d}$$

where the centimeter-gram-second system of units (CGS) is used (see Appendix ???? for Maxwell's equations written in the International System of Units (SI)), c is the speed of light in vacuum, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field and $\mathbf{D}$ is the electric displacement field (also referred to as the electric induction). We assume that the medium (dielectric) is non-magnetic meaning that the magnetic induction equals to the external magnetic field $\mathbf{B}$. Applying curl to both l.h.s. and r.h.s. of equation (2.106a) and using (2.106b) to exclude $\nabla \times \mathbf{B}$ results in

$$\nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2} \tag{2.107}$$

Recalling a vector identity

$$\text{curl curl } \mathbf{E} = \nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \text{grad div } \mathbf{E} - \nabla^2 \mathbf{E} \tag{2.108}$$

and assuming that

$$\text{div } \mathbf{E} = 0 \tag{2.109}$$

we obtain from equation (2.107) that

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2}. \tag{2.110}$$

Assumption (2.109) is usually well satisfied for laser beam propagation in dielectric media and is discussed below.

Equation (2.110) (as well as Maxwell equations (2.106)) is not closed because we have to additionally take into account the response of the dielectric to the electric field, i.e. a relation between $\mathbf{E}$ and $\mathbf{D}$.

Electric field $\mathbf{E}$ induces a polarization $\mathbf{P}$ in the dielectric such that

$$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}, \tag{2.111}$$

where $\mathbf{P} = (P_1, P_2, P_3)$ depends on $\mathbf{E} = (E_1, E_2, E_3)$ as a power series

(2.112)

$$P_j := P_j^{(LN)} + P_j^{(NL)} = \sum_{l=1}^{3} \hat{\chi}_{jl}^{(1)} E_l + \sum_{l,m=1}^{3} \hat{\chi}_{jlm}^{(2)} E_l E_m + \sum_{l,m,n=1}^{3} \hat{\chi}_{jlmn}^{(3)} E_l E_m E_n + \dots,$$

where $P_j^{(LN)} = \sum\limits_{l=1}^{3} \hat{\chi}_{jl}^{(1)} E_l$ is the linear part in $\mathbf{E}$ of polarization, $P_j^{(NL)}$ is the nonlinear part of polarization, the tensors $\hat{\chi}_{jl}^{(1)}$, $\hat{\chi}_{jlm}^{(2)}$ and $\hat{\chi}_{jlmn}^{(3)}$ are linear, quadratic and cubic susceptibilities of the media, respectively.

Generally $\chi^{(j)}$, $j = 1, 2, 3$, are integral operators over time and spatial coordinates to account for the finite response time of the medium and the nonlocality of medium response in space. However, the spatial nonlocality of response (called by the spatial dispersion) is usually very small [**LL84**] and neglected below. That smallness can be estimated as follows for dielectric media. A typical energy for the transition from the ground state to the excited state of atom under the action of electromagnetic wave is about the Borh energy $E_B = e^2/r_B$, where e is the charge of an electron, $r_B = \hbar^2/(m_e e^2)$ is the Bohr radius, m_e is the electron mass and $\hbar$ is the Planck's constant [**LL76**]. E_B is about the energy of the phonon, i.e. $E_B \sim \hbar c/\lambda$, where λ is the wavelength of electromagnetic wave. It follows from there that $r_B/\lambda \sim \alpha = \frac{e^2}{\hbar c} = 7.297\ldots \cdot 10^{-3} \ll 1$ (α is the fine-structure constant) ensuring smallness of the spatial dispersion.

After we neglect the spatial dispersion, the linear response takes the following general form

(2.113)
$$P_j^{(LN)}(\mathbf{r},t) = \sum_{l=1}^{3} \int_{-\infty}^{t} \chi_{jl}^{(1)}(t-t')E_l(\mathbf{r},t')dt' = \sum_{l=1}^{3} \int_{-\infty}^{\infty} \chi_{jl}^{(1)}(t-t')E_l(\mathbf{r},t')dt',$$

where $\chi_{jl}^{(1)}(t-t')$ is the kernel of the integral operator $\hat{\chi}_{jl}^{(1)}$. The causality of the response requires that $\chi_{jl}^{(1)}(t-t') = 0$ for $t < t'$ (i.e. there is no influence of future values of $E_l(\mathbf{r},t'),\ t < t'$ on the polarization at the time t) which allowed to extend integration to the entire real axis in r.h.-s. of (2.113) thus producing a convolution. We insure the causality by requiring that $\chi_{jl}^{(1)}(t-t') = \Theta(t-t')\chi_{jl}^{(1)}(t-t')$, where $\Theta(\cdot)$ is the *Heaviside step function* such that $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$.

Fourier transform (FT) over time

$$f_\omega(\mathbf{r}) := \int_{-\infty}^{\infty} f(\mathbf{r},t)e^{i\omega t}dt \tag{2.114}$$

transforms convolution of (2.113) into the local expression

$$P_{j,\omega}^{(LN)}(\mathbf{r}) = \sum_{l=1}^{3} \chi_{jl,\omega}^{(1)} E_{l,\omega}(\mathbf{r}). \tag{2.115}$$

Note that we chose for FT (2.114) to have zeroth power of 2π to be consistent with the standard definition of $\chi_{jl,\omega}^{(1)}$ in theoretical physics as e.g. in Ref. [**LL84**]. Respectively, the inverse FT is given by

$$f(\mathbf{r},t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f_\omega(\mathbf{r})e^{-i\omega t}d\omega, \tag{2.116}$$

see also Appendix A.4 for more discussion on FT.

Consider a propagation of a light wave with the linear polarization pointing along the unit vector $\mathbf{n}$. Then the electric field has the following form

$$\mathbf{E} = \mathbf{n}E, \tag{2.117}$$

where $E = |\mathbf{E}|$. Neglecting the generation of the other components of $\mathbf{E}$ beyond the direction $\mathbf{n}$, we reduce equation (2.115) to the scalar expression

$$P_\omega^{(LN)}(\mathbf{r}) := \mathbf{n} \cdot \mathbf{P}_\omega^{(LN)}(\mathbf{r}) = \chi_\omega^{(1)} E_\omega(\mathbf{r}), \tag{2.118}$$

where

$$\chi_\omega^{(1)} := \sum_{j,l=1}^{3} \chi_{jl,\omega}^{(1)} n_j n_l. \tag{2.119}$$

Neglecting the nonlinear polarization, $\mathbf{P}_j^{(NL)} = 0$, we obtain from equations (2.111) and (2.118) that

$$D_\omega(\mathbf{r}) := \mathbf{n} \cdot \mathbf{D}_\omega(\mathbf{r}) = \varepsilon(\omega)E_\omega(\mathbf{r}), \tag{2.120}$$

where

$$\varepsilon(\omega) := 1 + 4\pi\chi(\omega) \tag{2.121}$$

is called by the linear permittivity of the medium. FTs of equation over $\mathbf{r}$ and t together with equation (2.120) result in the *dispersion relation for dielectric medium*

$$k^2 = \frac{\omega^2\varepsilon(\omega)}{c^2}, \tag{2.122}$$

which describes the propagation of a monochromatic linearly polarized wave[2]

$$\mathbf{E} = \frac{\mathbf{n}}{2}\left[\mathcal{E}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} + c.c.\right] \tag{2.123}$$

along $\mathbf{k}$. Here $\mathcal{E}$ is the constant complex amplitude of wave and we consider the wavenumber $k(\omega) = |\mathbf{k}|$ to be the function of ω, as given by the dispersion relation (2.122). Comparison of (2.122) and (2.123) allows to conclude that the *phase velocity* of light propagation in the dielectric medium is given by $c_{phase} = \omega/k(\omega) = c/\epsilon_\omega^{1/2}$, where the phase velocity is defined as the speed of propagation of the phase $\phi := \mathbf{k}\cdot\mathbf{r} - \omega t$. Note that in optics is usually more convenient to express k through ω as in (2.122) which is in contrast with the dispersion law ω_k of general linear waves as e.g. in (2.99), where ω is assumed to be the function of k.

Consider a propagation of a quasi-monochromatic light wave along $z := x_3$ axis with the linear polarization along the unit vector $\mathbf{n}$ in $(x, y) := (x_1, x_2)$ plane. Then the electric field has the following form

$$\mathbf{E} = \frac{\mathbf{n}}{2}\left[\mathcal{E}(\mathbf{r}, t)e^{ik_0 z - i\omega_0 t} + c.c.\right], \tag{2.124}$$

where ω_0 is the carrier frequency, k_0 is given by

$$k_0^2 = \left.\frac{\omega^2\varepsilon(\omega)}{c^2}\right|_{\omega=\omega_0} \tag{2.125}$$

according to (2.122) and $\mathcal{E}(\mathbf{r}, t)$ is the slow function of $\mathbf{r}$ and t (envelope). Quasi-monochromaticity of light means that the width $\Delta\omega$ of $\mathcal{E}_\omega(\mathbf{r})$ is small compare with ω_0:

$$\Delta\omega/\omega_0 \ll 1 \tag{2.126}$$

as sketched in Figure ???. $\Delta\omega$ is defined e.g. by full width at half maximum (FWHM) of $|E_\omega|^2$. Limit $\Delta\omega \to 0$ recovers the monochromatic light wave (2.123).

Condition (2.126) allows Taylor series expansion of $k(\omega)^2$ for $(\omega - \omega_0)/\omega_0 \ll 1$ as follows

$$k(\omega)^2 = k_0^2 + 2k_0' k_0(\omega - \omega_0) + [(k_0')^2 + k_0'' k_0](\omega - \omega_0)^2 + O(\omega - \omega_0)^3 \tag{2.127}$$

where $k_0 := k(\omega_0)$, $k'(\omega_0) := \left(\frac{dk}{d\omega}\right)\big|_{\omega=\omega_0} = v_g^{-1}$, v_g is the group velocity at $k = k_0$ and

$$k_0'' := \left.\left(\frac{d^2k}{d\omega^2}\right)\right|_{\omega=\omega_0} \tag{2.128}$$

is the *group velocity dispersion* (GVD) which is also sometimes called by the second order cromatic dispersion or simply *dispersion*.

[2]See e.g. Ref. [**LL84**] for more discussion on polarizations of electromagnetic waves.

Assume that the medium is centro-symmetric one, i.e. the medium does not change its properties under the inversion $\mathbf{r} \rightarrow -\mathbf{r}$. That property requires that that $\chi^{(2)} \equiv 0$. We then add the cubic nonlinear response $\chi^{(3)}$ from (2.112) into (2.120) which gives

$$D(\mathbf{r}, t) = \hat{\varepsilon} E(\mathbf{r}, t) + 4\pi\chi^{(3)} E^3(\mathbf{r}, t), \tag{2.129}$$

where $\hat{\varepsilon}$ is the linear integral operator which is the inverse transform of (2.121). Last term in r.h.-s. of equation (2.129) is understood in a sense that we take into account the nonlinear response $\chi^{(3)}$ only near the carrier frequency ω_0. Generally $\chi^{(3)}$ is the integral operator in time. However for light with a narrow frequency band as given by (2.126) at leading order we evaluate $\chi^{(3)}$ only at $\omega = \omega_0$ thus it turns into constant. It implies that in cubic term

$$E^3 = \frac{3}{8}|\mathcal{E}|^2 \mathcal{E} e^{ik_0 z - i\omega_0 t} + \frac{1}{8}\mathcal{E}^3 e^{3ik_0 z - 3i\omega_0 t} + c.c. \tag{2.130}$$

we take into account only first term in r.h.-s. where we used equation (2.124). Plugging in (2.129) and (2.130) into equation (2.110) we obtain that

$$\nabla^2 [\mathcal{E} e^{ik_0 z - i\omega_0 t}] = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[(\hat{\varepsilon}\mathcal{E} + 3\pi\chi^{(3)}|\mathcal{E}|^2\mathcal{E}) e^{ik_0 z - i\omega_0 t}] \tag{2.131}$$

The nonlinear term in equation (2.131) is approximated at the leading order as

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}[3\pi\chi^{(3)}|\mathcal{E}|^2\mathcal{E} e^{ik_0 z - i\omega_0 t}] \simeq -\frac{3\pi\chi^{(3)}\omega_0^2}{c^2}|\mathcal{E}|^2\mathcal{E} e^{ik_0 z - i\omega_0 t}, \tag{2.132}$$

i.e. we differentiate over time only fast exponent with the frequency ω_0 and neglect time derivatives of $\mathcal{E}$.

FT over time of the first term in r.h.-s. of equation (2.131) gives $-k(\omega)^2 \mathcal{E}_{\omega-\omega_0} e^{ik_0 z}$. Using the expansion (2.127) for that term and performing inverse FT over ω via equation (A.500) of Appendix A.4 we obtain from equations (2.125), (2.131) and (2.125) that

$$2ik_0 \frac{\partial \mathcal{E}}{\partial z} + 2ik_0 k_0' \frac{\partial \mathcal{E}}{\partial t} + \nabla^2 \mathcal{E} - \left[(k_0')^2 + k_0'' k_0\right] \frac{\partial^2 \mathcal{E}(\mathbf{r}, t)}{\partial t^2} = -\frac{3\pi\chi^{(3)}\omega_0^2}{c^2}|\mathcal{E}|^2 \mathcal{E}. \tag{2.133}$$

Equation (2.133) has second partial derivatives both in t and z representing at the leading order the linear wave equation for the envelope $\mathcal{E}$ with the velocity of propagation v_g (the group velocity of light in the dielectric medium, $v_g = 1/k_0'$). First two terms in l.h.-s. of equation (2.133) correspond to unidirectional wave propagation in the form $\mathcal{E} = f(z - v_g t)$ with the constant velocity $v_g = 1/k_0'$, where f is the arbitrary continuously differentiable function and $v_g = 1/k_0'$ is the group velocity of light in the dielectric medium. Similar to the reduction of equation (1.189) to equation (1.191), we account for that fast dynamics by the transformation to the frame of reference moving with the the group velocity v_g. However, it is natural in optics to measure a distribution of light at different fixed spatial position rather than to move detector with the group velocity. This motivates to introduce a *retarded time* τ to account for the moving frame of reference as follows

$$\tau := t - \frac{z}{v_g} = t - k_0' z \tag{2.134}$$

while the longitudinal spatial coordinate z does not change,

$$\zeta := z. \tag{2.135}$$

Transforming to new variables τ and ζ we observe that the following two terms nearly cancel each other,

$$\left|\frac{\partial^2 \mathcal{E}}{\partial z^2} - k_0'^2 \frac{\partial^2 \mathcal{E}}{\partial t^2}\right| = \left|\left(\frac{\partial}{\partial \zeta} - 2k_0' \frac{\partial}{\partial \tau}\right) \frac{\partial \mathcal{E}}{\partial \zeta}\right| \ll k_0 \left|\frac{\partial \mathcal{E}}{\partial \zeta}\right|, \tag{2.136}$$

where we use that the amplitude $\mathcal{E}$ changes slowly at the spatial scale $1/k_0$.

Using equations (2.125) and (2.133)-(2.136) we obtain the nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial \mathcal{E}}{\partial \zeta} + \frac{1}{2k_0}\nabla_\perp^2 \mathcal{E} - \frac{k_0''}{2}\frac{\partial^2 \mathcal{E}(\mathbf{r}, t)}{\partial \tau^2} = -\frac{3\pi\chi^{(3)}k_0}{2\varepsilon(\omega_0)}|\mathcal{E}|^2\mathcal{E}, \tag{2.137}$$

where $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian. It follows that $\zeta = z$ plays the role of the evolutionary variable of the standard form of NLSE, while the retarder time τ plays the role of the third coordinate. Thus the role of time and longitudinal coordinates are reversed in optics with respect to NLSE.

We transform NLSE (2.137) into the dimensionless form by the scaling transform

$$\psi = \sqrt{\frac{3\pi|\chi^{(3)}|}{\varepsilon(\omega_0)}}\mathcal{E}, \ (\tilde{x}, \tilde{y}) = (x, y)k_0, \ \tilde{\tau} = \frac{\tau\sqrt{k_0}}{\sqrt{|k_0''|}}, \quad \tilde{\zeta} = \frac{k_0}{2}\zeta \tag{2.138}$$

which results, dropping tildes, in

$$i\frac{\partial}{\partial \zeta}\psi + \nabla_\perp^2\psi - \mathrm{sign}(k_0'')\frac{\partial^2}{\partial \tau^2}\psi + \mathrm{sign}(\chi^{(3)})|\psi|^2\psi = 0, \tag{2.139}$$

where we assume that $Re(\chi^{(3)}) = \chi^{(3)}$.

Negative GVD, $k_0'' < 0$ is called the *anomalous dispersion*. The word "anomalous" is historic and actually it is frequent for dielectric media to have $k_0'' < 0$. Equation (2.139) with the anomalous dispersion results in three-dimensional (3D) NLSE

$$i\frac{\partial}{\partial \zeta}\psi + \nabla_\perp^2\psi + \frac{\partial^2}{\partial \tau^2}\psi + \mathrm{sign}(\chi^{(3)})|\psi|^2\psi = 0. \tag{2.140}$$

Depending on sign of $\chi^{(3)}$, NLSE (2.140) is called by the *focusing NLSE*

$$i\frac{\partial}{\partial \zeta}\psi + \nabla_\perp^2\psi + \frac{\partial^2}{\partial \tau^2}\psi + |\psi|^2\psi = 0 \tag{2.141}$$

for $\chi^{(3)} > 0$ and the *defocusing NLSE*

$$i\frac{\partial}{\partial \zeta}\psi + \nabla_\perp^2\psi + \frac{\partial^2}{\partial \tau^2}\psi - |\psi|^2\psi = 0. \tag{2.142}$$

for $\chi^{(3)} < 0$. In focusing case the nonlinearity adds to the linear focusing of wave while in the defocusing case the nonlinearity contributes to the defocusing.

Positive GVD, $k_0'' > 0$ is called the *normal dispersion* and occurs in many media. Equation (2.137) then results in *hyperbolic NLSE*

$$i\frac{\partial}{\partial \zeta}\psi + \nabla_\perp^2\psi - \frac{\partial^2}{\partial \tau^2}\psi + \mathrm{sign}(\chi^{(3)})|\psi|^2\psi = 0, \tag{2.143}$$

where the usual 3D Laplacian as in (2.142) is replaced by the hyperbolic operator $\nabla_\perp^2 - \frac{\partial^2}{\partial\tau^2}$. Hyperbolic NLSE is also focusing for $\chi^{(3)} > 0$ and defocusing for $\chi^{(3)} < 0$.

For long pulses $\frac{\partial^2}{\partial\tau^2}\psi \to 0$ and NLSE (2.139) is reduced to the stationary NLSE

$$i\frac{\partial}{\partial\zeta}\psi + \nabla_\perp^2\psi + \operatorname{sign}(\chi^{(3)})|\psi|^2\psi = 0, \tag{2.144}$$

which is focusing for $\chi^{(3)} > 0$ and defocusing for $\chi^{(3)} < 0$.

Above we assumed that both ε_{ω_0} and $\chi^{(3)}$ are real constants. However, Kramers-Kronig relations [**LL84**] imply from causality that $\varepsilon(\omega)$ has both real and imaginary part. Consider light propagation through the medium in the frequency range where $\varepsilon(\omega)$ can be considered real, $Re[\varepsilon(\omega)] = \varepsilon(\omega)$, with high accuracy. That frequency domain is called by a *window of transparency* with light propagating without any losses which justify our above derivation of NLSE. Adding small imaginary components into ε_{ω_0} and $\chi^{(3)}$ would add dissipation to NLSE.

2.6. Hamiltonian form of Maxwell's equations for dielectric media with constant uniform linear medium susceptibility

Smallness of the losses in transparent optical media suggests to look for the Hamiltonian equations for the electromagnetic wave propagation in a window of transparency. Consider a simplest case of Maxwell equations in dielectric with constant and spatially uniform scalar susceptibility $\varepsilon = Re(\varepsilon) \equiv const$ such that

$$\mathbf{D} = \varepsilon\mathbf{E}. \tag{2.145}$$

We assume that dielectric medium occupies entire space $\mathbf{r} \in \mathbb{R}^3$. Dielectric medium implies that there are no free charges. For the electromagnetic field we introduce a scalar potential φ and a vector potential $\mathbf{A}$ as

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi \tag{2.146}$$

and

$$\mathbf{B} = \nabla\times\mathbf{A}. \tag{2.147}$$

The first and fourth Maxwell's equations (2.106a) and (2.106d) are identically satisfied for any φ and $\mathbf{A}$ as follows from (2.146) and (2.147). There is a freedom in choice of φ and $\mathbf{A}$ which satisfy two other Maxwell's equations (2.106b) and (2.106c). We choose the *Coulomb gauge*,

$$\operatorname{div}\mathbf{A} = 0. \tag{2.148}$$

Equations (2.106c), (2.145), (2.146) and (2.148) result in the Laplace equation $\nabla^2\varphi = 0$ which allows only a constant solution $\varphi \equiv \varphi_0$ remaining finite in $\mathbf{r} \in \mathbb{R}^3$, where $\varphi_0 = \text{const}$ is the arbitrary real constant as a function of $\mathbf{r}$. That constant does not affect the Maxwell's equation because it enters only through the spatial gradient in equation (2.146). That equation then reduces to

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}. \tag{2.149}$$

Using equations (2.147), (2.148) and (2.149) we obtain from the second Maxwell's equation (2.106b) that **A** satisfies the wave equation

$$\nabla^2 \mathbf{A} = \frac{\varepsilon}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}, \tag{2.150}$$

where we used the vector identity (2.108) (with **E** replaced by **A**).

The energy of electromagnetic wave in the medium with constant linear dielectric constant ε is given by [**LL84**]

$$H = \int \frac{1}{8\pi}\left(\mathbf{B}^2 + \varepsilon \mathbf{E}^2\right) d\mathbf{r}. \tag{2.151}$$

Assume that **A** is the canonical momentum for the Hamiltonian (2.151). Then the variational derivative

$$\begin{aligned} \frac{\delta H}{\delta \mathbf{A}} = \frac{\delta \int \frac{1}{8\pi}\left(\nabla\times\mathbf{A}\right)^2 d\mathbf{r}}{\delta \mathbf{A}} = \frac{\int \frac{1}{4\pi}\left(\nabla\times\delta\mathbf{A}\right)\cdot\left(\nabla\times\delta\mathbf{A}\right) d\mathbf{r}}{\delta \mathbf{A}} &= \frac{1}{4\pi}\nabla\times\nabla\times\mathbf{A} \\ &= \frac{1}{4\pi}\nabla\times\mathbf{B} = \frac{\varepsilon}{4\pi c}\frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{2.152}$$

allows to define the canonical coordinate

$$\mathbf{K} := -\frac{\varepsilon \mathbf{E}}{4\pi c}, \tag{2.153}$$

where we assumed that **E** is independent on **A** and used the second Maxwell's equation (2.106b) together with equation (2.145). We also performed the integration by parts for curl which gives $\int(\nabla\times\mathbf{f})^2 d\mathbf{r} = \int \mathbf{f}\cdot(\nabla\times\nabla\times\mathbf{f})d\mathbf{r}$ for the function **f** with the decaying boundary conditions $|\mathbf{f}| \to 0$ as $|\mathbf{r}| \to \infty$.

The variational derivative of H over **K** using the definition (2.153) and equation (2.149) gives that

$$\frac{\delta H}{\delta \mathbf{K}} = \frac{\delta \int \frac{\varepsilon}{8\pi}\mathbf{E}^2 d\mathbf{r}}{\delta \mathbf{K}} = \frac{4\pi c^2}{\varepsilon}\mathbf{K} = -c\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t}. \tag{2.154}$$

Thus we obtain from equations (2.152) and (2.154) that the Maxwell's equations (2.106) in the Coulomb gauge (2.148) for the dielectric with constant uniform linear medium susceptibility ε have the Hamiltonian form

$$\begin{aligned} \frac{\partial \mathbf{A}}{\partial t} &= \frac{\delta H}{\delta \mathbf{K}}, \\ \frac{\partial \mathbf{K}}{\partial t} &= -\frac{\delta H}{\delta \mathbf{A}}, \end{aligned} \tag{2.155}$$

where **A** is the canonical coordinate, **K** (2.153) is the canonical momentum and the Hamiltonian H is the total energy (2.151) rewritten in terms of the canonical coordinates as follows

$$H = \int \left[\frac{(\nabla\times\mathbf{A})^2}{8\pi} + \frac{2\pi c^2 \mathbf{K}^2}{\varepsilon(\rho_0)}\right] d\mathbf{r}. \tag{2.156}$$

2.7. Hamiltonian equation of the propagation of light wave in a single direction

If we take into account a time dependence of the linear susceptibility as well as the nonlinearity then recovering of the Hamiltonian structure of the Maxwell equations coupled with the media response is not straightforward as in section (2.6). As example consider 1D propagation of linear polarized electromagnetic wave along

z. Similar to previous section we look at the solution in a window of transparency. We assume that the nonlinear susceptibility $\chi^{(3)}$ is time independent (constant) so that $\chi^{(3)}$ equation (2.129) is valid and together with equation (2.110) it results in the closed dynamical equation

$$\frac{\partial^2}{\partial z^2}E = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\hat{\varepsilon}E + 4\pi\chi^{(3)}E^3\right]. \tag{2.157}$$

Here $\hat{\varepsilon}$ is the inverse transform of (2.121) which results in the linear integral operator over time

$$\hat{\varepsilon}E(z,t) = \int_{-\infty}^{\infty} \varepsilon(t-t')E(z,t')dt', \tag{2.158}$$

where $\varepsilon(t-t')$ is the kernel of the integral operator $\hat{\varepsilon}$ satisfying (similar to equation (2.121)) the causality condition $\varepsilon(t-t') = 0$ for $t < t'$. We recall that in optics the dispersion relation (2.122) looks on k as a function of ω rather than on ω as the function of k as we do for the general linear waves as e.g. in equation (2.99). Qualitatively similar, in the derivation of NLSE (2.137) the roles of the spatial variable ζ and the time variable τ are reversed (evolution along ζ) in comparison with the derivation of NLSE in Section ???? for the general nonlinear waves. It motivates to look for the Hamiltonian equations for light wave propagation in following form

$$\begin{aligned}\frac{\partial\rho}{\partial z} &= \frac{\delta H}{\delta\phi},\\ \frac{\partial\phi}{\partial z} &= -\frac{\delta H}{\delta\rho},\end{aligned} \tag{2.159}$$

where ρ and Φ are the canonical coordinate and momentum for the Hamiltonian dynamics in z. Comparison of equations (2.159) with (2.83) shows that we use z as the evolutionary variable instead of t in (2.83). We also notice that ct is somewhat similar to z in a sense that, assuming unidirectional propagation along increasing z in equation (2.157), we obtain that $\left|\left(\frac{\partial}{\partial z} + \frac{\partial}{c\partial t}\right)E\right| \ll \left|\frac{\partial}{\partial z}E\right|, \left|\frac{\partial}{c\partial t}E\right|$. Here we neglected the linear susceptibility, i.e. assumed that $\hat{\varepsilon}E \simeq E$ in equation (2.157). In other words, at leading order the partial derivatives over z and ct nearly cancel each other. Better cancelation would be achieved if instead of c we use the phase velocity in the medium, ω_0/k_0, as given by equation (2.125). That however requires quasi-monochromatic wave propagation as considered in Subsection 2.5.

There are no general recipes for the introduction of canonical variables in continuous media. To represent equation (2.157) in the Hamiltonian form (2.159) we use the analogy with the linearized equation of the compressible hydrodynamics (2.90) which reminds the linear part of equation (2.157). Equation (2.90) is equivalent to the system of equations (2.89) provided we use the additional variable ϕ there. It suggests to introduce the axillary variable ρ which is the rescaled the electric field E

$$\rho := \alpha E \tag{2.160}$$

and another axillary variable ϕ such that

$$\frac{\partial\rho}{\partial z} + \beta\frac{\partial^2\phi}{\partial t^2} = 0, \tag{2.161}$$

where nonzero real constants α and $\beta > 0$ are determined later at our convenience. Equation (2.161) is similar to first equation in system (2.89). We introduce the analog of the second equation in system (2.89) as follows

$$\frac{\partial \phi}{\partial z} + \frac{1}{\beta c^2}\left(\hat{\varepsilon}\rho + 4\pi\alpha^{-2}\chi^{(3)}\rho^3\right) = 0. \tag{2.162}$$

The system (2.160), (2.161) and (2.162) is equivalent to equation (2.157) if one excludes ϕ and expresses ρ through E according to (2.160). A comparison with the quadratic Hamiltonian (2.87) immediately suggests to choose the first term of the Hamiltonian for the system (2.161) and (2.162) as $\propto \int \left(\frac{\partial\phi}{\partial t}\right)^2 dt$ as well as it infers that two other ρ-dependent terms must be $\propto \int \rho\hat{\varepsilon}\rho\, dt$ and $\propto \int \chi^{(3)}\rho^4 dt$. Choosing the proper constants which multiply each of these three terms we obtain that the Hamiltonian

$$H = \int_{-\infty}^{\infty} \left\{ \frac{\beta}{2c}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2\beta c^3}\rho\hat{\varepsilon}\rho + \frac{\pi\chi^{(3)}}{\alpha^2\beta c^3}\rho^4 \right\} c\, dt \tag{2.163}$$

together with the Hamiltonian equations (2.159) recovers the system (2.161) and (2.162). In other words, Note that calculating variational derivative over ρ uses the symmetry between integration variables t and t' in the term

$$\int \rho\hat{\varepsilon}\rho\, dt = \int \rho(z,t)\varepsilon(t-t')\rho(z,t')\, dt' dt$$

of the Hamiltonian (2.160).

A comparison of the first term in integrand of equation (2.163) with the first term of the Hamiltonian (2.151) suggests to define the magnetic field as

$$B = \pm\left(\frac{4\pi\beta}{c}\right)^{1/2}\frac{\partial\phi}{\partial t}, \tag{2.164}$$

where the sign can be chosen at our convenience and we assume that $c\,dt$ factor of equation (2.163) is the analog of $d\mathbf{r}$ in the Hamiltonian (2.151).

Assume that $\mathbf{E}$ (and respectively $\mathbf{D}$) is parallel to x axis. Then $\mathbf{B}$ is parallel to y axis and the first two Maxwell's equations (2.106a) and (2.106b) take the following form

$$\frac{\partial E}{\partial z} = -\frac{1}{c}\frac{\partial B}{\partial t}, \tag{2.165a}$$

$$-\frac{\partial B}{\partial z} = \frac{1}{c}\frac{\partial D}{\partial t}, \tag{2.165b}$$

where $D = \hat{\varepsilon}E + 4\pi\chi^{(3)}E^3$. Using equations (2.160), (2.164), (2.165a) and (2.165b) we obtain that

$$\alpha = \pm\left(\frac{c^3\beta}{4\pi}\right)^{1/2}, \tag{2.166}$$

where the sign is the same as in equation (2.164). The Hamiltonian (2.163) take the following form

$$H = \frac{1}{8\pi}\int \left\{ B^2 + E\hat{\varepsilon}E + 2\pi\chi^{(3)}E^4 \right\} cdt \tag{2.167}$$

which does not depend explicitly on the remaining free parameter β. The Hamiltonian (2.167) is the natural generalization of the Hamiltonian (2.151) to the medium with both time-dependent linear response and cubic nonlinearity provided we interchange t and z. The Hamiltonian equations (2.159) and (2.167) together with the expressions (2.160), (2.164) and (2.166) are equivalent to Maxwell's equations (2.165). The free parameter $\beta > 0$ as well as sign in expressions (2.164) and (2.166) can be chosen at our convenience. E.g., it can be chosen such that

$$\rho = -\frac{E}{4\pi c}, \tag{2.168}$$

which is similar to the definition of $\mathbf{K}$ in equation (2.153) except that we do not include $\hat{\varepsilon}$ in the definition. Then equations (2.160), (2.164), (2.166) and (2.168) imply that

$$\begin{aligned} \alpha &= -\frac{1}{4\pi c}, \\ \beta &= \frac{1}{4\pi c^5}, \\ B &= -\frac{1}{c^3}\frac{\partial \phi}{\partial t}. \end{aligned} \tag{2.169}$$

It follows from the last equation in (2.169) that $A := -\frac{1}{c^2}\phi$ is the analog of the x component of the vector potential (2.147).

The quadratic Hamiltonian H_2 is immediately obtained from the general Hamiltonian (2.167) using equations (2.168) and (2.169) which gives after FT (2.116) over t that

$$H_2 = \frac{1}{2}\int \left[\frac{\omega^2}{4\pi c^5}|\phi_\omega|^2 + 4\pi c^3 \varepsilon(\omega)|\rho_\omega|^2\right] d\omega. \tag{2.170}$$

Here we multiplied H_2 after FT (2.116) by the factor 2π because FT (2.116) is not the canonical Hamiltonian transformation but requires that multiplication to preserve the canonical symplectic structure, see Appendix ??? for more discussion on that. Similar to equation (2.95), the dispersion law $k(\omega)$ is obtained by the product of terms multiplying $|\phi_\omega|^2$ and $|\rho_\omega|^2$ in H_2 (2.170) which results in the dispersion relation for dielectric medium (2.122).

A transformation to complex variables is given by

$$\begin{aligned} \rho_\omega &= \sqrt{\frac{\omega^2}{8\pi c^5 k(\omega)}}(a_\omega + \bar{a}_{-\omega}), \\ \phi_\omega &= -i\sqrt{\frac{2\pi c^5 k(\omega)}{\omega^2}}(a_\omega - \bar{a}_{-\omega}). \end{aligned} \tag{2.171}$$

Taking into account the fourth order term in the Hamiltonian (2.167) beyond H_2, we obtain through FT (2.116) by using equations (2.171) the following form

$$H = \int k(\omega)|a_\omega|^2 d\omega + \frac{1}{2}\int T_{\omega_1\omega_2\omega_3\omega_4}\bar{a}_{\omega_1}\bar{a}_{\omega_2}a_{\omega_3}a_{\omega_4}\delta(\omega_1+\omega_2-\omega_3-\omega_4)\prod_{i=1}^{4} d\omega_i + o.t.,$$

where the matrix element $T_{\omega_1\omega_2\omega_3\omega_4}$ of four-wave interactions is given by

$$T_{\omega_1\omega_2\omega_3\omega_4} = \frac{6\pi\chi^{(3)}}{c^5}\left[\frac{\omega_1^2\omega_2^2\omega_3^2\omega_4^2}{k_1k_2k_3k_4}\right]^{1/2}, \ k_i := k(\omega_i). \tag{2.172}$$

and *o.t.* means other terms of the type $a_{\omega_1}a_{\omega_2}a_{\omega_3}a_{\omega_4}$, $\bar{a}_{\omega_1}a_{\omega_2}a_{\omega_3}a_{\omega_4}$ and their complex conjugates. Similar to equation (2.170), we multiplied H in equation by the factor 2π because of noncanonicity of FT (2.116). ???

2.8. Nonlinear Schrödinger equation for the propagation of narrow wave packet in general media with cubic nonlinearity

??? We consider the cubic NLSE in the spatial dimension D with a power law nonlinearity

$$i\psi_t + \nabla^2\psi + |\psi|^2\psi = 0, \tag{2.173}$$

and decaying boundary conditions $\psi(\mathbf{r}, t) \to 0$ for $|\mathbf{r}| \to \infty$.

2.9. Hamiltonian Formalism in Continuous Media

Sections 2.4 and 2.7 provided examples of the introduction of the canonical Hamiltonian variables in continuous media such as compressible hydrodynamics and nonlinear optics. The introduction of a Hamiltonian structure for conservative nonlinear media is a generalization of the Hamiltonian formalism for systems with a finite number of degrees of freedom (1.1) (parameterized by a discrete set of indices like j) to systems with a continuum number of degrees of freedom parametrized by the continuous set of variables such as the continuous spatial variable $\mathbf{r}$ in Section 2.4 or even time in Section 2.7. In this section we consider a general case of a description of the dynamics of waves evolving in a continuous medium by means of canonical variables.

There are no general recipes for the introduction of canonical variables in continuous media. As we demonstrated in equation (2.85) of Section 2.4 and in more examples below throughout this book, ??? it is sometimes useful to make use of a Lagrangian formulation with constraints with the Lagrangian multiplier(s) used to enforce which a fraction of equation(s) of motion. This method, which was first introduced in the work of B.I. Davydov [**Dav49**] is justified when the expression for the Lagrangian without the constraints comes directly from mechanics or field theory. Such procedure of finding the canonical variables is especially successful in application to systems of hydrodynamic type that is considered throughout this book in Sections ???. It includes nonlinear waves in plasma, in hydrodynamics and magnetohydrodynamics.

In this section it assumed that the canonical variables are known so that the medium is described by a pair of canonical variables which are the generalized coordinate $q(\mathbf{r}, t)$ and the generalized momentum $p(\mathbf{r}, t)$ with the spatial coordinate $\mathbf{r} \in \mathbb{R}^D$. The evolution of these variables is given by the Hamiltonian equations

$$\frac{\partial p}{\partial t} = -\frac{\delta H}{\delta q}, \ \frac{\partial q}{\partial t} = \frac{\delta H}{\delta p}. \tag{2.174}$$

Here the Hamiltonian H is a functional of p and q. We assume that the Hamiltonian has a general form of a series in powers of the canonical variables as follows (2.175)

$$H = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\int G_n^j(\mathbf{r}_1, ..., \mathbf{r}_j, \mathbf{r}_{j+1}, ..., \mathbf{r}_n, t)p(\mathbf{r}_1)...p(\mathbf{r}_j)q(\mathbf{r}_{j+1})...q(\mathbf{r}_n)d\mathbf{r}_1...d\mathbf{r}_n,$$

where the integral kernels $G_n^j(\mathbf{r}_1, ..., \mathbf{r}_j, \mathbf{r}_{j+1}, ..., \mathbf{r}_n, t)$ are called the *matrix elements* and we omitted the common argument t in both p and q assuming the instantaneous response of the medium (i.e. G_n^j, p and q depend on the same t). Here n is the order in powers of both canonical variables so that for each given n one has the extra summation over all possible orders j of p such that $0 \le j \le n$. The terminology of the matrix elements originate from the perturbation theory of quantum mechanics (see e.g. [**LL76**]). For our purposes it essential that all physical properties of particular nonlinear problems are included into matrix elements. Thus different physical systems are mathematically equivalent if they have the same matrix elements although physical interpretations of the canonical variables $p(\mathbf{r},t)$ and $q(\mathbf{r},t)$ might be significantly different as we already encountered in Sections 2.4, 2.7 and many more examples are given in subsequent sections.

Without loss of generality we assume that all matrix elements $G_n^j(\mathbf{r}_1,...,\mathbf{r}_j,\mathbf{r}_{j+1},...,\mathbf{r}_n,t)$ are symmetric with respect to any permutation (interchange) of either the coordinates $\mathbf{r}_1,...,\mathbf{r}_j$ or the coordinates $\mathbf{r}_{j+1},...,\mathbf{r}_n$, i.e.

$$G_n^j(\mathbf{r}_1,...,\mathbf{r}_j,\mathbf{r}_{j+1},...,\mathbf{r}_n,t) = G_n^j(\mathbf{r}_{l_1},...,\mathbf{r}_{j_n},\mathbf{r}_{l_{j+1}},...,\mathbf{r}_{l_n},t), \tag{2.176}$$

where $(l_1,l_2,\dots l_j,l_{j+1},\dots,l_n)$ are all distinct integers such that $1 \le l_m \le j$ for $1 \le m \le j$ and $j+1 \le l_m \le n$ for $j+1 \le m \le n$.

If initially (after the derivation from a particular physical system) $G_n^j(\mathbf{r}_1,...,\mathbf{r}_j,\mathbf{r}_{j+1},...,\mathbf{r}_n,t)$ is not symmetric then without loss of generality we can bring it to the symmetric form (2.176) by the following symmetrization procedure:

$$\tilde{G}_n^j(\mathbf{r}_1,...,\mathbf{r}_j,\mathbf{r}_{j+1},...,\mathbf{r}_n,t) := \frac{1}{j!(n-j)!}\sum_{l_1,\dots,l_j}\sum_{l_{j+1},\dots,l_n} G_n^j(\mathbf{r}_{l_1},...,\mathbf{r}_{j_n},\mathbf{r}_{l_{j+1}},...,\mathbf{r}_{l_n},t), \tag{2.177}$$

where two sums are over all possible permutations of $l_1,l_2,\dots l_j$ and $l_{j+1},\dots,l_n$, respectively, with the range of values of $(l_1,l_2,\dots l_j,l_{j+1},\dots,l_n)$ defined after equation (2.176). Also $\frac{1}{j!(n-j)!}$ in equation (2.177) is the normalization factor to account for the total number of such permutations. We note that the permutations in equation (2.176) are compatible with (2.175) (it amounts just to relabelling of the two group of integration variables $(\mathbf{r}_1,...,\mathbf{r}_j)$ and $(\mathbf{r}_{j+1},...,\mathbf{r}_n)$. Below we omit ˜ sign over all G_n^j assuming that such symmetrization was performed in advance.

The zero order term $n=0$ in equation (2.175) does not enter into the dynamical equations 2.1 so it can be set to zero without loss of a generality. The first order term $n=1$ in equation (2.175) accounts for the possible existence of the external forces acting on the system (i.e. forces which do not depend neither on p or on q. Below we assume that the external forces are absent then without loss of generality we assume that the expansion (2.175) begins with quadratic terms in p and q. For spatially homogeneous media the structure functions G_n^k are functions of the differences $(\mathbf{r}_i - \mathbf{r}_j)$ only. We also from now on assume no explicit dependence of G_n^j on t. Then the quadratic term H_2 in the expansion (2.175) has the following general form:

$$H_2 = \frac{1}{2}\int \left[A(\mathbf{r}-\mathbf{r}')p(\mathbf{r})p(\mathbf{r}') + 2B(\mathbf{r}-\mathbf{r}')p(\mathbf{r})q(\mathbf{r}') + C(\mathbf{r}-\mathbf{r}')q(\mathbf{r})q(\mathbf{r}')\right] d\mathbf{r}d\mathbf{r}', \tag{2.178}$$

where $A(\mathbf{x})$, $B(\mathbf{x})$ and $C(\mathbf{x})$ are arbitrary continuous functions of $\mathbf{x}$. The assumed symmetrization procedure (2.177) requires that

$$A(-\mathbf{x}) = A(\mathbf{x}) \quad \text{and} \quad C(-\mathbf{x}) = C(\mathbf{x}). \tag{2.179}$$

We now diagonalize the quadratic Hamiltonian (2.178) to solve a problem of the stability of the medium under small perturbations using the expansion of p, q, A, B and C in the harmonics of the spatial FT (2.9) over the coordinate $\mathbf{r}$. Plugging in such expansions into equation (2.178) results in

$$H_2 = \frac{1}{2}\int [A_k p_k p_k^* + 2B_k p_k q_k^* + C_k q_k q_k^*] d\mathbf{k}. \tag{2.180}$$

The symmetry properties (2.179) as well as the real valued property $p, q, A, B, C \in \mathbb{R}$ imply the following symmetry property of Fourier harmonics

$$\begin{aligned} &p_{-k} = p_k^*, \quad q_{-k} = q_k^*, \\ &A_k = A_k^* = A_{-k}, \; C_k = C_k^* = C_{-k}, \\ &B_k = B_{-k}^* \equiv B_{r,k} + iB_{i,k} = B_{r,-k} - iB_{i,-k}. \end{aligned}$$

In the k-representation equations (2.174) then take the form

$$\frac{\partial p_k}{\partial t} = -\frac{\delta H}{\delta q_k^*}, \; \frac{\partial q_k}{\partial t} = \frac{\delta H}{\delta p_k^*}.$$

The equations for small perturbations are obtained from this by varying the Hamiltonian H_2. Analysis of these equations shows that waves can propagate in the medium with frequencies

$$\omega_{1,2} = -B_{i,k} \pm \sqrt{A_k C_k - B_{r,k}^2}.$$

The medium will be stable with respect to small perturbations if

$$A_k C_k - B_{r,k}^2 > 0 \tag{2.181}$$

and, respectively, unstable in the opposite case. The latter case, for instance, can be realized in a cold plasma with a monochromatic electron beam when plasma electrons and beam electrons can be considered as two independent fluids.

In the following we shall assume that the stability condition (2.181) is satisfied. For media that are invariant under reflection ($B(\mathbf{r}) = B(-\mathbf{r})$), one obtains

$$B_{i,k} = 0 \text{ and } \omega_k^2 = A_k C_k - B_{r,k}^2.$$

We further carry out a canonical transformation from the variables p_k and q_k to normal variables a_k and a_k^*:

$$p_k = U_k a_k + U_k^* a_{-k}^*, \; (U_k = U_{-k}), \tag{2.182}$$

$$q_k = V_k a_k + V_k^* a_{-k}^*, \; (V_k = V_{-k}).$$

in which the quadratic Hamiltonian is

$$H_2 = \int \omega_k a_k^* a_k d\mathbf{k}, \tag{2.183}$$

and the equations of motion have the form:

$$\frac{\partial a_k}{\partial t} = -i\frac{\delta H}{\delta a_k^*}. \tag{2.184}$$

Here ω_k denotes one of the functions $\omega_{1,2}$.

Substituting the transforms (2.182) into (2.178), from a comparison with (2.183) we get a system of equations for determining U_k and V_k. By requiring that this transformation is canonical, we get

$$U_k V_k^* - U_k^* V_k = -i \ ,$$

and find from this system

$$U_k = i\frac{(B_{r,k} - i\omega_{0k})}{\sqrt{2}A_k\omega_{0k}} \exp(i\varphi_k) \ ,$$

$$V_k = -i\sqrt{\frac{A_k}{2\omega_{0k}}} \exp(i\varphi_k) \ .$$

Here $\omega_{0k} := \mathrm{sign}[A_k](A_k C_k - B_{r,k}^2)^{1/2}$ and φ_k is an arbitrary phase factor, which we set equal to zero from now on (this corresponds to a simple redefinition $a_k \to a_k e^{i\varphi_k}$).

Let us now explicitly consider the frequency

$$\omega_k = -B_{i,k} + \mathrm{sign}(A_k)\sqrt{A_k C_k - B_{i,k}^2} \tag{2.185}$$

which is the dispersion law (dispersion relation) for the waves. It is essential that the sign of the frequency coincides with the sign of the wave energy in the nonlinear medium[3]. By this reason all waves can be divided in two big classes: waves with positive energy and waves with negative energy. All well-known waves (gravity and capillary waves on the fluid surface, acoustic and electromagnetic waves, and so on) belong to the first class. The waves with a negative energy typically appear in media with some current (it may be electron or ion beams in plasma, or flow of one fluid with respect to another, etc.) and in this case the origin of a negative frequency is connected with the Doppler effect. One should say that there is no principle difference in the nonlinear interaction between waves within their respective classes. This arises for the interaction between waves with positive and negative energies.

In the particular case $B_{\mathbf{k}} \equiv 0$, Eq. (2.185) is reduced to

$$\omega_{\mathbf{k}}^2 = A_{\mathbf{k}} C_{\mathbf{k}}. \tag{2.186}$$

Thus $\omega_{\mathbf{k}}^2$ in that particular case is immediately obtained from the quadratic Hamiltonian (2.180) as the product of the terms $A_{\mathbf{k}}$ and $C_{\mathbf{k}}$ multiplying $|p_{\mathbf{k}}|^2$ and $|q_{\mathbf{k}}|^2$ in teh integrand, respectively. It shows that *Remark 1* of Section 2.4 (that Eq. (2.99) is immediately obtained from the Hamiltonian as a product of similar integrands) applies to any quadratic Hamiltonian (2.180) with $B_{\mathbf{k}} \equiv 0$. We also note that the case $B_{\mathbf{k}} \equiv 0$ is the most common in applications including considered in Section 2.4.

In order to classify the nonlinear interaction between waves, let us consider the next terms in the expansion in powers of a and a^*, which is obtained after substitution of (2.182) into (2.175) as follows

$$H = H_2 + H_{int} = H_2 + H_3 + H_4 + H_5 + \dots, \tag{2.187}$$

where H_3 is the cubic term in powers of a and a^*, H_4 is the fourth order term in powers of a and a^*, etc. Here H_2 (2.183) is the quadratic part of the Hamiltonian responsible for the propagation of non-interacting linear waves while $H_{int} = H_3 +$

[3]Here we assume that the nonlinear interaction is weak so that the energy sign of the nonlinear medium coincides with the sign of its quadratic Hamiltonian.

$H_4 + H_5 + \dots$ is the *interaction Hamiltonian* which contains all the terms from the total Hamiltonian H responsive for the interaction between linear waves.

In particular, the cubic term H_3 has the general form

$$H_3 = \int (V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^*_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + c.c.)\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 \tag{2.188}$$

$$+\frac{1}{3}\int (U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^*_{\mathbf{k}} a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} + c.c.)\delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2$$

where the matrix elements of U and V have the following symmetry properties:

$$U_{kk_1k_2} = U_{kk_2k_1} = U_{k_2k_1k}, \; V_{kk_1k_2} = V_{kk_2k_1}. \tag{2.189}$$

Among the fourth-order terms H_4, we are most interested in the term of the following form

$$H_{4,int} = \frac{1}{2}\int T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_3) \prod_{i=1}^{4} d\mathbf{k}_i, \tag{2.190}$$

where the matrix element of T has the following symmetry properties:

$$T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = T_{\mathbf{k}_2\mathbf{k}_1\mathbf{k}_3\mathbf{k}_4} = T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_4\mathbf{k}_3} = T^*_{\mathbf{k}_3\mathbf{k}_4\mathbf{k}_1\mathbf{k}_2},$$

the last of them ensures that $H_{4,int}$ has the real value as required for the Hamiltonian.

Each term in the expansion of H in powers of a and a^* has a simple physical meaning. The equation of motion in the form (2.184) is the limit of the corresponding quantum equations for the Bose operators in the case of a classical wave field, where the variables a^* and a appear as analogs of the creation and annihilation operators. Thus the cubic term in the expansion of the Hamiltonian describes three-wave processes (the first term in H_3 is responsible for processes of decay of one wave into three waves, the second corresponds to simultaneous creation of three waves), the next term describes four-wave processes, etc.

It is necessary to say that a calculation of matrix elements in this scheme assumes a pure algebraic procedure that consists in a substitution of the transformation (2.182) into the corresponding Hamiltonian, a forthcoming simplification and a symmetrization of the final result.

For a medium described by several pairs of canonical variables and when H_2 is diagonalized, several wave branches can appear, with dispersion laws $\omega_i(k)$ and amplitudes $a_i(k)$. In this case a summation over all types of waves in each term of the expansion is needed.

2.10. Three-wave nonlinear processes. Decay instability of monochromatic wave

Consider a general three-wave Hamiltonian, i.e. it is assumed that

$$H = H_2 + H_3, \tag{2.191}$$

where

$$H_2 = \int \omega_{\mathbf{k}} |a_{\mathbf{k}}|^2 d\mathbf{k} \tag{2.192}$$

is the quadratic Hamiltonian with the dispersion law $\omega_{\mathbf{k}}$ and

$$H_3 = \int (V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^*_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + c.c.)\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2$$

$$+\frac{1}{3}\int (U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^*_{\mathbf{k}} a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} + c.c.)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 \tag{2.193}$$

is the cubic Hamiltonian which describes three-wave interactions. First term with $a^*_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2}$ in r.h.s of equation (2.193) describes the decay of two waves with wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ into a wave with a wavevector $\mathbf{k}$. This terminology originates from the analogy with (quasi)particles decay into a number of other quasi(particles) in quantum mechanics. The term with $a^*_{\mathbf{k}} a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2}$ can be interpreted as the classical analog of generation of three waves with wavevectors $\mathbf{k}$, $\mathbf{k}_1$ $\mathbf{k}_2$ from "vacuum" in quantum mechanics. This analogy motivates us to use $*$ in this section to designate the complex conjugation. Two over complex conjugated terms in r.h.s of equation (2.193) correspond to the opposite processes (decay of a single wave into two waves for the term $a_{\mathbf{k}} a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2}$ and annihilation of three waves for $a_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2}$). The same interpretation is obtained if we use the Hamiltonian equation

$$\partial_t a_{\mathbf{k}} = -i\frac{\delta(H_2+H_3)}{\delta a^*_{\mathbf{k}}} \tag{2.194}$$

together with equations (2.192) and (2.193).

We now consider general effects of three wave interactions. We start from the decay instability of the monochromatic wave

$$a_{\mathbf{k}} = A\delta(\mathbf{k}-\mathbf{k}_0)e^{-i\omega_{k_0}t}, \tag{2.195}$$

which is also often called by the *pump wave*. E.g. in optics pump wave corresponds to the laser beam entering into the medium. In plasma and hydrodynamics pump wave typically correspond to the generation of the powerful wave by different instabilities. The substitution of pump wave (2.195) into the equation (2.194) reveals that it is not the exact solution because the quadratic nonlinearity in equation (2.194) results in the formation of the second and zero harmonics, similar to what we obtained in Section 1.13. Then the second and zero harmonics results in the nonlinear frequency shift. However, that effect corresponds to the second-order perturbation theory which is neglected in this Section while we focus on the first-order perturbations.

In Sections 2.4 and 2.9 we considered the example of computation of matrix elements of three wave interactions corresponding to the cubic terms in the Hamiltonian and the general scheme of their derivation, respectively. Now we consider effects of three waves in the general Hamiltonian (2.193).

In this Section we consider a *decay instability of a monochromatic wave*. Assume that a nonlinear medium has the stable dispersion law

$$\omega_k \geq 0 \tag{2.196}$$

for all $\mathbf{k}$. A sign change of $\omega_{\mathbf{k}}$ corresponds to unstable media. Examples of such unstable media are plasma with beams, hydrodynamic systems with shear flow and many others, see e.g. Ref. ??? for more details. The condition (2.196) allows to exclude the matrix element $U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ in (2.193), i.e. neglect terms $\propto a^3, (a^*)^3$ in the Hamiltonian (2.193). These terms contribute only to fourth order terms in the Hamiltonian as explained in Section ???

Taking into account remaining terms $\propto V, V^*$ in (2.193) as well the quadratic term (2.192) we obtain the following reduced Hamiltonian

(2.197)
$$H = \int \omega_{\mathbf{k}} |a_{\mathbf{k}}|^2 d\mathbf{k} + \int (V_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3} a^*_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3} + c.c.) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,$$

where we added the symbol $|$ into the subscript of V to visualize the distinction between positive and negative signs in front of wavevector $\mathbf{k}_j$ in Dirac δ function.

The dynamical equation (2.194) with the truncated Hamiltonian (2.197) is given by

$$\begin{aligned} \frac{\partial a_{\mathbf{k}}}{\partial t} + i\omega_{\mathbf{k}} a_k = & -\mathrm{i} \int V_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 \\ & -2\mathrm{i} \int V^*_{\mathbf{k}_1|\mathbf{k}\mathbf{k}_2} a_{\mathbf{k}_1} a^*_{\mathbf{k}_2} \delta(\mathbf{k}_1 - \mathbf{k} - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2. \end{aligned} \tag{2.198}$$

Consider the stability of the solution (2.195) by looking at $a_{\mathbf{k}}$ in the following form

$$a_{\mathbf{k}} = A\delta(\mathbf{k} - \mathbf{k}_0) e^{-i\omega_{\mathbf{k}_0} t} + \alpha_{\mathbf{k}}, \tag{2.199}$$

where $\alpha_{\mathbf{k}}$ is the small perturbation. We substitute equation (2.199) into r.h.s. of equation and (2.198) and take into account only linear terms in $\alpha_{\mathbf{k}}$ (we say that we *linearize* equation (2.198) on the background on the monochromatic wave (2.195)). We also assume that A is constant, i.e. we neglect the depletion of the the monochromatic wave. Then the linearized equation takes the following form

(2.200)
$$\begin{aligned} \frac{\partial \alpha_{\mathbf{k}}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}} \alpha_{\mathbf{k}} = -2\mathrm{i} V_{\mathbf{k}|\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0} A e^{-i\omega_{\mathbf{k}_0} t} \alpha_{\mathbf{k}-\mathbf{k}_0} - 2\mathrm{i} V^*_{\mathbf{k}_0|\mathbf{k},\mathbf{k}_0-\mathbf{k}} A e^{-i\omega_{\mathbf{k}_0} t} \alpha^*_{\mathbf{k}_0-\mathbf{k}} \\ -2\mathrm{i} V^*_{\mathbf{k}_0+\mathbf{k}|\mathbf{k},\mathbf{k}_0} A^* e^{+\mathrm{i}\omega_{\mathbf{k}_0} t} \alpha_{\mathbf{k}_0+\mathbf{k}}. \end{aligned}$$

Each of three terms in r.h.s. of equation (2.200) have different physical meaning. We assume a weak nonlinearity (i.e. nonlinear frequency shift is small) then one can assume at the leading order of the nonlinearity that $\alpha_{\mathbf{k}} \propto e^{-\mathrm{i}\omega_{\mathbf{k}} t}$. It means that the first term in r.h.s. of equation (2.200) is $\propto e^{-\mathrm{i}\omega_{\mathbf{k}_0} t - \mathrm{i}\omega_{\mathbf{k}-\mathbf{k}_0} t}$. I.e. this first term corresponds to the process which adds the frequency $\omega_{\mathbf{k}-\mathbf{k}_0}$ to the main frequency $\omega_{\mathbf{k}_0}$ of the monochromatic wave (2.195) creating a new higher frequency $\omega_{\mathbf{k}}$ given by

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}-\mathbf{k}_0} \tag{2.201}$$

This process is not a decay but the new frequency $\omega_{\mathbf{k}}$ creation process with $\omega_{\mathbf{k}} > \omega_{\mathbf{k}_0}$, i.e. it results in the upshift (increase) of the frequency compared with the pump frequency $\omega_{\mathbf{k}_0}$. Such upshifted frequency (in comparison with the pump wave $\omega_{\mathbf{k}_0}$) is called *anti-Stokes wave* in nonlinear optics contrary to *Stokes wave* which is the downshifted frequency. This terminology named after Sir George Gabriel Stokes (1819-1903) who discovered the frequency downshift in the process of luminescence in the 19th century. The second new frequency $\omega_{\mathbf{k}-\mathbf{k}_0}$ in (2.201) generally can be either larger or smaller compare with the pump wave $\omega_{\mathbf{k}_0}$ according to equation (2.201) depending on the particular form of the dispersion law (2.284). Thus the wave with the frequency $\omega_{\mathbf{k}-\mathbf{k}_0}$ can be either anti-Stokes wave or Stokes wave. In optics the dispersion relation is often close to the linear one, $\omega_{\mathbf{k}} \simeq c|\mathbf{k}|$, with c been the speed of light. Such linear law implies that equation (2.201) in that particular case is reduced to $c|\mathbf{k}| = c|\mathbf{k}_0| + c|\mathbf{k}-\mathbf{k}_0|$. Thus both $\mathbf{k}_0$ and $\mathbf{k}-\mathbf{k}_0$ must be parallel to each other and point in the same direction which implies that $\omega_{\mathbf{k}-\mathbf{k}_0} < \omega_{\mathbf{k}_0}$, i.e. the second new frequency $\omega_{\mathbf{k}-\mathbf{k}_0}$ corresponds to Stokes wave for the linear dispersion relation.

The second term in r.h.s. of equation (2.200) is $\propto e^{-\mathrm{i}\omega_{\mathbf{k}_0}t}e^{+\mathrm{i}\omega_{\mathbf{k}_0-\mathbf{k}}t}$ turning into the resonance with $\alpha_{\mathbf{k}} \propto e^{-\mathrm{i}\omega_{\mathbf{k}}t}$ provided

$$\omega_{\mathbf{k}_0} = \omega_{\mathbf{k}_0-\mathbf{k}} + \omega_{\mathbf{k}}. \tag{2.202}$$

This process corresponds to the decay on the pumping wave $\propto e^{-\mathrm{i}\omega_{\mathbf{k}_0}t}$ into the two waves with smaller frequencies $\omega_{\mathbf{k}_0-\mathbf{k}}$ and $\omega_{\mathbf{k}}$. This is the second type of the resonance which is different from the first type (2.201). Then in terms of nonlinear optics both waves with frequencies $\omega_{\mathbf{k}_0-\mathbf{k}}$ and $\omega_{\mathbf{k}}$ are Stokes waves.

The third term in r.h.s. of equation (2.200) is $\propto e^{\mathrm{i}\omega_{\mathbf{k}_0}t-\mathrm{i}\omega_{\mathbf{k}_0+\mathbf{k}}t}$. It corresponds to the resonance given by

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}_0+\mathbf{k}} - \omega_{\mathbf{k}_0} \tag{2.203}$$

which is another process which is somewhat similar to (2.201) with the pump frequency $\omega_{\mathbf{k}_0}$ coupled with $\omega_{\mathbf{k}}$ and $\omega_{\mathbf{k}_0+\mathbf{k}}$. The positivity of $\omega_{\mathbf{k}_0}$ implies in equation (2.203) that $\omega_{\mathbf{k}_0+\mathbf{k}} > \omega_{\mathbf{k}_0}$, i.e. $\omega_{\mathbf{k}_0+\mathbf{k}}$ corresponds to anti-Stokes wave. Depending on the dispersion relation, the wave with the frequency $\omega_{\mathbf{k}}$ can be either anti-Stokes wave or Stokes wave.

Thus we have three different types (2.201)-(2.203) of resonances. Small widths of these resonances are determined by the smallness of the amplitude A in equation (2.195) as we will show below in this Section. Then the overlapping of the resonances (2.201)-(2.203) can be neglected and each of these three processes can be considered separately.

The resonances (2.201) and (2.203) do not produce instability as shown in Problem 1 of [2.10.1]. We consider in more details the second resonance (2.202). In that case equation (2.200) is reduced to

$$\frac{\partial\alpha_{\mathbf{k}}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}}\alpha_{\mathbf{k}} = -2\mathrm{i}V^*_{\mathbf{k}_0|\mathbf{k},\mathbf{k}_0-\mathbf{k}}Ae^{-\mathrm{i}\omega_{\mathbf{k}_0}t}\alpha^*_{\mathbf{k}_0-\mathbf{k}}. \tag{2.204}$$

To make a closed system of equations we have to add an expression for $\alpha^*_{\mathbf{k}_0-\mathbf{k}}$ in which we take into account the same corresponding second resonance (2.202) (by complex conjugation of equation (2.204) with replacing $\mathbf{k}\to\mathbf{k}_0-\mathbf{k}$) as follows

$$\frac{\partial\alpha^*_{\mathbf{k}_0-\mathbf{k}}}{\partial t} - \mathrm{i}\omega_{\mathbf{k}_0-\mathbf{k}}\alpha^*_{\mathbf{k}_0-\mathbf{k}} = 2\mathrm{i}V_{\mathbf{k}_0|\mathbf{k}_0-\mathbf{k},\mathbf{k}}A^*e^{\mathrm{i}\omega_{\mathbf{k}_0}t}\alpha_{\mathbf{k}}. \tag{2.205}$$

The time-dependent exponent is excluded from equations (2.204) and (2.205) by the following change of variables

$$\alpha^*_{\mathbf{k}_0-\mathbf{k}} \equiv \tilde{\alpha}^*_{\mathbf{k}_0-\mathbf{k}}e^{\mathrm{i}\omega_{\mathbf{k}_0}t}.$$

which results in the following homogeneous system of two linear ODEs over time t with constant coefficients

$$\begin{aligned}
&\frac{\partial\alpha_{\mathbf{k}}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}}\alpha_{\mathbf{k}} = -2\mathrm{i}V^*_{\mathbf{k}_0|\mathbf{k},\mathbf{k}_0-\mathbf{k}}A\tilde{\alpha}^*_{\mathbf{k}_0-\mathbf{k}},\\
&\frac{\partial\tilde{\alpha}^*_{\mathbf{k}_0-\mathbf{k}}}{\partial t} + \mathrm{i}(\omega_{\mathbf{k}_0} - \omega_{\mathbf{k}_0-\mathbf{k}})\tilde{\alpha}^*_{\mathbf{k}_0-\mathbf{k}} = 2\mathrm{i}V_{\mathbf{k}_0|\mathbf{k},\mathbf{k}_0-\mathbf{k}}A^*\alpha_{\mathbf{k}}.
\end{aligned} \tag{2.206}$$

for the variables $\alpha_{\mathbf{k}}$ and $\tilde{\alpha}^*_{\mathbf{k}_0-\mathbf{k}}$. Here we used the symmetry $V^*_{\mathbf{k}_0|\mathbf{k},\mathbf{k}_0-\mathbf{k}} = V^*_{\mathbf{k}_0|\mathbf{k}_0-\mathbf{k},\mathbf{k}}$ (see equation (2.189)) and $\mathbf{k}$ plays the role of the parameter of the ODE system (2.206).

Looking for the solution of the system (2.206) in the exponential form $\alpha_{\mathbf{k}},\ \tilde{\alpha}^*_{\mathbf{k}_0-\mathbf{k}} \propto e^{\lambda t}$, we obtain the following quadratic equation for λ

$$\lambda^2 + \mathrm{i}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_0} - \omega_{\mathbf{k}_0-\mathbf{k}})\lambda + \omega_{\mathbf{k}}(\omega_{\mathbf{k}_0-\mathbf{k}} - \omega_{\mathbf{k}_0}) - 4|V_0|^2|A|^2 = 0,$$

which has the solutions

$$\lambda_\pm = \frac{\mathrm{i}}{2}(\omega_{\mathbf{k}_0-\mathbf{k}} - \omega_{\mathbf{k}} - \omega_{\mathbf{k}_0}) \pm \sqrt{4|V_0|^2|A|^2 - \frac{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}_0-\mathbf{k}})^2}{4}}, \tag{2.207}$$

where we defined for brevity that

$$V_{\mathbf{k}_0|\mathbf{k},\mathbf{k}_0-\mathbf{k}} \equiv V_0. \tag{2.208}$$

The growth rate $\gamma_{\mathbf{k}} \equiv Re(\lambda_+)$ in equation (2.207) is given by

$$\gamma_{\mathbf{k}} = \sqrt{4|V_0|^2|A|^2 - \frac{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}_0-\mathbf{k}})^2}{4}} \tag{2.209}$$

provided

$$|\Delta\Omega| \le 4|V_0||A|, \tag{2.210}$$

where

$$\Delta\Omega \equiv \omega_{\mathbf{k}} - \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}_0-\mathbf{k}} \tag{2.211}$$

is called by the *frequency detuning* from the exact resonance (2.202).

The instability with the growth rate (2.209) is called by *decay instability* . The growth rate (2.209) reaches maximum $\gamma_{max} = 2|V_0||A|$ at the resonance manifold defined by the resonance condition (2.202). The width $\Delta\omega$ of that resonance in frequency domain is determined by the inequality (2.210) which gives

$$\Delta\omega = 8|V_0||A|, \tag{2.212}$$

i.e. $\Delta\omega$ exceeds γ_{max} in four times. Note that typically $V_0 \equiv V_{\mathbf{k}_0|\mathbf{k},\mathbf{k}-\mathbf{k}_0}$ is the slow function of $\mathbf{k}$ on the frequency scale of the width $\Delta\omega$ so at the leading order of nonlinearity one can evaluate V_0 at the exact resonance (2.202).

It is important that the resonance (2.202) is detuned (separated) from two other resonances (2.201) and (2.203). For example if $|\mathbf{k}| \sim |\mathbf{k}_0|$, then all resonances (2.201)-(2.203) are generally (except for very special choice of the dispersion relation $\omega = \omega_{\mathbf{k}}$) separated in frequency domain by a value $\sim \omega_{\mathbf{k}_0}$ because all these dispersion relations involve the difference $|\omega_{\mathbf{k}} - \omega_{\mathbf{k}_0}|$ which must be equal to $|\omega_{\pm\mathbf{k}\pm\mathbf{k}_0}|$ with different combinations of $+$ and $-$ for different resonances.

Often the resonances (2.202) and (2.203) have the smallest separation at $k \to 0$, i.e. at $k \ll k_0$. In that case Taylor series expansions result in $\omega_{\mathbf{k}} = \omega_{\mathbf{k}_0} - \omega_{\mathbf{k}_0-\mathbf{k}} = \mathbf{v}_{gr} \cdot \mathbf{k} + O(k^2)$ for (2.202) and $\omega_{\mathbf{k}} = \omega_{\mathbf{k}_0+\mathbf{k}} - \omega_{\mathbf{k}_0} = \mathbf{v}_{gr} \cdot \mathbf{k} + O(k^2)$ for (2.203), i.e. these two resonance manifolds approach each other separated only by small $O(k^2)$ terms. Here $\mathbf{v}_{gr} = \frac{d\omega_{\mathbf{k}_0}}{d\mathbf{k}_0}$ is the GVD. If the width (2.210) of the resonance exceeds that $O(k^2)$ separation, then both resonances (2.202) and (2.203) have to be considered together. In such particular case of the overlap of resonances, the decay instability is modified as was first considered in Ref. [**ZR72**], see also Ref. [**ZMR85**] for the review. ??? Give example for capillary waves.???

Problemss 2.10

2.10.1 Analyze the dynamics of the perturbation $\alpha_{\mathbf{k}}$ near the resonances (2.201) and (2.203), i.e. find equations similar to (2.207).

Solution. (a) We consider a solution for the perturbation $\alpha_{\mathbf{k}}$ near the resonance (2.201). In that case equation (2.200) is reduced to

$$\frac{\partial \alpha_{\mathbf{k}}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}}\alpha_{\mathbf{k}} = -2\mathrm{i}V_{\mathbf{k}|\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0} A e^{-i\omega_{\mathbf{k}_0} t}\alpha_{\mathbf{k}-\mathbf{k}_0}. \tag{2.213}$$

To make a closed system of equations we have to add an expression for $\alpha_{\mathbf{k}-\mathbf{k}_0}$ in which we take into account only the terms responsible for the first resonance (2.201). To do that we replace $\mathbf{k} \to \mathbf{k} - \mathbf{k}_0$ in equation (2.200) which results in

$$\begin{aligned} \frac{\partial \alpha_{\mathbf{k}-\mathbf{k}_0}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}-\mathbf{k}_0}\alpha_{\mathbf{k}-\mathbf{k}_0} &= -2\mathrm{i}V_{\mathbf{k}-\mathbf{k}_0|\mathbf{k}_0,\mathbf{k}-2\mathbf{k}_0} A e^{-i\omega_{\mathbf{k}_0} t}\alpha_{\mathbf{k}-2\mathbf{k}_0} \\ &- 2\mathrm{i}V^*_{\mathbf{k}_0|\mathbf{k}-\mathbf{k}_0,\mathbf{k}_0-2\mathbf{k}} A e^{-i\omega_{\mathbf{k}_0} t}\alpha^*_{2\mathbf{k}_0-\mathbf{k}} - 2\mathrm{i}V^*_{\mathbf{k}|\mathbf{k}-\mathbf{k}_0,\mathbf{k}_0} A^* e^{+\mathrm{i}\omega_{\mathbf{k}_0} t}\alpha_{\mathbf{k}}. \end{aligned} \tag{2.214}$$

Only the third term in r.h.s. of that equation participates in the resonance (2.201). Thus we reduce equation (2.214) to

$$\frac{\partial \alpha_{\mathbf{k}-\mathbf{k}_0}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}-\mathbf{k}_0}\alpha_{\mathbf{k}-\mathbf{k}_0} = -2\mathrm{i}V^*_{\mathbf{k}|\mathbf{k}-\mathbf{k}_0,\mathbf{k}_0} A^* e^{\mathrm{i}\omega_{\mathbf{k}_0} t}\alpha_{\mathbf{k}}. \tag{2.215}$$

The time-dependent exponent is excluded from equations (2.213) and (2.215) by the following change of variables

$$\alpha_{\mathbf{k}-\mathbf{k}_0} \equiv \tilde{\alpha}_{\mathbf{k}-\mathbf{k}_0} e^{\mathrm{i}\omega_{\mathbf{k}_0} t}.$$

which results in the following homogeneous system of two linear ODEs over time t with constant coefficients

$$\begin{aligned} &\frac{\partial \alpha_{\mathbf{k}}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}}\alpha_{\mathbf{k}} = -2\mathrm{i}V_{\mathbf{k}|\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0} A\tilde{\alpha}_{\mathbf{k}-\mathbf{k}_0}, \\ &\frac{\partial \tilde{\alpha}_{\mathbf{k}-\mathbf{k}_0}}{\partial t} + \mathrm{i}(\omega_{\mathbf{k}-\mathbf{k}_0} + \omega_{\mathbf{k}_0})\tilde{\alpha}_{\mathbf{k}-\mathbf{k}_0} = -2\mathrm{i}V^*_{\mathbf{k}|\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0} A^*\alpha_{\mathbf{k}} \end{aligned} \tag{2.216}$$

for the variables $\alpha_{\mathbf{k}}$ and $\tilde{\alpha}_{\mathbf{k}-\mathbf{k}_0}$. Here we used the symmetry $V^*_{\mathbf{k}|\mathbf{k}-\mathbf{k}_0,\mathbf{k}_0} = V^*_{\mathbf{k}|\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0}$ (see equation (2.189)) and $\mathbf{k}$ plays the role of the parameter of the ODE system (2.216). Looking for the solution of the system (2.216) in the exponential form $\alpha_{\mathbf{k}}$, $\tilde{\alpha}_{\mathbf{k}-\mathbf{k}_0} \propto e^{\lambda t}$, we obtain the following quadratic equation for λ

$$\lambda^2 + \mathrm{i}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}-\mathbf{k}_0})\lambda - \omega_{\mathbf{k}}(\omega_{\mathbf{k}-\mathbf{k}_0} + \omega_{\mathbf{k}_0}) + 4|V_0|^2|A|^2 = 0,$$

which has the solutions

$$\lambda_\pm = -\frac{\mathrm{i}}{2}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}-\mathbf{k}_0}) \pm \sqrt{-4|V_0|^2|A|^2 - \frac{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_0} - \omega_{\mathbf{k}-\mathbf{k}_0})^2}{4}},$$

where $V_{\mathbf{k}|\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0} \equiv V_0$. Here $\lambda_\pm$ are purely imaginary implying stability for all values of V_0, A and the frequency detuning $\Delta\Omega \equiv \omega_{\mathbf{k}} - \omega_{\mathbf{k}_0} - \omega_{\mathbf{k}-\mathbf{k}_0}$. Thus the growth rate is $\gamma_{\mathbf{k}} \equiv Re(\lambda_\pm) \equiv 0$.

(b) We consider a solution for the perturbation $\alpha_{\mathbf{k}}$ near the resonance (2.203). In that case equation (2.200) is reduced to

$$\frac{\partial \alpha_{\mathbf{k}}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}}\alpha_{\mathbf{k}} = -2\mathrm{i}V^*_{\mathbf{k}_0+\mathbf{k}|\mathbf{k},\mathbf{k}_0} A^* e^{+\mathrm{i}\omega_{\mathbf{k}_0} t}\alpha_{\mathbf{k}+\mathbf{k}_0}. \tag{2.217}$$

To make a closed system of equations we have to add an expression for $\alpha_{\mathbf{k}+\mathbf{k}_0}$ in which we take into account only the terms responsible for the third resonance (2.203). To do that we replace $\mathbf{k} \to \mathbf{k} + \mathbf{k}_0$ in equation (2.200) which results in

$$\begin{aligned}\frac{\partial \alpha_{\mathbf{k}+\mathbf{k}_0}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}+\mathbf{k}_0}\alpha_{\mathbf{k}+\mathbf{k}_0} &= -2\mathrm{i}V_{\mathbf{k}+\mathbf{k}_0|\mathbf{k}_0,\mathbf{k}} A e^{-i\omega_{\mathbf{k}_0}t}\alpha_{\mathbf{k}} \\ &- 2\mathrm{i}V^*_{\mathbf{k}_0|\mathbf{k}+\mathbf{k}_0,\mathbf{k}_0} A e^{-i\omega_{\mathbf{k}_0}t}\alpha^*_{-\mathbf{k}} - 2\mathrm{i}V^*_{2\mathbf{k}_0+\mathbf{k}|\mathbf{k}+\mathbf{k}_0,\mathbf{k}_0} A^* e^{+\mathrm{i}\omega_{\mathbf{k}_0}t}\alpha_{2\mathbf{k}_0+\mathbf{k}}.\end{aligned} \tag{2.218}$$

Only the first term in r.h.s. of that equation participates in the resonance (2.201). Thus we reduce equation (2.214) to

$$\frac{\partial \alpha_{\mathbf{k}+\mathbf{k}_0}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}+\mathbf{k}_0}\alpha_{\mathbf{k}+\mathbf{k}_0} = -2\mathrm{i}V_{\mathbf{k}+\mathbf{k}_0|\mathbf{k}_0,\mathbf{k}} A e^{-i\omega_{\mathbf{k}_0}t}\alpha_{\mathbf{k}}. \tag{2.219}$$

The time-dependent exponent is excluded from equations (2.217) and (2.219) by the following change of variables

$$\alpha_{\mathbf{k}+\mathbf{k}_0} \equiv \tilde{\alpha}_{\mathbf{k}+\mathbf{k}_0} e^{-\mathrm{i}\omega_{\mathbf{k}_0}t}.$$

which results in the following homogeneous system of two linear ODEs over time t with constant coefficients

$$\begin{aligned}\frac{\partial \alpha_{\mathbf{k}}}{\partial t} + \mathrm{i}\omega_{\mathbf{k}}\alpha_{\mathbf{k}} &= -2\mathrm{i}V^*_{\mathbf{k}_0+\mathbf{k}|\mathbf{k},\mathbf{k}_0} A^* \tilde{\alpha}_{\mathbf{k}+\mathbf{k}_0}, \\ \frac{\partial \tilde{\alpha}_{\mathbf{k}+\mathbf{k}_0}}{\partial t} + \mathrm{i}(\omega_{\mathbf{k}+\mathbf{k}_0} - \omega_{\mathbf{k}_0})\tilde{\alpha}_{\mathbf{k}+\mathbf{k}_0} &= -2\mathrm{i}V_{\mathbf{k}+\mathbf{k}_0|\mathbf{k}_0,\mathbf{k}} A \alpha_{\mathbf{k}}\end{aligned} \tag{2.220}$$

for the variables $\alpha_{\mathbf{k}}$ and $\tilde{\alpha}_{\mathbf{k}+\mathbf{k}_0}$. Here we used the symmetry $V^*_{\mathbf{k}|\mathbf{k}+\mathbf{k}_0,\mathbf{k}_0} = V^*_{\mathbf{k}|\mathbf{k}_0,\mathbf{k}+\mathbf{k}_0}$, (see equation (2.189)) and $\mathbf{k}$ plays the role of the parameter of the ODE system (2.220). Looking for the solution of the system (2.220) in the exponential form $\alpha_{\mathbf{k}}$, $\tilde{\alpha}_{\mathbf{k}+\mathbf{k}_0} \propto e^{\lambda t}$, we obtain the following quadratic equation for λ

$$\lambda^2 + \mathrm{i}(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}+\mathbf{k}_0})\lambda - \omega_{\mathbf{k}}(\omega_{\mathbf{k}+\mathbf{k}_0} - \omega_{\mathbf{k}_0}) + 4|V_0|^2|A|^2 = 0,$$

which has the solutions

$$\lambda_{\pm} = -\frac{\mathrm{i}}{2}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_0} + \omega_{\mathbf{k}+\mathbf{k}_0}) \pm \sqrt{-4|V_0|^2|A|^2 - \frac{(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_0} - \omega_{\mathbf{k}+\mathbf{k}_0})^2}{4}},$$

where $V_{\mathbf{k}|\mathbf{k}_0,\mathbf{k}+\mathbf{k}_0} \equiv V_0$. Here $\lambda_{\pm}$ are purely imaginary implying stability for all values of V_0, A and the frequency detuning $\Delta\Omega \equiv \omega_{\mathbf{k}} + \omega_{\mathbf{k}_0} - \omega_{\mathbf{k}+\mathbf{k}_0}$. Thus the growth rate is $\gamma_{\mathbf{k}} \equiv Re(\lambda_{\pm}) \equiv 0$.

2.10.1. Three wave decay process and three-wave system. In Section 2.10 we considered the decay instability of the monochromatic wave (2.195) with the frequency $\omega_0 \equiv \omega(\mathbf{k}_0) \equiv \omega_{\mathbf{k}_0}$ according to the dispersion law (2.284) and the wavevector $\mathbf{k} = \mathbf{k}_0$. This instability occurs near the resonant surface defined by the system

$$\omega_0 = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}, \tag{2.221a}$$

$$\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2, \tag{2.221b}$$

reaching the maximum of the growth rate $\gamma_{max} = \gamma_{\mathbf{k}}$ (2.209) at the surface (2.221). The width (2.212) of that resonance (i.e. the range of unstable frequencies, $\gamma_k > 0$) is of the same order as γ_{max}. Equations (2.204) and (2.205) have the same form as the system (1.174) for the parametric resonance provided one replaces $\omega_{\mathbf{k}_0}$ by Ω and assumes that $\omega_{\mathbf{k}} = \omega_{\mathbf{k}_0-\mathbf{k}} = \omega$ with the additional trivial renaming of complex constants. It means that the parametric resonance considered in Section (1.13.2) is the particular case of the decay instability of monochromatic wave which occurs near the resonance $\Omega = 2\omega_0$, where Ω is the frequency of the periodic variation of the oscillator frequency ω_0 according to equation (1.167).

The decay instability (2.221) results in the transfer of wave energy from the monochromatic wave (2.195) into the secondary waves. These secondary waves initially grow exponentially in time with the growth rate (2.209). When amplitudes of the secondary waves turns nonsmall compare with the pump wave amplitude, one has to take into account the modification of the pump wave by the the secondary waves, i.e. one has to consider the nonlinear stage of the development of the decay instability. It requires to go beyond the linearization of Section 2.10 and consider the nonlinear equation (2.198). The remaining simplification is that the width (both in wavenumber and frequency spaces) of the secondary waves produced by the decay instability are small and determined by equation (2.212).

It is then natural to consider three narrow wave packets such that their leading (carrier) wavevectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and frequencies $\omega(\mathbf{k}_1), \omega(\mathbf{k}_2), \omega(\mathbf{k}_3)$ satisfy the resonance conditions

$$\begin{aligned} \omega_{\mathbf{k}_1} &= \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3}, \\ \mathbf{k_1} &= \mathbf{k}_2 + \mathbf{k}_3 \end{aligned} \tag{2.222}$$

while wave packet widths $|\Delta\mathbf{k}_j|$ are small compare with values of their wavenumber carriers, $|\Delta\mathbf{k}_j| \ll |\mathbf{k}_j|,\ j = 1,2,3$ as sketched in Figure ???. We also assume that $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ are well separated in Fourier space, i.e.

$$|\mathbf{k}_i - \mathbf{k}_j| \gg max(|\Delta\mathbf{k}_1|, |\Delta\mathbf{k}_3|, |\Delta\mathbf{k}_3|) \text{ for any } i, j = 1,2,3. \tag{2.223}$$

Then one can expand the dispersion relation near each carrier value as follows

$$\omega(\mathbf{k}_j + \boldsymbol{\kappa}_j) = \omega_j + \boldsymbol{\kappa}_j \cdot \mathbf{v}_j + O(|\boldsymbol{\kappa}_j|^2), \tag{2.224}$$

where $\omega_j \equiv \omega(\mathbf{k}_j)$ is the leading frequency and $\mathbf{v}_j \equiv \partial\omega/\partial\mathbf{k}_j$ is the group velocity for jth wave packet. Also $\boldsymbol{\kappa}_j$ is the deviation of the wave vector from the carrier value $\mathbf{k}_j$ with $|\boldsymbol{\kappa}_j| \lesssim |\Delta\mathbf{k}_j|$ and $j = 1,2,3$. The condition (2.223) allows to introduce envelopes for each of three wave packets as $c_j(\boldsymbol{\kappa}, t) \equiv a(\mathbf{k}_j + \boldsymbol{\kappa}, t)$ and

$$\psi_j(\mathbf{r}, t) \equiv \frac{1}{(2\pi)^{D/2}} \int c_j(\boldsymbol{\kappa}, t) e^{\mathrm{i}\boldsymbol{\kappa}\cdot\mathbf{r}} d\boldsymbol{\kappa},\ j = 1,2,3 \tag{2.225}$$

which means that $a_{\mathbf{k}}$ can be represented as

$$a_{\mathbf{k}} = \sum_{j=1}^{3} c_j(\mathbf{k} - \mathbf{k}_j, t) \tag{2.226}$$

and its inverse FT results in

$$a(\mathbf{r}, t) = \sum_{j=1}^{3} \psi_j(\mathbf{r}, t) e^{\mathrm{i}\mathbf{k}_j\cdot\mathbf{r}}. \tag{2.227}$$

Equations for each envelope $\psi_j,\ j = 1,2,3$ can be obtained in the leading order by plugging in equation (2.226) into the Hamiltonian (2.197) and averaging of over fast linear oscillations $c_j \propto e^{-\mathrm{i}\omega_j t}$. Together with equations (2.222)-(2.225) it results in

$$H_2 = \sum_{j=1}^{3} \omega_j \int |c_j(\boldsymbol{\kappa}, t)|^2 d\boldsymbol{\kappa} + \sum_{j=1}^{3} \left(\int \mathbf{v}_j \cdot \boldsymbol{\kappa}\, c_j^*(\boldsymbol{\kappa}, t) c_j(\boldsymbol{\kappa}, t) d\boldsymbol{\kappa} \right)$$

$$= \sum_{j=1}^{3} \omega_j \int |\psi_j|^2 d\mathbf{r} - \mathrm{i} \sum_{j=1}^{3} \left(\mathbf{v}_j \cdot \int \psi_j^* \nabla \psi_j d\mathbf{r} \right) \tag{2.228}$$

for the quadratic part of the Hamiltonian and

$$H_3 = 2 \int \left[V_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3} c_1^*(\boldsymbol{\kappa}_1, t) c_2(\boldsymbol{\kappa}_2, t) c_3(\boldsymbol{\kappa}_3, t) + c.c. \right] \delta(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_3) d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3$$

$$= 2(2\pi)^{D/2} \int \left[V_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3} \psi_1^*(\mathbf{r}, t) \psi_2(\mathbf{r}, t) \psi_3(\mathbf{r}, t) + c.c. \right] d\mathbf{r} \tag{2.229}$$

for the cubic part of the Hamiltonian. The two terms inside the square brackets of equation (2.229) satisfy the resonance condition $\omega_1 = \omega_2 + \omega_3$ while all other terms were removed (averaged out) because they include fast oscillations in time. Also the matrix element $V_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3} = V_{-\mathbf{k}_2-\mathbf{k}_3|\mathbf{k}_2\mathbf{k}_3}$ is evaluated in the leading order at $\boldsymbol{\kappa}_j = 0,\ j = 1, 2, 3$. Generally, $V_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3}$ is the complex constant but the transformation $\psi_j \to \psi_j \exp\left[-\mathrm{i} Arg(V_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3})\right],\ j = 1, 2, 3$ turns $V_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3}$ into the positive constant. Thus below we assume $V_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3} > 0$ without loss of generality.

Combining equations (2.228) and (2.229) one obtain the averaged Hamiltonian

$$H = \sum_{j=1}^{3} \omega_j \int |\psi_j|^2 d\mathbf{r} - \mathrm{i} \sum_{j=1}^{3} \int \psi_j^* (\mathbf{v}_j \cdot \nabla) \psi_j d\mathbf{r} + V \int (\psi_1^* \psi_2 \psi_3 + \psi_1 \psi_2^* \psi_3^*) d\mathbf{r}, \tag{2.230}$$

where

$$V = 2(2\pi)^{D/2} V_{\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3}. \tag{2.231}$$

The dynamical equations for $\psi_j,\ j = 1, 2, 3$ are obtained from the variation of teh Hamiltonian (2.230) over ψ_j^* as

$$\frac{\partial \psi_j}{\partial t} = -\mathrm{i} \frac{\delta H}{\delta \psi_j^*}, \quad j = 1, 2, 3 \tag{2.232}$$

which results in the following closed system of equations for ψ_1, ψ_2 and ψ_3:

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} + \mathrm{i}\omega_1 \psi_1 + \mathbf{v}_1 \cdot \nabla \psi_1 &= -\mathrm{i} V \psi_2 \psi_3, \\ \frac{\partial \psi_2}{\partial t} + \mathrm{i}\omega_2 \psi_2 + \mathbf{v}_2 \cdot \nabla \psi_2 &= -\mathrm{i} V \psi_1 \psi_3^*, \\ \frac{\partial \psi_3}{\partial t} + \mathrm{i}\omega_3 \psi_3 + \mathbf{v}_3 \cdot \nabla \psi_3 &= -\mathrm{i} V \psi_1 \psi_2^*. \end{aligned} \tag{2.233}$$

The additional transformation $\psi_j \to \psi_j e^{-\mathrm{i}\omega_j t}$, $j = 1, 2, 3$ (it means that new ψ_j are envelopes both in space and time) removes terms with ω_j in equation (2.233) which gives

$$
\begin{aligned}
\frac{\partial\psi_1}{\partial t} + \mathbf{v}_1\cdot\nabla\psi_1 &= -\mathrm{i}V\psi_2\psi_3, \\
\frac{\partial\psi_2}{\partial t} + \mathbf{v}_2\cdot\nabla\psi_2 &= -\mathrm{i}V\psi_1\psi_3^*, \\
\frac{\partial\psi_3}{\partial t} + \mathbf{v}_3\cdot\nabla\psi_3 &= -\mathrm{i}V\psi_1\psi_2^*.
\end{aligned} \tag{2.234}
$$

Equations (2.234) are called by *three-wave system* as well as it is sometimes called by Bloembergen's system in nonlinear optics. Similar to (2.233), the system (2.234) has the Hamiltonian form (2.232) with the Hamiltonian given by

$$
H = -\mathrm{i}\sum_{j=1}^{3}\int \psi_j^*(\mathbf{v}_j\cdot\nabla)\psi_j d\mathbf{r} + V\int(\psi_1^*\psi_2\psi_3 + \psi_1\psi_2^*\psi_3^*)d\mathbf{r}. \tag{2.235}
$$

Note that both Hamiltonians (2.230) and (2.235) are real-valued which is verified by the integration by parts of the first term in r.h.s. of (2.235) as $\mathrm{i}\int\psi_j^*(\mathbf{v}_j\cdot\nabla)\psi_j d\mathbf{r} = (\mathrm{i}/2)\int\left[\psi_j^*(\mathbf{v}_j\cdot\nabla)\psi_j - \psi_j(\mathbf{v}_j\cdot\nabla)\psi_j^*\right]d\mathbf{r}$.

Beyond the conservation of the Hamiltonian (2.235), $dH/dt = 0$, three-wave system (2.234) has two additional conserved quantities (integrals of motion) $dI_1/dt = dI_2/dt = 0$, given by

$$
I_1 \equiv N_1 + N_2 \tag{2.236}
$$

and

$$
I_2 \equiv N_1 + N_3, \tag{2.237}
$$

where the integrals (squares of L_2 norm of ψ_j)

$$
N_j \equiv \int|\psi_j|^2 d\mathbf{r}, \quad j = 1,2,3 \tag{2.238}
$$

are called by either the number of particles or the wave action or the optical power for jth wave for different applications of nonlinear waves.

The three-wave system (2.234) has the infinite number of other conserved quantities (beyond (2.235)-(2.237)) and that the system (2.234) can be solved for generic initial conditions which decays at $|\mathbf{r}|\to 0$ using the *inverse scattering transform* technique, see e.g. Ref. ??? In this Section we focus on the simple reductions of the system (2.234). The first reduction corresponds to $\mathbf{r}$-independent (space-homogenous) solutions $\psi_j(\mathbf{r},t)\equiv\Psi_j(t)$ which gives the following system of ODEs

$$
\begin{aligned}
\frac{d\Psi_1}{dt} &= -\mathrm{i}V\Psi_2\Psi_3, \\
\frac{d\Psi_2}{dt} &= -\mathrm{i}V\Psi_1\Psi_3^*, \\
\frac{d\Psi_3}{dt} &= -\mathrm{i}V\Psi_1\Psi_2^*.
\end{aligned} \tag{2.239}
$$

These equations have similarities with Euler top equations described in Section???. Euler top equations conserve the kinetic energy and the angular momentum. In a similar way, equations (2.239) conserve the energy (the Hamiltonian)

$$
E = V(\Psi_1^*\Psi_2\Psi_3 + \Psi_1\Psi_2^*\Psi_3^*) \tag{2.240}
$$

and

$$I_1 = |\Psi_1|^2 + |\Psi_2|^2 = const,$$
$$I_2 = |\Psi_1|^2 + |\Psi_3|^2 = const \tag{2.241}$$

which result from the conservation of the integrals (2.235), (2.236) and (2.237) of the general three-wave system (2.234).

The conservation of I_1 and I_2 (2.241) (as well as its more general integral form (2.236),(2.237)) is called by the *Manley–Rowe relations*. These relations have a simple physical interpretation if one thinks about $|\Psi_j|^2$ as the density of quasiparticles (quanta) of the jth type with Ψ_j and Ψ_j^* being the classical analogs of the annihilation and the creation operators in quantum mechanics. Using the complex conjugation symbol $*$ for Ψ_j instead of the dagger $\Psi_j^\dagger$ indicates that one can neglect the commutation rules of quantum mechanical operators by taking the classical limit of large occupation numbers. In optical applications these quasiparticles are simply phonons. Then the first equation in (2.239) represents the rate of change of the number of quanta of the type 1 (described of l.h.s. of that equation) due to the annihilation of quanta of type 2 and 3 (described by r.h.s. of that equation). In a similar way, the rate of change of quanta of types 2 and 3 are determined by the annihilation of type 1 quanta and creation of type 3 and 2 quanta, respectively. Then the conservation of I_1 (**??**) means that the elementary decay process (2.222) includes the annihilation of one quantum of wave 1 and the creation of one quantum of wave 2 so the total number, I_1 is conserved. The same is true for waves 1 and 3 representing the conservation of I_2. These interpretations add a qualitative understanding of the Manley–Rowe relations while the conservation of I_1 and I_2 are immediately follow from equations (2.234).

The Manley–Rowe relations (**??**) provide tight constraints on possible values of the amplitude of pump wave $|\Psi_1|$ and the secondary waves $|\Psi_2|$ and $|\Psi_3|$. In particular, these amplitudes must remain bounded in time to satisfy (**??**) meaning that typically they experience oscillation-like dynamics with the periodic in time exchange of energy between three waves. There is an exception from that general behaviour for which the entire energy of the pump wave is transferred to the secondary waves as times evolves provided. That exception is ensured by specific relation between the phases of all three waves. In Section (2.10.2) we provide the particular example of such type of solution.

The simplest solution of the system (2.239) is given by $\Psi_1 = const, \Psi_2 = \Psi_3 = 0$. This is nothing more than the monochromatic wave (2.195) of Section 2.10 if we choose $A = (2\pi)^{D/2}\Psi_1$. The linearizaton of the system (2.239) about that solution results in the system of two linear ODEs

$$\begin{aligned}\frac{d\Psi_2}{dt} &= -\mathrm{i}(2\pi)^{-D/2} V A \Psi_3^*,\\ \frac{d\Psi_3^*}{dt} &= \mathrm{i}(2\pi)^{-D/2} V A^* \Psi_2.\end{aligned}$$

for two unknowns Ψ_2 and Ψ_3. Assuming $\Psi_2,\ \Psi_3 \propto e^{\gamma t}$ that systems results in

$$\gamma = \pm\frac{1}{(2\pi)^{D/2}}|VA| \tag{2.242}$$

with sign corresponding to the instability. Equation (2.242) corresponds to the maximum of the growth rate (2.209) at the exact resonance taking into account the normalizations (2.208) and (2.231).

The second reduction of the three-wave system (2.234) corresponds to time-independent (steady) waves $\psi_j(\mathbf{r},t) \equiv \Psi_j(\mathbf{r}),\ j=1,2,3$. In optics such situation occurs for the propagation of continuous wave (cw) monochromatic beam, i.e. the laser pump beam with time-independent power, in crystal with the nonzero quadratic susceptibility $\chi^{(2)}$. Such propagation of the cw beam with the frequency ω_1 results in the generation of the secondary beams with frequencies ω_2 and ω_3 satisfying the resonance condition (2.222). Time-independent solution of the system (2.234) implies that one has to solve the boundary-value problem. The simplest example of such problem occurs for the propagation of three infinitely wide beams in the same direction parametrized by the distance x which results in the ODE system

$$
\begin{aligned}
v_1 \frac{\partial \Psi_1}{\partial x} &= -iV\Psi_2\Psi_3, \\
v_2 \frac{\partial \Psi_2}{\partial x} &= -iV\Psi_1\Psi_3^*, \\
v_3 \frac{\partial \Psi_3}{\partial x} &= -iV\Psi_1\Psi_2^*,
\end{aligned}
\tag{2.243}
$$

which is mathematically equivalent to the system (2.239) if we replace x by t and assume that the group velocities of all harmonics have the same values $v_1 = v_2 = v_3$, where $|\mathbf{v}_j| := v_j, j = 1,2,3$.

We note that the system (2.243) can be also derived from the variation of the action $S = \int L dx$ by setting $\delta S = 0$, where

$$
L = -\mathrm{i}\sum_{j=1}^{3} v_j \Psi_j^* \frac{\partial}{\partial x}\Psi_j + V(\Psi_1^*\Psi_2\Psi_3 + \Psi_1\Psi_2^*\Psi_3^*)
\tag{2.244}
$$

is the Lagrangian and x plays the role of time.

Problems 2.10.1

2.10.1.1 Prove that I_1 and I_2 defined by (2.236)-(2.238) are the integrals of motion of the system (2.234), i.e. show that $\frac{dI_1}{dt} = \frac{dI_2}{dt} = 0$.

2.10.1.2 Show that the resonance condition (2.222) ensures the conservation of the first term in the Hamiltonian (2.230).???

2.10.1.3 Use the Lagrangian (2.244) to construct the Hamiltonian with the pairs the canonically conjugated coordinates and momenta. Show by the direct calculation that that the corresponding canonical Hamiltonian equations are equivalent to the system (2.243).

2.10.2. Second-harmonic generation. We now consider a second-harmonic generation (SHG, also sometimes called frequency doubling) which is the inverse process to the decay process of Section 2.10.1. In that case

$$
\omega(\mathbf{k}_0) + \omega(\mathbf{k}_0) = \omega(2\mathbf{k}_0),
\tag{2.245}
$$

i.e. two quanta of the pump wave with the frequency $\omega(\mathbf{k}_0)$ result in the generation of the double frequency wave $\omega(\mathbf{2k}_0)$. For the electromagnetic wave (EMW) such process is nearly allowed (i.e. equation (2.245) is nearly satisfied) because the dispersion law of EMW in many media is very close to the linear dependence

$$\omega = kc_m, \tag{2.246}$$

where c_m is the speed of light in the medium which typically depends only weakly on the frequency ω (or on k). However, if we take into account such weak dependence (i.e. take into account the dispersion of the medium) then the resonance (2.245) turns only approximate and is replaced by the mismatch

$$\Omega := \omega(2\mathbf{k}_0) - 2\omega(\mathbf{k}_0). \tag{2.247}$$

Then the weak dispersion of the medium means that the mismatch is small, $|\Omega| \ll |\omega(2\mathbf{k}_0)|, 2|\omega(\mathbf{k}_0)|$ and at the leading order we assume that $\Omega = 0$.

SHG is the particular case of three wave process. Compare with the condition (2.223), only two waves are now localized in Fourier space near $\mathbf{k} = \mathbf{k}_0, 2\mathbf{k}_0$. So we cannot use the results of Section 2.10.1 directly. Instead we define the amplitudes ψ_0 and ψ_2 of the pump wave and the second harmonic, respectively and rewrite the Hamiltonian (2.230) as follows

$$\begin{aligned} H = \omega(\mathbf{k}_0)\int |\psi_0|^2 d\mathbf{r} + \omega(2\mathbf{k}_0)\int |\psi_2|^2 d\mathbf{r} - i\int \psi_0^*(\mathbf{v}_0\cdot\nabla)\psi_0 d\mathbf{r} - i\int \psi_2^*(\mathbf{v}_2\cdot\nabla)\psi_2 d\mathbf{r} \\ + V\int (\psi_2^*\psi_0\psi_0 + \psi_2\psi_0^*\psi_0^*) d\mathbf{r}, \end{aligned} \tag{2.248}$$

where $\mathbf{v}_0$ and $\mathbf{v}_2$ are the corresponding group velocities of the first and the second harmonics and

$$V = 2(2\pi)^{D/2} V_{2\mathbf{k}_0|\mathbf{k}_0\mathbf{k}_0}. \tag{2.249}$$

Then using equations (2.232) and (2.248) we obtain that

$$\begin{aligned} \frac{\partial\psi_0}{\partial t} + i\omega(\mathbf{k}_0)\psi_0 + (\mathbf{v}_0\cdot\nabla)\psi_0 &= -2iV\psi_2\psi_0^*, \\ \frac{\partial\psi_2}{\partial t} + i\omega(2\mathbf{k}_0)\psi_2 + (\mathbf{v}_2\cdot\nabla)\psi_2 &= -iV\psi_0^2. \end{aligned} \tag{2.250}$$

Similar to the general three wave system (2.233), the system (2.250) has nontrivial integrals of motion but, contrary to the system (2.233), the reduction of two separate harmonics ψ_2 and ψ_3 of Section 2.10.1 into the single pump wave ψ_0 and the second harmonic ψ_2 implies only one independent *Manley–Rowe relation (integral)*

$$N = \int \left(|\psi_0|^2 + 2|\psi_2|^2\right) d\mathbf{r} = const. \tag{2.251}$$

A change of variables

$$\psi_0 \to \psi_0 e^{-i\omega(\mathbf{k}_0)t} \quad \text{and} \quad \psi_2 \to \psi_2 e^{-2i\omega(\mathbf{k}_0)t} \tag{2.252}$$

in the system (2.250) results in

$$\begin{aligned} \frac{\partial\psi_0}{\partial t} + (\mathbf{v}_0\cdot\nabla)\psi_0 &= -2iV\psi_2\psi_0^*, \\ \frac{\partial\psi_2}{\partial t} + i\Omega\psi_2 + (\mathbf{v}_2\cdot\nabla)\psi_2 &= -iV\psi_0\psi_0, \end{aligned} \tag{2.253}$$

where we used the definition (2.247) of the mismatch.

We now consider a particular case of the EMW with the frequency ω_0 and the time independent amplitude A which enters the nonlinear crystal and propagates along axis x with $x = 0$ being the location of the entrance surface of the crystal. We assume the zero mismatch $\Omega = 0$ and no dependence of EMW on other spatial coordinates. Then the system (2.253) for time-independent amplitudes $\psi_0 = \psi_0(x)$ and $\psi_2 = \psi_2(x)$ with the zero mismatch $\Omega = 0$ reduces to the system of two ODEs

$$\begin{aligned} v_0 \frac{d\psi_0}{dx} &= -2iV\psi_2\psi_0^*, \\ v_2 \frac{d\psi_2}{dx} &= -iV\psi_0^2 \end{aligned} \tag{2.254}$$

with the boundary conditions

$$\psi_0|_{x=0} = A, \quad \psi_2|_{x=0} = 0. \tag{2.255}$$

Here we use the notation $|\mathbf{v}_0| = v_0$, $|\mathbf{v}_2| = v_2$. Equation (2.254) is the analog of equation (2.243) for SHG. The Manley–Rowe relation (2.251) for the evolution along x (instead of the evolution over t in the system (2.250)) is modified into

$$N = v_0|\psi_0|^2 + 2v_2|\psi_2|^2 = const. \tag{2.256}$$

If A is purely real, then a change of variable

$$\psi_2 = i\varphi \tag{2.257}$$

reduces equation (2.254) to the real form given by

$$v_0 \tfrac{d\psi_0}{dx} = 2V\varphi\psi_0, \tag{2.258}$$

$$v_2 \tfrac{d\varphi}{dx} = -V\psi_0^2. \tag{2.259}$$

One can multiply equation in (2.258) on ψ_0 to obtain equations for ψ_0^2 and φ as follows

$$v_0 \frac{d\psi_0^2}{dx} = 4V\varphi\psi_0^2, \tag{2.260}$$

$$v_2 \frac{d\varphi}{dx} = -V\psi_0^2. \tag{2.261}$$

Excluding here ψ_0^2, we obtain for φ the following equation

$$\frac{d^2\varphi}{dx^2} = \frac{4V}{v_0}\varphi\frac{d\varphi}{dx},$$

which is immediately integrated over x giving

$$v_0 \frac{d\varphi}{dx} = 2V\varphi^2 + C, \tag{2.262}$$

where the real constant C is determined from the boundary conditions (2.255) and equation (2.259) evaluated at $x = 0$ which result in $C = -\frac{v_0}{v_2}VA^2$. Then solving equation (2.262) by the separation of variables gives the integral

$$\frac{2Vx}{v_0} = \int\limits_0^{\varphi} \frac{d\varphi'}{(\varphi')^2 - \frac{v_0A^2}{2v_2}} = -\left(\frac{v_0A^2}{2v_2}\right)^{-1/2} \operatorname{arctanh}\left[\varphi\left(\frac{v_0A^2}{2v_2}\right)^{-1/2}\right]$$

which is immediately inverted resulting in

$$\varphi = -A\sqrt{\frac{v_0}{2v_2}} \tanh\left(\frac{\sqrt{2}AV}{\sqrt{v_0v_2}}x\right). \tag{2.263}$$

To find the explicit expression for ψ_0^2 we differentiate φ from equation (2.263) over x and plug into equation (2.261) which gives that

$$\psi_0^2 = A^2 \operatorname{sech}^2\left(\frac{\sqrt{2}AV}{\sqrt{v_0 v_2}}x\right). \tag{2.264}$$

Using equations (2.257), (2.264) and (2.263) one obtains the solution of the boundary value problem (2.254) and (2.255) as follows

$$\begin{aligned} \psi_0 &= A \operatorname{sech}\left(\frac{\sqrt{2}AV}{\sqrt{v_0 v_2}}x\right). \\ \psi_2 &= -\mathrm{i}A\sqrt{\frac{v_0}{2v_2}} \tanh\left(\frac{\sqrt{2}AV}{\sqrt{v_0 v_2}}x\right). \end{aligned} \tag{2.265}$$

Typical plots of these solutions are shown in Figure 1.

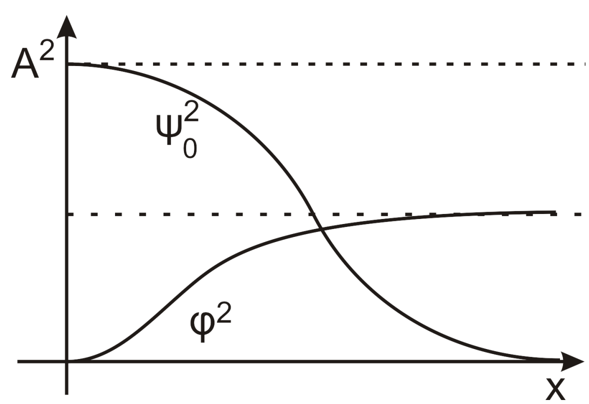

FIGURE 1. A typical spatial dependence of the first and the second harmonics for the exact resonance (2.245) of Section 2.10.2 for the particular case $v_0 = v_2$.

The Manley–Rowe relation (2.256) is trivially satisfied for the solutions (2.265). Thus in our particular case, the energy of the first harmonic is fully converted into the second-harmonic energy for $x \to \infty$.

For the crystal of the finite thickness L in the direction x, the coefficient

$$K = \operatorname{sech}^2\left(\frac{\sqrt{2}AV}{\sqrt{v_0 v_2}}x\right). \tag{2.266}$$

is called *the transformation coefficient* and $\alpha = \frac{\sqrt{2}AV}{\sqrt{v_0 v_2}}$ is called the *attenuation coefficient*. K means a fraction of optical energy transferred from the first harmonic to the second harmonic. ??? α has the dimension of the inverse length and is typically measured in terms of the inverse centimeters. Also $\sqrt{2}AV$ has the same scaling as the growth rate in equation (2.242) (dividing $\sqrt{2}AV$ by $\sqrt{v_0 v_2}$ we again obtain the dimension of the inverse centimeters). Equations (2.265) and (2.266) imply that the high intensity of the second harmonics at the exit of crystal is achieved provided L is large as well as that an increase of the nonlinear constant V

allows to make L smaller for the same K. The results of this section assume that there is no linear absorbtion of the light in the crystal which is justified if the L is significantly smaller than the linear absorbtion length of the crystal. Thus making V large (and, respectively L smaller) is beneficial to avoid losses from the linear absorbtion.

SHG has been widely used in commercial laser application with often near 100% achieved. SHG is the common source to make green 532 nm green laser pointer from a 1064 nm source embeded into the pointer.

2.10.3. Generation of the second harmonic for nonzero mismatch. Section 2.10.2 addressed a particular case $\mathrm{Im}\psi_0 = 0$, $i\psi_2 \in \mathbb{C}$ and $\Omega = 0$. In this section we allow $\Omega \neq 0$ and consider equation (2.253) which has the following Hamiltonian

$$H = \Omega \int |\psi_2|^2 d\mathbf{r} - i \int \psi_0^*(\mathbf{v}_0 \cdot \nabla)\psi_0 d\mathbf{r} - i \int \psi_2^*(\mathbf{v}_2 \cdot \nabla)\psi_2 d\mathbf{r} + V \int (\psi_2^* \psi_0 \psi_0 + \psi_2 \psi_0^* \psi_0^*) d\mathbf{r}. \tag{2.267}$$

corresponding to the change of variables (2.252) in the Hamiltonian (2.248).

We now consider a particular case of (2.253) there ψ_0 and ψ_2 are independent on the spatial coordinates resulting in

$$\begin{aligned} \frac{d\psi_0}{dt} &= -2iV\psi_2\psi_0^*, \\ \frac{d\psi_2}{dt} + i\Omega\psi_2 &= -iV\psi_0^2. \end{aligned} \tag{2.268}$$

We also note that in the particular (but quite practical case) $\mathbf{v}_0 = \mathbf{v}_2$ a change of variables to the moving frame with the velocity $\mathbf{v}_0$ in equation (2.253) also results in equation (2.268).

It is convenient to exclude V from equation (2.268) from the scaling transform $t \to t/V$ and $\Omega \to \Omega V$, as well as $\psi_0 \to \sqrt{2}\psi_0$ resulting in rescaled equations

$$\begin{aligned} \frac{d\psi_0}{dt} &= -2i\psi_2\psi_0^*, \\ \frac{d\psi_2}{dt} + i\Omega\psi_2 &= -2i\psi_0^2. \end{aligned} \tag{2.269}$$

which is easy to integrate using two motion integral which are the Manley–Rowe relations which take the following form in these rescaled coordinates

$$N = |\psi_0|^2 + |\psi_2|^2 \tag{2.270}$$

and the energy

$$E = \Omega|\psi_2|^2 + 2(\psi_2{\psi_0^*}^2 + \psi_2^*\psi_0^2). \tag{2.271}$$

It is convenient to additionally rescale ψ_0 and ψ_2 by $\sqrt{N}$ as follows

$$\psi_0 \to \sqrt{N}\psi_0, \psi_2 \to \sqrt{N}\psi_2,$$

as well as $E \to EN^{3/2}$, $\Omega \to \Omega\sqrt{N}$. It means that equations (2.270) and (2.271) take the following form

$$\begin{aligned} 1 &= |\psi_0|^2 + |\psi_2|^2, \\ E &= \Omega|\psi_2|^2 + 2(\psi_2{\psi_0^*}^2 + \psi_2^*\psi_0^2), \end{aligned}$$

Using these equations and defining the real amplitudes and phases of ψ_0 and ψ_2 as

$$\psi_0 = ue^{i\varphi_0}, \psi_2 = ve^{i\varphi_2},\ u \geq 0,\ v \geq 0,$$

we rewrite the system (2.269) as

$$\begin{aligned}\frac{du}{dt} + iu\frac{d\varphi_0}{dt} &= -2ie^{-2i\varphi_0 + i\varphi_2}uv,\\ \frac{dv}{dt} + i(\Omega + \frac{d\varphi_2}{dt})v &= -2ie^{2i\varphi_0 - i\varphi_2}u^2.\end{aligned}$$

Separating real and imaginary parts of these two equations we obtain the following system of four real equations

$$\begin{aligned}\frac{du}{dt} &= 2uv\sin(\varphi_2 - 2\varphi_0),\\ \frac{dv}{dt} &= 2u^2\sin(\varphi_2 - 2\varphi_0),\\ u\varphi_{0t} &= -2uv\cos(\varphi_2 - 2\varphi_0),\\ (\varphi_{2t} + \Omega)v &= -2u^2\cos(\varphi_2 - 2\varphi_0).\end{aligned} \tag{2.272}$$

One notices a simplification that the phases φ_0 and φ_2 enter into equations (2.272) only in a single combination $\chi = \varphi_2 - 2\varphi_0$. Also by combining the last two equations of (2.272) one obtains that

$$\frac{d\chi}{dt} + \Omega = 2\left(v - \frac{u^2}{v}\right)\cos\chi. \tag{2.273}$$

That dependence on χ allows to put together the first two equations from (2.272) and equation (2.273) to obtain the following system of three ODEs

$$\begin{aligned}\frac{du}{dt} &= 2uv\sin\chi,\\ \frac{dv}{dt} &= 2u^2\sin\chi,\\ \frac{d\chi}{dt} + \Omega &= 2\frac{v^2 - u^2}{v}\cos\chi\end{aligned} \tag{2.274}$$

for the unknowns u, v and χ. That system has two integrals of motion (2.270) and (2.271) which in the variables u, v and χ take the following form

$$\begin{aligned}1 &= u^2 + v^2,\\ E &= \Omega v^2 + 4vu^2\cos\chi.\end{aligned}$$

It is convenient to express $\cos\chi$ and u through these equations as follows

$$u^2 = 1 - v^2, \tag{2.275}$$

$$\cos\chi = \frac{E - \Omega v^2}{4vu^2}. \tag{2.276}$$

so that we can now solve the second equation in the system (2.274).

Assume that the amplitude of the second harmonic is zero at the initial time $t = 0$, i.e. $v = 0$. Then equation (2.275) implies that $u = 1$ while equation (2.276) requires that $E = 0$, and, respectively,

$$\cos\chi = \frac{-\Omega v}{4(1 - v^2)}.$$

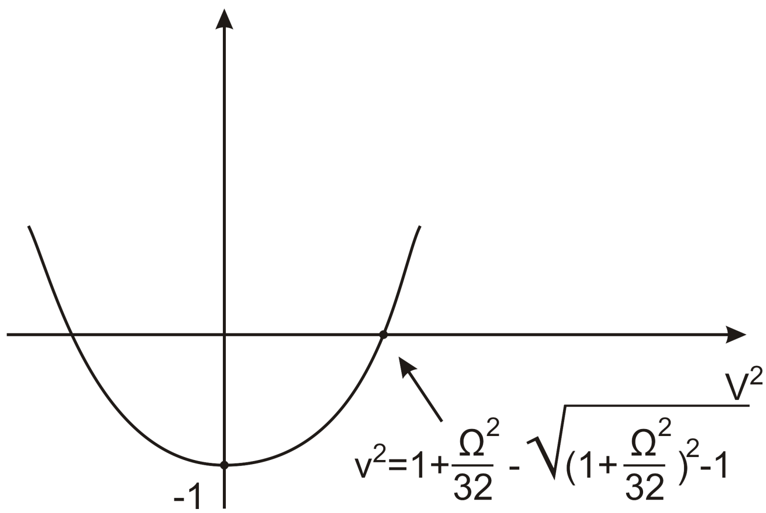

FIGURE 2. A plot the function $U(v)$ from equation (2.279).

Then

$$\sin\chi = \pm\left[1 - \frac{\Omega^2}{16}\frac{v^2}{(1-v^2)^2}\right]^{1/2}. \tag{2.277}$$

Equations (2.275), (2.277) and the second equation in the system (2.274) result in the closed equation

$$\frac{dv}{dt} = \pm 2(1-v^2)\left[1 - \frac{\Omega^2}{16}\frac{v^2}{(1-v^2)^2}\right]^{1/2}.$$

It is convenient to take a square of both sides of that equation to easily understand the qualitative properties of its solutions $v(t)$ which gives that

$$\left(\frac{dv}{dt}\right)^2 + U(v) = 0, \tag{2.278}$$

where

$$U(v) := \frac{\Omega^2}{4}v^2 - 4(1-v^2)^2 \tag{2.279}$$

can be treated as the effective potential energy for the motion of the effective Newtonian particle (see e.g. equation (1.4) with $N = 1$) with the coordinate v, the mass $m = 2$ and the energy $E = H = 0$. A plot $U(v)$ is shown in Fig. 2.

Depending on the initial value of sign(χ), this effective effective particle initially either moves to the right or left according to the second equation in the system (2.274) stopping at one of two turning (reflection) points $v = \pm v_{reflection}$, and going back after that, where

$$v_{reflection} = \left[1 + \frac{\Omega^2}{32} - \sqrt{\left(1+\frac{\Omega^2}{32}\right)^2 - 1}\right]^{1/2} \leq 1 \tag{2.280}$$

is determined from the condition that $U(\pm v_{reflection}) = 0$. After the reflection, the particle moves to the opposite direction until reaching the second reflection points etc. It means that $v(t)$ experience periodic oscillations with the period T found from the solution of equation (2.278) as

$$T = 4 \int_0^{v_{reflection}} \frac{dv}{\sqrt{-U(v)}}. \tag{2.281}$$

The limit $\Omega \to 0$ recovers the solution (2.265) with $v^2_{reflection} = 1$ (with a trivial replacement of x by t and $v_0,\ v_2$ by 1) reached at $t \to \infty$. Thus a full conversion of the first harmonic into the second one is achieved during the infinite time for $\Omega = 0$. Any nonzero value of Ω results in the oscillations of $v(t)$ with the full conversion never achieved.

Qualitative the same effect of the lack of the full transformation from the fundamental harmonic to other harmonics occurs for the interaction of three harmonics provided the mismatch Ω is nonzero, i.e.

$$\Omega \quad = \quad \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3) \neq 0,$$

while $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$. The three-wave system in that case is similar to equations (2.234) but with the added nonzero mismatch Ω take the following form

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} + i\Omega\psi_1 + (\mathbf{v}_1 \cdot \nabla)\psi_1 &= -iV\psi_2\psi_3, \\ \frac{\partial \psi_2}{\partial t} + (\mathbf{v}_2 \cdot \nabla)\psi_2 &= -iV\psi_1\psi_3^*, \\ \frac{\partial \psi_3}{\partial t} + (\mathbf{v}_3 \cdot \nabla)\psi_3 &= -iV\psi_1\psi_2^*. \end{aligned} \tag{2.282}$$

The applicability conditions of the system (2.282) is that mismatch is small compare with all frequencies of each wave, i.e. $|\Omega| \ll \omega_1, \omega_2, \omega_3$. If that condition is not satisfied then one cannot reduce the general three wave Hamiltonian (2.191)-(2.193) to the reduced Hamiltonian (2.197). We remind that such reduction assumes the nonresonant terms in the Hamiltonian average out because of fast nearly linear oscillations.

Problems 2.10.2

2.10.1 Find the growth rate of the decay instability from the analysis of the system (2.282). Compare that growth rate with equation (2.209).
Hint: Consider the spatially uniform solution ($\nabla\psi_1 \equiv 0$) such that $\psi_1 = Ae^{-i\Omega t}$, $\psi_{2,3} = 0$ (the steady-state solution). Then linearize about that solution.

2.10.2 Show that the stationary solutions ($\frac{\partial}{\partial t} \equiv 0$) of the three-wave system (2.282) for parallel group velocities $\mathbf{v}_1 \parallel \mathbf{v}_2 \parallel \mathbf{v}_3$ are mathematically equivalent to the spatially uniform solutions ($\nabla \equiv 0$) of the three-wave system (2.282).
Hint: use the scaling transformations for the amplitudes $\psi_j,\ j = 1, 2, 3$.

2.10.4. Quasi-phase-matching. Quasi-phase-matching is a technique in nonlinear optics based on the design of periodic structures in the quadratic nonlinear media. Often it is achieved by the periodic in space flipping of the crystal axis by 180 degrees. Assuming that such spatial variation with the period $2\pi/\Delta k$ occurs in the direction of the wavevector $\Delta\mathbf{k}$, we conclude that the quadratic nonlinear susceptibility $\chi^{(2)}$ in equation (2.112) has the periodic spatial variation $\propto \cos(\Delta\mathbf{k}\cdot\mathbf{r})$ (as well as generally $\chi^{(2)}$ has higher harmonics $2\Delta\mathbf{k}\cdot\mathbf{r}, 3\Delta\mathbf{k}\cdot\mathbf{r}, \ldots$ which we neglect at the leading order), where $\Delta k = |\Delta\mathbf{k}|$ is the wavenumber. Then equation (2.221a) of the resonant surface (2.221) is replaced by

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 + \Delta\mathbf{k}, \tag{2.283}$$

while equation (2.221b) remains valid. Equation (2.283) allows the additional flexibility in achieving three wave interactions by varying $\Delta\mathbf{k}$. For example, the interaction between the counterpropagating first and second harmonics can be routinely achieved instead of co-propagation described in Section 2.10.2. ???

2.11. Four-wave nonlinear interactions

2.11.1. Decay vs. non-decay dispersion laws. Assume that the medium allows propagation of linear waves with the dispersion law

$$\omega \equiv \omega_{\mathbf{k}} \equiv \omega(\mathbf{k}) \tag{2.284}$$

so the the quadratic Hamiltonian in complex variable $a_{\mathbf{k}}$, is given by

$$H_2 = \int \omega_k |a_{\mathbf{k}}|^2 d\mathbf{k}. \tag{2.285}$$

The dispersion law which allows solution of a system

$$\begin{aligned} \omega_{\mathbf{k}} &= \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}, \\ \mathbf{k} &= \mathbf{k}_1 + \mathbf{k}_2 \end{aligned} \tag{2.286}$$

is called the *decay dispersion law* because it allows a wave with wave vector $\mathbf{k}$ and frequency $\omega_{\mathbf{k}}$ to decay into two waves with wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and frequencies $\omega_{\mathbf{k}_1}$, $\omega_{\mathbf{k}_2}$, respectively. Such decay dispersion law was assumed in Section 2.10. We remind that the terminology of "decay" comes from the analogy with (quasi)particles decay into a number of other quasi(particles) in quantum mechanics. In the particular case of it can be interpreted as decay of one quasiparticle to tow other quasiparticles.

Another possibility to have the decay dispersion law is to satisfy another system,

$$\begin{aligned} \omega_{\mathbf{k}} + \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} &= 0, \\ \mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 &= 0, \end{aligned} \tag{2.287}$$

which requires to have different signs of the dispersion $\omega_{\mathbf{k}}$ for different $\mathbf{k}$ which is possible only for systems with the linear instability at least in some range of values of $\mathbf{k}$. Such situation occur e.g. for propagation of particle beams in plasma (see .eg. Ref. ???)===???Add more discussion on that???==== We do not consider the case of equation (2.287) because it is lett common. In particular such case is forbidden for the linearly stable media. If there are no solutions of either system (2.286) or (2.287) then $\omega_{\mathbf{k}}$ is called the *non-decay dispersion law.*

The decay condition (2.286) is satisfied for wide classes of dispersion law $\omega(\mathbf{k})$. Consider the particular example of the Landau dispersion law (the Landau spectrum) of the excitations $\omega = \omega(\mathbf{k}) \equiv \omega(k)$ in the superfluid Helium 4 (see e.g. Ref.

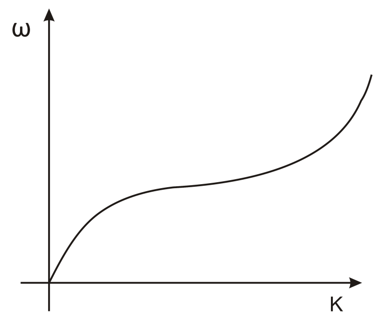

FIGURE 3. Landau spectrum of excitations $\omega(k)$.

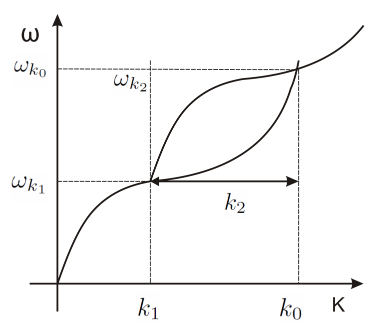

FIGURE 4. Visual proof that the Landau spectrum of excitations from Fig. 3 corresponds to the decay dispersion law.

[**LLP13**]).The qualitative form of that spectrum is shown in Fig. 4 with the sub-linear growth for small k and super-linear growth for larger k. One can graphically show that this spectrum corresponds to decay dispersion law as follows. We take a copy of the curve $\omega = \omega(k)$ from Fig. 3 and translate it in (k, ω) plane to have the origin $(k, \omega) = (0, 0)$ moved to the point $(k, \omega) = (k_1, \omega(k_1))$ as shown in Fig. 4. We also keep the original curve $w = \omega(k)$ in that Fig. It is clear Fig. 4 that these two curves intersect at the second point $(k_0, \omega(k_0))$ (in addition to $(k_1, \omega(k_1))$ which means that both resonance conditions (2.286) are satisfied as follows

$$\begin{aligned} \omega_{\mathbf{k}_0} &= \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}, \\ \mathbf{k}_0 &= \mathbf{k}_1 + \mathbf{k}_2, \end{aligned}$$

where $\mathbf{k}_0$, $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel to each other and point in the same direction. It provide the visual proof that the Landau spectrum of excitations is the decay dispersion law. from Fig. 3 corresponds to the decay dispersion law.

Following the same approach we now prove that

$$\omega = Ak^{\alpha} \tag{2.288}$$

is the decay dispersion law for $\alpha > 1$ while for $0 < \alpha < 1$ it is the non-decay dispersion law. Here we assume that $\mathbf{k} = (k_x, k_y) \in \mathbb{R}^2$. Instead of the (k, ω) plane

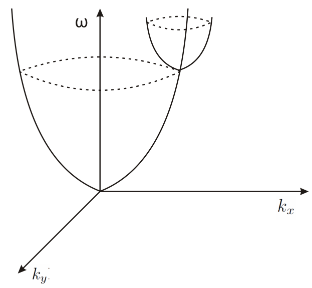

FIGURE 5. Visual proof that $\omega = Ak^{\alpha}$ with $\alpha > 1$ is the decay dispersion law.

of Fig. 3, we consider $3D$ plots with the coordinates (k_x, k_y, ω) as shown in Figs. 5 and 6. If $\alpha > 1$ then the surface $\omega = \omega(\mathbf{k})$ has the form of cup as schematically shown in Fig. 5. Then it is visually clear that if we move such cup (the dispersion surface) to any nonzero point $\mathbf{k} = \mathbf{k_1}$, $\omega = \omega(\mathbf{k_1})$ of the dispersion curve, then both cups inevitably intersect over the closed curve as schematically shown in Fig. 5. Thus the conditions (2.286) are satisfied at that curve implying that $\alpha > 1$ corresponds to the decay dispersion law. If $0 < \alpha < 1$ then the surface $\omega = \omega(\mathbf{k})$ has the form of inverted cone bowl (cocktail-like glass) as schematically shown in Fig. 6. Then it is visually clear that if we move such bowl (the dispersion surface) to any nonzero point $\mathbf{k} = \mathbf{k_1}$, $\omega = \omega(\mathbf{k_1})$ of the dispersion curve, then both bowls never intersect again (beyond the point $\mathbf{k} = \mathbf{k_1}$, $\omega = \omega(\mathbf{k_1})$) as schematically shown in Fig. 6. Thus the conditions (2.286) are never satisfied implying that $0 < \alpha < 1$ corresponds to the non-decay dispersion law. By setting $k_y \equiv 0$ we obtain the same result for 1D case $\mathbf{k} \in \mathbb{R}$ that $\alpha > 0$ and $0 < \alpha < 1$ correspond to the decay and non-decay dispersion laws, respectively. It is not difficult to show (using the triangular inequalities) that the same result remains valid in any dimension $\mathbf{k} \in \mathbb{R}^D$.

It is shown below in Section ??? that the capillary waves on the surface of the infinitely deep fluid has the dispersion law $\omega = \sqrt{\sigma k^3}$, where σ is the surface tension coefficient. It implies that $\alpha = \frac{3}{2} > 1$. Thus the capillary waves have the decay dispersion law. The gravity waves on the surface of the infinitely deep fluid have the dispersion law $\omega = \sqrt{gk}$, where g is the acceleration due to gravity Thus $\alpha = 1/2 < 1$ in that case implying that the gravity waves have the non-decay dispersion law.

Problems 2.11

2.11.1 Consider the dispersion law $\omega = \sqrt{\omega_p^2 + k^2c^2}$ of the electromagnetic waves in isotropic plasma, where c is the speed of light and ω_p is the plasma frequency. Find if that dispersion law is decay or non-decay.

2.11.2 Consider the dispersion law $\omega = \sqrt{gk\tanh(kh)}$ of the surface gravity waves of the fluid with the finite depth h. Find if that dispersion law is decay or non-decay.

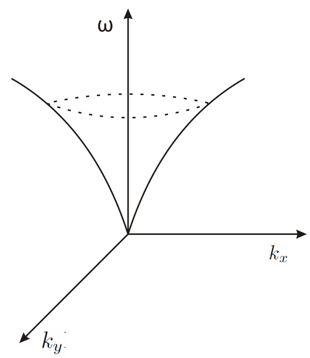

FIGURE 6. Visual proof that $\omega = Ak^{\alpha}$ with $0 < \alpha < 1$ is the non-decay dispersion law.

2.11.2. Removal of three wave interaction terms for non-decay dispersion law for the quasi-monochromatic waves. It was obtained in Section 1.13 for the particular example of a single nonlinear oscillation in one spatial dimension, that cubic terms in the Hamiltonian (and, respectively, quadratic terms in the dynamic equations) contribute only to the next order terms (the forth order terms in the Hamiltonian) with respect to the small parameter which is the amplitude of the oscillations at the fundamental frequency ω. Such next-order-only contribution results from the nonresonance character of these modes (only the fundamental mode correspond to the linear resonance). These nonresonant modes can be removed at the leading order of that small parameter from both the Hamiltonian and the dynamic equations resulting in the renormalization of the fourth order interaction constant T (1.143) thus contributing to the nonlinear frequency shift as given by equation (1.139).

In a similar way one can remove the nonresonant terms in the general case of the waves in continuous media. It represents the limit of the infinite number of degrees of freedom, i.e. waves. The general case of such removal is given in Section 2.15 while in this Section we consider a simple particular case of the contribution to the nonlinear frequency shift from the removal the cubic terms in the Hamiltonian for quasi-monochromatic wave (wave packet) centered at $\mathbf{k} = \mathbf{k}_0$ with the characteristic width Δk in $\mathbf{k}$ space.

The condition that the wave is quasi-monochromatic means the existence of the extra small parameter $\Delta k/k_0 \ll 1$ in addition to the weak nonlinearity assumption used in the general Hamiltonian series expansion (2.187). That extra small parameter makes sufficient at the leading order to evaluate the contribution to the nonlinear frequency shift from the fourth order Hamiltonian $H_{4,int}$ (2.190) near $\mathbf{k} = \mathbf{k}_0$. It means approximating the matrix element $T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}$ from (2.190) by $T_0 := T_{\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}$. Then equation (2.190) takes the following form

$$
\begin{aligned}
H_{4,int} &= \frac{1}{2}\int T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_3) \prod_{i=1}^{4} d\mathbf{k}_i. \\
&\simeq \frac{T_0}{2}\int a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_3) \prod_{i=1}^{4} d\mathbf{k}_i. \qquad (2.289)
\end{aligned}
$$

If the matrix elements V and U in (2.188) are identically zero, i.e. $H_3 \equiv 0$, then the Hamiltonian dynamical equations (2.184) in our approximation $H_4 \simeq H_{4,int}$ (with neglecting higher order terms H_5 etc. in equation (2.187)) take the following form

$$\begin{aligned}\partial_t a_{\mathbf{k}} &= -i\frac{\delta(H_2 + H_{4,int})}{\delta a^*_{\mathbf{k}}}\\ &= -i\omega_{\mathbf{k}} a_{\mathbf{k}} - i\,T_0 \int a^*_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4} \delta(\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_3) d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4. \end{aligned} \tag{2.290}$$

where we used equations (2.183) and (2.289).

For monochromatic wave

$$a_{\mathbf{k}} = A(t)\delta(\mathbf{k} - \mathbf{k}_0), \tag{2.291}$$

equation (2.290) reduces to

$$\frac{\partial A}{\partial t} + i\omega(\mathbf{k}_0)A = -iT_0|A|^2 A \tag{2.292}$$

while for the quasi-monochromatic wave one has to expand $\omega_{\mathbf{k}}$ near $\mathbf{k} = \mathbf{k}_0$ which is done in the next section.

Beyond the interaction Hamiltonian (2.289) we generally have to take into account the cubic interaction Hamiltonian H_3 (2.188). The only exception is if the matrix elements V and U in (2.188) are identically zero. Such exception is however not rare, e.g. it corresponds to the nonlinear dielectrics which are centro-symmetric with respect to the inversion of the spatial coordinates $\mathbf{r} \to -\mathbf{r}$ which results in equation (2.129) for the dielectric response (Kerr effect) as discussed in Section 2.5.

Consider a contribution of three-wave process into four-wave processes for the monochromatic wave (2.291). In the leading order of the nonlinearity $A(t)$ is determined by the linear dynamics, i.e.

$$A(t) \propto e^{-i\omega_{\mathbf{k}_0} t}. \tag{2.293}$$

The three-wave interactions result in the generation of the secondary harmonics with $\mathbf{k} = 2\mathbf{k}_0$ and $\mathbf{k} = 0$, i.e. equation (2.291) is replaced by

$$a_{\mathbf{k}} \simeq A\delta(\mathbf{k} - \mathbf{k}_0) + A_2\delta(\mathbf{k} - 2\mathbf{k}_0) + A_0\delta(\mathbf{k}) + A_{-2}\delta(\mathbf{k} + 2\mathbf{k}_0). \tag{2.294}$$

To find $A_2(t)$ and $A_0(t)$ we plug in the leading order contribution $a_{\mathbf{k}} = A(t)\delta(\mathbf{k} - \mathbf{k}_0)$ into r.h.s. of equations (2.193) and (2.194) which gives that

$$\begin{aligned}\frac{\partial a_{\mathbf{k}}}{\partial t} + i\omega_{\mathbf{k}} a_{\mathbf{k}} &= -i \int U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2\\ &\quad -2i \int V^*_{\mathbf{k}_1|\mathbf{k}\mathbf{k}_2} a_{\mathbf{k}_1} a^*_{\mathbf{k}_2} \delta(\mathbf{k}_1 - \mathbf{k} - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2\\ &\quad -\mathrm{i} \int V_{\mathbf{k}|\mathbf{k}_1\mathbf{k}_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2\\ \simeq -iU_{-2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}(A^*)^2\delta(\mathbf{k} + 2\mathbf{k}_0) &- 2iV^*_{\mathbf{k}_0|0\mathbf{k}_0}|A|^2\delta(\mathbf{k}) - iV_{2\mathbf{k}_0|\mathbf{k}_0\mathbf{k}_0}A^2\delta(\mathbf{k} - 2\mathbf{k}_0). \end{aligned} \tag{2.295}$$

Because of the leading order time dependence (2.293), the three terms at the last line of equation (2.295) are oscillating in time with frequencies $\pm 2\omega_0, 0$ and have the general form (2.294). These three frequencies are generally not in resonance with the linear dispersion relation $\omega = \omega_{\mathbf{k}}$ so we can solve for $A_{\pm 2},\ A_0$ at each corresponding wave number $\mathbf{k} = \pm\mathbf{k}_0, 0$ and plug in the result into equation (2.294) which gives that

$$\begin{aligned} a_{\mathbf{k}} \simeq & A\delta(\mathbf{k}-\mathbf{k}_0) - \frac{U_{-2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}}{2\omega(\mathbf{k}_0)+\omega(2\mathbf{k}_0)}(A^*)^2\delta(\mathbf{k}+2\mathbf{k}_0) \\ & - 2\frac{V^*_{\mathbf{k}_0|0\mathbf{k}_0}}{\omega(0)}|A|^2\delta(\mathbf{k}) + \frac{V_{2\mathbf{k}_0|\mathbf{k}_0\mathbf{k}_0}}{2\omega(\mathbf{k}_0)-\omega(2\mathbf{k}_0)}A^2\delta(\mathbf{k}-2\mathbf{k}_0), \end{aligned} \tag{2.296}$$

i.e. by comparison with equation (2.294) we obtain that

$$\begin{aligned} A_{-2} = -\frac{U_{-2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}}{2\omega(\mathbf{k}_0)+\omega(2\mathbf{k}_0)}(A^*)^2, \ A_0 = -2\frac{V^*_{\mathbf{k}_0|\mathbf{k}_0 0}}{\omega(0)}|A|^2\delta(\mathbf{k}), \\ A_2 = \frac{V_{2\mathbf{k}_0|\mathbf{k}_0\mathbf{k}_0}}{2\omega(\mathbf{k}_0)-\omega(2\mathbf{k}_0)}A^2. \end{aligned} \tag{2.297}$$

Then, similar to Section 1.13, one can take into account a modification of $A(t)$ by the influence of the secondary harmonics $\pm 2\omega_0, 0$ on the fundamental harmonic $\mathbf{k} = \mathbf{k}_0$ by plugging equation (2.296) into equations (2.193) and (2.194) (which modifies the last line of equation (2.295)) and collect terms $\propto \delta(\mathbf{k}-\mathbf{k}_0)$. It results in

$$\begin{aligned} & \frac{\partial A}{\partial t} + i\omega(\mathbf{k}_0)A = -2iU_{-2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}A^*A^*_{-2} \\ & - 2iV^*_{\mathbf{k}_0|\mathbf{k}_0 0}AA^*_0 - 2iV^*_{2\mathbf{k}_0|\mathbf{k}_0\mathbf{k}_0}A_2A^* - 2iV_{\mathbf{k}_0|\mathbf{k}_0 0}AA_0 \\ & = \frac{2i|U_{-2k_0k_0k_0}|^2}{2\omega(k_0)+\omega(2k_0)}|A|^2A + 8i\frac{|V_{k_0|0k_0}|^2}{\omega(0)}|A|^2A - 2i\frac{|V_{2k_0|k_0k_0}|^2}{2\omega(k_0)-\omega(2k_0)}|A|^2A \\ & = -2i\left[-\frac{|U_{-2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}|^2}{2\omega(\mathbf{k}_0)+\omega(2\mathbf{k}_0)} - 4\frac{|V_{\mathbf{k}_0|0\mathbf{k}_0}|^2}{\omega(0)} + \frac{|V_{2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}|^2}{2\omega(\mathbf{k}_0)-\omega(2\mathbf{k}_0)}\right]|A|^2A. \end{aligned} \tag{2.298}$$

By a comparison of equations (2.292) and (2.298), we conclude that equation provides the effective contribution to the four-wave matrix element $T_0 = T_{\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}$ by the three-wave processes as follows

$$\tilde{T}_0 = T_0 + 2\left[-\frac{|U_{-2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}|^2}{2\omega(\mathbf{k}_0)+\omega(2\mathbf{k}_0)} - 4\frac{|V_{\mathbf{k}_0|0\mathbf{k}_0}|^2}{\omega(0)} + \frac{|V_{2\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0}|^2}{2\omega(\mathbf{k}_0)-\omega(2\mathbf{k}_0)}\right]. \tag{2.299}$$

Then adding contribution of both four-wave and three wave process we obtain from equations (2.292) and (2.298) that

$$\frac{\partial A}{\partial t} + i\omega(\mathbf{k}_0)A = -i\tilde{T}_0|A|^2A. \tag{2.300}$$

which is similar to equation (2.292) with T_0 replaced by the renormalized matrix element $\tilde{T}_0$ (2.299). In a similar way the renormalization of the interaction Hamiltonian (2.289) by taking into account the non-resonant three-wave interaction means replacing T_0 by $\tilde{T}_0$ (2.299).

It is important to note that the third term in equation (2.298) has the frequency $\omega(\mathbf{k})$ for $\mathbf{k} = 0$. According to the Goldstone theorem (Ref. ???), the frequency $\omega(\mathbf{k})$ for $k \to 0$ has the limiting value which is either the nonzero constant or zero. If $\omega(0) = 0$ then it is necessary to consider the limit $V_{\mathbf{k}|0\mathbf{k}}/\omega(k)$ as $k \to 0$. (Ref. ???) E.g. for the gravity waves on the surface of deep fluid this limit is zero. However, if that limit turns infinite, then equation (2.299) for the renormalization of T_0 is not applicable and one has to write a separate equation for low frequency mode. E.g.

such situation occurs if $\omega(\mathbf{k})$ for $k \to 0$ has is $\propto k$, i.e. has the asymptotic of sound wave (Ref ???)

2.11.3. Nonlinear Schrödinger equation and modulational instability. Below we assume that $\tilde{T_0}$ has the finite value (2.299) meaning that we can take into account only the quasi-monochromatic wave near $\mathbf{k} = \mathbf{k}_0$ without a separate consideration of the low frequency mode as discussed at the end of Section 2.11.2. Also in this section we assume that the non-resonant three-wave interactions are included into the renormalized matrix element $\tilde{T_0}$ (2.299) while we remove tilde sign below.

Similar to Section 2.1, we introduce the envelope ψ as inverse FT and forward FT of $a_{\mathbf{k}_0+\kappa}$ over $\kappa := \mathbf{k} - \mathbf{k}_0$ as follows

$$\psi(\mathbf{r}) := \frac{1}{(2\pi)^{D/2}} \int a_{\mathbf{k}_0+\kappa} e^{i\kappa\cdot\mathbf{r}} d\kappa \text{ and } a_{\mathbf{k}_0+\kappa} = \frac{1}{(2\pi)^{D/2}} \int \psi(\mathbf{r}) e^{-i\kappa\cdot\mathbf{r}} d\mathbf{r}. \tag{2.301}$$

Using equation (2.301), we obtain from equation (2.289) that

$$H_{4,int} = \frac{T_0}{2}(2\pi)^D \int |\psi|^4 d\mathbf{r}. \tag{2.302}$$

A quasi-monochromatic wave assumes the replacements of the delta function (2.291) by the narrow distribution near $\mathbf{k} = \mathbf{k}_0$ of the width $\Delta k/k_0 \ll 1$. Expanding the quadratic Hamiltonian H_2 near $\mathbf{k} = \mathbf{k}_0$ and using equation (2.301) to take into account the linear propagation of wave coupled with the interaction Hamiltonian (2.302) result in

$$\begin{aligned} H = H_2 + H_{4,int} = \omega_0 \int |\psi|^2 d\mathbf{r} - i\mathbf{v}_{gr} \cdot \int \psi^* \nabla\psi d\mathbf{r} + \\ + \frac{1}{2}\sum_{j,l=1}^{D} \int \omega_{jl} \nabla_j \psi \nabla_l \psi^* d\mathbf{r} + \frac{1}{2}(2\pi)^D T_0 \int |\psi|^4 d\mathbf{r}, \end{aligned} \tag{2.303}$$

where $\omega_{jl} = \frac{\partial^2 \omega}{\partial k_j \partial k_l}\Big|_{\mathbf{k}=\mathbf{k}_0}$ is the dispersion tensor and $\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}}|_{\mathbf{k}=\mathbf{k}_0}$ is the group velocity. The first two terms in r.h.s. of equation (2.303) are the integrals of motion: $N = \int |\psi|^2 d\mathbf{r}$ is number of particles (wave action) and $\mathbf{P} = -i\int \psi^* \nabla\psi d\mathbf{r}$ is the total momentum of wave packet. The third term in r.h.s. of equation (2.303) is responsible for the dispersive broadening of wave packet (in optics application it also includes the diffraction, see Section 2.5). The fourth term in r.h.s. of equation (2.303) correspond to the action of nonlinear four-wave interactions.

The dynamic equation for ψ is obtained from (2.303) by the variational derivative of the Hamiltonian (2.303) as

$$\psi_t = -i\frac{\delta H}{\delta \psi^*} = -i\omega_0\psi + \mathbf{v}_{gr} \cdot \nabla\psi + i\frac{1}{2}\sum_{j,l=1}^{D} \omega_{jl} \nabla_j \nabla_l \psi - i(2\pi)^D T_0 |\psi|^2 \psi.$$

In this equation the term $-i\omega_0\psi$ can be removed by the adding the rotating phase as $\psi \to \psi e^{-i\omega_0 t}$ and the term $\mathbf{v}_{gr} \cdot \nabla\psi$ vanishes in the moving frame $\mathbf{r} \to (\mathbf{r} + \mathbf{v}_{gr} t)$, which results in the *nonlinear Schrödinger equation* (NLSE)

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\sum_{j,l=1}^{D} \omega_{jl} \nabla_j \nabla_l \psi - (2\pi)^D T_0 |\psi|^2 \psi = 0, \tag{2.304}$$

For the spatially homogenous solution ($\nabla\psi \equiv 0$), NLSE (2.304) turns into ODE with the nonlinear frequency shift $\Delta\omega = (2\pi)^d T_0|\psi|^2$. Thus the qualitative structure is NLSE is amazingly simple: it is the combination of the dispersive addition to the main frequency and the nonlinear frequency shift. Both of the contributions are small in comparison with the linear frequency ω_0 so they can be found independently as we have done that above for the nonlinear frequency shift Section 2.11.2 and for the dispersive term in equation (2.303). The correspondign two independent small parameters are the amplitude of ψ and $\Delta k/k_0 \ll 1$. However, these two contributions can be of the same order of magnitude which results in the diverse physical phenomena as described below in this book.

The tensor $\omega_{\alpha\beta}$ is the symmetric one which ensures that can can be diagonalized by the linear coordinate transform so that NLSE (2.304) can be rewritten in the following diagonal form

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\sum_{j=1}^{D}\lambda_j\frac{\partial^2\psi}{\partial x_j^2} - \chi|\psi|^2\psi = 0, \tag{2.305}$$

where λ_j $(j = 1, \ldots, D)$ are the eigenvalues of the matrix $\omega_{\alpha\beta}$ and $\chi := (2\pi)^d T_0$.

If the dispersion is strictly positive, e.g. the matrix the matrix ω_{jl} is positive-definite then $\lambda_j > 0$, $j = 1, \ldots, D$. In such a case the trivial stretching of the coordinate $r \to r_j\lambda_j^{-\frac{1}{2}}$ turns that equation (2.305) into the standard NLSE form with the Laplacian $\Delta = \nabla^2$ as follows

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\Delta\psi - \alpha|\psi|^2\psi = 0. \tag{2.306}$$

This equation can be interpreted as the Schrödinger equation for the motion of the "particle"' in the self-consistent potential $U = \alpha|\psi|^2$. If $\alpha > 0$, then U corresponds to the repulsion between particles. In nonlinear optics such case is called by the *defocusing nonlinearity.* If $\alpha < 0$, then U corresponds to the attraction and in nonlinear optics it is called by the *focusing nonlinearity.* In that case the increasing of the amplitude $|\psi|$ result in making U more negative, i.e. increasing the attraction. Such attraction is the reason for the formation of self-localized structures such as solitons and collapses described below in Sections ???

NLSE was derived from Maxwell equations in Section ??? with the propagation distance z playing the role of time compared with equation (2.306) while retarder time playing the role of the coordinate. Otherwise equations (2.140) and (2.306) are mathematically equivalent. We also notice that the complex conjugation of equation (2.306) (the change of the sign of time) results in the change of sign of both dispersion term and nonlinearity. However, such change does not change the properties of NLSE. What is really important is the relative sign in front of the dispersion and nonlinearity. Add discussion of self-focusing and Ref. to Askayan???

We now consider the stability of four-wave process in the framework of NLSE (2.306). The simplest solution of that equation is strictly monochromatic wave (2.291) which corresponds to the identically zero equation Laplacian, i.e. $i\frac{\partial\psi}{\partial t} + \frac{1}{2}\Delta\psi - \alpha|\psi|^2\psi = 0$ which has the exact solution $\psi = Ae^{-i\alpha|A|^2 t}$, where $\alpha|A|^2$ is the nonlinear frequency shift. We look for the linear stability of that solution assuming that $\psi = (A + a)e^{-i\alpha|A|^2 t}$, where a is spatially dependent perturbation such that

$|a| \ll |A|$. Linearized equations for a and a^* are obtained from NLSE (2.306) giving

$$\begin{aligned} ia_t + \alpha|A|^2 a + \frac{1}{2}\Delta a - 2\alpha|A|^2 a - \alpha A^2 a^* &= 0, \\ -ia_t^* + \alpha|A|^2 a^* + \frac{1}{2}\Delta a^* - 2\alpha|A|^2 a^* - \alpha A^{*2} a &= 0 \end{aligned}$$

Because coefficients of these linear equations are time independent, we look for the solution in the follow form $a = be^{\gamma t + i\kappa \mathbf{r}}$ which results in the system of linear equations for b and b^* as follows

$$\begin{aligned} (i\gamma - \alpha|A|^2 - \frac{1}{2}\kappa^2)b &= \alpha A^2 b^*, \\ (-i\gamma - \alpha|A|^2 - \frac{1}{2}\kappa^2)b^* &= \alpha A^{*2} b. \end{aligned}$$

The solvability of that homogenous system of two linear equations is determined by the zero determinant which gives

$$\gamma^2 + (\alpha|A|^2 + \frac{1}{2}\kappa^2)^2 = \alpha^2|A|^4$$

or

$$\gamma = \pm\sqrt{\alpha^2|A|^4 - (\alpha|A|^2 + \frac{1}{2}\kappa^2)^2} = \pm\sqrt{-\alpha|A|^2\kappa^2 - \frac{1}{4}\kappa^4}. \tag{2.307}$$

It follows from equation (2.307) that for $\alpha > 0$ (defocusing medium), γ is purely imaginary which gives the oscillations with the frequency

$$\Omega = \kappa\sqrt{\alpha|A|^2 + \frac{\kappa^2}{4}}. \tag{2.308}$$

This is the Bogolyubov's dispersion law (Ref. ???), which was obtained by N.N. Bogolyubovfor the spectrum of oscillations of the weakly Bose-Einstein condensate of non-ideal Bose gas. This law is linear in κ for small κ, i.e. it corresponds to the acoustic waves with the speed $c_s = \sqrt{\alpha|A|^2}$. As κ increases, the positive dispersion of these sounds waves becomes important. In the opposite limit of large κ, the Bogolyubov's dispersion law (2.308) turns in to the dispersion law $\Omega = \frac{\kappa^2}{2}$ for the free particles. Discussion on the relation to the minimal phase velocity Ω/κ ???

If $\alpha < 0$ (the focusing nonlinearity) then equation (2.308) results in the instability with the grow rate

$$\Gamma = \kappa\sqrt{|\alpha||A|^2 - \frac{\kappa^2}{4}}. \tag{2.309}$$

This instability is called *modulational instability.* The qualitative dependence of Γ on κ is shown in Fig. 7. The origin of instability is from the attractive nonlinearity: quasiparticles tend to gather together while the size of such domains should not be too large, namely such that κ^2 must not to be larger than $4|\alpha||A|^2$.

2.12. Stimulated Brillouin scattering

Electromagnetic (EM) waves including light waves are routinely scattered by by numerous materials. When photons of EM waves are scattered many of them are elastically scattered which is called by Rayleigh scattering (see e.g. Ref. ???). Elastic scattering means that that the scattered photons have the frequency as the incident photons but propagate in different direction. In this section we consider

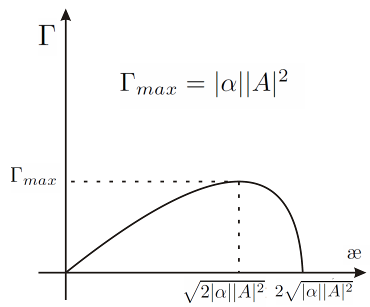

FIGURE 7. Growth rate (2.308) of the modulational instability vs. κ for $\alpha < 0$.

the nonlinear interaction of the electromagnetic (EM) waves with acoustic waves (phonons) which is called by *Brillouin scattering* or *Mandelstam-Brillouin scattering* (MBS). This scattering is inelastic because the energy is transferred from incident EM wave to both scattered EM wave and acoustic wave.

(Historic comment: Leonid Mandelstam has recognised the possibility of such scattering and allegedly wrote a paper on that in 1918 as well as he delivered multiple scientific presentations around that time on that topic, but his paper [**Man26**] was published only in 1926 because of the turmoil of the Russian revolution during that period of history. In 1922 Léon Brillouin published the work [**Bri22**] on the same type of scattering. Since then in former Soviet Union and Russian scientific literatures that scattering has been referred as Mandelstam-Brillouin scattering while in Western literature it has been often called by *Brillouin scattering* omitting the name of Leonid Mandelstam). MBS implies that we have to consider two types of waves simultaneously. EM waves in dielectric were considered in Section 2.5 and acoustic waves in compressible hydrodynamics were addressed in Section 2.4. For both waves the dispersion laws are usually very close to liner ones and are given by

$$\omega(\mathbf{k}) = \frac{|\mathbf{k}|c}{n} \tag{2.310}$$

for the EM wave and

$$\Omega(\mathbf{k}) = |\mathbf{k}|c_s \tag{2.311}$$

for the acoustic wave as considered in Section 2.4. Notice that Eq. (2.311) is the same as Eq. (2.99) with ω replaced by Ω to distinguish EM and sound wave. Here c is the speed of light in vacuum, n is the refraction index (so that the speed of light in the medium is given by c/n) and c_s is the speed of sound. These two speeds are strongly different in magnitude satisfying the condition

$$c_s \ll c/n \tag{2.312}$$

which ensures that Brillouin scattering can involve only the decay of the EM wave with the wave vector $\mathbf{k}$ into another electromagnetic wave with the wave vector $\mathbf{k}'$ and the acoustic wave with the wavevector $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$. The resonance between these three waves occurs provided

$$\omega(\mathbf{k}) = \omega(\mathbf{k}') + \Omega(\mathbf{k} - \mathbf{k}'). \tag{2.313}$$

The condition (2.312) ensures that the phonon frequency $\Omega(\mathbf{k}-\mathbf{k}') = |\mathbf{k}-\mathbf{k}'|c_s$ is much smaller than the frequencies of both electromagnetic waves, $\Omega(\mathbf{k}-\mathbf{k}') \ll |\mathbf{k}'|c_s, |\mathbf{k}-\mathbf{k}'|c_s$. There are two limiting case of Brillouin scattering. First is *forward Brillouin scattering* when $\mathbf{k} \simeq \mathbf{k}'$, i.e. the electromagnetic wave scatters almost in the same direction. Equations (2.312) and (2.313) then imply that $\mathbf{k}-\mathbf{k}' \perp \mathbf{k}$ at the leading order in c_s/c, i.e. the acoustic wave propagate in the perpendicular direction to $\mathbf{k}$. Second case is *backward Brillouin scattering* when $\mathbf{k} \simeq -\mathbf{k}'$, i.e. the electromagnetic wave scatters in the backward direction. Equations (2.312) and (2.313) then imply that $\mathbf{k}-\mathbf{k}' \simeq 2\mathbf{k}$, i.e. the acoustic wave propagates in the forward direction. The resonance condition (2.313) implies that the scattered electromagnetic wave is downshifted (has a smaller frequency) by $\Omega(\mathbf{k}-\mathbf{k}')$. Tha downshift reaches maximum $\Omega(\mathbf{k}-\mathbf{k}') \simeq 2|\mathbf{k}|c_s$ for backward Brillouin scattering while it is nearly zero $\Omega(\mathbf{k}-\mathbf{k}') = |\mathbf{k}-\mathbf{k}'|c_s \ll 2|\mathbf{k}|c_s$ for forward Brillouin scattering.

The quadratic Hamiltonian for linear waves

$$H_2 = \int \omega(\mathbf{k})|a_{\mathbf{k}}|^2 d\mathbf{k} + \int \Omega(\mathbf{k})|b_{\mathbf{k}}|^2 d\mathbf{k},$$

is the sum of the energy of EM waves with complex amplitude $a_{\mathbf{k}}$ and the energy of acoustic waves with the complex amplitude $b_{\mathbf{k}}$.

The interaction of between waves occurs generally from the cubic and higher order terms in the Hamiltonian which are sometimes called by the *interaction Hamiltonian*, H_{int}.

The resonance condition (2.313) implies that only terms $\propto a_{\mathbf{k}_1} a^*_{\mathbf{k}_2} b^*_{\mathbf{k}_3}$ and their complex conjugates are important in the interaction Hamiltonian H_{int} because only they do not average out to zero at large times

$$t \gg 1/\Omega(\mathbf{k}_2) \tag{2.314}$$

at the resonance manifold

$$\begin{aligned} \omega_{\mathbf{k}_1} &= \omega_{\mathbf{k}_2} + \Omega_{\mathbf{k}_3}, \\ \mathbf{k}_1 &= \mathbf{k}_2 + \mathbf{k}_3. \end{aligned} \tag{2.315}$$

The general form of the interaction Hamiltonian for the scattering of the type (2.315) is given by

$$H_{int} = \int \left(V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} a^*_{\mathbf{k}_1} a_{\mathbf{k}_2} b_{\mathbf{k}_3} + V^*_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} a_{\mathbf{k}_1} a^*_{\mathbf{k}_2} b^*_{\mathbf{k}_3} \right) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$

Here the matrix element $V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ has to be determined from the interaction between EM and acoustic wave. We note that the first and the second equations Eq. in (2.315) represent the conservation of energy and momentum, respectively, in each elementary scattering process. Thus this process is inelastic with respect to EM waves because a part of energy is transferred into the energy of the acoustic wave. The scattering process is linear with respect to each wave type but nonlinear overall.

The energy of EM waves in the dielectric medium is given

$$\mathcal{E}_{EM} = \int \left(\frac{\mathbf{H}^2}{8\pi} + \varepsilon \frac{\mathbf{E}^2}{8\pi} \right) d\mathbf{r}, \tag{2.316}$$

which is the same expression as equation (2.151) with the linear dielectric permittivity $\varepsilon(\mathbf{r}, t)$ allowed to change not only in time as in definition (2.121) but also in space. Here we assumed that the linear integral operator $\hat{\varepsilon}(\mathbf{r}, t)$ as in equation

(2.129) is replaced by the trivial multiplication on the function $\varepsilon(\mathbf{r},t)$. This is justified provided $\varepsilon(\mathbf{r},t)$ is the slow function of t compared with the response time t_r of the medium, i.e. $|\varepsilon^{-1}\frac{\partial\varepsilon}{\partial t}| \gg 1/t_r$. As discussed in Section 2.5 and Ref. [**LL84**], a spatial dispersion of ε is often very small so the similar slowness of $\varepsilon(\mathbf{r},t)$ over $\mathbf{r}$ is usually well satisfied for most continuous media (i.e. the linear spatial dispersion is slow on the spatial scale above interatomic scale).

A propagation of EM waves results in the change of the medium density $\rho(\mathbf{r},t)$ through the dependence of ε on ρ which is called *electrostriction*. Electrostriction is the physical phenomenon in all dielectric materials and it results in in the change of the density under the action of electric field. We assume that the variation of the density ρ_1 with respect to the average density ρ_0 is small, $|\rho_1|/\rho_0 \ll 1$, such that we keep only linear term in ρ_1 for the Taylor series expansion, i.e. $\varepsilon(\rho) = \varepsilon(\rho_0) + \rho_1\frac{\partial\varepsilon}{\partial\rho_0} + O(\rho_1^2)$ while neglecting $O(\rho_1^2)$ terms. Then the EM energy (2.95) results in the quadratic term

$$\mathcal{E}_{EM,0} = \int \left(\frac{\mathbf{H}^2}{8\pi} + \varepsilon(\rho_0)\frac{\mathbf{E}^2}{8\pi}\right) d\mathbf{r}, \tag{2.317}$$

and the cubic term

$$H_{int} = \int \frac{\partial\varepsilon}{\partial\rho_0}\rho_1\frac{\mathbf{E}^2}{8\pi}d\mathbf{r}. \tag{2.318}$$

We combine the quadratic Hamiltonian (2.95) of the linear sounds waves with the Hamiltonians (2.317) and (2.318) as follows

$$H = \int \left[\frac{\rho_0(\nabla\phi)^2}{2} + \frac{c_s^2}{2}\frac{\rho_1^2}{\rho_0}\right] d\mathbf{r} + \int \left(\frac{\mathbf{H}^2}{8\pi} + \varepsilon\frac{\mathbf{E}^2}{8\pi}\right) d\mathbf{r} + \int \frac{\partial\varepsilon}{\partial\rho_0}\rho_1\frac{\mathbf{E}^2}{8\pi}d\mathbf{r}. \tag{2.319}$$

Two first two terms in that Hamiltonian correspond to the propagation of linear acoustic and EM waves, respectively. The third term is responsible for the interaction between these two types of waves. We use the canonical Hamiltonian variables $\mathbf{A}$ and

$$\mathbf{K} = -\frac{\varepsilon(\rho_0)\mathbf{E}}{4\pi c} \tag{2.320}$$

introduced Section 2.6 for EM waves in the liner media. Here $\mathbf{A}$ is the vector potential defined by $\mathbf{B} = \nabla\times\mathbf{A}$ with the Coulomb gauge $\operatorname{div}\mathbf{A} = 0$. Then the Maxwell equations take th following form

$$\frac{\partial\mathbf{A}}{\partial t} = -c\mathbf{E},$$
$$\varepsilon(\rho_0)\frac{\partial\mathbf{E}}{\partial t} = -\Delta\mathbf{A}.$$

The canonical Hamilton equations for the media with the dielectric permittivity $\epsilon(\rho_0)$ are given by

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{\delta H_{EM}}{\delta\mathbf{P}}, \tag{2.321}$$

$$\frac{\partial\mathbf{K}}{\partial t} = -\frac{\delta H_{EM}}{\delta\mathbf{A}}, \tag{2.322}$$

where

$$H_{EM} = \int \left[\frac{(\nabla\times\mathbf{A})^2}{8\pi} + \frac{2\pi c^2\mathbf{K}^2}{\varepsilon(\rho_0)}\right] d\mathbf{r} \tag{2.323}$$

is the energy (2.317) of the linear EM waves rewritten through the canonical variables $\mathbf{A}$ and $\mathbf{K}$. Thus $\mathbf{A}$ is the generalized coordinate and $\mathbf{K}$ is the generalized momentum.

Using the inverse FT (2.10) we rewrite the Hamiltonian (2.323) through the Fourier harmonics $\mathbf{A_k}$ and $\mathbf{K_k}$ as follows

$$H_{EM} = \frac{1}{2}\int \left[\frac{k^2|\mathbf{A_k}|^2}{4\pi} + \frac{4\pi c^2}{\varepsilon(\rho_0)}|\mathbf{K}_k|^2\right] d\mathbf{k}. \tag{2.324}$$

Similar to ???, the dispersion relation (2.122) of the linear EM waves is obtained from (2.324) by the product of the factors in front of both $|\mathbf{A_k}|^2$ and $|\mathbf{K_k}|^2$ in the integrand giving

$$\omega_{\mathbf{k}}^2 = \frac{k^2}{4\pi}\frac{4\pi c^2}{\varepsilon(\rho_0)} = \frac{k^2 c^2}{\varepsilon(\rho_0)}.$$

Similar to Sections 1.11 and 2.4, we define the normal complex variables $a_{\mathbf{k}\lambda}$ and $a^*_{\mathbf{k}_\lambda}$ of the linear EM waves as follows

$$\mathbf{A}_k = \sqrt{\frac{4\pi\omega_{\mathbf{k}}}{k^2}}\sum_{\lambda=1}^{2}\mathbf{S}_{\mathbf{k}\lambda}(a_{\mathbf{k}\lambda} + a^*_{-\mathbf{k}\lambda}),$$

where the subscript $\lambda = 1, 2$ distinguishes two linear polarizations of EM waves with the polarization vectors $\mathbf{S}_{\mathbf{k}\lambda}$ and $\mathbf{S}_{\mathbf{k}\lambda}$ which satisfy the orthogonality relations

$$\mathbf{S}_{\mathbf{k}\lambda}\cdot\mathbf{S}_{\mathbf{k}\lambda'} = \delta_{\lambda\lambda'} \text{ and } \mathbf{S}_{\mathbf{k}\lambda}\cdot\mathbf{k} = 0.$$

Respectively, $\mathbf{K_k}$ is given by

$$\mathbf{K_k} = -i\sqrt{\frac{\varepsilon\omega_{\mathbf{k}}}{4\pi c^2}}\sum_{\lambda=1}^{2}\mathbf{S}_{\mathbf{k}\lambda}(a_{\mathbf{k}\lambda} - a^*_{-\mathbf{k}\lambda}). \tag{2.325}$$

We also use the normal complex variable $b_{\mathbf{k}}$ for the sound waves defined in Eq. (2.101) of Section 2.4 (with $a_{\mathbf{k}}$ of Section 2.4 replaced by $b_{\mathbf{k}}$ in this section) as follows

$$\rho_{\mathbf{k}} = \sqrt{\frac{\Omega_{\mathbf{k}}\rho_0}{2c_s^2}}(b_{\mathbf{k}} + b^*_{-\mathbf{k}}),\ \Omega_k = kc_s. \tag{2.326}$$

The quadratic Hamiltonian of the sound waves is given by

$$H_s = \int \Omega_{\mathbf{k}}|b_{\mathbf{k}}|^2 d\mathbf{k}.$$

The interaction Hamiltonian (2.318) rewritten through the canonical variable (2.320) is given by

$$H_{int} = \int \frac{2\pi c^2}{\varepsilon(\rho_0)^2}\frac{\partial\varepsilon}{\partial\rho_0}\rho_1\mathbf{K}^2 d\mathbf{r}. \tag{2.327}$$

We use the inverse FT in Eq. (2.327) together with (2.325) and (2.326) to express H_{int} through the complex canonical variables $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$. Assuming at the leading order of the weak nonlinearity that $a_{\mathbf{k}} \propto e^{-\mathrm{i}\omega_{\mathbf{k}}t}$ and $b_{\mathbf{k}} \propto e^{-\mathrm{i}\Omega_{\mathbf{k}}t}$, we obtain that at the time scale (2.314) the only contribution to H_{int} which is not average out by integration must approximately satisfy the resonance condition which gives that (=====check coefficients=====)

$$H_{int} = \frac{1}{\varepsilon(\rho_0)} \frac{\partial \varepsilon}{\partial \rho_0} \sqrt{\frac{\rho_0}{2c_s^2}} \int \sum_{\lambda,\lambda_1=1}^{2} \sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{k}_1} \Omega_{\mathbf{k}_2}} (\mathbf{S}_{\mathbf{k}\lambda} \mathbf{S}_{\mathbf{k}_1\lambda_1}) a^*_{\mathbf{k}\lambda} a_{\mathbf{k}_1\lambda_1} (b_{\mathbf{k}_2} + b^*_{-\mathbf{k}_2})$$
$$\times \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2.$$

It implies that the matrix element for the scattering EM wave→ EM wave+Sound wave (i.e. the scattering of one EM wave into another EM wave and sound wave) is given by

$$V^{\lambda\lambda_1}_{kk_1k_2} = \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial \rho} \sqrt{\frac{\rho_0}{2c_s^2}} \sqrt{\omega_k \omega_{k_1} \Omega_{k_2}} (\mathbf{S}_{k\lambda} \cdot \mathbf{S}_{k_1\lambda_1}). \tag{2.328}$$

???===Provide details====????Similar to the analysis of Section 2.10, the maximum growth rate Γ_{max} of the decay instability (linear Brillouin scattering instability) is determined by that matrix element as follows

$$\Gamma_{max} = |V^{\lambda\lambda_1}_{\mathbf{k}_0\mathbf{k}_1\mathbf{k}_2}||A_0|, \tag{2.329}$$

where wavevectors are chosen to satisfy the resonance conditions

$$\begin{aligned} \omega_{\mathbf{k}_0} &= \omega_{\mathbf{k}_1} + \Omega_{\mathbf{k}_2}, \\ \mathbf{k}_0 &= \mathbf{k}_1 + \mathbf{k}_2. \end{aligned} \tag{2.330}$$

Eq. (2.329) implies that $\Gamma_{max} \propto \Omega^{1/2}_{k_2}$. A maximal value of Ω_{k_2} corresponds to $\mathbf{k}_0 \approx -\mathbf{k}_1$ with $k_2 \approx 2k_0$. Such process is called *backscattering* because the scattered EM wave $\mathbf{k}_1$ points in the opposite direction to the EM pump wave $\mathbf{k}_0$. A scalar product $(\mathbf{S}_{\mathbf{k}\lambda} \cdot \mathbf{S}_{\mathbf{k}_1\lambda_1})$ in (2.328) also ensures that the maximum value Γ_{max} corresponds to the same polarization vector direction for both incident and scattered EM waves, i.e. $\lambda = \lambda_1$. Thus the stimulated Brillouin scattering is at maximum for the backscattering of EM waves with the same polarization. In terms of experimentally observed EM wave frequency spectra, it means that the pump wave results in the formation of the extra peak in spectra which is shifted downwards in freque This effect was first observed by Leonid Mandelstam in (1928). ???

2.13. Stimulated Raman scattering

The stimulated Brillouin scattering considered in Section 2.12 corresponds to the scattering of EM waves on acoustic phonons. The second type of phonons routinely observed in condensed matter are the optical phonons (see e.g. Ref. ??? discussion of such phonons). A scattering of EM waves on optical phonons is called the *Raman scattering* or the *Raman effect*. This is another example of inelastic scattering. Raman scattering was discovered by Sir Chandrasekhara Venkata Raman (Ref. ???) in 1928 and awarded him the Nobel Prize in 1930. That scattering was also discovered by Leonid Mandelstam in??? but published [**LM28**] later than the paper of Raman, see Ref. for historic accounts [**Fab03**] ???

??? The inelastic scattering of light was predicted by Adolf Smekal in 1923[3] and in older German-language literature it has been referred to as the Smekal-Raman-Effekt.[4] In 1922, Indian physicist C. V. Raman published his work on the "Molecular Diffraction of Light", the first of a series of investigations with his collaborators that ultimately led to his discovery (on 28 February 1928) of the radiation effect that bears his name. The Raman effect was first reported by Raman and his coworker K. S. Krishnan,[5] and independently by Grigory Landsberg and Leonid Mandelstam, in Moscow on 21 February 1928 (one week earlier than Raman and Krishnan). In the former Soviet Union, Raman's contribution was always disputed; thus in Russian scientific literature the effect is usually referred to as "combination scattering" or "combinatory scattering". Raman received the Nobel Prize in 1930 for his work on the scattering of light.[6]

2.14. Stimulated Raman scattering in high temperature plasmas

???

2.15. Removal of cubic terms in the Hamiltonian for nonlinear system with non-decay dispersion law

If in the previous sections we dealt with introducing the Hamiltonian structure, then further we will suppose that we were able by some way to introduce canonical variables and together with them complex variables diagonalizing a quadratic part of the Hamiltonian. In this section we turn to the classical perturbation theory for the wave Hamiltonian systems which is based on an assumption about smallness of wave amplitudes. Difference of the wave systems from the finite-dimensional ones consists in that application of the perturbation theory to the wave systems leads to appearance of the resonant dominators not at separate points, as it is for finite-dimensional equations, but on the whole manifolds. By their classification, we arrive at the whole set of standard Hamiltonians and corresponding equations.

Suppose that in a medium there is one type of wave with non-decay dispersion law ω_k (see Section 2.11.1) and amplitudes $a_{\mathbf{k}}$, whose evolution is determined by the Hamiltonian equations

$$\frac{\partial a_{\mathbf{k}}}{\partial t} = -i\frac{\delta H}{\delta \bar{a}_{\mathbf{k}}}. \tag{2.331}$$

We assume that the Hamiltonian H in equation (2.331) has a form of power series

$$H = H_2 + H_3 + H_4 + \dots, \tag{2.332}$$

where

$$H_2 = \int \omega_k |a_{\mathbf{k}}|^2 d\mathbf{k}$$

is the quadratic Hamiltonian with the dispersion law ω_k,

$$\begin{aligned} H_3 = &\int (V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \bar{a}_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + c.c.)\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 \\ &+ \frac{1}{3}\int (U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \bar{a}_{\mathbf{k}} \bar{a}_{\mathbf{k}_1} \bar{a}_{\mathbf{k}_2} + c.c.)\delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 \end{aligned} \tag{2.333}$$

is the cubic Hamiltonian, H_4 is fourth-order Hamiltonian etc.

Consider a transformation from the variables $a_{\mathbf{k}}$ to new variables $c_{\mathbf{k}}$ in the form of integral power series,

$$\begin{aligned} a_{\mathbf{k}} = a_{\mathbf{k}}^{(1)} + a_{\mathbf{k}}^{(2)} + \dots = c_{\mathbf{k}} &+ \int L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} c_{\mathbf{k}_1} c_{\mathbf{k}_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 \\ &+ \int M_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \bar{c}_{\mathbf{k}_1} c_{\mathbf{k}_2} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 \\ &+ \int N_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \bar{c}_{\mathbf{k}_1} \bar{c}_{\mathbf{k}_2} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 + \dots, \end{aligned} \tag{2.334}$$

where $L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$, $M_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $N_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ are matrix elements (functions) which depend on integration variables $\mathbf{k}$, $\mathbf{k}_1$ and $\mathbf{k}_2$. Also $a^{(1)}_{\mathbf{k}} := c_{\mathbf{k}}$, and $a^{(2)}_{\mathbf{k}}$ includes all terms quadratic over a, $\bar{a}$ in equation (2.334), etc. One can perform symmetrization of $L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $N_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ over $\mathbf{k}_1$ and $\mathbf{k}_2$ because they enter symmetrically in the first and the third integrals in r.h.-s. of equation (2.334). Thus without loss of generality we assume that

$$L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = L_{\mathbf{k}\mathbf{k}_2\mathbf{k}_1}, \quad N_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = N_{\mathbf{k}\mathbf{k}_2\mathbf{k}_1}. \tag{2.335}$$

Our goal is to choose such a transformation (2.334) to eliminate the third order terms H_3 from the Hamiltonian (2.331). That transformation must be canonical to ensure that the Hamiltonian equations for the transformed Hamiltonian are in canonical form,

$$\frac{\partial c_{\mathbf{k}}}{\partial t} = -i\frac{\delta H}{\delta \bar{c}_{\mathbf{k}}}. \tag{2.336}$$

The canonicity of transformation is ensured by the requirement

$$\{a_{\mathbf{k}}, \bar{a}_{\mathbf{k}'}\} = \delta(\mathbf{k}-\mathbf{k}'), \;\; \{a_{\mathbf{k}}, a_{\mathbf{k}'}\} = \{\bar{a}_{\mathbf{k}}, \bar{a}_{\mathbf{k}'}\} = 0, \tag{2.337}$$

where the Poisson bracket for complex variables $c_{\mathbf{k}}$ and $\bar{c}_{\mathbf{k}'}$ which are defined as the continuous analog of the definition (1.120)

$$\{F, G\} = \int \left(\frac{\partial F}{\partial c_{\mathbf{k}''}}\frac{\partial G}{\partial \bar{c}_{\mathbf{k}''}} - \frac{\partial F}{\partial \bar{c}_{\mathbf{k}''}}\frac{\partial G}{\partial c_{\mathbf{k}''}}\right) d\mathbf{k}''. \tag{2.338}$$

We obtain from equations (2.334), (2.338) and (2.337)

$$\begin{aligned}\{a_{\mathbf{k}}, a_{\mathbf{k}'}\} &= \{a^{(1)}_{\mathbf{k}}, a^{(1)}_{\mathbf{k}'}\} + \{a^{(2)}_{\mathbf{k}}, a^{(1)}_{\mathbf{k}'}\} + \{a^{(1)}_{\mathbf{k}}, a^{(2)}_{\mathbf{k}'}\} + \dots \\ &= 0 + M_{\mathbf{k}'|\mathbf{k}|\mathbf{k}+\mathbf{k}'}c_{\mathbf{k}+\mathbf{k}'} + 2N_{\mathbf{k}'|\mathbf{k}|-\mathbf{k}-\mathbf{k}'}\bar{c}_{-\mathbf{k}-\mathbf{k}'} - M_{\mathbf{k}|\mathbf{k}'|\mathbf{k}+\mathbf{k}'}c_{\mathbf{k}+\mathbf{k}'} \\ &\quad - 2N_{\mathbf{k}|\mathbf{k}'|-\mathbf{k}-\mathbf{k}'}\bar{c}_{-\mathbf{k}-\mathbf{k}'} + h.o.t. = 0,\end{aligned} \tag{2.339}$$

where $h.o.t.$ means cubic and higher order terms in a, $\bar{a}$. We also added vertical bars into subscripts of matrix elements to distinguish different arguments of matrix elements were necessary. Equation has to be satisfied for any $\mathbf{k}$ and $\mathbf{k}'$ which requires

$$M_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = M_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2} \tag{2.340}$$

and

$$N_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = N_{\mathbf{k}_1\mathbf{k}\mathbf{k}_2}. \tag{2.341}$$

We conclude from equations (2.335) and (2.341) that $N_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ is symmetric with respect to all three indexes.

We obtain from equations (2.334), (2.338) and (2.337) that

$$\begin{aligned}\{a_{\mathbf{k}}, \bar{a}_{\mathbf{k}'}\} &= \{a^{(1)}_{\mathbf{k}}, \bar{a}^{(1)}_{\mathbf{k}'}\} + \{a^{(2)}_{\mathbf{k}}, \bar{a}^{(1)}_{\mathbf{k}'}\} + \{a^{(1)}_{\mathbf{k}}, \bar{a}^{(2)}_{\mathbf{k}'}\} + \dots \\ &= \delta(\mathbf{k}-\mathbf{k}') + 2\bar{L}_{\mathbf{k}'|\mathbf{k}|\mathbf{k}'-\mathbf{k}}\bar{c}_{\mathbf{k}-\mathbf{k}'} + \bar{M}_{\mathbf{k}'|\mathbf{k}-\mathbf{k}'|\mathbf{k}}c_{\mathbf{k}-\mathbf{k}'} + 2L_{\mathbf{k}|\mathbf{k}'|\mathbf{k}-\mathbf{k}'}c_{\mathbf{k}-\mathbf{k}'} \\ &\quad + M_{\mathbf{k}|\mathbf{k}'-\mathbf{k}|\mathbf{k}'}\bar{c}_{\mathbf{k}-\mathbf{k}'} + h.o.t. = 0.\end{aligned} \tag{2.342}$$

Equation (2.342) is identically satisfied provided

$$M_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = -2\bar{L}_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1}, \tag{2.343}$$

where we also used the symmetry conditions (2.335) and (2.340).

Equations (2.334) and (2.343) results in

$$a_{\mathbf{k}} = c_{\mathbf{k}} + \int L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} c_{\mathbf{k}_1} c_{\mathbf{k}_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2$$
$$- \int 2\bar{L}_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1} \bar{c}_{\mathbf{k}_1} c_{\mathbf{k}_2} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2$$
$$+ \int N_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \bar{c}_{\mathbf{k}_1} \bar{c}_{\mathbf{k}_2} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 + \dots . \tag{2.344}$$

Plugging in equation (2.344) into the Hamiltonian (2.332),(2.333) and requiring that the cubic terms H_3 to vanish identically in new variables c, and $\bar{c}$ we obtain that

$$L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = -\frac{V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}}, \tag{2.345a}$$

$$N_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = -\frac{U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}, \tag{2.345b}$$

where we performed symmetrization of ω_k over $\mathbf{k}_1$, $\mathbf{k}_2$ to obtain (2.345a) and symmetrization of ω_k over $\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$ to obtain (2.345b).

Equations (2.344) and (2.345) imply that the canonical transformation which removes the cubic terms H_3 in the Hamiltonian (2.332),(2.333) is given by

$$a_{\mathbf{k}} = c_{\mathbf{k}} - \int \frac{V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}} c_{\mathbf{k}_1} c_{\mathbf{k}_2} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2$$
$$+ 2 \int \frac{\bar{V}_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1}}{\omega_k - \omega_{k_1} - \omega_{k_2}} \bar{c}_{\mathbf{k}_1} c_{\mathbf{k}_2} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2$$
$$- \int \frac{U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}} \bar{c}_{\mathbf{k}_1} \bar{c}_{\mathbf{k}_2} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 + \dots . \tag{2.346}$$

Here the first two integral terms guarantee the cancelation in the Hamiltonian (2.344) of two terms with the matrix elements $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $\bar{V}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$, while the last term gives the cancelation of the other two terms, proportional to $a^*a^*a^*$ and aaa. These two transformations (elimination of both pairs from H) are independent and can be carried out separately. This procedure for successive elimination of perturbation terms in the Hamiltonian expansion by means of canonical transformations is called classical perturbation theory. In constructing such a theory we quickly come up against the problem of "small denominators", related in the present case to the appearance of nonintegrable singularities near the manifolds

$$\omega_k + \dots + \omega_{k_i} - \omega_{k_{i+1}} - \dots - \omega_{k_n} = 0,$$
$$\mathbf{k} + \dots + \mathbf{k}_i - \mathbf{k}_{i+1} - \dots - \mathbf{k}_n = 0,$$

which give the condition for an nth order resonance. The simplest manifolds already appear in the elimination of the three-wave Hamiltonian H_1 (2.333), when (cf. (2.346)):

$$\omega_k + \omega_{k_1} + \omega_{k_2} = 0,$$

$$\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 = 0 \tag{2.347}$$

and

$$\omega_k - \omega_{k_1} - \omega_{k_2} = 0,$$

$$\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 = 0. \tag{2.348}$$

Satisfying the first condition is possible if waves with negative energy exist in the medium, and then one of the frequencies ω_k must be negative. Such a situation, as a rule, occurs in unstable media, for example, in a plasma with a current. If there are no waves with negative energy in the medium, then the terms proportional to $a^*a^*a^*$ and aaa can be eliminated from H_1 by a canonical transformation, and in this sense they are unimportant (non-resonant).

The possible existence of solutions of the system (2.348) depends on the form of the functions $\omega(\mathbf{k})$. For isotropic media, in which $\omega(k)$ depends only on $\mid \mathbf{k} \mid$, there is no solution if $\omega(0) = 0$ and $\omega''(k) < 0$. Such a situation is realized, for example, for surface gravitational waves. For capillary waves the resonance conditions (2.348) are satisfied.

If the conditions (2.347), (2.348) have no solutions then the three-wave terms are eliminated. Among the fourth order terms the important one is the Hamiltonian in the form

$$H_3 = \int T_{k_1k_2k_3k_4} a^*_{k_1} a^*_{k_2} a_{k_3} a_{k_4} \delta_{k_1+k_2-k_3-k_4} \Pi d\mathbf{k}_i, \tag{2.349}$$

for which the resonance condition

$$\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4} = 0,$$
$$\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 = 0$$

has solutions independent of the form of $\omega(k)$. Here the three-wave interaction leads to a re-normalization of the vertex $T_{kk_1k_2k_3}$ in (2.349):

$$\begin{aligned} T_{kk_1k_2k_3} = T^{(0)}_{kk_1k_2k_3} &- 2\frac{U_{-k_2-k_3,k_2k_3} U^*_{-k-k_1,kk_1}}{\omega_{k+k_1} + \omega_k + \omega_{k_1}} \\ +2\frac{V_{k_2+k_3k_2k_3} V^*_{k+k_1kk_1}}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} &- 2\frac{V_{kk_2k-k_2} V^*_{k_3k_1k_3-k_1}}{\omega_{k_3-k_1} + \omega_{k_1} - \omega_{k_3}} \\ -2\frac{V_{k_1k_3k_1-k_3} V^*_{k_2kk_2-k}}{\omega_{k_2-k} + \omega_k - \omega_{k_2}} &- 2\frac{V_{k_1k_2k_1-k_2} V^*_{k_3-kk_3-k}}{\omega_{k_3-k} + \omega_k - \omega_{k_3}} \\ &-2\frac{V_{kk_3k-k_3} V^*_{k_2k_1k_2-k_1}}{\omega_{k_2-k_1} + \omega_{k_1} - \omega_{k_2}}. \end{aligned} \tag{2.350}$$

CHAPTER 3

Hamiltonian variables in hydrodynamics

3.1. Simple waves in Hydrodynamics

We now consider a particular $1D$ version of Euler equations (2.66)-(2.68) of compressible hydrodynamics of barotropic fluid by assuming a dependence only on a single spatial coordinate $x \in \mathbb{R}$ (which replaces $\mathbf{r} \in \mathbb{R}^D$ of Section 2.4) with the velocity $\mathbf{v} \in \mathbb{R}^D$ replaced by the single component $v \in \mathbb{R}$ alon x which result in

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \lambda(\rho)\frac{\partial \rho}{\partial x} = 0, \\ \lambda(\rho) := \frac{1}{\rho}\frac{\partial p}{\partial \rho} = \frac{1}{\rho}c^2(\rho), \end{aligned}$$

where r.h.s. of Eq. (2.68) is rewritten in the equivalent form through $\lambda(\rho)$ using the chain rule of differentiation and by analogy with Eq. (2.88) we defined the "nonlinear" speed of sound $c(\rho)$ as

$$\frac{\partial p}{\partial \rho} = c^2(\rho).$$

In contrast, Eq. (2.88) defines the linear speed of sound of the average background density ρ_0.

We study a special class of solutions of system (3.1) when the velocity is defined by the density

$$v = v(\rho), \tag{3.1}$$

which is called by a *simple wave*.

In this case, the first equation in (3.1) can be rewritten as

$$\left(\frac{\partial}{\partial t} + S(\rho)\frac{\partial}{\partial x}\right)\rho = 0, \tag{3.2}$$

where

$$S(\rho) := \frac{\partial}{\partial \rho}\left[\rho v(\rho)\right] = v(\rho) + \rho\frac{\partial v(\rho)}{\partial \rho}, \tag{3.3}$$

with $v(\rho)$ is undetermined by Eq. (3.2). To find an expression for $v(\rho)$ we use the second equation in (3.1) together with (3.1) which takes the following form

$$\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t} + v(\rho)\frac{\partial \rho}{\partial x}\right) + \lambda(\rho)\frac{\partial \rho}{\partial x} = 0. \tag{3.4}$$

Equations (3.2) and (3.4) is the system of two linear homogeneous equations for the unknowns $\frac{\partial\rho}{\partial t}$ and $\frac{\partial\rho}{\partial x}$. A solvability condition for that system is the zero value of the determinant of its coefficients,

$$\begin{vmatrix} 1 & S(\rho) \\ \frac{\partial v}{\partial\rho} & \frac{\partial v}{\partial\rho}v+\lambda(\rho) \end{vmatrix} = \frac{\partial v}{\partial\rho}v+\lambda(\rho)-\frac{\partial v}{\partial\rho}S(\rho)=0, \tag{3.5}$$

which gives that

$$S(\rho)=v+\frac{\lambda}{\frac{\partial v}{\partial\rho}}. \tag{3.6}$$

Equations (3.3) and (3.6) result in

$$\left(\frac{\partial v}{\partial\rho}\right)^2=\frac{1}{\rho}\lambda(\rho)=\frac{c^2}{\rho^2} \tag{3.7}$$

which implies that

$$\frac{\partial v}{\partial\rho}=\pm\frac{c}{\rho}$$

resulting in

$$v=\pm\int_{\rho_0}^{\rho}\frac{c(\xi)}{\xi}d\xi,$$

where the refence density ρ_0 can be chosen at our convenience. The characteristic velocity S apparently takes two values $S=S_\pm=\pm S$

$$S(\rho)=c(\rho)=v(\rho)=c(\rho)\pm\int_{\rho_0}^{\rho}\frac{c(\xi)}{\xi}d\xi \tag{3.8}$$

For the special case of a polytropic gas,

$$P=\frac{1}{\gamma}c_0^2\rho_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$$
$$c^2=c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$
$$c=c_0\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}$$

Note that c_0 is the sound speed if $\rho=\rho_0$. So, for a polytropic gas,

$$S_\pm=\pm\left[\frac{\gamma+1}{\gamma-1}c_0\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}-\frac{2}{\gamma-1}c_0\right] \tag{3.9}$$

In the special case $\gamma=-1$,

$$P=-\frac{c_0^2\rho_0^2}{\rho}$$
$$S=\pm c_0$$

Note that S is constant and independent of density, and the pressure P is negative. This is the "Chaplygin gas" which can be used as a model for dark energy in the Universe. In the case of the Chaplygin gas, $c_0 = c$, the speed of light.

In the general case,

$$v(\rho) = \frac{2}{\gamma - 1} c_0 \left[\left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}} - 1 \right] \tag{3.10}$$

In the limit $\gamma \to 1$,

$$S = v = c_0 \, ln \left(\frac{\rho}{\rho_0} \right)$$

If $\gamma \geq 1$, then $\frac{\partial S}{\partial \rho} > 0$. In this case, the "'simple wave,"' which is a solution of equation (1.2), has a tendency to create a compressional shock wave.

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In the case $\gamma < 1$, $\frac{\partial S}{\partial \rho} < 0$ and the simple wave generates a shock wave of rarification.

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In both cases, solutions of equation (3.2) become multivalued as $t \to \pm\infty$.

Equation (3.2) can be solved exactly. First, we seek a particular solution of the form

$$\rho = x - A(\rho)t \tag{3.11}$$

After differentiating by t and x, we get

$$\rho_t = -A(\rho) - tA'(\rho)\rho_t$$

$$\rho_x = 1 - tA'(\rho)\rho_x$$

Hence,

$$\rho_t = \frac{-A(\rho)}{1 + tA'(\rho)}$$

$$\rho_x = \frac{1}{1 + tA'(\rho)}$$

Now we can choose $A(\rho) = S(\rho)$. Doing so gives $\rho_t + S(\rho)\rho_x = 0$.

A general solution of (3.2) has the form

$$\rho = F(x - A(\rho)t) \tag{3.12}$$

where $F(\chi)$ is an arbitrary function of one variable. An alternative form of this solution is

$$x - A(\rho)t = f(\rho) \tag{3.13}$$

Here, $f(\rho)$ is another arbitrary function of one variable.

Let us address the following question: can we find a function $A = A(\rho, v)$ that satisfies

$$\frac{\partial A}{\partial t} + S(\rho, v)\frac{\partial A}{\partial x} \tag{3.14}$$

Equation (1.9) can be rewritten as follows:

$$\frac{\partial A}{\partial \rho}\frac{\partial \rho}{\partial t} + \frac{\partial A}{\partial v}\frac{\partial v}{\partial t} + S\left(\frac{\partial A}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial A}{\partial v}\frac{\partial v}{\partial x}\right) = 0$$

Using (1.1) to find the time derivative gives

$$\left(v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x}\right)\frac{\partial A}{\partial \rho} + \left(v\frac{\partial \rho}{\partial x} + \lambda\frac{\partial \rho}{\partial x}\right)\frac{\partial A}{\partial v} = S\left(\frac{\partial A}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial A}{\partial v}\frac{\partial v}{\partial x}\right)$$

The coefficients of $\frac{\partial \rho}{\partial x}$ and $\frac{\partial v}{\partial x}$ must vanish; this gives us

$$\lambda\frac{\partial A}{\partial v} = (S - v)\frac{\partial A}{\partial \rho}$$
$$\rho\frac{\partial A}{\partial \rho} = (S - v)\frac{\partial A}{\partial v}$$

Multiplying these equations yields the relation $(S-v)^2 = \lambda\rho = c^2$. There are two solutions, $S_{\pm} = v \pm c$. This means that

$$A_{\pm} = v \pm \int_{\rho_0}^{\rho} \frac{c(\rho')}{\rho'} d\rho' \tag{3.15}$$

Therefore we have the following system of equations:

$$\frac{\partial A_{\pm}}{\partial t} + S_{\pm}\frac{\partial A_{\pm}}{\partial x} = 0 \tag{3.16}$$

which is just another form of the initial system, (1.1). Suppose $A_- = 0$. Then,

$$v = \int_{\rho_0}^{\rho} \frac{c(\rho')}{\rho'} = v(\rho)$$

in accordance with (1.4). Also, in this case, $A_+ = 2v(\rho)$. This is a simple wave solution. In general, $A_{\pm}$ are called Riemann invariants. Now, let $\gamma = 2$. Then,

$$\int_{\rho_0}^{\rho} \frac{c(\rho')}{\rho'} d\rho' = c(\rho) - c_0$$
$$A_+ = v + c - c_0 = S_+ - c_0$$
$$A_- = v - c + c_0 = S_- + c_0$$

or

$$S_+ = c_0 + A_+$$
$$S_- = -c_0 + A_-$$

Now, system (1.13) splits in a pair of independent equations for $A_{\pm}$. In the general case, the velocities $S_{\pm}$ depend on both Riemann invariants.

For any dependence of $S_\pm$ on $A_\pm$, system (1.13) has an infinite number of conservation laws. Let us try to find a pair of functions $F = F(A_+, A_-)$ and $G = G(A_+, A_-)$ satisfying the relation

$$\frac{\partial F}{\partial t} = \frac{\partial G}{\partial x} \tag{3.17}$$

Then,

$$\begin{aligned} \frac{\partial F}{\partial t} = \frac{\partial F}{\partial A_-}\frac{\partial A_-}{\partial x} + \frac{\partial F}{\partial A_+}\frac{\partial A_+}{\partial t} &= -\left(S_-\frac{\partial F}{\partial A_-}\frac{\partial A_-}{\partial x} + S_+\frac{\partial F}{\partial A_+}\frac{\partial A_+}{\partial x}\right) \\ \frac{\partial G}{\partial x} &= \frac{\partial G}{\partial A_-}\frac{\partial A_-}{\partial x} + \frac{\partial G}{\partial A_+}\frac{\partial A_+}{\partial x} \end{aligned} \tag{3.18}$$

By plugging (1.15) into (1.14) and canceling terms before $\frac{\partial A_\pm}{\partial x}$, one obtains a system of hyperbolic equations

$$\begin{aligned} S_+\frac{\partial F}{\partial A_+} + \frac{\partial G}{\partial A_+} &= 0 \\ S_-\frac{\partial F}{\partial A_-} + \frac{\partial G}{\partial A_-} &= 0 \end{aligned} \tag{3.19}$$

Any solution of this system generates a conservative quantity for system (1.13). Finally, we consider the case of weak nonlinearity. Suppose

$$\begin{aligned} \rho &= \rho_0 + \delta\rho \\ S(\rho) &= S_0 + S_1\frac{\delta\rho}{\rho_0} + ... \end{aligned}$$

Then system (1.2) becomes the Hopf equation,

$$\frac{\partial V}{\partial \tau} + V\frac{\partial V}{\partial x} = 0$$

Here,

$$\begin{aligned} \frac{\partial}{\partial \tau} &= \frac{\partial}{\partial t} + S_0\frac{\partial}{\partial x} \\ V &= S_1\frac{\delta\rho}{\rho_0} \end{aligned}$$

3.1.1. Hamiltonian systems. Let $q(x,t)$ and $p(x,t)$ be real functions on $\mathbb{R}^1$. Let $H[q,p]$ be a functional. We will study the dynamical system

$$\begin{aligned} \frac{\partial q}{\partial t} &= -\frac{\delta H}{\delta p} \\ \frac{\partial p}{\partial t} &= \frac{\delta H}{\delta q} \end{aligned}$$

This is a Hamiltonian system, where H is the Hamiltonian, and q and p are canonically-conjugated variables. If H does not depend on t explicitly, it is a constant of the motion. Indeed,

$$\frac{dH}{dt} = \int \left(\frac{\delta H}{\delta q}\frac{\partial q}{\partial t} + \frac{\delta H}{\delta p}\frac{\partial p}{\partial t} \right) dx = \int \left(-\frac{\delta H}{\delta q}\frac{\delta H}{\delta p} + \frac{\delta H}{\delta p}\frac{\delta H}{\delta q} \right) dx = 0$$

The system of hydrodynamic equations, (1.1), is a Hamiltonian system. To prove this, we introduce the hydrodynamic potential $v = \frac{\partial \phi}{\partial x}$. Then the second equation in (1.1) can be integrated by x, which gives the system

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho \frac{\partial \phi}{\partial x} \right) = 0 \tag{3.20}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x} \right)^2 + W(\rho) = 0 \tag{3.21}$$

Here,

$$W(\rho) = \int_{\rho_0}^{\rho} \lambda(\rho) d\rho = \int_{\rho_0}^{\rho} \frac{c^2(\rho)}{\rho} d\rho = \int_{\rho_0}^{\rho} \frac{1}{\rho}\frac{\partial P}{\partial \rho} d\rho \tag{3.22}$$

Let us introduce the Hamiltonian

$$\begin{aligned} H &= T + U & T &= \frac{1}{2}\int \rho \phi_x^2 dx \\ U &= \int \varepsilon(\rho) d\rho & \varepsilon(\rho) &= \int_{\rho_0}^{\rho} W(\rho) d\rho \end{aligned} \tag{3.23}$$

Then equations (2.19) and (2.20) can be rewritten as

$$\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \phi} \qquad \frac{\partial \phi}{\partial t} = -\frac{\delta H}{\delta \rho} \tag{3.24}$$

All the equations mentioned above can be generalized to the two and three dimensional cases. If we assume that the fluid flow is irrotational, in other words $curl\, v = 0$, then $v = \nabla \phi$. Then equations (2.19) and (2.20) become

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + div\,(\rho \nabla \phi) = 0 \\ &\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + W(\rho) = 0 \end{aligned} \tag{3.25}$$

These equations are Hamiltonian, with $T = \frac{1}{2}\int \rho (\nabla \phi)^2 d\vec{r}$.

3.1.2. Hamiltonian structure of simple waves. Notice that equation (3.2) can be presented in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\frac{\delta H}{\delta \rho} = 0 \tag{3.26}$$

Where H is a local functional of ρ:

$$H = \int E(\rho) d\rho \qquad E = \int S(\rho) d\rho$$

System (3.26) is Hamiltonian for any choice of functional H. To prove this, let us define $f(x) = \frac{\delta H}{\delta \rho}$. Then,

$$\begin{aligned} \frac{\partial \rho(x)}{\partial t} + \frac{\partial f(x)}{\partial x} &= 0 \\ \frac{\partial \rho(-x)}{\partial t} - \frac{\partial f(-x)}{\partial x} &= 0 \end{aligned} \tag{3.27}$$

One can split $\rho(x)$ into symmetric and skew-symmetric parts,

$$\rho(x) = \frac{1}{\sqrt{2}} \left(q(x) + \frac{\partial p}{\partial x} \right)$$

Now,

$$\delta H = \int_{-\infty}^{\infty} \frac{\delta H}{\delta \rho} \delta \rho dx = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} f(x) \left[\delta q(x) + \frac{\partial \delta p(x)}{\partial x} \right] dx$$
$$= \frac{1}{\sqrt{2}} \int_{0}^{\infty} (f(x) + f(-x)) \delta q(x) dx - \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} (f'(x) - f'(-x)) \delta p(x) dx$$

Hence,

$$\begin{aligned} \frac{1}{\sqrt{2}}(f(x) + f(-x)) &= \frac{\delta H}{\delta q} \\ \frac{1}{\sqrt{2}}(f'(x) + f'(-x)) &= -\frac{\delta H}{\delta p} \end{aligned} \tag{3.28}$$

Then,

$$\begin{aligned} q(x) &= \frac{1}{\sqrt{2}}(\rho(x) + \rho(-x)) \\ p(x) &= \frac{1}{\sqrt{2}} \partial^{-1} (\rho(x) - \rho(-x)) \end{aligned}$$

which gives us

$$\frac{\partial q}{\partial t} + \frac{1}{2}(f'(x) - f'(-x)) = \frac{\delta H}{\delta p}$$

and, in the same way,

$$\frac{\partial p}{\partial t} = -\frac{\delta H}{\delta q}$$

What is the difference between Hamiltonian systems (3.20) and (3.26)? They have different phase spaces. The phase space of system (3.20) is composed of a pair of functions defined on the entire x axis, $-\infty < x < \infty$. In the case of system (3.26), the pair of canonically-conjugated functions $q(x)$, $p(x)$ is defined on the half-axis $0 < x < \infty$. In practice, it is more convenient to work with the single function $\rho(x)$. To make this possible, we must generalize the definition of a Hamiltonian system.

3.2. Dynamics of ideal fluid with free surface

Consider the dynamics of an incompressible ideal fluid with free surface and constant depth. Fluid occupies the region

$$-h \le z \le \eta(\mathbf{x},t), \quad \mathbf{x}=(x,y), \tag{3.29}$$

where (x, y) are the horizontal coordinates and z is the vertical coordinate pointing upwards. The time dependent vertical location of the free surface $\eta(\mathbf{x},t)$ is called the *surface elevation* and assumed to the the smooth function of $\mathbf{x}$. A particular case when $\eta(\mathbf{x},t)$ looses smoothness (but remains continuous) along a line in $\mathbf{x}$ plane such as the limiting Stokes (see Section ???) is considered separately below in Section ???.

Viscosity is assumed to be absent and the fluid's velocity $\mathbf{v}$ is potential one,

$$\mathbf{v} = \nabla\Phi, \tag{3.30}$$

where $\Phi(\mathbf{r},z,t)$ is the velocity potential and $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x})$. Incompressibility condition $\rho \equiv const$ implies in the first Euler equation (2.67) that

$$\nabla\cdot\mathbf{v} = 0. \tag{3.31}$$

Together with (3.30), equation (3.31) results in the Laplace Eq.

$$\nabla^2\Phi = 0. \tag{3.32}$$

The second Euler equation (2.68) with the acceleration due to gravity term $\mathbf{g}$ (points in the negative direction of z axis) added to the r.h.-s. of (2.68) for the potential Φ satisfies the Bernoulli equation:

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\big(\nabla\Phi\big)^2 + p + gz = 0, \tag{3.33}$$

where p is the pressure, $g = |\mathbf{g}|$ is the acceleration of gravity, and we set density of fluid to unity $\rho \equiv 1$.

There are two types of boundary conditions at free surface for Eqs. (3.32), (3.33). First one is the kinematic boundary condition which states that each fluid particle at the free surface moves with the surface. This statement can be written as that the total (material) derivative $\frac{d\eta}{dt}$ of η equals to the vertical fluid velocity at the free surface $\frac{\partial\Phi}{\partial z}\Big|_{z=\eta}$, i.e.

$$\frac{d\eta}{dt} := \frac{\partial\eta}{\partial t} + (\mathbf{v}\cdot\nabla)\eta\Big|_{z=\eta} = \frac{\partial\Phi}{\partial z}\Big|_{z=\eta}. \tag{3.34}$$

Solving equation (3.34) for $\frac{\partial\eta}{\partial t}$ we obtain the equivalent form of the kinematic BC

$$\frac{\partial\eta}{\partial t} = \Big(\frac{\partial\Phi}{\partial z} - \nabla\eta\cdot\nabla\Phi\Big)\Big|_{z=\eta} = v_n\sqrt{1+(\nabla\eta)^2}, \tag{3.35}$$

where

$$v_n = \mathbf{n}\cdot\nabla\Phi \tag{3.36}$$

is the normal component of fluid's velocity at free surface, and $\mathbf{n} = (-\nabla\eta, 1)\big[1 + (\nabla\eta)^2\big]^{-1/2}$ is the unit normal vector to the free surface pointing to outside of the fluid.

Kinematic boundary condition results from the physical condition that a free surface propagates with fluid particles. Consider Lagrangian variables for the description of hydrodynamics (see Section 3.10), $\mathbf{r}(\mathbf{a},t) = (\mathbf{x}(\mathbf{a},t), z(\mathbf{a},t))$. A velocity of fluid particles in these variables is given by

$$\mathbf{v} = \left(\frac{d\mathbf{x}}{dt}, \frac{dz}{dt}\right). \tag{3.37}$$

Only a normal component of the fluid's particle velocity affects the motion of free surface. Then the kinematic condition that the fluid particle at the free surface moves with the the surface is given by???

$$\mathbf{v}\cdot\mathbf{n} = \left(\frac{d\mathbf{x}}{dt}, \frac{dz}{dt}\right)\cdot\mathbf{n} = v_n. \tag{3.38}$$

Second type is the dynamic boundary condition at free surface

$$p\Big|_{z=\eta} = -\sigma\nabla\cdot\frac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}}, \tag{3.39}$$

where σ is the surface tension coefficient which determines the jump of the pressure at free surface from zero value outside of the fluid to $p\Big|_{z=\eta}$ value inside fluid.

Boundary condition at the bottom is

$$\Phi_z|_{z=-h} = 0 \tag{3.40}$$

which means that there is no flow through the bottom, $\Phi_z|_{z=-h} = v_n|_{z=-h} = 0$. The absence of viscosity in the ideal fluid implies that the fluid velocity can have a nonzero components $v_x|_{z=-h}$ and $v_y|_{z=-h}$ along the bottom.

The total energy, H, of the fluid consists of the kinetic energy, K, and the potential energy, U:

$$H = K + U, \tag{3.41}$$

$$K = \frac{1}{2}\int\limits_{-h\le z\le\eta,\ \mathbf{x}\in\mathbb{R}^2}(\nabla\Phi)^2 d\mathbf{r} = \frac{1}{2}\int d\mathbf{x}\int_{-h}^{\eta}\left(\nabla\Phi\right)^2 dz,\ \mathbf{r} := (\mathbf{x}, z), \tag{3.42}$$

$$U = \frac{1}{2}g\int\eta^2\,d\mathbf{x} + \sigma\int\left[\sqrt{1+\left(\nabla\eta\right)^2} - 1\right]d\mathbf{x}. \tag{3.43}$$

Here $\mathbf{x} = (x, y)$ is the horizontal coordinate.

It is convenient to introduce the value of the velocity potential at interface as

$$\Phi\Big|_{z=\eta} \equiv \Psi(\mathbf{x}, t). \tag{3.44}$$

Then

$$\frac{\partial\Psi}{\partial t} = \frac{\partial\Phi(\mathbf{x},\eta(\mathbf{x},t),t)}{\partial t} = \left[\frac{\partial\Phi(\mathbf{x},z,t)}{\partial t} + \frac{\partial\eta}{\partial t}\frac{\partial\Phi(\mathbf{x},z,t)}{\partial z}\right]\Bigg|_{z=\eta(\mathbf{x},t)}. \tag{3.45}$$

Here $\frac{\partial\Phi}{\partial z}\Big|_{z=\eta}$ can be expressed from the kinematic BC (3.34).

We now show that that the free surface problem (3.32) – (3.40) can be written in the canonical Hamiltonian form

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi}, \quad \frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta}. \tag{3.46}$$

with the Hamiltonian H defined in (3.41) so it coincides with the total (potential and kinetic) energy of fluid.

We convert the integral (3.42) into the integral over free surface using integration by parts, the boundary conditions (3.40) and taking into account the Laplace equation (3.32) to remove the volume part of the integral. It results in

$$K = \frac{1}{2}\int \Psi v_n ds = \frac{1}{2}\int \Psi v_n \sqrt{1+\left(\nabla\eta\right)^2} d\mathbf{x}, \tag{3.47}$$

where v_n is the normal component of velocity at free surface defined in equation (3.36) and we used that the integral over elementary surface area ds is related to $d\mathbf{x}$ as follows

$$ds = \sqrt{1+\left(\nabla\eta\right)^2} d\mathbf{x}. \tag{3.48}$$

Equations (3.43) and (3.47) result in the Hamitonian (3.41) rewritten as the surface integral

$$H = \frac{1}{2}\int \Psi v_n \sqrt{1+\left(\nabla\eta\right)^2} d\mathbf{x} + \frac{1}{2} g\int \eta^2\, d\mathbf{x} + \sigma\int \left[\sqrt{1+\left(\nabla\eta\right)^2} - 1\right] d\mathbf{x}. \tag{3.49}$$

The Hamiltonian equations (3.46) require to express the Hamiltonian (3.49) through the canonical variables η and Ψ, i.e. we have to find v_n through η and Ψ. That problem can be understood as finding a Dirichlet-Neumann operator $\hat{G}$ for the Laplace equation (3.32) which relates the Dirichlet boundary condition (3.44) to the Neumann boundary condition (3.36) as follows

$$v_n \sqrt{1+\left(\nabla\eta\right)^2} = \hat{G}\Psi, \tag{3.50}$$

where the factor $\sqrt{1+\left(\nabla\eta\right)^2}$ is chosen for convenience to a little more compact expressions because v_n enters together with that factor in Eq. (3.49). We conclude that while the Laplace equation (3.32) does not appear explicitly in the Hamiltonian equations, we still have to solve the Laplace equation (3.32) at each time t to find the Dirichlet-Neumann operator (3.50).

Because the Laplace equation cannot be solved explicitly for the general form of the free surface, then the Hamiltonian (3.49) cannot be expressed in a closed form as a function of surface variables η, Ψ. Instead we solve the Laplace equation (3.32) by perturbations to presented the Hamiltonian (3.49) by the infinite series in powers of surface steepness $|\nabla\eta|$:

$$H = H_2 + H_3 + H_4 + \ldots. \tag{3.51}$$

Here H_2, H_3, H_4 are quadratic, cubic and quartic terms, respectively.

Fourier transform (FT),

$$\Psi_{\mathbf{k}} = \frac{1}{2\pi}\int \exp(-i\mathbf{k}\cdot\mathbf{x})\Psi(\mathbf{x})d\mathbf{x}, \tag{3.52}$$

is the canonical transformation which conserves the Hamiltonian structure and Eqs. (3.46) take the following form:

$$\frac{\partial\eta_{\mathbf{k}}}{\partial t} = \frac{\delta H}{\delta\Psi_{-\mathbf{k}}}, \quad \frac{\partial\Psi_{\mathbf{k}}}{\partial t} = -\frac{\delta H}{\delta\eta_{-\mathbf{k}}}, \quad \bar{\eta}_{\mathbf{k}} = \eta_{-\mathbf{k}}, \quad \bar{\Psi}_{\mathbf{k}} = \Psi_{-\mathbf{k}}. \tag{3.53}$$

FT (3.53) of the Laplace equation (3.32) results in

$$\left(-k^2+\frac{\partial^2}{\partial z^2}\right)\Phi_{\mathbf{k}}(z,t)=0, \tag{3.54}$$

where $\Phi_{\mathbf{k}}(z,t)$ is FT of $\Phi_{\mathbf{k}}(\mathbf{r},z,t)$ over the horizontal coordinates $\mathbf{r}=(x,y)$. A general solution of equation (3.54) is given by

$$\Phi_{\mathbf{k}}(z,t)=C_{\mathbf{k}}(t)e^{kz}+D_{\mathbf{k}}(t)e^{-kz}, \tag{3.55}$$

where the amplitudes $C_{\mathbf{k}}(t)$ and $D_{\mathbf{k}}(t)$ are the functions of $\mathbf{k}$ and t. The boundary condition at the bottom (3.40) together with equation (3.55) relates $C_{\mathbf{k}}(t)e^{kz}$ and $D_{\mathbf{k}}(t)$ as follows

$$\left.\frac{\partial}{\partial z}\Phi_{\mathbf{k}}(z,t)\right|_{z=-h}=k\left[C_{\mathbf{k}}(t)e^{-kh}-D_{\mathbf{k}}(t)e^{kh}\right]=0. \tag{3.56}$$

Assume that $\Phi_{\mathbf{k}}(0,t)$ is known. Then equations (3.55) and (3.56) result in

$$C_{\mathbf{k}}(t)=e^{2kh}D_{\mathbf{k}}(t)=\frac{\Phi_{\mathbf{k}}(0,t)}{1+e^{-2kh}}, \tag{3.57}$$

which implies that

$$\Phi_{\mathbf{k}}(z,t)=\frac{\Phi_{\mathbf{k}}(0,t)\cosh(k[z+h])}{\cosh(kh)}. \tag{3.58}$$

3.2.1. Linear surface waves. By limiting in equation (3.51) to the quadratic Hamiltonian H_2 we find the leading term for v_n in the the Hamiltonian (3.49) as follows

$$v_n=\left.\frac{\partial}{\partial z}\Phi_{\mathbf{k}}(z,t)\right|_{z=0}+O(|k\eta|)=\Psi_{\mathbf{k}}(t)k\tanh(kh)+O(|k\eta|), \tag{3.59}$$

where $O(|k\eta|)$ means terms with the additional smallness $\sim|k\eta|\sim|\nabla\eta|$. That smallness has the meaning of the typical surface steepness and it plays the role of the small parameter in the perturbation series (3.51). Here k and η are the typical wavenumber and typical elevation of surface perturbation above plane $\mathbf{r}=0$.

Equations (3.49) and (3.59) result in the quadratic Hamiltonian

$$\begin{gathered}H_2=\frac{1}{2}\int\left\{A_k|\Psi_{\mathbf{k}}|^2+B_k|\eta_{\mathbf{k}}|^2\right\}d\mathbf{k},\\ A_k=k\tanh(kh),\ B_k=g+\sigma k^2,\ k=|\mathbf{k}|.\end{gathered} \tag{3.60}$$

The dispersion relation ω_k of gravity-capillary waves is immediately obtained from equation (3.60) as

$$\omega_k^2=A_kB_k=k(g+\sigma k^2)\tanh(kh). \tag{3.61}$$

It follows from equation (3.61) by looking at the factor $(g+\sigma k^2)$ that the gravity effect is more important at $g>\sigma k^2$ while the surface tension is dominant for $g<\sigma k^2$. It defines the gravity-capillary scale

$$\lambda_{gc}=\frac{2\pi}{k_{gc}},\quad k_{gc}=\sqrt{\frac{g}{\sigma}}, \tag{3.62}$$

at which the gravity and capillary forces equally contribute to the dispersion relation (3.61). For the surface of water at room temperature $\lambda_{gc}\simeq 1.7$cm.

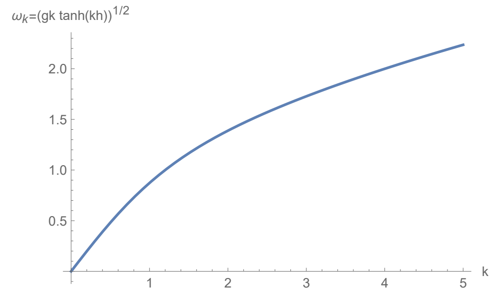

FIGURE 1. A dispersion relation ω_k (3.63) of the surface gravity wave of the depth h vs. k for $g = h = 1$. It is seen that ω_k has the sub-linear growth at all k which ensures the non-decay dispersion law.

At $k \ll k_{gc}$, the dispersion relation (3.61) is reduced to

$$\omega_k^2 = kg \tanh(kh). \tag{3.63}$$

This dispersion relation has the non-decay type as seen Fig. 1, see Section 2.11.1.

Small fluid depth/shallow water limit. Assuming that $kh \ll 1$ we obtain from (3.63) that

$$\omega_k = k(gh)^{1/2}\left[1 - \frac{(kh)^2}{6} + O(kh)^4\right], \tag{3.64}$$

i.e. the dispersion relation is almost linear with the small negative dispersion because of the term $-\frac{(kh)^2}{6}$ which ensures non-decay dispersion law in that limit.

Deep fluid limit. Assuming that $kh \gg 1$ we obtain from equation (3.61) that

$$\omega_k^2 = gk + \sigma k^3 \tag{3.65}$$

which is the dispersion law of gravity-capillary wave on the surface of deep fluid. This dispersion law is decay type because $\omega_k \propto k^{3/2}$ at the capillary-dominated scales $\lambda \ll 2\pi/k_{gc}$, (i.e. $k \gg k_{gc}$) reducing equation (3.65) to the dispersion law of the capillary wave,

$$\omega_k^2 = \sigma k^3, \tag{3.66}$$

with $\alpha = 1/2$ in equation (2.288). For the surface of ocean capillary waves often show up near the crests of large oceanic waves.

The opposite limit $\lambda \gg 2\pi/k_{gc}$, (i.e. $k \ll k_{gc}$) results in the dispersion law of surface gravity waves

$$\omega_k^2 = gk. \tag{3.67}$$

These waves are most common in deep ocean with their typical scales ranging from tenth of cm to many hundreds meter. The dispersion law (3.67) is of non-decay type with $\alpha = 1/2$ in equation (2.288). When the oceanic gravity waves propagate towards the coastline or reach other shallower areas of the ocean, then the dispersion law (3.67) typically has to be replace by the more general expression (3.63).

The analysis of the dispersion relation???

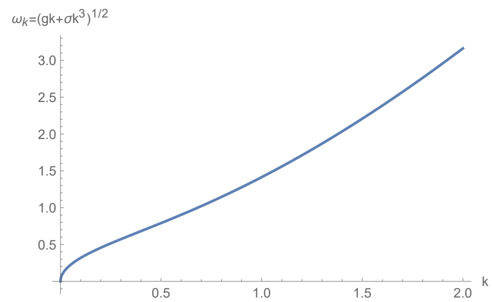

FIGURE 2. A dispersion relation ω_k (3.65) of the surface gravity-capillary wave of the infinite vs. k for $g = \sigma = 1$. It is seen that ω_k has the decay dispersion law ensured by $\omega_k \propto k^{3/2}$ at $k \gg k_{gc}$.

3.2.2. Nonlinear surface waves. To consider higher order terms $H_3, H_4, \dots$ in the Hamiltonian (3.49), we introduce a shift operator $\hat{L}_z$ as follows

(3.68)
$$f(\mathbf{x}, z, t) = \hat{L}_z f := \left(1 + z\frac{\partial}{\partial z} + \frac{z^2}{2!}\frac{\partial^2}{\partial z^2} + \ldots + \frac{z^n}{n!}\frac{\partial^n}{\partial z^n} + \ldots\right) f(\mathbf{x}, z, t)\Big|_{z=0}$$

which relates $f(\mathbf{x}, z = 0, t)$ to $f(\mathbf{x}, z, t)$ for the harmonic function $f(\mathbf{x}, z, t)$ by the Taylor series expansion over z at $z = 0$. Plugging $z = \eta(\mathbf{x}, t)$ into equation (3.68) results in the *shift operator*

(3.69)
$$f(\mathbf{x}, \eta(\mathbf{x}, t), t) = \hat{L}_\eta f := \left(1 + \eta\frac{\partial}{\partial z} + \frac{\eta^2}{2!}\frac{\partial^2}{\partial z^2} + \ldots + \frac{\eta^n}{n!}\frac{\partial^n}{\partial z^n} + \ldots\right) f(\mathbf{x}, z, t)\Big|_{z=0}$$

which relates $f(\mathbf{x}, z = 0, t)$ to $f(\mathbf{x}, z = \eta(\mathbf{x}, t), t)$ by the Taylor series expansion over z at $z = 0$.

We assume that $f(\mathbf{x}, z, t)$ is harmonic in the domains

$$z \le \eta(\mathbf{x}, t) \text{ and } z \le 0 \tag{3.70}$$

which ensures a convergence of the Taylor series in (3.69) by the real analyticity of any harmonic function. We notice that the solution of the Laplace equation (3.32) with the Dirichlet BC (3.44) (called by Dirichlet problem, see e.g. Ref. ???) generally guarantees that $\Phi(\mathbf{x}, z, t)$ is harmonic only for $z < \eta(\mathbf{x}, t)$ and continuous at $z = \eta(\mathbf{x}, t)$. It means that for the generic function $\Psi(\mathbf{x}, t)$ which is the continuous in $\mathbf{x}$ and for $\mathbf{x}$ such that $\eta(\mathbf{x}, t) \le 0$ we cannot generally expect a convergence of the Taylor series in (3.69) because $z = 0$ is located either outside of water for $\eta(\mathbf{x}, t) < 0$ or exactly at the free surface for $\eta(\mathbf{x}, t) = 0$. However, in practical physical situation (with $\Phi(\mathbf{r}, z, t)$ specific to water dynamics) a lack of convergence of the Taylor series in (3.69) implies that the continuous extension of $\Phi(\mathbf{r}, z, t)$ to outside of fluid (which is $3D$ analog of the analytical continuation in the complex analysis) has singularities just outside of the fluid surface thus invalidating the expansion (3.51) of the Hamiltonian in powers of the nonlinearity. Such strongly nonlinear solutions are not considered in this section (see Sections ??? for examples of such strongly nonlinear solutions which require different approaches that (3.51)). Thus below in this section we assume a convergence of (3.69) for any harmonic function f.

In practical simulations one can use the explicit expression of $\Phi_{\mathbf{k}}(z,t)$ through $\Phi_{\mathbf{k}}(0,t)$ as follows from equation (3.58). Then choosing $z = \eta_{max} := \max_{\mathbf{x}} \eta(\mathbf{x},t) > 0$ and applying inverse FT

$$f(\mathbf{x}) = \frac{1}{2\pi}\int \exp(i\mathbf{k}\cdot\mathbf{x}) f_{\mathbf{k}} d\mathbf{k}, \tag{3.71}$$

to both sides of equation (3.58) results in

$$\Phi(\mathbf{x},\eta_{max},t) = \frac{1}{2\pi}\int \exp(i\mathbf{k}\cdot\mathbf{x})\frac{\Phi_{\mathbf{k}}(0,t)\cosh(k[\eta_{max}+h])}{\cosh(kh)} d\mathbf{k}. \tag{3.72}$$

A convergence of the integral over $\mathbf{k}$ in r.h.s. of equation (3.72) in the limit $kh \to \infty$ is ensured if $|\Phi_{\mathbf{k}}(0,t)|$ decays faster than $e^{-k\eta_{max}}$ (here we used that $\frac{\Phi_{\mathbf{k}}(0,t)\cosh(k[\eta_{max}+h])}{\cosh(kh)} \to \Phi_{\mathbf{k}}(0,t)e^{k\eta_{max}}$ in that limit).

We consider a particular case of infinitely deep of fluid, $h \to \infty$. Then the dispersion relation (3.61) and the quadratic Hamiltonian (3.60) turn into

$$\omega_k^2 = A_k B_k = k(g + \sigma k^2) \tag{3.73}$$

and

$$H_2 = \frac{1}{2}\int \Big\{ k|\Psi_{\mathbf{k}}|^2 + (g+\sigma k^2)|\eta_{\mathbf{k}}|^2 \Big\} d\mathbf{k}. \tag{3.74}$$

Taking limit $h \to \infty$ in equation (3.58) gives that

$$\Phi_{\mathbf{k}}(z,t) = \Phi_{\mathbf{k}}(0,t)e^{kz}. \tag{3.75}$$

Equation (3.75) allows to explicitly find all derivatives of $\Phi_{\mathbf{k}}(z,t)$ over z at $z=0$. Then the inverse FT of these derivatives can be plugged into equation (3.69) for the shift operator $\hat{L}_\eta$ with $f = \Phi$. It results in

$$\Phi(\mathbf{x},\eta(\mathbf{x},t),t) = \Psi = \left(1 + \eta\hat{k} + \frac{\eta^2}{2!}\hat{k}^2 + \dots\right)\Phi(\mathbf{x},0,t) := \hat{L}_\eta \Phi(\mathbf{x},0,t), \tag{3.76}$$

where $\hat{k}$ is the positive definite linear integral operator which corresponds to the multipication on k in Fourier space.

It is important to note that the ordering of both η and $\hat{k}$ is important in the definition of the shift operator $\hat{L}_\eta$ in equation (3.76). To better understand that ordering we can expand r.h.s. of (3.75) in Taylor series over z at $z=0$ and apply inverse FT (3.71) to both sides of (3.75) to obtain that

$$\Phi(\mathbf{x},z,t) = \left(1 + z\hat{k} + \frac{z^2}{2!}\hat{k}^2 + \dots\right)\Phi(\mathbf{x},0,t). \tag{3.77}$$

Here we moved each power of z to the left from each power of $\hat{k}$. It is always possible because the linear operator $\hat{k}$ does not depend on z. Only *after* such ordering we were able plug in $z = \eta(\mathbf{x},t)$ into equation (3.77) which results again in equation (3.76).

Inverting the shift operator $\hat{L}_\eta$ (3.76) we obtain that

$$\Phi(\mathbf{x},0,t) = \hat{L}_\eta^{-1}\Psi(\mathbf{x},t). \tag{3.78}$$

Such inversion can be done using a formal operator identity

$$(1+\varepsilon\hat{A})^{-1} = 1 - \varepsilon\hat{A} + \varepsilon^2\hat{A}^2 - \varepsilon^3\hat{A}^3 + \dots,\ 0 < \varepsilon \ll 1 \tag{3.79}$$

which can be verified at each order of ε by multiplying both sides of equation (3.79) on $(1+\varepsilon\hat{A})$. To apply (3.79) to $\hat{L}_\eta$, we set that $\varepsilon\hat{A}=\hat{L}_\eta-1$. Here the small parameter $\varepsilon=|k_0\eta_0|\sim|\nabla\eta_0|\ll 1$ means small slopes of free surface perturbations, where k_0 and η_0 are typical wavenumber and height of surface perturbations, respectively. Then equations (3.76) and (3.79) result in

$$\hat{L}_\eta^{-1}=1-\eta\hat{k}-\frac{\eta^2}{2}\hat{k}^2+\eta\hat{k}\eta\hat{k}+\dots \tag{3.80}$$

To calculate $\frac{\partial}{\partial z}\Phi_{\mathbf{k}}(z,t)\big|_{z=\eta}$ we differentiate equation (3.75) over z and apply inverse FT (3.71) to obtain, similar to equations (3.77) and (3.76) that

$$\Phi(\mathbf{x},z,t)=\left(1+z\hat{k}+\frac{z^2}{2}\hat{k}^2+\dots\right)\hat{k}\Phi(\mathbf{x},0,t) \tag{3.81}$$

and

$$\frac{\partial}{\partial z}\Phi_{\mathbf{k}}(z,t)\bigg|_{z=\eta}=\hat{L}_\eta\hat{k}\Phi(\mathbf{x},0,t)=\hat{L}_\eta\hat{k}\hat{L}_\eta^{-1}\Psi(\mathbf{x},t), \tag{3.82}$$

where we used that any derivative of the harmonic function is also the harmonic function as well as we took into account equation (3.78).

In a similar way, by taking gradient of equation (3.75) over $\mathbf{x}$, we obtain that

$$(\nabla\eta)\cdot\nabla\Phi_{\mathbf{k}}(z,t)|_{z=\eta}=\hat{L}_\eta\hat{k}\Phi(\mathbf{x},0,t)=(\nabla\eta)\cdot\hat{L}_\eta\nabla\hat{L}_\eta^{-1}\Psi(\mathbf{x},t), \tag{3.83}$$

Using the kinematic boundary condition (3.35), equations (3.82) and (3.83) we find that

$$v_n\sqrt{1+(\nabla\eta)^2}=\Big(\Phi_z-\nabla\eta\cdot\nabla\Phi\Big)\Big|_{z=\eta}=\Big[\hat{L}_\eta\hat{k}-(\nabla\eta)\cdot\hat{L}_\eta\nabla\Big]\hat{L}_\eta^{-1}\Psi(\mathbf{x},t). \tag{3.84}$$

Respectively, the Dirichlet-Neumann operator (3.50) is given by

$$\hat{G}=\Big[\hat{L}_\eta\hat{k}-(\nabla\eta)\cdot\hat{L}_\eta\nabla\Big]\hat{L}_\eta^{-1} \tag{3.85}$$

and the Hamitonian (3.49) takes the following form

$$H=\frac{1}{2}\int\Psi\Big[\hat{L}_\eta\hat{k}-(\nabla\eta)\cdot\hat{L}_\eta\nabla\Big]\hat{L}_\eta^{-1}\Psi\,d\mathbf{x}+\frac{1}{2}g\int\eta^2\,d\mathbf{x}+\sigma\int\Big[\sqrt{1+\left(\nabla\eta\right)^2}-1\Big]d\mathbf{x}. \tag{3.86}$$

Expanding the Hamiltonian (3.86) into the power series over the small parameter $\varepsilon\sim|\nabla\eta_0|$ and using equation (3.80), we obtain after FT over $\mathbf{x}$ and taking into account the definition of $\hat{k}$ that the quadratic Hamiltonian H_2 is given by (3.74), the cubic Hamiltonian is

$$H_3=\frac{1}{4\pi}\int L^{(1)}_{\mathbf{k}_1,\mathbf{k}_2}\Psi_{\mathbf{k}_1}\Psi_{\mathbf{k}_2}\eta_{\mathbf{k}_3}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3 \tag{3.87}$$

and the quartic Hamiltonian is

$$\begin{aligned}H_4=\frac{1}{2(2\pi)^2}\int\Big[&L^{(2)}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}\Psi_{\mathbf{k}_1}\Psi_{\mathbf{k}_2}-\frac{\sigma}{4}(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_3\cdot\mathbf{k}_4)\eta_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\Big]\\&\times\eta_{\mathbf{k}_3}\eta_{\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4,\end{aligned} \tag{3.88}$$

where the matrix elements $L^{(1)}_{\mathbf{k}_1,\mathbf{k}_2}$ and $L^{(2)}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}$ are given by

$$
\begin{aligned}
L^{(1)}_{\mathbf{k}_1,\mathbf{k}_2} &= -\mathbf{k}_1\cdot\mathbf{k}_2 - k_1k_2,\\
L^{(2)}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} &= \frac{1}{4}k_1k_2\Big(k_{1+3}+k_{2+3}+k_{1+4}+k_{2+4}\Big) - \frac{1}{2}(k_1^2k_2+k_2^2k_1),\\
&\qquad k_j := |\mathbf{k}_j|,\ k_{j+l} := |\mathbf{k}_j+\mathbf{k}_l|.
\end{aligned}
\tag{3.89}
$$

The corresponding dynamical equations for the canonical variables η and Ψ follow from (3.46),(3.51),(3.74),(3.87),(3.88) and (3.89) giving

$$
\begin{aligned}
\frac{\partial\Psi}{\partial t} &= -g\eta + \sigma\nabla^2\eta + \frac{1}{2}\Big[\big(\hat{k}\Psi\big)^2 - \big(\nabla\Psi\big)^2\Big] - (\hat{k}\Psi)\hat{k}\big[\eta(\hat{k}\Psi)\big]\\
&\quad - (\nabla^2\Psi)(\hat{k}\Psi)\eta - \frac{\sigma}{2}\nabla\cdot\Big[\nabla\eta(\nabla\eta\cdot\nabla\eta)\Big],\\
\frac{\partial\eta}{\partial t} &= \hat{k}\Psi - \nabla\cdot\big[(\nabla\Psi)\eta\big] - \hat{k}\big[\eta\hat{k}\Psi\big] + \hat{k}\Big\{\eta\hat{k}\big[\eta\hat{k}\Psi\big]\Big\}\\
&\quad + \frac{1}{2}\nabla^2\big[\eta^2\hat{k}\Psi\big] + \frac{1}{2}\hat{k}\big[\eta^2\nabla^2\Psi\big],
\end{aligned}
\tag{3.90}
$$

where we used the inverse FT (3.71).

The expressions for the cubic and quartic terms of the Hamiltonian (3.49) for the general depth h and corresponding dynamic equations which replace (3.90) are given in Appendix ???

3.3. Dynamics of ideal fluid with free surface in two dimensions

Consider the motion of ideal fluid with free surface in $\mathbb{R}^3$ which does not depend (uniform) in one direction. Then Euler equations (2.67), (2.68) are reduced to equations in two spatial coordinates $(x, y) \in \mathbb{R}^2$ while be independent on the third coordinate. The resulting Euler equations describes $2D$ hydrodynamics of ideal fluid. Boundary conditions have to be also uniform along the third coordinate which implies that third coordinate has to coincide with one of orthogonal horizontal directions of unperturbed (flat) fluid's free surface. Similar to Section (3.2), we define the fluid domain by

$$
-h \le y \le \eta(x), \quad \mathbf{r} = (x, y),
\tag{3.91}
$$

where is x is the horizontal coordinate and y is the vertical coordinates pointing upwards (i.e. in the opposite direction to the acceleration due to gravity vector $\mathbf{g}$). Note the change of notation in comparison with Section (3.2) that y is now the vertical coordinate in comparison with z being the vertical coordinate in Section (3.2). The reason for such change of notation is historical for 2D hydrodynamics as well as the letter z is used below for other important quantity.

The fluid's velocity has two components $\mathbf{v} = (v_x, v_y)$ which are defined for the potential flow through the velocity potential $\Phi(x, y, t)$ as

$$
\mathbf{v} = \nabla\Phi
\tag{3.92}
$$

with $\nabla = (\partial_x, \partial_y)$. similar to equation (3.31), the incompressibility condition $\rho \equiv$ *const* implies the Laplace equation

$$
\nabla^2\Phi = 0.
\tag{3.93}
$$

The second Euler equation (2.68) with the acceleration due to gravity term $\mathbf{g}$ (points in the negative direction of z axis) added to the r.h.-s. of (2.68) for the potential Φ satisfies the Bernoulli equation:

$$\Phi_t + \frac{1}{2}\big(\nabla\Phi\big)^2 + p + gy = 0, \tag{3.94}$$

where p is the pressure, $g = |\mathbf{g}|$ is the acceleration of gravity, and we set density of fluid to unity $\rho \equiv 1$.

Similar to equations (3.35) and (3.39), we have the kinematic boundary condition

$$\frac{\partial\eta}{\partial t} = \Big(\Phi_y - \partial_x\eta\,\partial_x\Phi\Big)\Big|_{y=\eta} = v_n\sqrt{1+(\partial_x\eta)^2}, \tag{3.95}$$

and the dynamic boundary condition at free surface

$$p\Big|_{y=\eta} = -\sigma\partial_x\frac{\partial_x\eta}{\sqrt{1+(\partial_x\eta)^2}}, \tag{3.96}$$

where

$$v_n = \mathbf{n}\cdot\nabla\Phi \tag{3.97}$$

is the normal component of fluid's velocity at free surface, $\mathbf{n} = (-\partial_x\eta, 1)\big[1 + (\partial_x\eta)^2\big]^{-1/2}$ is the interface normal vector and σ is the surface tension coefficient which determines the jump of the pressure at free surface from zero value outside of the fluid to $p\Big|_{y=\eta}$ value inside fluid.

Boundary condition at the bottom is

$$\Phi_y|_{y=-h} = 0 \tag{3.98}$$

which means that there is no flow through the bottom, $\Phi_y|_{y=-h} = v_n|_{y=-h} = 0$. While the ideal fluid implies that the fluid velocity can have a nonzero component $v_x|_{y=-h}$ along the bottom.

The total energy, H, of the fluid

$$H = K + U, \tag{3.99}$$

consists of the kinetic energy, T,

$$K = \frac{1}{2}\int dx\int_{-h}^{\eta}\big(\nabla\Phi\big)^2 dy, \tag{3.100}$$

and the potential energy, U,

$$U = \frac{1}{2}g\int \eta^2\,dx + \sigma\int\Big[\sqrt{1+\big(\nabla\eta\big)^2} - 1\Big]dx. \tag{3.101}$$

Similar to the definition (3.44), we introduce the value of the velocity potential at interface as

$$\Phi\Big|_{y=\eta} \equiv \Psi(x,t). \tag{3.102}$$

The reduction of equations (3.46) to 2D flow results in the canonical Hamiltonian form

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi}, \quad \frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta}. \tag{3.103}$$

of 2D hydrodynamics with free surface.

So far this Section was the immediate adaption of 3D flow results to 2D flow. However, study 2D flow offers significant advantage if we introduce the stream function $\Theta(x, y, t)$ as follows

$$\frac{\partial}{\partial x}\Theta = -\frac{\partial}{\partial y}\Phi = -v_y \quad \text{and} \quad \frac{\partial}{\partial y}\Theta = \frac{\partial}{\partial x}\Phi = v_x \tag{3.104}$$

which allows to satisfy the incompressibility condition identically by

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}v_x + \frac{\partial}{\partial y}v_y = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\Theta + \frac{\partial}{\partial y}\left[-\frac{\partial}{\partial x}\Theta\right] = 0. \tag{3.105}$$

We define the *complex velocity potential* Π as

$$\Pi = \Phi + \mathrm{i}\Theta. \tag{3.106}$$

with the definition (3.104) representing the Cauchy-Riemann conditions for analyticity of $\Pi(z, t)$ in complex variable

$$z := x + \mathrm{i}y, \tag{3.107}$$

where x and y are in the fluid domain (3.91). In other words, $\Pi(z, t)$ is the complex analytic function of z but not of its complex conjugate $\bar{z}$. It implies that the complex analysis can be used to address 2D hydrodynamics with t playing the role of parameter. Analyticity of $\Pi(z, t)$ ensures that its real part Φ and imaginary part Θ are harmonically conjugated functions.

It is convenient to consider the conformal map $z(w, t)$ at each time t from the strip $-\tilde{h} \leq Im(w) \leq 0$ into the area occupied by the fluid. Here $\tilde{h}(t)$ is the thickness of that strip which is generally time-dependent. The conformal map ensures that the Laplace equation in (x, y) variables turns into the Laplace equation

$$(\partial_u^2 + \partial_v^2)\Phi = 0 \tag{3.108}$$

for $\Phi(u, v, t) := \Phi(x(u, v, t), y(u, v, t), t)$.

The Laplace equation in the strip have the explicit solution for any smooth boundary conditions which provides the explicit form for the Dirichlet-Neumann operator (3.50)???

3.4. KHI for the interface between ideal fluid 1 below and viscous fluid 2 above the interface

We consider incompressible $2D$ motion of two fluids of infinite depth in the plane (x, y) in the gravity field $\mathbf{g}$. These fluids are separated by the interface

$$y = \eta(x, t) \tag{3.109}$$

as schematically shown in Fig. ??? with the surface tension coefficient σ at that interface. The constant vector $\mathbf{g}$ is pointing downwards, $\mathbf{g} = -g\hat{y}$, where $\hat{y}$ is the unit vector in the positive direction along axis y. The fluid 1 is located below the interface (3.109) and is assumed to have the negligible viscosity. The fluid 1 is located below the interface (3.109) and is assumed to have the negligible kinematic viscosity. The fluid 2 is located above the interface (3.109) and is assumed to have the nonzero dynamic viscosity η_2 and the density ρ_2. We notice that η_2 is completely different quantity compared with η in equation (3.109) but we choose to keep the notation η_2 to be consistent with the notation of the book [**LL89a**]. We define the kinematic viscosity as $\nu_2 = \frac{\eta_2}{\rho_2}$.

The dynamics of the lower ideal fluid is given by the Euler equations for velocity $\mathbf{v}_1$,

$$\partial_t \mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 = \mathbf{g} - \frac{1}{\rho_1}\nabla p_1, \tag{3.110}$$

and the incompressiblity condition

$$\nabla \cdot \mathbf{v}_1 = 0 \tag{3.111}$$

where the subscript "1" refers to the fluid below the interface, p_1 is the fluid 1 pressure and $\rho_1 \equiv \mathit{const}$ is the fluid density.

The dynamics of the upper viscous fluid is governed by the Navier-Stokes equations for the velocity $\mathbf{v}_2$,

$$\partial_t \mathbf{v}_2 + (\mathbf{v}_2 \cdot \nabla)\mathbf{v}_2 = \frac{\eta_2}{\rho_2}\nabla^2\mathbf{v}_2 + \mathbf{g} - \frac{1}{\rho_2}\nabla p_2, \tag{3.112}$$

and the incompressibility condition

$$\nabla \cdot \mathbf{v}_2 = 0 \tag{3.113}$$

where the subscript "2" refers to the fluid above the interface, p_2 is the fluid 2 pressure, $\rho_2 \equiv \mathit{const}$ is the fluid density, and η_2 is the dynamic viscosity.

Assuming background uniform and time independent flows $\mathbf{U}_1 = U_1\hat{x}$ for fluid 1 and $\mathbf{U}_2 = U_2\hat{x}$ for fluid 2 along the horizontal direction x (here $\hat{x}$ is the unit vector in the positive direction along axis x) with the plane interface $\eta \equiv 0$, we obtain the exact solution of Eqs. (3.112)-(3.111) as follows

$$\begin{aligned}
\mathbf{v}_1 &= U_1\hat{x},\\
\mathbf{v}_2 &= U_2\hat{x},\\
p_1 &= \rho_1 g y,\\
p_2 &= \rho_2 g y,
\end{aligned} \tag{3.114}$$

where we assumed without loss of generality that the pressure at the unperturbed interface $\eta \equiv 0$ is zero.

We look for the perturbation of the flow (3.115) as

$$\begin{aligned}
\mathbf{v}_1 &= U_1\hat{x} + \delta\mathbf{v}_1,\\
\mathbf{v}_2 &= U_2\hat{x} + \delta\mathbf{v}_2,\\
p_1 &= \rho_1 g y + \delta p_1,\\
p_2 &= \rho_2 g y + \delta p_2,
\end{aligned} \tag{3.115}$$

where $\delta\mathbf{v}_1, \delta\mathbf{v}_2, \delta p_1$ and δp_2 are small perturbations. A linearization of Eqs. (3.112)-(3.111) with respect to these small perturbations together with the assumption that

$$\delta\mathbf{v}_1, \delta\mathbf{v}_2, \delta p_1, \delta p_2 \propto e^{\mathrm{i}k_x x - \mathrm{i}\omega t} \tag{3.116}$$

results in

$$\begin{aligned}
-\mathrm{i}\omega\delta\mathbf{v}_1 + U_1\mathrm{i}k_x\delta\mathbf{v}_1 &= -\frac{1}{\rho_1}\nabla\delta p_1,\\
-\mathrm{i}\omega\delta\mathbf{v}_2 + U_2\mathrm{i}k_x\delta\mathbf{v}_2 &= \frac{\eta_2}{\rho_2}(-k_x^2 + \partial_y^2)\mathbf{v}_2 - \frac{1}{\rho_2}\nabla\delta p_2,\\
\mathrm{i}k_x\delta v_{1,x} + \partial_y\delta v_{1,y} &= 0,\\
\mathrm{i}k_x\delta v_{2,x} + \partial_y\delta v_{2,y} &= 0.
\end{aligned} \tag{3.117}$$

We apply divergence to the first Eqs. in (3.117) which removes all terms with $\delta\mathbf{v}_1$ and $\delta\mathbf{v}_2$ by the incompressibility equations (3.113) and (3.111). In particular, equations for pressure fluctuations take the following form

$$\begin{aligned}\frac{1}{\rho_1}(-k_x^2+\partial_y^2)\delta p_1 &= 0,\\ \frac{1}{\rho_2}(-k_x^2+\partial_y^2)\delta p_2 &= 0\end{aligned} \tag{3.118}$$

which implies that

$$\begin{aligned}\frac{\delta p_1}{\rho_1} &= c_1 e^{\mathrm{i}k_x x-\mathrm{i}\omega t+|k_x|y} + c_2 e^{\mathrm{i}k_x x-\mathrm{i}\omega t-|k_x|y},\\ \frac{\delta p_2}{\rho_2} &= c_3 e^{\mathrm{i}k_x x-\mathrm{i}\omega t+|k_x|y} + c_4 e^{\mathrm{i}k_x x-\mathrm{i}\omega t-|k_x|y},\end{aligned} \tag{3.119}$$

where c_1, c_2, c_3, c_4 are constants.

We assume the decaying boundary conditions

$$\delta\mathbf{v}_1, \delta p_1 \to 0 \text{ as } y \to -\infty \text{ and } \delta\mathbf{v}_2, \delta p_2 \to 0 \text{ as } y \to +\infty, \tag{3.120}$$

i.e. deep inside fluid. It implies that $c_2 = c_3 = 0$ and equation (3.119) is then reduced to

$$\frac{\delta p_1}{\rho_1} = c_1 e^{\mathrm{i}k_x x-\mathrm{i}\omega t+|k_x|y}, \tag{3.121a}$$

$$\frac{\delta p_2}{\rho_2} = c_4 e^{\mathrm{i}k_x x-\mathrm{i}\omega t-|k_x|y}. \tag{3.121b}$$

Plugging (3.121a) into the first Eq. in (3.117) results in

$$\delta v_{1,x} = A_1 e^{\mathrm{i}k_x x-\mathrm{i}\omega t+|k_x|y}, \tag{3.122a}$$

$$\delta v_{1,y} = -\mathrm{i}\,\mathrm{sign}(k_x) A_1 e^{\mathrm{i}k_x x-\mathrm{i}\omega t+|k_x|y}. \tag{3.122b}$$

where A_1 is the arbitrary complex constant with the arbitrary constant c_1 of equation (3.121a) absorbed into A_1. The incompressibility is satisfied by the factor $-\mathrm{i}\,\mathrm{sign}(k_x)$ in equation (3.122b).

Plugging (3.121b) into the second Eq. in (3.117) results in the inhomogeneous system of two linear second order ODEs over y with constant coefficients for the unknowns $\delta v_{2,x}$ and $\delta v_{2,y}$. These equations are however decoupled and we consider ODE for $\delta v_{2,x}$ only while the expression for $\delta v_{2,y}$ is recovered later from the incompressibility condition. The inhomogeneous part of the solution of that ODE results from the gradient of the pressure variations (3.121b). The general solution of that ODE is the sum of a particular solution of the full inhomogeneous system and a general solution of the homogeneous system resulting in

$$\delta v_{2,x} = A_2 e^{\mathrm{i}k_x x-\mathrm{i}\omega t-|k_x|y} + B_2 e^{\mathrm{i}k_x x-\mathrm{i}\omega t-m_2 y}, \tag{3.123}$$

where

$$m_2 \equiv \left(|k_x|^2 - \frac{\mathrm{i}\omega}{\nu_2} + U_2\frac{\mathrm{i}k_x}{\nu_2}\right)^{1/2}, \tag{3.124}$$

and A_2, B_2 are two arbitrary constants with the arbitrary constant c_4 of equation (3.121b) absorbed into A_2. We assume that the square root in (3.124) have the principal branch such that the term $B_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t - m_2 y}$ in equation (3.123) decays as $y \to +\infty$. We also removed another term $\propto e^{\mathrm{i}k_x x - \mathrm{i}\omega t + m_2 y}$ from (3.124) to take into account the decaying boundary conditions (3.124).

Equation (3.123) and the incompressibility condition in the fourth equation of (3.117) imply that

$$\delta v_{2,y} = \mathrm{i}\,\mathrm{sign}(k_x) A_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t - |k_x| y} + \frac{\mathrm{i}k_x}{m_2} B_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t - m_2 y}. \tag{3.125}$$

Plugging equations (3.121),(3.122),(3.123) and (3.125) into (3.117) allows to recover the constants c_1 and c_4 in (3.121) to obtain the expressions for the pressure fluctuations as follows

$$\begin{aligned} \frac{\delta p_1}{\rho_1} &= A_1 \left(\frac{\omega}{k_x} - U_1 \right) e^{\mathrm{i}k_x x - \mathrm{i}\omega t + |k_x| y}, \\ \frac{\delta p_2}{\rho_2} &= A_2 \left(\frac{\omega}{k_x} - U_2 \right) e^{\mathrm{i}k_x x - \mathrm{i}\omega t - |k_x| y}. \end{aligned} \tag{3.126}$$

Equations (3.122)-(3.126) provide general solutions of the system (3.117) which satisfy the decaying boundary conditions (3.124) with the arbitrary complex constants A_1, A_2 and B_2. We use these arbitrary constants to satisfy boundary conditions at the interface (3.109). We need to take into account both dynamic and kinematic boundary conditions.

The dynamic boundary condition for viscous fluid involve the continuity of both tangential stress and normal stress at the interface. The full expressions are provided in the book [**LL89a**] (give them here ???). The tangential stress of the ideal fluid 1 is zero. In the linear approximation we replace (3.109) by the condition at the flat surface $y \equiv 0$ which gives together with equations (3.123) and (3.125) that

$$\begin{aligned} \frac{1}{\eta_2}\sigma_2'|_{y=0} &= \partial_y v_{x,2} + \partial_x v_{y,2}|_{y=0} = -|k_x| A_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t} - m_2 B_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t} \\ &\quad + \mathrm{i}^2 k_x\,\mathrm{sign}(k_x) A_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t} + \frac{\mathrm{i}^2 k_x^2}{m_2} B_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t} \\ &= -2|k_x| A_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t} + \frac{-B_2}{m_2}(m_2^2 + k_x^2) e^{\mathrm{i}k_x x - \mathrm{i}\omega t} \\ &= -2|k_x| A_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t} - \frac{B_2}{m_2}\left(2k_x^2 - \frac{\mathrm{i}\omega}{\nu_2} + U_2 \frac{\mathrm{i}k_x}{\nu_2} \right) e^{\mathrm{i}k_x x - \mathrm{i}\omega t} = 0. \end{aligned} \tag{3.127}$$

The viscous part of the normal stress is zero at the interface because the fluid 1 is ideal. The pressure contribution is nonzero and must be continuous across the interface. For the normal stress we have to take into account the unperturbed pressure evaluated at the interface which all together with equations (3.115),(3.122)-(3.126) gives that

$$
\begin{aligned}
&p_2 - p_1 - 2\nu_2\rho_2\partial_y v_{2,y}|_{y=0} = -(\rho_2-\rho_1)g\eta + \delta p_2 - \delta p_1 - 2\nu_2\rho_2\partial_y v_{2,y}|_{y=0}\\
&= -(\rho_2-\rho_1)g\eta + \rho_2 A_2\left(\frac{\omega}{k_x} - U_2\right)e^{\mathrm{i}k_x x - \mathrm{i}\omega t} - \rho_1 A_1\left(\frac{\omega}{k_x} - U_1\right)e^{\mathrm{i}k_x x - \mathrm{i}\omega t}\\
&- 2\nu_2\rho_2\left[(-|k_x|)\mathrm{i}\,\mathrm{sign}(k_x)A_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t} + (-m_2)\frac{\mathrm{i}k_x}{m_2}B_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t}\right]\\
&= -(\rho_2-\rho_1)g\eta + \rho_2 A_2\left(\frac{\omega}{k_x} - U_2\right)e^{\mathrm{i}k_x x - \mathrm{i}\omega t} - \rho_1 A_1\left(\frac{\omega}{k_x} - U_1\right)e^{\mathrm{i}k_x x - \mathrm{i}\omega t}\\
&- 2\nu_2\rho_2(-\mathrm{i}k_x)\left[A_2 + B_2\right]e^{\mathrm{i}k_x x - \mathrm{i}\omega t} = \sigma(-k_x^2)\eta,
\end{aligned}
\tag{3.128}
$$

where

$$
\eta \equiv \eta_1 e^{\mathrm{i}k_x x - \mathrm{i}\omega t}. \tag{3.129}
$$

The kinematic boundary condition is given by $\partial_t\eta = v_{1(2),y} - (\mathbf{v}_{1(2)}\cdot\nabla)\eta|_{y=\eta}$ which if written separately for each fluid together with $\partial_t\eta = -\mathrm{i}\omega\eta$ give in the linear approximation that

$$
\begin{aligned}
&-\mathrm{i}\omega\eta = v_{1,y} - (\mathbf{v}_1\cdot\nabla)\eta|_{y=0} = \delta v_{1,y}|_{y=0} - U_1\mathrm{i}k_x\eta,\\
&-\mathrm{i}\omega\eta = v_{2,y} - (\mathbf{v}_2\cdot\nabla)\eta|_{y=0} = \delta v_{2,y}|_{y=0} - U_2\mathrm{i}k_x\eta.
\end{aligned}
\tag{3.130}
$$

Eqs. (3.122a),(3.122b),(3.123),(3.125),(3.127),(3.128),(3.129),(3.130) form a homogeneous system of 4 linear equations for the unknowns A_1, A_2, B_2, η_1. Its solvability condition gives the following dispersion relation

$$
(\rho_1-\rho_2)g|k_x| + \sigma|k_x|^3 = \rho_1(\omega - U_1k_x)^2 + \rho_2\left(\omega - U_2k_x + \mathrm{i}2\nu_2k_x^2\right)^2 + 4\rho_2\nu_2^2|k_x|^3 m_2. \tag{3.131}
$$

Particular case is $U_1 = U_2 = \rho_1 = 0$ which gives the following dispersion relation

$$
\left(2 - \frac{\mathrm{i}\omega}{\nu_2k_x^2}\right)^2 - \frac{g}{k_x^3\nu_2^2} = 4\sqrt{1 - \frac{\mathrm{i}\omega}{\nu_2k_x^2}} \tag{3.132}
$$

It give the same expression as on page 136 of Ref. [**LL89a**] if we replace $g\to -g$ there because our liquid 2 is located above the surface (3.109) while in Ref. [**LL89a**] the single fluid is located below the free surface.

3.5. KHI for the free surface of Helium II with both normal and superfluid components between ideal fluid 1 below and viscous fluid 2 above the interface

The difference with section 3.4 is that the viscous fluid 2 is now also below the interface (free surface). We now replace index 2 by "n" and index "1" by "s". Navier Stokes equations for the normal component of fluid

$$
\partial_t\mathbf{v}_n + (\mathbf{v}_n\cdot\nabla)\mathbf{v}_n = \frac{\eta_n}{\rho_n}\nabla^2\mathbf{v}_n - \mathbf{g} - \frac{1}{\rho_n}\nabla p_n, \tag{3.133}
$$

and the incompressibility condition

$$\nabla \cdot \mathbf{v}_n = 0 \tag{3.134}$$

where η_n means the dynamics viscosity of the normal component. In a similar way, the ideal fluid is given by the Euler equations

$$\partial_t \mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = -\mathbf{g} - \frac{1}{\rho_s}\nabla p_s, \tag{3.135}$$

and the incompressiblity condition

$$\nabla \cdot \mathbf{v}_s = 0. \tag{3.136}$$

Here in the usual way we separated pressure to the superfluid and normal components p_s and p_n, respectively such that each of the satisfy Navier-Stokes eqs. and the actual (total) pressure in the fluid is given by $p = p_s + p_n$. Assuming a background uniform flow $\mathbf{U}_{1,2} = U_{1,2}\hat{x}$ along the horizontal direction x we represent solution in the following form

$$\begin{aligned} \mathbf{v}_s &= U_s\hat{x} + \delta\mathbf{v}_s, \\ \mathbf{v}_n &= U_n\hat{x} + \delta\mathbf{v}_n, \\ p_{s,n} &= -\rho_{s,n} g y + \delta p_{s,n}. \end{aligned} \tag{3.137}$$

Then the linearization of the above eqs. together with the assumption $\delta\mathbf{v}_s, \delta\mathbf{v}_n, \delta p_{s,n} \propto e^{\mathrm{i}k_x x - \mathrm{i}\omega t}$ results in

$$\begin{aligned} &-\mathrm{i}\omega\delta\mathbf{v}_s + U_s \mathrm{i}k_s \delta\mathbf{v}_1 = -\frac{1}{\rho_s}\nabla\delta p_s, \\ &-\mathrm{i}\omega\delta\mathbf{v}_2 + U_n \mathrm{i}k_x \delta\mathbf{v}_n = \frac{\eta_2}{\rho_2}(-k_x^2 + \partial_y^2)\mathbf{v}_n - \frac{1}{\rho_n}\nabla\delta p_n, \\ &\mathrm{i}k_x\delta v_{s,x} + \partial_y \delta v_{s,y} = 0, \\ &\mathrm{i}k_x\delta v_{n,x} + \partial_y \delta v_{n,y} = 0. \end{aligned} \tag{3.138}$$

We define the kinematic viscosity as

$$\nu_n = \frac{\eta_n}{\rho_n}, \tag{3.139}$$

where we note that we defined here ν_n in a little NONSTANDARD way because we normalized the dynamic viscosity by ρ_n, not by $\rho \equiv \rho_s + \rho_n$.

We apply divergence to first two eqs. in (3.138) and use incompressibility to obtain the equations for $\delta\mathbf{v}_s, \delta p_s$ and separately for $\delta\mathbf{v}_n, \delta p_n$. In particular, equations for pressure fluctuations take the following form

$$\begin{aligned} \frac{1}{\rho_s}(-k_x^2 + \partial_y^2)\delta p_s &= 0, \\ \frac{1}{\rho_n}(-k_x^2 + \partial_y^2)\delta p_n &= 0 \end{aligned} \tag{3.140}$$

which implies that

$$\begin{aligned} \frac{\delta p_s}{\rho_s} &= c_1 e^{\mathrm{i}k_x x - \mathrm{i}\omega t + |k_x| y} + c_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t - |k_x| y}, \\ \frac{\delta p_n}{\rho_n} &= c_3 e^{\mathrm{i}k_x x - \mathrm{i}\omega t + |k_x| y} + c_4 e^{\mathrm{i}k_x x - \mathrm{i}\omega t - |k_x| y}. \end{aligned} \tag{3.141}$$

Assuming now exponential dependence on y as $\delta\mathbf{v}_s, \delta\mathbf{v}_n, \delta p_{s,n} \propto e^{\mathrm{i}k_x x - \mathrm{i}\omega t + \kappa_{s,n} y}$ and setting the condition that solutions decays at $y \to \infty$ we obtain that

$$
\begin{aligned}
&\kappa_s = |k_x|, \\
&\kappa_n = -|k_x|, \text{ and } \kappa_n = \left(|k_x|^2 - \frac{\mathrm{i}\omega}{\nu_n} + U_n \frac{\mathrm{i}k_x}{\nu_n}\right)^{1/2} \equiv m_n,
\end{aligned} \tag{3.142}
$$

I.e. addition to solutions for $\delta\mathbf{v}_s, \delta\mathbf{v}_s$ with pressure given by (3.121) we also for the normal fluid the second solution without pressure with the corresponding values of κ_n given in Eq. (3.124).

Assume that

$$
\delta v_{s,x} = A_s e^{\mathrm{i}k_x x - \mathrm{i}\omega t + |k_x| y}, \tag{3.143}
$$

where A_1 is the arbitrary complex constant. Then from the incompressibility we conclude that

$$
\delta v_{s,y} = -\mathrm{i}\,\mathrm{sign}(k_x) A_s e^{\mathrm{i}k_x x - \mathrm{i}\omega t + |k_x| y}. \tag{3.144}
$$

In a similar way using both solutions in the second Eq. in (3.142) and incompressibility, we obtain that

$$
\delta v_{n,x} = A_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t + |k_x| y} + B_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t + m_n y}, \tag{3.145}
$$

and

$$
\delta v_{n,y} = -\mathrm{i}\,\mathrm{sign}(k_x) A_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t + |k_x| y} - \frac{\mathrm{i}k_x}{m_n} B_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t + m_n y}, \tag{3.146}
$$

where A_n and B_n are the arbitrary complex constants. Note that contrary to Eqs. (3.123) and (3.125), we assumed in Eqs. (3.145) and (3.146) that both components of superfluid are located below a free surface then only decaying Fourier components at $y \to -\infty$ are taken into account. Then plugging in equations (3.141),(3.143),(3.144),(3.145),(3.146) we obtain the explicit expressions for the pressure fluctuations as follows

$$
\begin{aligned}
\frac{\delta p_s}{\rho_s} &= A_s \left(\frac{\omega}{k_x} - U_s\right) e^{\mathrm{i}k_x x - \mathrm{i}\omega t + |k_x| y}, \\
\frac{\delta p_n}{\rho_n} &= A_n \left(\frac{\omega}{k_x} - U_n\right) e^{\mathrm{i}k_x x - \mathrm{i}\omega t + |k_x| y}.
\end{aligned} \tag{3.147}
$$

To satisfy boundary conditions we need to take into account both dynamic and kinematic boundary conditions.

Tangential stress gives

$$
\begin{aligned}
&\frac{1}{\eta_n}\sigma'_n|_{y=0} = \partial_y v_{x,n} + \partial_x v_{y,n}|_{y=0} = |k_x| A_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t} + m_n B_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t} \\
&- \mathrm{i}^2 k_x\,\mathrm{sign}(k_x) A_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t} - \frac{\mathrm{i}^2 k_x^2}{m_n} B_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t} \\
&= 2|k_x| A_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t} + \frac{B_n}{m_n}(m_n^2 + k_x^2) e^{\mathrm{i}k_x x - \mathrm{i}\omega t} \\
&= 2|k_x| A_n e^{\mathrm{i}k_x x - \mathrm{i}\omega t} + \frac{B_n}{m_n}\left(2k_x^2 - \frac{\mathrm{i}\omega}{\nu_n} + U_n \frac{\mathrm{i}k_x}{\nu_n}\right) e^{\mathrm{i}k_x x - \mathrm{i}\omega t} = 0.
\end{aligned} \tag{3.148}
$$

For the normal stress we have to take into account the unperturbed pressure evaluated at the interface which all together gives

$$
\begin{aligned}
&-p_n - p_s + 2\nu_n\rho_n\partial_y v_{n,y}|_{y=0} = (\rho_n+\rho_s)g\eta - \delta p_n - \delta p_s + 2\nu_n\rho_n\partial_y v_{n,y}|_{y=0}\\
&= (\rho_n+\rho_s)g\eta - \rho_n A_n\left(\frac{\omega}{k_x} - U_n\right)e^{\mathrm{i}k_x x - \mathrm{i}\omega t} - \rho_s A_s\left(\frac{\omega}{k_x} - U_s\right)e^{\mathrm{i}k_x x - \mathrm{i}\omega t}\\
&+ 2\nu_n\rho_n\left[(-|k_x|)\mathrm{i}\,\mathrm{sign}(k_x)A_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t} + (-m_2)\frac{\mathrm{i}k_x}{m_2}B_2 e^{\mathrm{i}k_x x - \mathrm{i}\omega t}\right]\\
&= (\rho_n+\rho_s)g\eta - \rho_n A_n\left(\frac{\omega}{k_x} - U_n\right)e^{\mathrm{i}k_x x - \mathrm{i}\omega t} - \rho_s A_s\left(\frac{\omega}{k_x} - U_s\right)e^{\mathrm{i}k_x x - \mathrm{i}\omega t}\\
&+ 2\nu_2\rho_2(-\mathrm{i}k_x)\left[A_2 + B_2\right]e^{\mathrm{i}k_x x - \mathrm{i}\omega t} = \sigma(-k_x^2)\eta,
\end{aligned}
\tag{3.149}
$$

where

$$
\eta \equiv \eta_1 e^{\mathrm{i}k_x x - \mathrm{i}\omega t}. \tag{3.150}
$$

Kinematic boundary condition is given by $\partial_t\eta = v_{1(2),y} - (\mathbf{v}_{1(2)}\cdot\nabla)\eta|_{y=\eta}$ which if written separately for each fluid together with $\partial_t\eta = -\mathrm{i}\omega\eta$ give in the linear approximation that

$$
\begin{aligned}
-\mathrm{i}\omega\eta &= v_{1,y} - (\mathbf{v}_1\cdot\nabla)\eta|_{y=0} = \delta v_{1,y}|_{y=0} - U_1\mathrm{i}k_x\eta,\\
-\mathrm{i}\omega\eta &= v_{2,y} - (\mathbf{v}_2\cdot\nabla)\eta|_{y=0} = \delta v_{2,y}|_{y=0} - U_2\mathrm{i}k_x\eta.
\end{aligned}
\tag{3.151}
$$

Eqs. (3.122a),(3.122b),(3.123),(3.125),(3.127),(3.149),(3.150),(3.151) form a homogeneous system of 4 linear equations for the unknowns A_1, A_2, B_2, η_1. Its solvability condition gives the following dispersion relation

$$
(\rho_s+\rho_n)g|k_x| + \sigma|k_x|^3 = \rho_s(\omega - U_s k_x)^2 + \rho_n\left(\omega - U_n k_x + \mathrm{i}2\nu_n k_x^2\right)^2 + 4\rho_n\nu_n^2|k_x|^3 m_n. \tag{3.152}
$$

Particular case is $U_1 = U_2 = \rho_1 = 0$ which gives the following dispersion relation

$$
\left(2 - \frac{\mathrm{i}\omega}{\nu_2 k_x^2}\right)^2 + \frac{g}{k_x^3\nu_2^2} = 4\sqrt{1 - \frac{\mathrm{i}\omega}{\nu_2 k_x^2}} \tag{3.153}
$$

which is the same on page 136. of Ref. [**LL89a**].

By comparison of dispersion relations (3.131) and (3.152) we conclude that they are obtained from each other if we change the sign of ρ_2 (or the sign of the corresponding ρ_n) in their l.h.s.

3.6. Hamiltonian formalizm for the motion of the interface of two ideal fluids

Consider the motion of boundary surface separating two ideal fluids in a gravitational field $\mathbf{g}$. We assume that axes z has opposite to $\mathbf{g}$ direction.

The boundary surface coordinates are given by eqn. $z = \eta(x, y, t)$. The first heavier fluid (index 1) occupies the region $-\infty < z < \eta$ and the second slight fluid (index 2) occupies the region $\eta < z < \infty$. Normal surface vector is given by the equation

$$\vec{n} = (-\nabla\eta, 1)\frac{1}{\sqrt{1+(\nabla\eta)^2}}.$$

It is assumed that fluids motion is irrotational one and that the upper fluid moves as a whole relatively to the lower with average velocity $\vec{V}$, which is parallel to the axes x :

$$\vec{V_1} = \nabla\phi_1, \quad \vec{V_2} = \vec{V} + \nabla\phi_2 \ . \tag{3.154}$$

Fluids incompressibility condition leads to Laplace equation for velocity potentials $\phi_{1,2}$:

$$\Delta\phi_{1,2} = 0. \tag{3.155}$$

The boundary conditions at infinity are:

$$\phi_1 \to 0 \text{ for } z \to -\infty \ ,$$

$$\phi_2 \to 0 \text{ for } z \to +\infty \ . \tag{3.156}$$

The boundary conditions of equations (3.155) on the boundary surface $z = \eta$ are separated on kinematic and dynamic ones.

Kinematic condition on $z = \eta$

$$\frac{d\eta}{dt} = \frac{\partial\eta}{\partial t} + (\vec{V}\nabla)\eta = V_z, \tag{3.157}$$

is given by the equality of fluids velocities components, normal to the boundary surface:

$$V_{n,1}|_\Gamma = V_{n,2}|_\Gamma$$

or

$$\left(\frac{\partial\phi_1}{\partial z} - (\nabla\phi_1, \nabla\eta)\right)|_{z=\eta} = \left(\frac{\partial\phi_2}{\partial z} - (\vec{V}, \nabla\eta) - (\nabla\phi_2, \nabla\eta)\right)|_{z=\eta} \ . \tag{3.158}$$

The dynamic boundary conditions is determined by the residual of the pressure of two fluids on the boundary surface which equals to the surface tension term:

$$p_2 - p_1 = \alpha div\frac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}} \ . \tag{3.159}$$

Meanwhile the pressures can be expressed by Bernoulli equation through the $\phi_{1,2}$:

$$\left(\frac{\partial\phi_1}{\partial t} + \frac{(\nabla\phi_1)^2}{2} + g\eta\right)|_{z=\eta} + \alpha div\frac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}} =$$

$$= \epsilon\left(\frac{\partial\phi_2}{\partial t} + \frac{(\nabla\phi_2)^2}{2} + (\vec{V}, \nabla\phi_2) + g\eta\right)|_{z=\eta} \ . \tag{3.160}$$

Here ϵ is the densities ratio $\epsilon = \frac{\rho_2}{\rho_1}$. The equations (3.155-3.160) form the closed system of equations.

Let us introduce the boundary values of velocity potentials $\Psi_1 = \phi_1 \ |_{z=\eta}$, $\Psi_2 = \phi_2 \ |_{z=\eta}$ and their linear combination

$$\Psi = \rho_1\Psi_1 - \rho_2\Psi_2. \tag{3.161}$$

The known values of $\Psi_{1,2}$ ensure solvability of the Laplace equations (3.155). Then it is possible to find by direct calcination that a set of equations (3.155-3.160) can be written in hamiltonian form [11, 12] :

$$\frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \ \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi} \ , \tag{3.162}$$

where Hamiltonian

$$H = \frac{1}{2} \int\limits_{z \geq \eta} \epsilon \left[(\nabla \phi_2)^2 + 2(\nabla \phi_2, \vec{V}) \right] d^3 r + \frac{1}{2} \int\limits_{z \leq \eta} (\nabla \phi_1)^2 d^3 r +$$

$$+ \int \left[\frac{g\eta^2}{2}(1-\epsilon) + \alpha(\sqrt{1+(\nabla\eta)^2} - 1) \right] d^2 r \ . \tag{3.163}$$

with the accuracy of a constant coincides with the total fluids energy. Here and below we set $\rho_1 = 1$.

For the subsequent calculations it is convenient to write Hamiltonian (3.163) as a surface integral:

$$H = \frac{1}{2} \int \Big[V_n \Psi \sqrt{1+(\nabla\eta)^2} + \epsilon \Psi_2 (\vec{V}, \nabla)\eta +$$

$$+ (1-\epsilon) g \eta^2 + 2\alpha(\sqrt{1+(\nabla\eta)^2} - 1) \Big] d^2 \vec{r} \ . \tag{3.164}$$

Her and below $\vec{r} = (x, y)$ is a two dimensional vector in horizontal plain, ∇ is a gradient operator with respect to x and y.

In (3.164) we need to express the normal component of velocity V_n and potential Ψ_2 through the canonical variables Ψ and η. To do so first it is necessary to salve Laplace equations (3.155) with boundary conditions (3.156):

$$\phi_{1,2}(\vec{r}, z) = \int \phi_{1,2k}(0) e^{\pm kz + i\vec{k}\vec{r}} d^2 k, \tag{3.165}$$

where

$$\phi_{1,2k}(0) = \frac{1}{(2\pi)^2} \int \phi_{1,2}(\vec{r}, 0) e^{-i\vec{k}\vec{r}} d^2 r \ , \ \vec{k} = (k_x, k_y),$$

and then it is necessary to find V_n, Ψ_2 with the help of boundary conditions (3.158), (3.161). It is evident that explicit expressions for V_n and Ψ_2 can be obtained only by expansion of (3.158), (3.161) in series by small parameter $\mid \nabla\eta \mid$ which is a typical boundary surface slope angle. As a result Hamiltonian will be expressed in a power series of canonical variables Ψ and η.

3.7. Linear Kelvin-Helmholtz Instability

Since the 19th century the Kelvin-Helmholtz instability (KHI) has been considered as one of the main mechanism of surface waves generation by wind (see e.g. [1]). The capillary forces and gravitation cause the threshold appearance of this instability. The surface waves generation exists only if the wind velocity V exceeds the critical value V_{cr} connected with the minimal phase velocity $V_{min} = min\frac{\omega_k}{k}$ of gravitational-capillary forces in the absence of wind:

$$V_{cr} = \frac{1}{\sqrt{\epsilon}} V_{min}. \tag{3.166}$$

Here

$$\omega_k^2 = \frac{k}{1+\epsilon}\Big[g(1-\epsilon) + \alpha k^2\Big] \tag{3.167}$$

is the dispersion law of gravitational-capillary waves, g is the acceleration of gravity, α is the surface tension, $\epsilon = \frac{\rho_2}{\rho_1}$ is the ratio of upper ρ_2 and lower ρ_1 fluids densities. We assume that $\epsilon \ll 1$. E.g. for air and water $\epsilon = 1.24 \cdot 10^{-3} \ll 1$ and critical wind velocity $V_{cr} = 6.4 \frac{m}{sec}$ significantly exceeds the minimal phase velocity $V_{min} = 23 \frac{cm}{sec}$. It is necessary to note that KHI is aperiodic one. The structure of linear waves spectrum radically changes near to increment maximum of KHI. In Fourier space this maximum lies in the intermediate regime between gravitational and capillary fields of dispersion law: $k \sim k_0 = \sqrt{\frac{g}{\alpha}}$. However well beyond the instability region the dispersion law coincides with (3.167). If the wind velocity slightly exceeds the threshold value V_{cr}, then only small wave disturbances having wave vectors near to $\vec{k} = \vec{k}_0$ are generated, where the direction of $\vec{k}_0$ coincides with the direction of wind $\vec{V}$ and $|\vec{k}_0| = \sqrt{\frac{g}{\alpha}}$.

KHI exists only for ideal fluids without viscosity. Just from this fact the critical velocity (3.166) exceeds the minimal phase velocity V_{min} of gravitational-capillary waves for large parameter $\frac{1}{\sqrt{\epsilon}}$. The surface waves spectrum (3.167) looks like to the Landau excitation spectrum of liquid helium [2]. And similarly to the superfluidity destruction of liquid helium it is possible to expect the surface waves generation for $V > V_{min}$. A linear theory of similar type was created by Miles [3]. According to this theory the surface wave generation for $V > V_{min}$ is possible due to fluids viscosity. The viscosity leads to the boundary layer formation in the air near to the water surface. The surface waves generation is connected with the existence of the shear flow $\vec{V} = \vec{V}(z)$ in the boundary layer, where z is a vertical coordinate. The increment of Miles instability near to V_{min} is small in comparison with ω_k. This increment grows with the growth of the wind velocity and it has no singularity for the $V \sim V_{cr}$. These features are different for Miles instability and KHI.

Recently Zakharov and Newell [4] developed nonlinear theory of surface waves generation by Miles instability. This theory is based on the kinetic equations of weak turbulence. For $V > V_{min}$ the instability region lies in the gravitational part of spectrum. The growth of waves amplitudes is bounded by the cascade processes.These processes form two Kolmogorov type spectra [5]. First is the cascade of constant flux of wave action in the range of long gravitational waves. Second is the cascade of constant energy flux on small scale of capillary waves. In the intermediate gravitational-capillary region these two spectra have to fit each other. But this is possible only if wind velocity does not exceed some value of the order of V_{cr} (3.166). If wind velocity exceed this critical value then the energy flux from gravitational region does not equal to the energy flux to capillary region. As a result waves accumulation and waves condensate formation take place in the intermediate region. According to [4] this leads to the rough sea foam formation. This prediction agrees with the experimental observation that near to the wind velocity $V \sim 6 \frac{m}{sec}$ the part of the sea surface covered by the foam grows very fast (see $[6-10]$). But we can note that KHI arises at the same wind velocities. Is it possible that KHI also leads to the foam formation? In the present work we try to answer this question by studying nonlinear stage of KHI development. The perturbation theory by small parameter, which is the typical slope of boundary surface between two fluids, is created. The using small parameter does not give the complete picture of foam formation because from the mathematical point of view the foam generation means the singularity appearance on the boundary surface where slope becomes of the order of unity. In present work only the general tendency on instability development is obtained. It is shown that the first nonvanishing order nonlinearity by wave amplitudes results in the explosive amplitude growth. Moreover if the wind velocity is slightly less critical one (3.166) then rigid mechanism of instability is realized. It means that small perturbations decay but dig ones blow up in a finite time.

In section 2 basic hydrodynamic equations in hamiltonian form describing KHI are obtained for irrotational fluids motions. In section 3 the perturbation theory by small parameter, which is a typical boundary surface slope, is developed. Near to the instability threshold, when

$$|\delta| = \frac{|V^2 - V_{cr}^2|}{V_{cr}^2},$$

the perturbation theory is simplified because only narrow (in Fourier space) wave packets are generated. On this bases the nonlinear relativistic $(2+1)$ Klein-Gordon equation $|\phi|^4$ is obtained, where $-\delta$ corresponds to the mass square and nonlinear interaction corresponds to attraction. The nonlinearity sign results in explosive wave amplitudes growth. In section 4 the space-uniform solution and automodel asymptotics of this eqn. is found. Using the integral estimation method the sufficient collapse criterion is established. This criterion follows from the second order differential inequality for the square norm of wave envelope. By simple substitution studying this inequality reduces to the analysis of the motion of Newton "particle" in some potential. In last section the comparison with satellite observations is discussed and comparison KHI with Miles instability is made.

3.8. Nonlinear Kelvin-Helmholtz instability

The linear theory of KHI can be expressed by quadratic Hamiltonian in k-representation (see [**?**]):

$$H^{(2)} = \frac{1}{2}\int \left[k|\Psi_k|^2 + \frac{1}{k}\left(\omega_k^2 - \frac{\epsilon\left(\vec{k},\vec{V}\right)^2}{1+\epsilon}\right)|\eta_k|^2\right] d^2k, \tag{3.168}$$

where ω_k is given by (3.167). According to (3.168) the plain boundary surface becomes unstable under the condition:

$$-\Gamma_k^2 = \omega_k^2 - \frac{\epsilon\left(\vec{k},\vec{V}\right)^2}{1+\epsilon} < 0. \tag{3.169}$$

This instability is known as linear KHI. Its increment is equal to Γ_k.

Everywhere below we assume that the fluids density ratio is small $\epsilon \ll 1$. This condition significantly simplifies the subsequent analysis. According to (3.169) increment Γ_k is positive if wind velocity exceeds critical value $V > V_{cr}$, where

$$V_{cr} = \frac{1}{\sqrt{\epsilon}} min \frac{\omega_k}{k} = \sqrt{\frac{2g}{\epsilon k_0}}.$$

Here $k_0 = \sqrt{\frac{g}{\alpha}}$ is a wave number of neutral perturbation on the instability threshold. If the supercriticity is small:

$$\delta = \frac{V^2 - V_{cr}^2}{V_{cr}^2} \ll 1,$$

then the instability is quasimonochromatic one. Linear waves are generated only in a small neighbourhood $\Delta k \sim k_0 \delta^{1/2}$ of $\vec{k} = \vec{k}_0$, where $\vec{k}_0$ is parallel one to the wind velocity $\vec{V}$. In this neighbourhood increment Γ_k can be written as follows:

$$\Gamma_k^2 = \omega_0^2 \left[\delta - \frac{1}{2}\frac{q_x^2}{k_0^2} - \frac{q_y^2}{k_0^2} \right], \tag{3.170}$$

where

$$\vec{k} = \vec{k}_0 + \vec{q}, \quad |q| \ll k_0, \quad \mathbf{V} = (V, 0, 0), \quad \omega_0^2 = 2gk_0.$$

So, near to KHI threshold it is natural to use wave envelopes:

$$\begin{aligned} \Psi(\vec{r}, t) &= \psi_1(\vec{r}, t) e^{i(\vec{k}_0, \vec{r})} + \psi_1^*(\vec{r}, t) e^{-i(\vec{k}_0, \vec{r})} \\ \eta(\vec{r}, t) &= \eta_1(\vec{r}, t) e^{i(\vec{k}_0, \vec{r})} + \eta_1^*(\vec{r}, t) e^{-i(\vec{k}_0, \vec{r})}, \end{aligned}$$

where ψ_1, η_1 are slow function of $\vec{r}$,

$$\psi_1(\vec{r}, t) = \frac{1}{(2\pi)^2} \int \Psi(\vec{k}_0 + \vec{q}) e^{i(\vec{q}, \vec{r})} d^2q \ , \quad \eta_1(\vec{r}, t) = \frac{1}{(2\pi)^2} \int \eta(\vec{k}_0 + \vec{q}) e^{i(\vec{q}, \vec{r})} d^2q \ .$$

The small values of parameters δ, η result in the following estimates for Ψ_1, Psi_2 in the frame of linear theory:

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \frac{\omega_0 \eta}{k_0} \end{pmatrix} \sim \begin{pmatrix} \sqrt{\delta} - i\sqrt{\varepsilon} \\ -\frac{i}{\sqrt{\epsilon}} \\ 1 \end{pmatrix} . \tag{3.171}$$

This estimation shows that velocity fluctuations of upper fluid exceeds velocity fluctuation of lower fluid an large parameter 1 ϵ. From physical point of view it means that upper slight fluid rapidly follows the slow movement of lower heavy one. It is simply to understand that this velocity fluctuations ratio retains also for nonlinear stage of KHI development. In particular it means that in the first order by ϵ the normal velocity component V_n of the boundary surface equals zero:

$$\frac{\partial \phi_2}{\partial n} \sqrt{1 + (\nabla \eta)^2} = (\vec{V}, \nabla)\eta. \tag{3.172}$$

In the next order by ϵ we obtain the usual equality of normal velocities of two fluids (3.158).

Using the approximate boundary condition (3.172) and Laplace equations (3.155)we can express potential Ψ_2 only through η. This result drastically simplifies the perturbation theory. Additional simplification comes from the smallness of the term $\frac{1}{2}\int V_n \Psi \sqrt{1 + (\nabla\eta)^2} d^2\vec{r}$ by parameter δ of the Hamiltonian (3.163) (see [**?**]). That is the Hamiltonian (3.163 is reduced to:

$$H = \frac{1}{2} \int \left[\epsilon \Psi_2 (\vec{V}, \nabla)\eta + + (1 - \epsilon) g \eta^2 + 2\alpha (\sqrt{1 + (\nabla\eta)^2} - 1) \right] d^2\vec{r} \ . \tag{3.173}$$

The nonlinear interaction results in harmonics appearance. In addition to the main space harmonic $\vec{k}=\vec{k}_0$ the whole number of harmonics are generated. Instead of (3.171) we can write for $\Psi(\vec{r},t)$ and $\eta(\vec{r},t)$:

(3.174)

$$\Psi(\vec{r},t)=\psi_0+\psi_1(\vec{r},t)e^{i(\vec{k}_0,\vec{r})}+\psi_1^*(\vec{r},t)e^{-i(\vec{k}_0,\vec{r})}+\psi_2(\vec{r},t)e^{2i\vec{k}_0,\vec{r})}+\psi_2^*(\vec{r},t)e^{-2i(\vec{k}_0,\vec{r})}+...,$$
$$\eta(\vec{r},t)=\eta_0+\eta_1(\vec{r},t)e^{i(\vec{k}_0,\vec{r})}+\eta_1^*(\vec{r},t)e^{-i(\vec{k}_0,\vec{r})}+\eta_2(\vec{r},t)e^{2i(\vec{k}_0,\vec{r})}+\eta_2^*(\vec{r},t)e^{-2i(\vec{k}_0,\vec{r})}+... .$$

Here ψ_i and η_i are slow functions of coordinate. Their widths in k-space are of the order of the width of the main harmonic $\vec{k}=\vec{k}_0$.

Up to the forth order of perturbation theory it is necessary to take into account the zeroth, the first and the second harmonics. In this approximation it is possible to obtain the closed equation on envelope of main harmonic η_1 (see [**?**]). This equation can be written in dimension variables (after rescalinng $\omega_0 t\to t$, $\sqrt{2}k_0x\to x$, $k_0y\to y$, $\sqrt{\frac{11}{4}}k_0\eta\to\eta$) as follows:

$$\eta_{tt}=\delta\eta+\triangle\eta+|\eta|^2\eta\ . \tag{3.175}$$

where we omit the subscript 1 for η_1. Equation (3.175) is the nonlinear relativistic-invariant Klein-Gordon equation with the Hamiltonian:

$$H=\int(|\eta_t|^2-\delta|\eta|^2+|\nabla\eta|^2-\frac{1}{2}|\eta|^4)d^2r, \tag{3.176}$$

Main feature of this Klein-Gordon equation and the Hamiltonian (3.176) is that a nonlinearity in the first nonvanishing order doesn't lead to the instability stabilization. Moreover it is possible to consider the nonlinearity in equation (3.175) as a negative correction of the critical velocity V_{cr}, that is nonlinearity enlarges instability.

It should be noted that equation (3.175) describes the evolution of surface waves envelope as for positive $\delta>0$, when linear instability exists, as for negative $\delta<0$, when linear instability is absent. Although for the case $V<V_{cr}$ and small $|\delta|\ll 1$ the system is linearly stable one, the nonlinearity can result in the instability for the final amplitude perturbations. In this case waves generation is rigid one, wave amplitudes tend to infinity in a finite time. This statement is almost trivial one for space-independent solutions of (3.175) but for space-localized solutions it needs the special consideration.

3.9. Wave collapse criteria for nonlinear KHI

To find the condition of rigid waves generation it is necessary to study solutions of equation (3.175). The simplest one is the space-uniform solution $\eta=\eta(t)$. In this case the equation (3.175) is Newton equation of particle movement in central-symmetric potential. If we write $\eta(t)=Re^{i\phi},\ R\geq 0$ then is possible to obtain the closed equation on R :

$$\ddot{R}=-\frac{\partial U_{eff}}{\partial R}\ , \tag{3.177}$$

where effective potential is given by equation

$$U_{eff}=-\frac{\delta}{2}R^2-\frac{R^4}{4}+\frac{M^2}{2R^2}\ , \tag{3.178}$$

and $M = R_0^2\dot{\phi} = const$ is an angular momentum, $R_0 = R(t = 0)$. Equation (3.177) has energy integral

$$E = \frac{\dot{R}^2}{2} + U_{eff}$$

Then it is simple to understand for which values of E and M the value of R becomes infinite in a finite time. Is such cases time t_0 of singularity achievement is given by the integral:

$$t_0 = \int\limits_{R_0}^{\infty} \frac{dR}{\sqrt{2(E - U_{eff})}} \ .$$

In a small neighbourhood of $t = t_0$ $R(t)$ the dependence $R(t)$ has the following asymptotic:

$$R(t) \simeq \frac{\sqrt{2}}{(t_0 - t)} \ . \tag{3.179}$$

(IF $\delta = M = 0$ this is exact solution of (3.177)). In addition to the space-uniform solution, the automodel substitution in equation (3.175) exists near to the singularity point then nonlinearity dominates ($|\eta|^2 \gg \delta$) :

$$\eta = \frac{1}{t_0 - t} g\left(\frac{r}{t_0 - t}\right) . \tag{3.180}$$

In a case of cylindrical symmetry $g(\xi)$ can be found from ordinary differential equation:

$$(1 - \xi^2)g'' + \frac{1 - 4\xi^2}{\xi}g' + g(|g|^2 - 2) = 0 \ .$$

By simple substitution $g = Re^{i\varphi}$ this equation can be written as

$$(1 - \xi^2)(R'' - (\varphi')^2)R) + \frac{1 - 4\xi^2}{\xi}R' + R(R^2 - 2) = 0 \ ,$$

$$(1 - \xi^2)(2\varphi' R' + \varphi'' R) + \frac{1 - 4\xi^2}{\xi}\varphi' R = 0 \ .$$

The second equation of this system can be explicitly integrated:

$$\varphi' = \frac{C}{R^2\xi|1 - \xi^2|^{3/2}} \ .$$

The finiteness condition φ in the origo results in $C = 0$. Then $\varphi = const$ and R is given by equation:

$$R'' + \frac{1 - 4\xi^2}{\xi(1 - \xi^2)}R' + R\frac{R^2 - 2}{1 - \xi^2} = 0 \ , R'(0) = 0 \ . \tag{3.181}$$

The solution of eqn. (3.181) for $\xi \to \infty$ has the following asymptotic:

$$R \simeq c_1\xi^{-2} + c_2\xi^{-1} \ .$$

This solution has a finite norm ($\int R^2 d^2\xi < \infty$) if $c_2 = 0$. In this case all integral in the Hamiltonian H (3.176) are also finite. But it means that the total Hamiltonian H equals zero, because in other case it would depend on time. The norm square $\int R^2 d^2\xi$ is also independent on time on automodel ($c_2 = 0$) solution. But in general it is not true. This leads to the conclusion that automodel solution can not be the general asymptotic for arbitrary initial conditions. What kind of solution is realized could be found by numeric integration of (3.175). It should be also noted that other principal question about the stability of above considered solutions is still open. It is clear from physical point of view that if the initial conditions of (3.175) have smooth plato then, in the field of plato, the solution of (3.175) is near to the space uniform one. As it is shown below the integral wave collapse criteria of equation (3.175) support this qualitative arguments. Moreover it is found that these criteria weekly depends on the space dimension (in contrast with e.g. sufficient collapse criteria in the nonlinear Schrödinger equation [13, 14].

Consider the time evolution of norm square $B = \int |\eta|^2 d^2 r > 0$. According to (3.175) it is possible to write:

$$\frac{d^2 B}{dt^2} = \int \left\{ 2|\eta_t|^2 + \eta_{tt}\eta^* + \eta^*_{tt}\eta \right\} d^2 r =$$
$$= -4H + \int \left\{ 6|\eta_t|^2 + 2|\vec{\nabla}\eta|^2 - 2\delta|\eta|^2 \right\} d^2 r.$$

If we multiply both sides of this equation by B and use the Cauchy-Schwarz inequality then it is possible to obtain:

$$B_{tt}B - \frac{3}{2}B_t^2 \geq -4HB - 2\delta B^2. \tag{3.182}$$

This type of majoring differential inequality is the most general one used for deriving of sufficient collapse condition in many nonlinear partial differential equations [15]. The simplest form of (3.182) is the inequality

$$B_{tt}B - \frac{3}{2}B_t^2 \geq 0, \tag{3.183}$$

following from (3.182) under the conditions $H < 0$ and $\delta < 0$. The inequality of the type (3.183) was first used in [16] and later in Kalantarov and Ladyzhenskaya paper [17] for the obtaining of the sufficient collapse criterion in 1D Boussinesq equation. It is convenient to write (3.182) in the form of second Newton law by substitution $B = A^{-2}$. In terms of A we have

$$A_{tt} \leq -\frac{\partial V(A)}{\partial A} , \tag{3.184}$$

where A has a meaning of some "particle" coordinate, and $V(A) = -\frac{HA^4}{2} - \delta\frac{A^2}{2}$ is a potential energy of "particle" (compare with (3.177)).

Achievement by B of infinity in a finite time means loss of smoothness and singularity appearance in the solution of (3.175) before or at the same time as B becomes infinite. In terms of A it means that "particle" reaches the origo $A = 0$ in a finite time. If the "velocity" A_t is negative then inequality (3.184) can be integrated once:

$$E(t) = \frac{A_t^2}{2} + V(A) \geq E(0) . \tag{3.185}$$

The sign of this inequality means that "particle energy increases as "particle" tends to origo. According to (3.185) it is then possible to obtain the sufficient collapse conditions. The wave collapse takes place in the following cases:
1)$H < 0,\ \delta < 0,\ A_t(0) < 0$;
2) $H < 0,\ \delta > 0,\ E(0) > 0,\ A_t(0) < 0$;
3) $H > 0,\ \delta > 0,\ A_t(0) < 0$ and either $A^2(0) < \frac{\delta}{2H},\ E(0) < \frac{\delta^2}{8H}$, or $A_t(0) < 0,\ E(0) > \frac{\delta^2}{8H}$.

In all these cases collapse time t_0 is estimated from above by the integral:

$$t_0 \leq \int\limits_0^{A(0)} \frac{dA}{\sqrt{2(E(0)-V(A)}} \ .$$

Note that the inequality (3.184) sign means that "particle" moves not only under the influence of potential force $-\frac{\partial V(A)}{\partial A}$, but also under the influence of some nonpositive force directed to the origin $A=0$. It follows that the sufficient criteria 1)-3) can be improved. E.g. the condition $A_t(0)<0$ in cases 1) and 2) can be omitted. In these cases with $A_t \geq 0$ the "particle" reaches the origo after reflection on the potential. For more detail of similar generalization of collapse criteria see [?].

Near to the singularity point the time dependence of square norm $B=\int|\eta|^2 d\vec{r}$ is given by:

$$\int |\eta|^2 d\vec{r} \geq \frac{C}{(t_0-t)^2} \ . \tag{3.186}$$

This estimation stems from the finiteness of "particle" velocity near to the origo $A=0$. Then for $t \to t_0$ A value of A tends to zero at least according to the linear law, $A \leq C_1(t_0-t)$ resulting in (3.186) for B.

It should be noted that estimation (3.186) is dimension independent and corresponds to the time-dependence $\eta \sim (t_0-t)^{-1}$ of space-uniform solution (3.179). The space compression of initial distribution of η looks improbable one. In any case, if the compression exists it is certainly weaker one then the compression of the automodel solution (3.180).

Thus it was shown that surface waves generation by KHI is explosive one on nonlinear stage. This explosive growth of waves amplitudes takes place up to the value of the boundary surface slope angle of the order of unity, where equation (3.175) loses its applicability. It should be remembered that (3.175) was obtained by assumption of small slope angle $k\eta$. Then the forth order terms of perturbation theory can be compensated by terms of next orders. This automatically gives the angle range $k\eta \sim 1$, where capillary forces are crucial ones. It is very natural to suppose that explosive surface waves growth results in wave breaking and in foam formation on the wave crests. This suggestion is in accordance with the satellite and plain observations of the sea surface. According to $[6-10]$ if wind velocity is near to the critical one (3.166), then the dependence of the part of sea surface covered by the foam is very sharp one.

Consider now the significance of other mechanism of surface waves generation by wind which is connected with Miles instability [4]. As it was already pointed out in Introduction, KHI suggest the existence of tangential discontinuity. But in real fluids (e.g. in air and water) the viscosity results in destruction of the tangential discontinuity and in boundary layer formation near to the boundary surface. Miles instability is connected with the resonance between the surface waves with dispersion law ω_k (3.167) and shear flow in the critical layer $z=z^*$, where phase velocity of surface waves is equal to the local fluid velocity:

$$\frac{\omega}{k} = V(z^*).$$

This instability and ray instability in plasma [18] are similar ones. Miles instability is analog of kinetic ray instability when ray partition function is wide one by energy and its increment is strongly depend on partition function. The instability of rather cold ray is hydrodynamic one, it has no dependence on the structure of ray partition function. The analog o hydrodynamic instability is KHI, because KHI does not depend on the structure of boundary layer but it only depends on the wind velocity outside the boundary layer. The problem of quantitative comparing of Miles instability and KHI lies beyond the scope of this paper. We can only note that for the significance of KHI one condition is necessary to satisfy. It is necessary that the wavelength of generated wave significantly exceeds the boundary layer thickness. In this case the flow outside the boundary layer with good accuracy is potential one which leads to the KHI theory. If the wind velocity is of the order of $6\ \frac{m}{sec}$ (i.e. $V \sim V_{cr}$) then the capillary and gravitational scales of wave spectrum are significantly separated. Consider as a typical wavelength of gravitational wave $\lambda \sim 1\ m$. It is possible to estimate the boundary layer thickness h, which is formed in the air blowing over the crest of gravitational wave. The boundary thickness is given by (see e.g. [1]):

$$h \sim \frac{\lambda}{\sqrt{Re}}\ ,$$

where $Re = \frac{h\ V_{cr}}{\nu}$ is Reynolds number,ν is air viscosity. If $\lambda = 1\ m, \qquad V = 6\frac{m}{sec}$, then $h \simeq 0.16 sm$. This thickness is small in comparison with wavelength $\lambda_0 = \frac{2\pi}{k_0} = 1.7sm$. It means that KHI is significant one in this case. If $\lambda \ll \lambda_0$ then Miles instability dominates and respectively Zakharov-Newell theory [5] dominates in nonlinear regime. In most real situations both mechanisms are likely significant ones because they both explain foam formation for the same wind velocities.

3.10. Lagrangian description of hydrodynamics

Lagrangian description of hydrodynamics is based on the tracking the trajectories of individual particles $\mathbf{r}(\mathbf{a}, t)$ in the fluid which is different from *Eulerian description of hydrodynamics* which deals with the velocity field $\mathbf{v}(\mathbf{r}, t)$ but not with any trajectories of particles. Here the vector $\mathbf{a} \in \mathbb{R}^D$ parametrizes (labels) different fluid particle and satisfies the initial condition

$$\mathbf{r}(\mathbf{a}, t_0) = \mathbf{a}. \tag{3.187}$$

Below without loss of generality we choose $t_0 = 0$. Each particle in the fluid moves with the velocity $\mathbf{v}(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}(\mathbf{a}, t), t) := \mathbf{v}(\mathbf{a}, t)$. Lagrangian description has the independent variables $(\mathbf{a}, \tau = t)$ while Eulerian description has the independent variables $(\mathbf{r}, t)$. We occasionally use τ instead of t if we want to stress that we work in Lagrangian variables. A change of independent variables in Euler equations (2.67) and (2.69) from $(\mathbf{r}, t)$ into $(\mathbf{a}, \tau)$ results in the Lagrangian form of the hydrodynamics of ideal fluid as follows

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial \tau} = -\frac{1}{\rho}\frac{\partial \mathbf{a}}{\partial \mathbf{r}}\nabla_{\mathbf{a}} p, \\ &\operatorname{div} \mathbf{v} = 0, \end{aligned} \tag{3.188}$$

3.11. Vorticity-velocity form of Euler equations

Incompressible Euler equations (2.67) and (2.69) for the incompressible fluid with the constant density $\rho \equiv const$ takes the following form

$$\begin{aligned} &\frac{D\mathbf{v}}{Dt} := \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p, \\ &\operatorname{div}\mathbf{v} = 0, \end{aligned} \tag{3.189}$$

where $\frac{D}{Dt} := \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)$ is the *material derivative* (also called by the total derivative or the substantial derivative). We look for the solution of the Euler equation in the domain Ω which can be either bounded or unbounded.

We now exclude the pressure p from Euler equations by defining the *vorticity*

$$\omega := \operatorname{curl}\mathbf{v} \equiv \nabla\times\mathbf{v} \tag{3.190}$$

and taking curl of the first equation in (3.189) which results in

$$\frac{\partial \omega}{\partial t} = \nabla\times(\mathbf{v}\times\omega), \tag{3.191}$$

where and we used the vector identities

$$\mathbf{v}\times\omega = \mathbf{v}\times(\nabla\times\mathbf{v}) = \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{v}\cdot\nabla)\mathbf{v} \tag{3.192}$$

to express the advection term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ while $\nabla\times\nabla$ ensures that all terms with gradients vanish.

We can additionally transform equation (3.191) by using a vector identity

$$\nabla\times(\mathbf{v}\times\omega) = \mathbf{v}(\nabla\cdot\omega) - \omega(\nabla\cdot\mathbf{v}) + (\omega\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\omega \tag{3.193}$$

together with the incompressibility condition from the second equation in (3.189) and the identity $\nabla\cdot\omega \equiv 0$ (follows from the definition (3.190)) which results in

$$\frac{\partial \omega}{\partial t} + (\mathbf{v}\cdot\nabla)\omega = (\omega\cdot\nabla)\mathbf{v}. \tag{3.194}$$

Here l.h.s. represents the material derivative $\frac{D\omega}{Dt}$ which corresponds to the advection of the vorticity by the flow. R.h.s of equation (3.194) describes the stretching or tilting of the vorticity by the velocity gradients.

Any of equation (3.191) or (3.194) together with the definition (3.190) form a closed set of equation of incompressible ideal hydrodynamics if additionally we define the boundary conditions.

We apply curl to equation (3.190) and use a vector identity

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v} \tag{3.195}$$

together with the incompressibility condition to obtain the Poisson equation

$$-\nabla^2\mathbf{v} = \nabla\times\omega \tag{3.196}$$

for $\mathbf{v}$ provided $\nabla\times\omega$ is known. For either the unbounded domain with $\mathbf{v}\to 0$ for $|\mathbf{r}|\to 0$ or the box with the periodic boundary conditions we obtain from FT of equation (3.196) the explicit solution

$$\mathbf{v}_{\mathbf{k}} = \frac{\mathbf{i}\mathbf{k}\times\omega_{\mathbf{k}}}{\mathbf{k}^2}, \tag{3.197}$$

where we assumed the zero value of the zeroth Fourier harmonic of $\mathbf{v}$. It means that we choose a reference frame where the integral of velocity over the spatial domain Ω is zero, $\int\limits_{\Omega} \mathbf{v} d\mathbf{r} = \mathbf{0}$.

3.12. Canonical variables in hydrodynamics

In this section we generalize the canonical variables introduced in section 2.4 for the potential motion of ideal fluid to include vortex motion in an ideal fluid. We start from the full Euler equations of hydrodynamics of bartoropic fluid (2.67), (2.68) and (2.72), which we reproduce her for the convenience,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{3.198}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p(\rho) = -\nabla w(\rho). \tag{3.199}$$

We know that, for the Euler equations in accordance with the Kelvin theorem, the circulation of the fluid velocity around any closed contour moving together with the fluid is conserved. In other words, in such a system there is a certain scalar function $\mu(\mathbf{r}, t)$ which is convected by the fluid and described by the following equation:

$$\frac{d\mu}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right)\mu = 0. \tag{3.200}$$

Therefore, in formulating the variational principle for the vortex motion we generalize the Lagrangian from equation (2.85) to include equation (3.200) as an additional constraint which results in

$$L = \int \left[\frac{\rho \mathbf{v}^2}{2} - \varepsilon(\rho) + \varphi\left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v})\right) - \lambda\left(\frac{\partial \mu}{\partial t} + \mathbf{v}\nabla\mu\right)\right] d\mathbf{r}, \tag{3.201}$$

λ is the Lagrangian multiplier. A vanishing of the variation of the action $S = \int L dt$ with respect to the variable $\mathbf{v}$ results in

$$\mathbf{v} = \frac{\lambda}{\rho}\nabla\mu + \nabla\varphi. \tag{3.202}$$

Here the term $\nabla\varphi$ recovers equation (2.69) if $\mu \equiv 0$ while the term $\frac{\lambda}{\rho}\nabla\mu$ generally has nonzero vorticity,

$$\omega = \nabla \times \mathbf{v} = \nabla \times \left[\frac{\lambda}{\rho}\nabla\mu + \nabla\varphi\right] = \nabla \times \left[\frac{\lambda}{\rho}\nabla\mu\right], \tag{3.203}$$

where we used the vector identity $\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + (\nabla f) \times \mathbf{A}$ and that $\nabla \times \nabla \equiv 0$. Equation (3.203) is called the *Clebsch representation* of the general vector field $\mathbf{v}$ while λ and μ are the *Clebsch variables*. We notice that generally $\frac{\lambda}{\rho}\nabla\mu$ is not solenoidal, i.e. $\operatorname{div}\left[\frac{\lambda}{\rho}\nabla\mu\right]$ is not identically zero. Thus equation (3.203) generally does not represent the Helmholtz's decomposition (also called by the fundamental theorem of vector analysis) into the sum of the an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field.

A vanishing of the variation of the action S for equation (3.201) with respect to the variable ρ results in

$$\frac{\partial\varphi}{\partial t}+(\mathbf{v}\nabla\varphi)-\frac{\mathbf{v}^2}{2}+w(\rho)=0, \tag{3.204}$$

which is the generalization of the unsteady Bernoulli equation (2.74) to the non-potential flows. A vanishing of the variation of the action S for equation (3.201) with respect to the variable μ provides a temporal dynamics for λ as follows

$$\frac{\partial\lambda}{\partial t}+\operatorname{div}(\lambda\mathbf{v})=0. \tag{3.205}$$

H the last one governs the dynamics of a new variable λ.

The choice of λ and μ for a given value of $\mathbf{v}$ is not unique. Let us consider two sets of potentials λ,μ,φ, giving the same value for the velocity $\mathbf{v}$ with the help of Eq.(3.202). Multiplying (3.202) by the differential $d\mathbf{r}$ (for fixed time t), we get a relation between the new and old variables:

$$d\varphi+\frac{\lambda}{\rho}d\mu=d\varphi'+\frac{\lambda'}{\rho}d\mu'$$

or

$$df\equiv d(\varphi-\varphi')=-\frac{\lambda}{\rho}d\mu-\frac{\lambda'}{\rho}d\mu' \tag{3.206}$$

The last relation shows that $\varphi'-\varphi$ is the generating function f of a gauge transformation, depending on μ and μ'. The old and new canonical coordinates are then expressed in terms of the generating function f as follows

$$\lambda=-\rho\frac{\partial f}{\partial\mu},\ \lambda'=\rho\frac{\partial f}{\partial\mu'} \tag{3.207}$$

determining the nonuniqueness in the choice of Clebsch variables.

Substituting the velocity $\mathbf{v}$ expressed in terms of the variables λ,μ and φ directly into the Euler equation (3.199), we verify that

$$\frac{\lambda}{\rho}\nabla\left(\frac{\partial\mu}{\partial t}+(\mathbf{v}\nabla)\mu\right)+\nabla\mu\left(\frac{\partial}{\partial t}\frac{\lambda}{\rho}+(\mathbf{v}\nabla)\frac{\lambda}{\rho}\right)+\nabla\left(\frac{\partial\varphi}{\partial t}+(\mathbf{v}\nabla)\varphi-\frac{\mathbf{v}^2}{2}+w(\rho)\right)=0.$$

Thus this equation is satisfied if Eqs. (3.204), (3.205) are also imposed. If it is so the system of equations of hydrodynamics can be said to be equivalent to the system (3.198), (3.200), (3.204), and (3.205). ??? This is based on the uniqueness of the solution of the Cauchy problem for the original system and the one obtained (that is, rigorously speaking, an assumption). In doing this, we must in addition, by means of the velocity $\mathbf{v}$ given at the initial time, construct some set of functions λ_0,μ_0 and φ_0, appearing as initial conditions for the system (3.200), (3.204), (3.205).

Now changing to a Hamiltonian description, we obtain that

$$\frac{\partial\rho}{\partial t}=\frac{\delta H}{\delta\varphi},\ \frac{\partial\varphi}{\partial t}=-\frac{\delta H}{\delta\rho}; \tag{3.208}$$

$$\frac{\partial\lambda}{\partial t}=\frac{\delta H}{\delta\mu},\ \frac{\partial\mu}{\partial t}=-\frac{\delta H}{\delta\lambda},$$

where the Hamiltonian

$$H=\int\left[\frac{\rho\mathbf{v}^2}{2}+\varepsilon(\rho)\right]d\mathbf{r},$$

is the total energy of the system, see also Eq. (2.76) for the particular case of the potential flow. For potential flows ($\lambda = \mu = 0$) we again arrive at a pair of canonical variables (ρ, φ). The enthalpy w is determined from $\epsilon(\rho)$ by equation (2.77) as $\frac{\partial \varepsilon(\rho)}{\partial \rho} = w(\rho)$.

Canonical variables for the equations of relativistic hydrodynamics,

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0,$$

$$\left(\frac{\partial}{\partial t} + (\mathbf{v}\nabla)\right)\mathbf{p} + m\nabla w\,(\rho) = 0,$$

$$\mathbf{p} = m\mathbf{v}\left(1 - v^2/c^2\right)^{-1/2},$$

are introduced in analogy to (3.202). In this case,

$$\frac{\mathbf{p}}{m} = \frac{\lambda}{\rho}\nabla\mu + \nabla\varphi.$$

Just as in the preceding example, the variables (λ, μ) and (ρ, φ) form pairs of canonically conjugate quantities, subjected to Eqs (3.204), with the Hamiltonian

$$H = \int \left[\frac{\rho}{m}\left(m^2 c + p^2 c^2\right)^{1/2} + \varepsilon\,(\rho)\right] d\mathbf{r}.$$

A natural generalization of the Clebsch formulation (3.199) is the introduction of canonical variables for nonbarotropic flows [**SW68**] when ε depends on the density as well as on the entropy S. For this the equations of motion (3.200), (3.202) are supplemented by the equation for the entropy advected by the fluid,

$$\left(\frac{\partial}{\partial t} + (\mathbf{v}\nabla)\right) S = 0,$$

and the thermodynamic relation

$$d\varepsilon = \rho T dS + w d\rho,$$

with T as the temperature. In this case the transition to the new variables is accomplished by the formula

$$\mathbf{v} = \nabla\varphi + \frac{\lambda}{\rho}\nabla\mu + \frac{\beta}{\rho}\nabla S. \tag{3.209}$$

For such flows $(\varphi, \rho,\ \lambda, \mu)$ and (S, β) are pairs of canonically conjugate quantities:

$$\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \varphi} = -\mathrm{div}\rho\mathbf{v},$$

$$\frac{\partial \varphi}{\partial t} = -\frac{\delta H}{\delta \rho} = \frac{\mathbf{v}^2}{2} - \mathbf{v}\nabla\varphi - w,$$

$$\frac{\partial \lambda}{\partial t} = \frac{\delta H}{\delta \mu} = -\mathrm{div}\lambda\mathbf{v},$$

$$\frac{\partial \mu}{\partial t} = -\frac{\delta H}{\delta \lambda} = -\mathbf{v}\nabla\mu,$$

$$\frac{\partial \beta}{\partial t} = \frac{\delta H}{\delta S} = -\mathrm{div}\beta\mathbf{v} + \rho T,$$

$$\frac{\partial S}{\partial t} = -\frac{\delta H}{\delta \beta},$$

where $H = \int[\rho(\mathbf{v}^2/2) + \varepsilon(\rho, s)]d\mathbf{r}$. The equivalence of these equations to the equations of hydrodynamics is verified by direct substitution of the velocity in the Euler equation (3.199). Thus, in comparison with the barotropic case the number of canonical variables increases by two.

Now let us raise the natural question: What is the minimal number of the canonical conjugated pairs for describing any flow? As we saw above, introducing new canonical variables in the framework of the Lagrangian approach was connected with the addition of new constraints into the Lagrangian. For example, for the Lagrangian (3.201) they were the continuity equation for the density and the equation for the Lagrangian (material) invariant μ advected by the fluid. In the nonbarotropic case the new Lagrangian invariant, i.e, the entropy S, was added.

To describe the fluid in terms of the Lagrangian (material) variables it is enough to give three values $(a_1, a_2, a_3) = \mathbf{a}$ which, in the simplest case, coincide with the initial positions of each fluid particles, so that the coordinate of the particle at time t will be equal to

$$\mathbf{r} = \mathbf{r}(\mathbf{a}, t). \tag{3.210}$$

Hence it becomes clear that originally there are three independent Lagrangian invariants,

$$\mathbf{a} = \mathbf{a}(\mathbf{r},t),$$

that are the inverse map to (3.210). All other Lagrangian invariants are functions of $\mathbf{a}$. If now we assign the equations for $\mathbf{a}$

$$\frac{d\mathbf{a}}{dt} \equiv \frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v}\nabla)\mathbf{a} = 0$$

as constraints [1] in the Lagrangian for the fluid we immediately come to three new pairs of the canonical variables $\{\lambda_l, a_l\}$ $(l = 1, 2, 3)$ with the velocity in the form,

$$\mathbf{v} = u_l \nabla a_l. \tag{3.211}$$

Here $u_l = \lambda_l/\rho$ and the density ρ is expressed through $\mathbf{a}$ my means of

$$\rho = 1/J$$

with $J = \det||\hat{J}_{ij}||$ as the Jacobian of the mapping (3.210), and $\hat{J}_{ij} = \partial x_i/\partial a_j$ being the Jacoby matrix (for more details, see two next sections). The vector $\mathbf{u}$ in this formula is expressed in terms of the velocity components v_i by

$$\mathbf{u} = \hat{J}^t \mathbf{v}$$

where subscript t means transpose.

The representation (3.211) is the most general one. In particular, all the changes of variables presented above follow from this formula. It can be simplified although remaining general.

Let us consider reversible smooth changes of variables

$$\mathbf{a} = \mathbf{a}(\tilde{\mathbf{a}}).$$

Under these changes the representation (3.211) remains invariant,

$$\mathbf{v} = \tilde{u}_l \nabla \tilde{a}_l,$$

[1] These constraints are called often as the Lin's constraints [?]

but the vector $\mathbf{u}$ transforms as

$$\tilde{u}_l = u_k \frac{\partial a_k}{\partial \tilde{a}_l}.$$

If we now require that one of the component, say u_3, is equal to 1, the representation (3.211) becomes (compare with [**Boo**])

$$\mathbf{v} = \nabla \phi + \frac{\lambda_1}{\rho} \nabla \mu_1 + \frac{\lambda_2}{\rho} \nabla \mu_2. \tag{3.212}$$

If now in this equation we put for μ_2 the entropy S, then we come back to the transformation (3.209). Note that such a reduction is possible if the surface family of the constant entropy $S(\mathbf{r}, t) = \text{const}$ are homotopic to the one of the constant surfaces, related to, e.g., $a_1(\mathbf{r}, t) = \text{const}$. Hence, in particular, it follows that in the barotropic case it is enough to take two pairs of the Clebsch variables in order to describe any fluid flow. One pair of the Clebsch variables, as we will see further, describes a partial type of flows. Nevertheless, locally any flow can be parametrized by one pair of the Clebsch variables [**?**].

3.13. Noncanonical Poisson Brackets in ideal hydrodynamics

Now let us consider how one introduces a Hamiltonian structure into hydrodynamics in terms of the natural physical variables. To do so, it is sufficient to construct a Poisson bracket that satisfies all the necessary requirements. The simplest way of constructing such brackets is to convert the Poisson bracket expressed in terms of canonical variables to a bracket in terms of the natural variables. Note that in this case the symplectic operator appears to be local in those variables. As an example we carry out the conversion of the formula for barotropic flows of an ideal fluid. The calculations for other models can be done in exactly the same way.

According to (3.208), the Poisson bracket is given by the expression:

$$\{F, G\} = \int \left\{ \left[\frac{\delta F}{\delta \rho} \frac{\delta G}{\delta \varphi} - \frac{\delta F}{\delta \varphi} \frac{\delta G}{\delta \rho} \right] + \left[\frac{\delta F}{\delta \lambda} \frac{\delta G}{\delta \mu} - \frac{\delta F}{\delta \mu} \frac{\delta G}{\delta \lambda} \right] \right\} d\mathbf{r}. \tag{3.213}$$

Here the velocity is expressed in terms of λ, μ and ρ, φ by the formula

$$\mathbf{v} = \nabla \varphi + \frac{\lambda}{\rho} \nabla \mu,$$

by means of which one can calculate the variational derivatives of F with respect to ρ, φ, λ and μ:

$$\frac{\delta F}{\delta \rho}\bigg|_{\lambda} = \frac{\delta F}{\delta \rho}\bigg|_{v} - \frac{\lambda \nabla \mu}{\rho} \frac{\delta F}{\delta \mathbf{v}}, \quad \frac{\delta F}{\delta \varphi} = -\text{div} \frac{\delta F}{\delta \mathbf{v}}, \tag{3.214}$$

$$\frac{\delta F}{\delta \lambda} = \frac{\nabla \mu}{\rho} \frac{\delta F}{\delta \mathbf{v}}, \quad \frac{\delta F}{\delta \mu} = -\text{div} \left(\frac{\lambda}{\rho} \frac{\delta F}{\delta \mathbf{v}} \right).$$

In these formulas the variational derivatives on the left of the equality signs are taken with fixed $\lambda, \mu, \rho, \varphi$, and those on the right for constant ρ and $\mathbf{v}$. Substitution of these relations in (3.213) leads us to the bracket [**?**]

$$\{F, G\} = \int \left\{ \left(\nabla \frac{\delta F}{\delta \rho}, \frac{\delta G}{\delta \mathbf{v}} \right) - \left(\nabla \frac{\delta G}{\delta \rho}, \frac{\delta F}{\delta \mathbf{v}} \right) \right\} d\mathbf{r} \tag{3.215}$$

$$+ \int \left(\frac{\text{curl} \mathbf{v}}{\rho}, \left[\frac{\delta F}{\delta \mathbf{v}} \times \frac{\delta G}{\delta \mathbf{v}} \right] \right) d\mathbf{r}$$

the Jacobi identity (**??**) being satisfied automatically.

In terms of this bracket, the continuity and Euler equations have the form

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}(\rho \mathbf{v}) = \{\rho, H\}\,,$$

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}, \nabla)\mathbf{v} - \nabla w\,(\rho) = \{\mathbf{v}, H\}\,,$$

where $H = \int [\rho \mathbf{v}^2/2 + \varepsilon(\rho)] d\mathbf{r}$.

The bracket (3.215) has a more obvious meaning if we make a transformation to the new variable $\mathbf{p} = \rho \mathbf{v}$ which is the momentum density. In the new set of the variables $(\rho, \mathbf{p})$, this bracket is changed to the Berezin-Kirillov-Kostant (BKK) bracket [**?**]:

$$\{F, G\} = \int \rho \left\{ \left(\nabla \frac{\delta F}{\delta \rho}, \frac{\delta G}{\delta \mathbf{p}} \right) - \left(\nabla \frac{\delta G}{\delta \rho}, \frac{\delta F}{\delta \mathbf{p}} \right) \right\} d\mathbf{r} + \int \left(\mathbf{p}, \left[\left(\frac{\delta G}{\delta \mathbf{p}} \nabla \right) \frac{\delta F}{\delta \mathbf{p}} - \left(\frac{\delta F}{\delta \mathbf{p}} \nabla \right) \frac{\delta G}{\delta \mathbf{p}} \right] \right) d\mathbf{r}. \tag{3.216}$$

Using (3.216) to calculate brackets between components of $\mathbf{p}$ and ρ, we find that

$$\{p_i(\mathbf{r}), p_j(\mathbf{r}')\} = (p_j(\mathbf{r}') \nabla'_i - p_i(\mathbf{r}) \nabla_j)\, \delta(\mathbf{r} - \mathbf{r}')\,, \quad \{p_i(\mathbf{r}), \rho(\mathbf{r}')\} = \rho \nabla_i \delta(\mathbf{r} - \mathbf{r}')\,. \tag{3.217}$$

In accordance with (1.206), these relations give a Lie algebra, which coincides with the algebra of vector fields [**?, ?**] in this case.

The brackets (3.216), (3.217) can also be obtained in other ways. The simplest method is to regard the Poisson bracket as the classical limit of the corresponding quantum commutators, which were first calculated for hydrodynamics by L.D.Landau [**?**]. Another method for calculating the Poisson brackets for hydrodynamic models, proposed by G.E.Volovik and I.E.Dzyaloshinskii [**?**], is based on the fact that $\mathbf{p}$ and ρ are the densities of the generators of translations and gauge transformations.

The Poisson bracket (3.215) is generalized to the Poisson brackets for the hydrodynamic equations of ideal fluids for arbitrary dependence of the pressure on both the density and the entropy [**?**]

$$\{F, G\} = \int \left\{ \left(\nabla \frac{\delta F}{\delta \rho}, \frac{\delta G}{\delta \mathbf{v}} \right) - \left(\nabla \frac{\delta G}{\delta \rho}, \frac{\delta F}{\delta \mathbf{v}} \right) \right\} d\mathbf{r} + \int \left(\frac{\mathrm{curl}\mathbf{v}}{\rho}, \left[\frac{\delta F}{\delta \mathbf{v}} \times \frac{\delta G}{\delta \mathbf{v}} \right] \right) d\mathbf{r} + \int \left(\frac{\nabla S}{\rho}, \left[\frac{\delta F}{\delta \mathbf{v}} \frac{\delta G}{\delta S} - \frac{\delta G}{\delta \mathbf{v}} \frac{\delta F}{\delta S} \right] \right) d\mathbf{r}. \tag{3.218}$$

We want to repeat once more that the introduction of the Poisson brackets to a system means that such systems possess the Hamiltonian structure in the weakest sense. For example, for the above equations of ideal hydrodynamics it reflects in the fact that the brackets expressed in terms of natural variables are degenerate, i.e, there exist annulators of these Poisson brackets (Casimirs) which, as we will see in the next sections, are connected with a specific gauge symmetry of the hydrodynamic equations, providing, in particular, the conservation of the fluid velocity circulation. Besides, it means that a direct conversion, as, i.e., passing from (3.215) or to (3.218) to the canonical basis is impossible in general. For this case at first we need to resolve all our constraints (Casimirs). A typical example just consists in introducing Clebsch variables. This is all the more interesting as, so far, we have not explicitly known how these Casimirs invariants look like.

Of particular interest is the introduction of a Hamiltonian structure for the incompressible fluid. In this case ρ is no longer an independent variable, and can be eliminated by using the relation div$\mathbf{v} = 0$. Thus in the limit of the incompressible fluid there is only one pair of canonical variables λ and μ, and the Poisson bracket in this case takes the form

$$\{F, G\} = \int \left[\frac{\delta F}{\delta \lambda}\frac{\delta G}{\delta \mu} - \frac{\delta F}{\delta \mu}\frac{\delta G}{\delta \lambda}\right] d\mathbf{r}.$$

By means of relations analogous to (3.214), one can derive

$$\frac{\delta F}{\delta \lambda} = \left(\frac{\nabla \mu}{\rho}, \frac{\delta F}{\delta \mathbf{v}} - \nabla \frac{1}{\Delta}\mathrm{div}\frac{\delta F}{\delta \mathbf{v}}\right),$$

$$\frac{\delta F}{\delta \mu} = -\mathrm{div}\frac{\lambda}{\rho}\left(\frac{\delta F}{\delta \mathbf{v}} - \nabla \frac{1}{\Delta}\mathrm{div}\frac{\delta F}{\delta \mathbf{v}}\right).$$

As a result, we arrive at the equation

$$\{F, G\} = \int \left(\mathrm{curl}\ \mathbf{v}, \left[\left(\frac{\delta F}{\delta \mathbf{v}} - \nabla \frac{1}{\Delta}\mathrm{div}\frac{\delta F}{\delta \mathbf{v}}\right) \times \left(\frac{\delta G}{\delta \mathbf{v}} - \nabla \frac{1}{\Delta}\mathrm{div}\frac{\delta G}{\delta \mathbf{v}}\right)\right]\right) d\mathbf{r}. \tag{3.219}$$

(Here we put $\rho = 1$.) This expression shows that the manifold G coincides with the algebra of vector fields $\mathbf{A}(\mathbf{r})$ for which div$\mathbf{A} = 0$. This bracket is expressed in a more compact form using $\mathbf{\Omega}$ =curl$\mathbf{v}$ [**KM80**] as follows

$$\{F, G\} = \int \left(\mathbf{\Omega} \cdot \left[\mathrm{curl}\frac{\delta F}{\delta \mathbf{\Omega}} \times \mathrm{curl}\frac{\delta G}{\delta \mathbf{\Omega}}\right]\right). \tag{3.220}$$

As a result, the Euler equation for $\mathbf{\Omega}$,

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \mathrm{curl}\,[\mathbf{v} \times \mathbf{\Omega}] \tag{3.221}$$

becomes the Hamiltonian one [**Arn69, KM80**]:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \{\mathbf{\Omega}, H\},$$

where

$$H = \int \frac{\mathbf{v}^2}{2} d\mathbf{r}.$$

The bracket (3.220) also gives a Hamiltonian structure for two-dimensional hydrodynamics. In this case $\mathbf{\Omega}$ has a single component, which is conveniently expressed in terms of the stream function ψ:

$$\Omega = -\Delta\psi \ \left(v_x = \frac{\partial \psi}{\partial y}, v_y = -\frac{\partial \psi}{\partial x}\right).$$

In the two-dimensional case the equation of motion (3.221) and the Poisson bracket (3.220) have the form:

$$\frac{\partial \Omega}{\partial t} = \{\Omega, H\} = -\frac{\partial \Omega}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial \Omega}{\partial y}\frac{\partial \psi}{\partial x} \equiv -\frac{\partial\,(\Omega, \psi)}{\partial\,(x, y)},$$

$$\{F, G\} = \int \Omega \frac{\partial\,(\delta F/\delta\Omega, \delta G/\delta\Omega)}{\partial\,(x, y)} dxdy, \tag{3.222}$$

$$H=\frac{1}{2}\int(\nabla\psi)^2\,dxdy.$$

A Hamiltonian structure is introduced analogously into the Rossby equation, which differs from (3.220) in having the additional term $\beta(\partial\psi/\partial x)$ entering [**?**]:

$$\frac{\partial}{\partial t}\Delta\psi+\beta\frac{\partial\psi}{\partial x}=-\frac{\partial\left(\Delta\psi,\psi\right)}{\partial\left(x,y\right)}. \tag{3.223}$$

It is then easy to see that the change $\Omega\to\Omega-\beta y$ reduces this equation to (3.222). Thus, the Poisson bracket for (3.223) is given analogously by

$$\{F,G\}=\int(\Omega+\beta y)\frac{\partial\left(\delta F/\delta\Omega,\delta G/\delta\Omega\right)}{\partial\left(x,y\right)}dxdy, \tag{3.224}$$

while the Hamiltonian H is still defined by the earlier expression

$$H=\frac{1}{2}\int(\nabla\psi)^2\,dxdy.$$

To the said above one should add that the Poisson brackets (3.222) and (3.224) for flows with closed stream lines can be reduced to the Gardner-Zakharov-Faddeev brackets used in the theory of integrable equations [**?**]. Details of such consideration can be found in the original papers [**?, ?**].

Thus, introducing noncanonical Poisson brackets on the base of canonical ones represents the most simple way of their finding. Moreover the Hamiltonian structure given by means of these brackets is the weakest Hamiltonian formulation of the equations. In this formulation, in particular, it is impossible to write explicitly the variational principle. From another side, as it will be shown later, the representation of the hydrodynamic type equations by means of the noncanonical Poisson bracket can be written for arbitrary flows. However, for the arbitrariness one should pay by the bracket degeneracy, i.e., by existence of Casimirs invariants annulling noncanonical brackets.

3.14. Ertel's Theorem

In this section and in the next ones we show, by following mainly results expounded in [**?, ?**], that for perfect fluids with arbitrary dependence of the pressure on the fluid density and entropy the Ertel's theorem as well as the Kelvin theorem about the conservation of the velocity circulation are a consequence of the specific gauge symmetry connected with the relabeling of fluid particles. We discuss also the role played by this symmetry in the Hamiltonian structures.

The Ertel's theorem [**?**] for a perfect fluid says that the quantity

$$I_L=\frac{(\mathbf{\Omega}\nabla S)}{\rho} \tag{3.225}$$

is the Lagrangian invariant. Here $\Omega=\mathrm{curl}\mathbf{v}$ is the vorticity, $\mathbf{v}$ is the fluid velocity which satisfies the Euler equation,

$$\frac{\partial\mathbf{v}}{\partial t}-(\mathbf{v}\nabla)\mathbf{v}=-\frac{\nabla p}{\rho} \tag{3.226}$$

and S the specific entropy advected by the fluid,

$$\frac{\partial S}{\partial t}+(\mathbf{v}\nabla)S=0. \tag{3.227}$$

The density ρ is defined from the continuity equation,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0. \tag{3.228}$$

We omit a proof of this theorem validity of which can be checked by direct calculations (see, for instance, [**?**]).

The invariance of I_L means that I_L depends only on the Lagrangian coordinates $\mathbf{a}$, and it does not change in time moving together with a fluid particle.

As was mentioned before, the choice of the Lagrangian variables is arbitrary: they label each fluid particle. Therefore often these coordinates are called as the Lagrangian markers. Usually the Lagrangian coordinates are chosen to coincide with the initial positions of fluid particles, $\mathbf{r}|_{\mathbf{t=0}} = \mathbf{a}$. Thus, a transition from one (Euler) description to another (Lagrangian) one is accomplished by means of the change of variables

$$\mathbf{r} = \mathbf{r}(\mathbf{a}, t) \tag{3.229}$$

with $\mathbf{a}$ being a label of each fluid particle. Velocity of the particle at the point $\mathbf{r}$ is given by the usual formula

$$\mathbf{v}(\mathbf{r}, \mathbf{t}) = \dot{\mathbf{r}}|_{\mathbf{a}} \tag{3.230}$$

where dot means derivative with respect to time t. In terms of the Lagrangian variables, the solution to the equations (3.228) and (3.227) can be written as follows,

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{a})/J, \quad S(\mathbf{r}, t) = S_0(\mathbf{a}), \tag{3.231}$$

where $J = \det||\hat{J}_{ij}||$ is a Jacobian and

$$\hat{J}_{i\alpha} = \frac{\partial x_i}{\partial a_\alpha}$$

is a Jacobi matrix of the mapping (5.91), which is assumed to be one-to-one. Further we will suppose $J \neq 0$ everywhere, that guarantee the existence of the mapping inverse to (5.91). The Jacobi matrix plays the basic role. Its knowledge allows to determine not only the main flow parameters but also its geometrical characteristics, in particular the metric tensor. The equation of motion for the Jacobi matrix follows directly from the definition of the velocity (3.230). Consider the vector $\delta\mathbf{r}$ connected two nearest fluid particles,

$$\delta\mathbf{r} = \mathbf{r}(\mathbf{a} + \delta\mathbf{a}, \mathbf{t}) - \mathbf{r}(\mathbf{a}, \mathbf{t}).$$

Using the definition (3.230) it is easy to get the equation for this quantity,

$$\frac{d\delta\mathbf{r}}{dt} = (\delta\mathbf{r}, \nabla)\mathbf{v}. \tag{3.232}$$

Expanding then $\delta\mathbf{r}$ relative to the small vector $\delta\mathbf{a}$,

$$\delta x_i = \hat{J}_{ij}\delta_j, \tag{3.233}$$

we arrive at the equation of motion for the Jacobi matrix,

$$\frac{d}{dt}\hat{J} = U\hat{J}, \tag{3.234}$$

containing the matrix elements

$$U_{ij} = \frac{\partial v_i}{\partial x_j}.$$

The symmetric part of U,

$$B = \frac{1}{2}(U + U^T),$$

is a stress tensor, and its antisymmetric part corresponds to the vorticity,

$$\Omega = \frac{1}{2}(U - U^T).$$

Hence the equation for the matrix inverse to $\hat{J}$ is

$$\frac{d}{dt}\hat{J}^{-1} = -\hat{J}^{-1}U \tag{3.235}$$

that in the component notation has the form

$$\frac{d}{dt}\frac{\partial a_\alpha}{\partial x_i} = -\frac{\partial a_\alpha}{\partial x_j}\frac{\partial v_j}{\partial x_i}. \tag{3.236}$$

The metric tensor is defined by means of distances between two nearest Lagrangian particles,

$$(\delta x_i)^2 = g_{ik}\delta a_i \delta a_k,$$

and equal to

$$g_{ik} = \hat{J}_{li}\hat{J}_{lk}.$$

The invariant I_L is local in Lagrangian variables. Therefore if one takes its convolution with arbitrary function $f(\mathbf{a})$, then one can get the infinite family of the conservation laws in the integral form

$$I_i = \int I_L(\mathbf{a}) f(\mathbf{a}) d\mathbf{a}. \tag{3.237}$$

To begin with, we show that for barotropic fluids (when pressure p depends only on the density ρ) the Kelvin theorem follows from this relation. Notice that in this case there is one additional freedom: the entropy S has no link with the pressure and therefore instead of S in (3.226) and (3.231) we can take an arbitrary function of Lagrangian markers $\mathbf{a}$. Also one should note that in the first equation of (3.231), without any loose of generality, one can set $\rho_0(\mathbf{a}) = 1$ [2], so that

$$\rho(\mathbf{r}, t) = 1/J. \tag{3.238}$$

Substitute (3.225) into (3.237) and integrate once by parts. With account of (3.231) and $Jd\mathbf{a} = d\mathbf{r}$, we get

$$I_i = \int (\mathbf{v}, [\nabla f \times \nabla S]) d\mathbf{r}. \tag{3.239}$$

Here the gradient is taken with respect to $\mathbf{r}$, but functions f and S_0 are functions of $\mathbf{a} = \mathbf{a}(\mathbf{x},t)$. Therefore come back again to the integration against $\mathbf{a}$. As a result of simple algebra we arrive at the expression,

$$I_i = \int \dot{x}_i J \epsilon_{ijk} \frac{\partial a_\alpha}{\partial x_j}\frac{\partial a_\beta}{\partial x_k}\frac{\partial f(a)}{\partial a_\alpha}\frac{\partial S_0(a)}{\partial a_\beta} d\mathbf{a}.$$

Taking then into account the identity

$$J\epsilon_{ijk}\frac{\partial a_\alpha}{\partial x_j}\frac{\partial a_\beta}{\partial x_k} = \epsilon_{\alpha\beta\gamma}\frac{\partial x_i}{\partial a_\gamma} \tag{3.240}$$

[2] It corresponds to such change of variables $\mathbf{b} = \mathbf{b}(\mathbf{a})$ which eliminates ρ_0: $J_{ab} = \rho_0$

the integral is transformed into

$$I_i = \int A_j(\mathbf{a})\dot{x}_i \frac{\partial x_i}{\partial a_j} d\mathbf{a}. \tag{3.241}$$

Here the vector function $\mathbf{A}(\mathbf{a})$ reads:

$$\mathbf{A}(\mathbf{a}) = [\nabla f \times \nabla S_0]. \tag{3.242}$$

It has a zero divergence:

$$\mathrm{div}\mathbf{A}(\mathbf{a}) = 0. \tag{3.243}$$

Note that till now we have never used the fact that the fluid is barotropic, i.e., the equation (3.241) is applicable for any equation of state including the general dependence of the pressure on both the density and the entropy. For the barotropic case the entropy S_0 can be considered as an arbitrary function of $\mathbf{a}$. Therefore $A(a)$ can be considered also as arbitrary with the only constraint (3.243).

Let this vector function $\mathbf{A}(\mathbf{a})$ be concentrated on some closed curve: it is equal to zero everywhere outside this curve. We will parameterize the curve by the arc length s,

$$\mathbf{a} = \mathbf{a}(s) \;\text{ with }\; \mathbf{a}(s + l) = \mathbf{a}(\mathbf{s}) \tag{3.244}$$

where l is the curve length.

It is then easy to check that the function

$$\mathbf{A} = \int_0^l \frac{d\mathbf{a}(s)}{ds}\delta(\mathbf{a} - \mathbf{a}(s))ds$$

satisfies all necessary conditions: it concentrates on the curve $\mathbf{a} = \mathbf{a}(s)$ and has zero divergence. Plugging this formula into the integral (3.241), after simple integration, we come to the Kelvin theorem for the barotropic fluid:

$$I_K = \int_C (\mathbf{v}(\mathbf{r}, \mathbf{t}), d\mathbf{l}). \tag{3.245}$$

Here the contour C, moving together with the fluid, is the image of the closed curve (3.244). Thus, we have shown that the Kelvin's theorem is a direct consequence of the Ertel's theorem applied to the case of barotropic fluids.

The Kelvin theorem is valid also for arbitrary dependence $p(\rho, S)$. This property is not widely known in the literature, for instance, it is absent in the Landau-Lifshits course [**?**]. Curiously, the answer in this case will have the same form as (3.245). The only difference will be connected with a choice of contour. For the barotropic case, as we saw before, the only restriction was connected with the condition (3.243) which provides the closure of the contour. For the general dependence $p = p(\rho, S)$ in addition to (3.243) one needs to satisfy the condition (5.69). According to the latter the lines of the vector field $\mathbf{A}$ must lie on the surfaces of the constant entropy $S_0(\mathbf{a})$. Therefore if we choose the closed contour lying on this (fluid!) surface we immediately arrive at the Kelvin theorem (3.245). Thus, the Kelvin theorem in the general case says that the velocity circulation is conserved in time if the fluid contour lies on the surface $S(\mathbf{a}(\mathbf{r},t)) = \text{const}$ advected by the fluid.

To the end of this section we pay attention to one interesting interpretation of the Kelvin theorem. According to [**?**] conservation of the velocity circulation can been considered as a sequence of conservation of the relative Poincare invariant

$$\oint \mathbf{p}d\mathbf{q}. \tag{3.246}$$

For barotropic flows to each fluid particle one can correspond the Hamiltonian

$$h = \frac{p^2}{2} + w(\rho),$$

where $\mathbf{p} = \dot{\mathbf{r}}$, and the enthalpy w plays a role of its potential energy.

If now one takes instead of the contour in (3.246) the fluid one then it is seen that the Poincare invariant will coincide with the velocity circulation

$$\oint \mathbf{v}d\mathbf{r},$$

and, thus, the Kelvin theorem becomes a direct consequence of a conservation of the relative Poincare invariant.

This concept has been occurred to be very useful for other hydrodynamic systems, in particular, for some problems in plasma physics [**?, ?**], when the motion of a fluid particle can be reduced to the Hamilton equation for a charge particle in a magnetic field in a presence of a self-consistent potential. In such cases the analog of the Kelvin theorem is simply a sequence of conservation of the relative Poincare invariant.

3.15. Gauge Symmetry - Relabeling Group

In this section we consider how the conservation of the Ertel invariants follows from the variational principle.

To begin, we make two remarks.

Firstly, let $\mathbf{I_l} = (I_1, ..., I_n)$ be a set of Lagrangian invariants, each of them moving with the fluid and respectively

$$\frac{dI_k}{dt} = \frac{\partial I_k}{\partial t} + \mathbf{v}\nabla I_k = 0.$$

Then any function of $\mathbf{I_l}$ will also be a Lagrangian invariant. To construct an Eulerian conservative density from the given Lagrangian one it is enough to be convinced that the quantity $I_{eu} = \rho I_k$ obeys the continuity equation

$$\frac{\partial I_{eu}}{\partial t} + \mathrm{div}(I_{eu}\mathbf{v}) = 0.$$

The equations of ideal hydrodynamics, as we saw above, have two Lagrangian invariants, i.e., the Ertel invariant I_L (3.225) and s [3]. Both these integrals generate the following conservation law

$$I_i = \int \rho f(I_L, s)d\mathbf{r} \tag{3.247}$$

with $f(I_L, s)$ being an arbitrary function of its arguments.

Secondly, the Euler equation (3.226) in terms of the Lagrangian variables is nothing else but the Newton equation for a fluid particle,

$$\ddot{x}_i = -\frac{\nabla_i p}{\rho}. \tag{3.248}$$

[3] To avoid a confusion only in this section we denote the entropy as s, everywhere outside this section the entropy remains the previous notation S.

Acting by the Jacobi matrix $\hat{J}$ to the both sides of this equation gives

$$\frac{\partial x_i}{\partial a_k}\ddot{x}_i = -\frac{1}{\rho}\frac{\partial p(\rho, s)}{\partial a_k}. \tag{3.249}$$

This equation in the form (3.248) or (3.249) is closed by means of Eqs. (3.231) and (3.238).

The action in terms of the Lagrangian (material) variables is written in the same form as in classical mechanics,

$$S = \int dtL = \int dtd\mathbf{r}\left(\rho\frac{\dot{x}_i^2}{2} - \varepsilon(\rho, s)\right) \tag{3.250}$$

where ε is the internal energy density connected with the enthalpy w by means of the thermodynamic relation,

$$d\varepsilon = \rho T ds + w d\rho \tag{3.251}$$

with T as the temperature.

Let us now check that varying the action, $\delta S = 0$, is equivalent to the equation of motion (3.249).

At first let us pass in (3.250) from integration over $\mathbf{r}$ to $\mathbf{a}$. As a result, the action can then be transformed as follows,

$$S = \int dtd\mathbf{a}\left(\frac{\dot{x}_i^2}{2} - \tilde{\varepsilon}(\rho, s)\right). \tag{3.252}$$

Here the time derivative of x is taken for fixed a, $\tilde{\varepsilon} = \varepsilon/\rho$ is the function of ρ and s which are defined with the help of relations (3.231), (3.238). Because only ρ in the internal energy $\tilde{\varepsilon}$ contains the dependence of x through the Jacobian (3.238), the main difficulty with a variation will be connected with the second term in (3.252).

Using both the identity (3.240) and the formula

$$J = \frac{1}{6}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\frac{\partial x_i}{\partial a_\alpha}\frac{\partial x_j}{\partial a_\beta}\frac{\partial x_k}{\partial a_\gamma}$$

one can get

$$\begin{aligned} \delta S &= \int dtd\mathbf{a}\left(-\ddot{x}_i\delta x_i + \rho^2\frac{\partial\tilde{\varepsilon}}{\partial\rho}\delta J\right) \\ &= \int dtd\mathbf{a}\left(-\ddot{x}_i - \frac{1}{2}\frac{\partial}{\partial a_\alpha}\left(\rho^2\frac{\partial\tilde{\varepsilon}}{\partial\rho}\right)\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}\frac{\partial x_j}{\partial a_\beta}\frac{\partial x_k}{\partial a_\gamma}\right)\delta x_i \\ &= \int dtd\mathbf{a}\left(-\ddot{x}_i - \frac{1}{\rho}\frac{\partial}{\partial a_\alpha}\left(\rho^2\frac{\partial\tilde{\varepsilon}}{\partial\rho}\right)\frac{\partial a_\alpha}{\partial x_i}\right)\delta x_i = 0 \end{aligned} \tag{3.253}$$

or

$$\ddot{x}_i = -\frac{1}{\rho}\frac{\partial}{\partial a_\alpha}\left(\rho^2\frac{\partial\tilde{\varepsilon}}{\partial\rho}\right)\frac{\partial a_\alpha}{\partial x_i}.$$

Hence it is seen that the resulting equation coincides with the equation of motion (3.249) if one puts

$$p(\rho, s) = \rho^2\frac{\partial\tilde{\varepsilon}}{\partial\rho}.$$

(The last equality is a direct sequence of the thermodynamic relation (3.251).)

Thus, we have proved that the equations of motion of ideal fluid in the Lagrangian form follow directly from the variational principle.

The simplest conservation laws, i.e., the conservation of momentum

$$\mathbf{P} = \int \dot{\mathbf{x}} d\mathbf{a} = \int \rho \mathbf{v}(\mathbf{r},t) d\mathbf{r}$$

and the conservation of energy,

$$E = \int \left(\frac{\dot{\mathbf{x}}^2}{2} + \tilde{\varepsilon}(\rho, S) \right) d\mathbf{a} = \int \left(\frac{\rho v^2}{2} + \varepsilon(\rho, S) \right) d\mathbf{r}$$

follow as a result of invariance of the action relative to two independent symmetries, translations in space and time.

The equations of hydrodynamics, as it was shown at first in [**?**], have an additional nontrivial symmetry connected with arbitrariness in the possible choices of the Lagrangian markers. Nothing has to depend on this choice: the fluid dynamics as well as the equations of motion remain the same. From all possible relabeling transformations, the action invariance requirement restrains some certain class. In the case of the barotropic fluids the action appears to be invariant if transformations $\mathbf{b} = \mathbf{b}(\mathbf{a})$ are incompressible, i.e., for which the Jacobian equal to 1:

$$J = \det \| \partial b_i / \partial a_j \| = 1. \tag{3.254}$$

All these transformations form the group of diffeomorphisms preserving the volume. (It is interesting to note that the same group governs the motion of an incompressible fluid.) This symmetry, in accordance with the Noether theorem, generates new conservation laws. To find them it is enough to consider infinitesimal transformations. In the given case those are defined by

$$\mathbf{b} = \mathbf{a} + \delta \mathbf{a}$$

where the function $\delta \mathbf{a} = \alpha(\mathbf{a})$ satisfies the condition

$$\frac{\partial \alpha_i(\mathbf{a})}{\partial a_i} = 0 \tag{3.255}$$

which is a direct sequence of Eq. (3.254).

For the general equation of state $p = p(\rho, s)$ the invariance of the action implies that the transformations should keep the surface $s = s(\mathbf{a})$ to remain, being simultaneously incompressible. As a result, on the function $\alpha(\mathbf{a})$ we have one additional constraint

$$[\nabla s \times \alpha] = 0. \tag{3.256}$$

If in the first case Eq.(3.255) can be resolved by introducing the vector potential

$$\alpha = \mathrm{curl} \zeta,$$

for example, with the Coulomb gauge $\mathrm{div} \zeta = 0$, then in the general case both equations (3.255) and (3.256) are satisfied if one puts

$$\alpha = [\nabla s \times \nabla \psi].$$

Here ψ is a scalar function and gradient is taken with respect to $\mathbf{a}$.

Omitting all the intermediate derivation of the conservation law (it is the standard procedure, for reference see, for instance, [**?**]) we present only the final answers:

i) For the **barotropic** fluid the conservation law has the form

$$\frac{d}{dt} [\nabla_a \dot{x}_i \times \nabla_a x_i] = 0$$

or it gives the whole conserved vector

$$\mathbf{I_L} = [\nabla_a \dot{x}_i \times \nabla_a x_i]. \tag{3.257}$$

This integral was known since the last century: it was found by Cauchy [**?**] (see also [**?, ?**]).

Matrix notation of the equation (3.257) has the form:

$$\hat{J}\frac{d\hat{J}^T}{dt} - \hat{J}^T\frac{d\hat{J}}{dt} = \Omega^{(0)},$$

where index T means transposition, and the matrix $\Omega^{(0)}$ is expressed through the vector invariant $\mathbf{I_L}$ with the help of formula

$$\Omega^{(0)}_{ij} = \epsilon_{ijk} I_{Lk}.$$

Recently this matrix representation of the equation (3.257) was used by the authors of the paper [**?**] to construct a set of exact three-dimensional solutions for the Euler equation for incompressible fluids.

Returning to the Euler description and using the identity (3.240) this vector integral can be transformed into the form

$$\mathbf{I_L} = J(\mathbf{\Omega}, \nabla)\mathbf{a} \equiv \frac{\rho_0(\mathbf{a})}{\rho}(\mathbf{\Omega}, \nabla)\mathbf{a}. \tag{3.258}$$

Here $\mathbf{a}$ is considered as a function of $\mathbf{r}$ and t. If $\mathbf{a}$ are the initial coordinates of fluid particles, then the vector (3.258) can be expressed through the initial distributions $\mathbf{\Omega_0}(\mathbf{a})$ and $\rho_0(\mathbf{a})$ as follows

$$\mathbf{I_L} = \frac{\mathbf{\Omega_0}(\mathbf{a})}{\rho_0(\mathbf{a})}.$$

From (3.258) it follows immediately for the vector $\mathbf{B} \equiv \mathbf{\Omega}/\rho$ that

$$\mathbf{B}(\mathbf{r}, t) = \hat{J}\mathbf{B_0}(\mathbf{a}).$$

Thus, the Jacobi matrix becomes the evolution operator for the vector $\mathbf{\Omega}/\rho$.

The invariants (3.258), indeed, are well-known in hydrodynamics but in a slightly different form. Let us write down the equation of motion for the fraction $\mathbf{\Omega}/\rho$ which directly follows from Eqs. (3.226, 3.228),

$$\frac{d}{dt}\mathbf{B} = (\mathbf{B}\nabla)\,\mathbf{v}. \tag{3.259}$$

Comparing this equation with the equation (3.232) for $\delta\mathbf{r}$ one can see that both quantities $\mathbf{B}$ and $\delta\mathbf{r}$ obey the same equation. This means that the vorticity is frozen into a fluid, the well-known statement in hydrodynamics. Sometimes this property is called as the frozenness of the vorticity into a fluid. If now one makes the next step, namely, multiplying Eq.(3.259) from the right by $\hat{J}^{-1}$ and Eq. (3.235) from the left by $\mathbf{\Omega}/\rho$, after summation of the obtained results we arrive at the conservation of the vector invariant (3.258). These integrals just consist in the mathematical formulation of the frozenness of the vorticity into a fluid. The corresponding equation for the vector field $\mathbf{B}$ is called by the frozenness equation.

ii) In the **general** case (for arbitrary dependence of pressure on both density and entropy) from this vector invariant the only scalar that survives is a projection of $\mathbf{I_L}$ to the vector ∇s:

$$I_L = (\nabla_a s[\nabla_a \dot{x}_i \times \nabla_a x_i]).$$

Here all derivatives are taken with respect to **a**. Passing in this expression to the Eulerian variables and using the identity

$$\epsilon_{\alpha\beta\gamma}\frac{\partial x_i}{\partial a_\alpha}\frac{\partial x_j}{\partial a_\beta}\frac{\partial x_k}{\partial a_\gamma} = \epsilon_{ijk}J$$

one can get

$$I_L = \frac{(\mathbf{\Omega}\nabla s)}{\rho}.$$

This integral is just the Ertel invariant (3.225). Thus, the conservation of the Ertel invariant as well as the Kelvin theorem about the conservation of the velocity circulation are a sequence of a specific gauge symmetry - the relabeling group.

It is interesting to follow how all the above formulas transform in two dimensions. In this case the Ertel invariant is identically equal to zero, due to the orthogonality of the vectors $\mathbf{\Omega}$ and ∇s. Therefore the nontrivial answers appear only for the barotropic fluid.

Applying the identity to (3.257) the identity

$$\epsilon_{\alpha\beta}\frac{\partial x_i}{\partial a_\alpha}\frac{\partial x_j}{\partial a_\beta} = \epsilon_{ij}J,$$

it is easy to get that the Cauchy invariant transforms into the well-known Lagrangian invariant:

$$\frac{\Omega}{\rho} = \mathrm{const}(a).$$

Important to pay attention to that, unlike three-dimensional case, this relation does not contain the Jacobi matrix.

Let us turn to the incompressible fluid. In this case the obtained formulas are simplified. For example, the relation (3.258) in three dimensions is written in the form

$$\mathbf{I_L} = (\mathbf{\Omega}, \nabla)\mathbf{a}. \tag{3.260}$$

In the formula (3.260) $\mathbf{I_l}$ coincides with

$$\mathbf{\Omega_0}(\mathbf{a}) = \mathrm{curl}_a\mathbf{u},$$

where the vector $\mathbf{u}$ is defined by means of (3.211). This, in particular, means that the transverse part of the vector $\mathbf{u}$ conserves (being the Lagrangian invariant), and its temporal varying is due to its longitudinal part. Moreover, as pointed out in the third section, the choice of this vector is arbitrary due to the arbitrariness in the Lagrangian markers choice. The same takes place also to the vector $\Omega_0(\mathbf{a})$. If one performs the contact transformations $\mathbf{b} = \mathbf{b}(\mathbf{a})$ under the condition $\partial(b_1b_2b_3)/\partial(a_1a_2a_3) = 1$, then the vector $\Omega_0(\mathbf{a})$ will be transformed by the law:

$$\tilde{\Omega}_{0i}(\mathbf{b}) = \frac{\partial b_i}{\partial a_j}\Omega_{0j}(\mathbf{a}). \tag{3.261}$$

This is the transformation of the gauge type, being the generalization [**?**] of the gauge transformations for the Clebsch variables (3.207) [4].

[4]Another approach to the gauge transformations in hydrodynamics was developed in [**?**].

Let, as a result of these transformations, the vector $\tilde{\mathbf{\Omega}}_{\mathbf{0}}(\mathbf{b})$ have one nonzeroth component, say, z-component, equal to 1:

$$\tilde{\Omega}_{01} = (\tilde{\mathbf{\Omega}}_{\mathbf{0}}\nabla_a)b_1 = 0, \tag{3.262}$$

$$\tilde{\Omega}_{02} = (\tilde{\mathbf{\Omega}}_{\mathbf{0}}\nabla_a)b_2 = 0, \tag{3.263}$$

$$\tilde{\Omega}_{03} = (\tilde{\mathbf{\Omega}}_{\mathbf{0}}\nabla_a)b_3 = 1. \tag{3.264}$$

These relations within the given "vorticity" $\mathbf{\Omega}_{\mathbf{0}}(\mathbf{a})$ represent the equations to determine the dependence $\mathbf{b}(\mathbf{a})$. These are the linear differential equations of the first order, which allow to be applied the method of characteristics. Equations for characteristics here are the same for all three equations of the system (3.262),

$$\frac{d\mathbf{a}}{ds} = \mathbf{\Omega}_{\mathbf{0}}(\mathbf{a}),$$

that define the "vortex" line for the $\mathbf{\Omega}_{\mathbf{0}}(\mathbf{a})$. (Here s may be understood as the arc length of the "vortex" line.) Equations on the characteristic (for the component of the $\mathbf{b}$ are then given by:

$$\frac{db_1}{ds} = 0, \tag{3.265}$$

$$\frac{db_2}{ds} = 0, \tag{3.266}$$

$$\frac{db_3}{ds} = 1. \tag{3.267}$$

Two first components b_1 and b_2 are constant along the characteristic. Therefore b_1 and b_2 can be chosen as two independent integrals c_1 and c_2 of the system for characteristics (3.265), and the third component is a linear function of the arc length s. It is important to notice that a solution of the system (3.262) can be found always, at least, locally in the vicinity of some nonsingular surface provided by the coordinate system given by the constants c_1 and c_2. Rigorously speaking this is not a global solution as it is usual when one uses the method of characteristics.

Hence, by using the equation $\mathrm{curl}_b\tilde{\mathbf{u}} = \tilde{\mathbf{\Omega}}_{\mathbf{0}}(\mathbf{b})$, one can reconstruct the velocity $\tilde{\mathbf{u}}$:

$$\tilde{u}_1 = \frac{\partial\phi}{\partial b_1}, \tag{3.268}$$

$$\tilde{u}_2 = \frac{\partial\phi}{\partial b_2} + b_1, \tag{3.269}$$

$$\tilde{u}_3 = \frac{\partial\phi}{\partial b_3}. \tag{3.270}$$

After substitution of these expressions into the equation (3.211) we come back to the Clebsch representation with one pair of canonical variables (for more details, see [**?**]) which yields

$$\mathbf{v} = b_1\nabla b_2 + \nabla\phi.$$

So, the vorticity $\mathbf{\Omega}(\mathbf{r},t)$ takes the form,

$$\mathbf{\Omega}(\mathbf{r},t) = [\nabla b_1 \times \nabla b_2] = \frac{\partial\mathbf{r}}{\partial b_3}(\mathbf{b},t). \tag{3.271}$$

The last equality is a direct sequence of the property following which the transformation $\mathbf{b} = \mathbf{b}(\mathbf{r},t)$ is a diffeomorphism preserving the volume. It is easy to check also that the same expression as (3.271) where $\mathbf{r}$ is replaced by $\mathbf{b}$ satisfies the system (3.262). In this case the first equation of the system becomes the equation $\partial(b_1 b_2 b_3)/\partial(a_1 a_2 a_3) = 1$.

Thus, locally any flow of incompressible fluid can be paratmerized by one pair of the Clebsch variables. In the general situation one needs two pairs of such variables.

3.16. The Hopf Invariant and Degeneracy of the Poisson Brackets

So far we have not discussed the question of which classes of flows are described by the canonical variables introduced in the preceding sections.

To begin with, we consider this question for the example of an ideal incompressible fluid.

Let a flow be parameterized in terms of Clebsch variables in some simply connected domain,

$$\mathbf{v} = \lambda \nabla\mu + \nabla\varphi.$$

Take some point inside this domain and draw through this point some closed curve. Starting from this point and constructing continiousely Clebsch variables on each peice of this curve we come back to the point. Generally speaking, Clebsch variables will take different values. Thus, Clebcsh variables will be multi-valued functions of space coordinates. One partial case of fluid flows with multi-valued Clebsch variables allows the following geometrical interpretation.

Consider a compact oriented two-dimensional manifold M^2 and suppose that λ and μ are local coordinates on this manifold.

The gauge transformations associated with the nonuniqueness of choice of the Clebsch variables lead to the appearance of a whole family of gauge-equivalent manifolds, obtainable from one another by continuous deformations, preserving the surface element:

$$d\lambda d\mu = d\lambda' d\mu'.$$

It is therefore sufficient to select from each of such a family one representative. For example, among the surfaces of genus zero having the same area, it is natural to select the sphere S^2.

It is easy to understand that the inverse image of any point of M^2 in R^3 is a closed curve coinciding with a vortex line. This follows directly from the expression for the curl of the velocity:

$$\boldsymbol{\Omega} = \mathrm{curl}\mathbf{v} = [\nabla\lambda \times \nabla\mu]\,. \tag{3.272}$$

The vortex line is the intersection of the two surfaces $\lambda(\mathbf{r}) = const, \mu(\mathbf{r}) = const$. If the variables λ and μ are single-valued functions, then the manifold M^2 cannot be a closed surface of genus g. Then the flows given by such variables have no nodes. This fact can also be proved differently.

It is known [**?**],[**?**] that the degree of knottiness of a flow is characterized in ideal hydrodynamics by the conserved quantity

$$I = \int (\mathbf{v}, \mathrm{curl}\mathbf{v})\, d\mathbf{r}. \tag{3.273}$$

The conservation of this integral follows immediately from the Kelvin theorem. In order to illustrate this statement, following [**?**] we consider two closed vortex lines

$$\boldsymbol{\Omega} = \int \kappa_1 \mathbf{n}_1 \delta\left(\mathbf{r} - \mathbf{l}_1(s_1)\right) ds_1 + \int \kappa_2 \mathbf{n}_2 \delta\left(\mathbf{r} - \mathbf{l}_2(s_2)\right) ds_2$$

where $\mathbf{n}_{1,2}$ are the tangents and $ds_{1,2}$ the arc elements of these curves.

Calculating the velocity circulation around the contours $\mathbf{r} = \mathbf{l}_1(\mathbf{s}_1)$ and $\mathbf{r} = \mathbf{l}_2(\mathbf{s}_2)$, we find

$$\oint (\mathbf{v}, d\mathbf{l}_1) = m\kappa_2, \quad \oint (\mathbf{v}, d\mathbf{l}_2) = m\kappa_1$$

where m is the linking number of these two curves. Multiplying the first equation by κ_2 and the second by κ_1, and adding the results, we get the integral I:

$$\int (\mathbf{v}, \kappa_1 d\mathbf{l}_1 + \kappa_2 d\mathbf{l}_2) = \int (\mathbf{v}, \mathrm{curl}\mathbf{v})\, d\mathbf{r} = 2m\kappa_1\kappa_2.$$

This formula is generalized without difficulty to a vortex, and then to a continuous distribution. The conservation law (3.273) is valid not only for the infinite region but for the finite one when the vorticity lines are tangent to the boundary.

This integral is thus identically equal to zero for a flow with the trivial topology, in particular, for flows parameterized in terms of single-valued Clebsch variables.

We shall show that the Clebsch variables in the formulation (3.272) described knotted flows, and illustrate their topological meaning.

Suppose that the variables λ and μ are local coordinates on S^2. In this case λ and μ are expressed in terms of the polar and azimuthal angles, θ and φ, so that

$$\mathbf{\Omega} = 2\mathrm{A}\, [\nabla \cos\theta \times \nabla\varphi]$$

where A is a dimensional constant. Now the Clebsch variables are no longer single-valued functions, and on a circuit around the z axis the angle φ acquires an addition 2π. It is also convenient to go over, in the expression for the vector field Ω, from the angles θ and φ to the $\mathbf{n}$-field ($\mathbf{n}^2 = 1$) [**?**]:

$$\Omega_\alpha = \varepsilon_{\alpha\beta\gamma} \left(\mathbf{n}, [\partial_\beta \mathbf{n} \times \partial_\gamma \mathbf{n}]\right). \tag{3.274}$$

We shall limit our considerations to the flows for which $\mathbf{n}$ tends sufficiently rapidly at the infinity to a constant vector $\mathbf{n}_0$. For this class on flows R^3 is isomorphic to the four-dimensional sphere S^3. Thus the classification of the flows is a problem of classification of smooth mappings $S^3 \to S^2$. Such mappings are characterized by the homotopy group $\pi_3(S^2) = \mathsf{Z}$, i.e., any class of flows is characterized by the linked number that coincides with the winding number of two any lines $\mathbf{n}(\mathbf{r}) = \mathbf{n}_1$ and $\mathbf{n}(\mathbf{r}) = \mathbf{n}_2$ ($\mathbf{n}_{1,2} = \mathrm{const}$). The index N for smooth mappings is called the Hopf invariant [**?**]. One can show that the Hopf invariant coincides with the integral I up to a constant factor [**?**]:

$$I = \int (\mathbf{v}, \mathbf{\Omega})\, d\mathbf{r} = 64\pi^2 N A^2.$$

The derivation of this relation is based on the well-known formula of Gauss for the linking number of two curves.

It should be mentioned that in the quantum case, according to [**?**], $A = \hbar/2m$. The remaining manifolds are od secondary interest from the point of view of topology. So a manifold M^2, which is a surface with boundary, is homotopically to a bouquet of circles. Therefore its homotopic group π_3 is trivial. The groups π_3 are also trivial for closed surfaces of genus $g \geq 1$. Topological nontrivial situations occur only for surfaces with zero genus.

We now give an example of a nontrivial mapping with $N = 1$ (Hopf mapping):

$$(\mathbf{n}, \sigma) = q^+ \sigma_3 q, \tag{3.275}$$

$$q = (1 - i\mathbf{r}\sigma)(1 + i\mathbf{r}\sigma)^{-1}$$

where σ are the Pauli matrices.

In toroidal coordinates, one has

$$x + iy = \tfrac{\sinh U}{\cosh U + \cos\beta} e^{i\alpha},\ z = \tfrac{\sin\alpha}{\cosh U + \cos\beta},$$
$$(0 \le U < \infty, 0 < \alpha, \beta < 2\pi),$$

and Eq. (3.275) reads

$$\arctan(n_y/n_x) = \alpha - \beta,\ n_z = 1 - 2\tanh^2 U.$$

These formulas show that the flow looks as follows: the whole space is sliced up by the tori $U = \text{const}$, while any vortex line coils up on a torus, making one loop. Thus any vortex line links once. The expressions for $\boldsymbol{\Omega}$ and $\mathbf{v}$, calculated from (3.275) are not solution of the stationary Euler equations, and can therefore be used as initial conditions for (3.221). It is obvious that the evolution of such a distribution does not take the solution out of the given class with Hopf invariant $N = 1$. The evolution of the vector field $\mathbf{n}$ is determined from the equation

$$\mathbf{n}_t + (\mathbf{v}\nabla)\mathbf{n} = 0 \tag{3.276}$$

which is equivalent to the evolution equation for the variables λ and μ. Equations (3.276) are also Hamiltonian,

$$\mathbf{n_t} = 2A\left[\mathbf{n}\times\frac{\delta H}{\delta\mathbf{n}}\right]$$

and differ from the familiar Landau-Lifshitz equations only by the choice the Hamiltonian H.

The Poisson bracket in this case coincides with the BKK bracket (**??**), (1.205):

$$\{F, G\} = 2A\int\left(\mathbf{n}\left[\frac{\delta F}{\delta\mathbf{n}}\times\frac{\delta H}{\delta\mathbf{n}}\right]\right)d\mathbf{r}.$$

When we go over in this bracket from the $\mathbf{n}$-field to $\boldsymbol{\Omega}$ according to formula (3.274) we get the Poisson bracket (3.220). It is important to note that (3.220) is a degenerate bracket with respect to the invariant I: $\{I, ...\} = 0$, which again shows its origin. From one side, it is connected with its topology, from another side, with the Kelvin theorem. One should remind that the latter is a sequence of the gauge symmetry of the Lagrangian markers.

As we see below, the question about degeneracy of the Poisson brackets for arbitrary equation of state directly is connected the gauge symmetry.

Let us discuss in more details this question for the Poisson hydrodynamic brackets. For this aim, we consider the most general form of the bracket for ideal hydrodynamics, namely, for the nonbarotropic fluids. The bracket in this case has the form of (3.218)

$$\{F, G\} = \int\left\{\left(\nabla\frac{\delta F}{\delta\rho}, \frac{\delta G}{\delta\mathbf{v}}\right) - \left(\nabla\frac{\delta G}{\delta\rho}, \frac{\delta F}{\delta\mathbf{v}}\right)\right\}d\mathbf{r} \tag{3.277}$$
$$+\int\left(\frac{\operatorname{curl}\mathbf{v}}{\rho}, \left[\frac{\delta F}{\delta\mathbf{v}}\times\frac{\delta G}{\delta\mathbf{v}}\right]\right)d\mathbf{r} + \int\left(\frac{\nabla S}{\rho}, \left[\frac{\delta F}{\delta\mathbf{v}}\frac{\delta G}{\delta S} - \frac{\delta G}{\delta\mathbf{v}}\frac{\delta F}{\delta S}\right]\right)d\mathbf{r}.$$

By substituting the integral (3.247), $I_i = \int\rho f(I_l, S)d\mathbf{r}$, into this expression one can verify that the integral commutes with any functional,

$$\{I_i, .\} = 0.$$

In accordance with the definition of the section 1, this integral represents the Casimir against the bracket (3.277).

One should remind that the fact of the conservation of the integral (3.247) is a sequence of the special gauge symmetry of the ideal hydrodynamics equations, that is, as we see, responsible also or the degeneracy of the Poisson brackets.

In order to transform from this bracket to the canonical one it is necessary to resolve the integral (3.247) by introducing new coordinates. We have already known one answer to the question how to do it. If we take the expression (3.209) for the velocity and put there instead of μ the Ertel invariant I_l then the integral (3.247) transforms into the dynamical conservation law with respect to the canonical bracket

$$\{F,G\} = \int \left\{ \left[\frac{\delta F}{\delta \rho}\frac{\delta G}{\delta \varphi} - \frac{\delta F}{\delta \varphi}\frac{\delta G}{\delta \rho}\right] + \left[\frac{\delta F}{\delta \lambda}\frac{\delta G}{\delta I_l} - \frac{\delta F}{\delta I_l}\frac{\delta G}{\delta \lambda}\right] + \left[\frac{\delta F}{\delta \beta}\frac{\delta G}{\delta s} - \frac{\delta F}{\delta s}\frac{\delta G}{\delta \beta}\right] \right\} d\mathbf{r}$$

so that

$$\{I_i, H\} = 0.$$

We can also remark that, as it was shown by van Saarlos [**?**], the transition from the Lagrangian description in terms of the action (3.252) to the canonical variables is determined through the change (**??**) or (3.212).

(*====================================*)

3.17. Two mechanisms of surface waves generation: Kelvin-Helmholtz instability vs. Miles instability

Since the 19th century the problem of surface waves generation by wind has been intensively studied. At least two main mechanisms of air-water interface instability are known up to now. The first one is the Kelvin-Helmholtz instability of the interface of two ideal fluids [**?**]. The equation for the complex phase velocity of surface waves c has the form:

$$k^2c^2 - \omega_k^2 = -\epsilon(\vec{k}, \vec{V})^2, \tag{3.278}$$

where $\vec{V}$ is a wind velocity,

$$\omega_k^2 = \frac{k}{1+\epsilon}\left[g(1-\epsilon) + \alpha k^2\right] \tag{3.279}$$

is the dispersion law of gravitational-capillary waves, g is the acceleration of gravity, α is the surface tension, $\epsilon = \frac{\rho_2}{\rho_1}$ is the ratio of upper ρ_2 and lower ρ_1 fluids densities. We assume that $\epsilon \ll 1$. E.g. for air and water $\epsilon = 1.24 \cdot 10^{-3} \ll 1$. According to (3.278) the Kelvin-Helmholtz instability arises if the wind velocity V exceeds the critical value

$$V_{cr} = \frac{1}{\sqrt{\epsilon}} V_{min}, \tag{3.280}$$

which is connected with the minimal phase velocity $V_{min} = min\frac{\omega_k}{k}$ of surface waves under the absence of wind. If the wind velocity exceeds this threshold $V > V_{cr}$ then the instability growth rate has maximum in the intermediate gravitational-capillary branch of spectrum, where $k \sim k_0 = \sqrt{\frac{g}{\alpha}}$. In this case the instability is aperiodic one due to vanishing of real part of surface waves phase velocity.

The second mechanism of surface waves generation by wind is the Miles instability [**?, ?**] arising from finite viscosity of upper fluid. Viscosity results in boundary layer formation in the upper slight fluid near to interface between two fluids. The instability is due to the shear flow $V(y)$ in the boundary layer, where y is a vertical co-ordinate. It is assumed that mean flow in heavy lower fluid is negligible, i.e. $V(0) = 0$, and that the upper fluid velocity outside the boundary layer is $V = V_0 = Const$. The Miles instability arises for wind velocity far below the critical value V_{cr}. The equation for complex phase velocity is given by:

$$k^2(c + 2ik\nu_1)^2 - \omega_k^2 = -\epsilon(\vec{k}, \vec{V})^2 + \gamma_r + i\gamma_i, \tag{3.281}$$

where ν_1 is a kinematic viscosity of lower fluid and values $\gamma_1,\ \gamma_2$ depend on kinematic viscosity of upper fluid ν_2 and on phase velocity c. If $V > V_{cr},\ |\gamma_{r,i}| \ll \epsilon k^2(V_0 - V_{cr})^2,\ k \sim k_0$ (from which it is followed $Re(c) \ll Im(c)$), then we will say that the Kelvin-Helmholtz instability dominates, in all other cases it will be assumed that the Miles instability prevails.

It can be noted that in gravitational spectrum range the Phillips instability [**?**] is very important one. But in present work we consider only perturbation at scales $k \sim k_0 = \sqrt{\frac{g}{\alpha}}$, where Phillips instability is negligible one.

Miles theory gives only linear stability analysis, but in terms of Kelvin-Helmholtz it is possible to study the nonlinear stage of instability development. In [**?**] the nonlinear theory of the Kelvin-Helmholtz instability of ideal fluids was developed. The perturbation theory by small parameter, which is the typical slope of boundary surface between two fluids, was created. It was shown that the first nonvanishing order of nonlinearity by wave amplitudes results in the explosive amplitude growth, which was interpreted as sea foam formation in a finite time. This prediction agrees with the experimental observation that near to the wind velocity $V \sim 6\frac{m}{sec}$ the part of the sea surface covered by the foam has very sharp growth (see [**?, ?, ?, ?, ?**]). In contrast with Kelvin-Helmholtz instability, the Miles instability has no threshold for $V \sim 6\frac{m}{sec}$, it grows continuously with growth of wind velocity. So it seems very important to compare the growth rates of the Kelvin-Helmholtz instability with the Miles instability for $V \sim V_{cr}$ and this is the aim of the present paper. Particularly, it is found that the Kelvin-Helmholtz instability starts prevail as supercriticity $\delta = \frac{V_0^2 - V_{cr}^2}{V_{cr}^2}$ growths and the necessary for this dominance value of supercriticity crucially depends on the boundary layer thickness and on the kinematic viscosity of the upper fluid (which is typically the air).

The plan of the paper is the following: in §2 the boundary problem for the Orr-Sommerfeld equation is formulated. On this bases the equation of the type (3.281) for complex phase velocity c of surface waves is found. In §3 the boundary problem is solved for the particular choice of velocity profile $V(y) = V_0(1 - e^{-x/h})$ under the additional condition $khR \gg 1$, where $R = (V_0 h)/\nu_2$ is a Reynolds, ν_2 is kinematic viscosity of upper fluid. It is found that if $kh \sim 1$ then the Miles instability dominates. In §4 it is shown that the solutions of §3 can be significantly simplified in the limit of thin, in comparison with perturbation wave length, boundary layer, i.e. $kh \ll 1$, but still $khR \gg 1$. In this case the parameter range of Kelvin-Helmholtz instability dominance is found to be $kh/(khR)^{1/3} \ll \delta$. In §5 the limit of very small boundary layer thickness $khR \ll 1,\ kh \ll 1$ is studied for arbitrary wind velocity profile $V(y)$. The condition of Kelvin-Helmholtz instability dominance is $2/R \ll \delta$. In concluding section the application to the particular case of air blowing over water is investigated. Also we give the results of numerical calculation of dependence $c(k)$ for intermediate case $khR \sim 1$, where there are no analytical results. In particular the boundary layer thickness is found for which the Kelvin-Helmholtz instability dominates for the smallest supercriticity δ.

Let us consider two-dimensional flow, where x and y are horizontal and vertical Cartesian co-ordinates respectively, unperturbed wind velocity is $\vec{V} = (V, 0)$, wave vector of perturbations is $\vec{k} = (k, 0)$. Below the dimensionless variables are used for which $h = 1,\ V_0 = 1$, where h is a typical boundary layer thickness, V_0 is an asymptotic value of wind velocity $V(y)|_{y\to\infty} = V_0$. In linear approximation the harmonic disturbance of the interface between two fluids is given by:

$$y = a(t)e^{ikx}, \qquad a(t) = a_0 e^{-ikct}. \tag{3.282}$$

To describe such disturbances on the background of shear flow $V(y)$ Benjamin [?] introduced curvilinear co-ordinates:

$$\xi = x - iae^{ik(x+iy)}, \qquad \eta = y - ae^{ik(x+iy)}. \tag{3.283}$$

In these co-ordinates equation (3.282) transforms to $\eta = 0$ within a factor $1+O(ka)$. If we write stream function in the form:

$$\psi(\xi,\eta) = \int_0^\eta [V(\eta) - c]\, d\eta + \left[F(\eta) + [U(\eta) - c]e^{-kh}\right] ae^{ik\xi}, \tag{3.284}$$

then linearization of the Navier-Stokes equations relative to the parallel shear flow $V(y)$ leads to the nonuniform Orr-Sommerfeld equation for $F(\eta)$:

$$(V - c)(F'' - k^2F) - U''F = \frac{1}{ikR}\left[F^{IV} - 2k^2F'' + k^4F + (U^{IV} - 2kU''')e^{-k\eta}\right], \tag{3.285}$$

where $R = \frac{V_0 h}{\nu_2}$ is the Reynolds number, ν_2 is kinematic viscosity of upper fluid. If we neglect mean flow in heavy lower fluid then kinematic condition of continuity of fluids velocity on interface between two fluids and the dynamic conditions for stress tensor σ :

$$\sigma_1^{xy} = \sigma_2^{xy}, \qquad \sigma_1^{yy} = \sigma_2^{yy} - \rho_1\alpha k^2 ae^{ik\xi}, \tag{3.286}$$

where ω_k is given by (3.279), subscripts $1, 2$ are due to the lower and upper fluids respectively, lead to the boundary conditions for F (see [**?**, **?**]):

$$F(0) = c, \qquad F'(0) = -V'(0) + kc \tag{3.287}$$

and to the equation (3.281) in the form [**?**]:

$$(c + 2ik\nu_1)^2 - \frac{\omega_k^2}{k^2} = \frac{\epsilon}{k}(P + 2ik\frac{F'(0)}{R} + iT), \tag{3.288}$$

where quantities

$$\begin{gathered} P = k^2 \int_0^\infty (V - c)F d\eta - \tfrac{i}{kR}\left[k^2F'(0) + \int_0^\infty (k^4F - k^2V''e^{-k\eta})d\eta\right], \\ T = \tfrac{1}{R}\left[F''(0) + k^2c + V''(0)\right] \end{gathered} \tag{3.289}$$

are connected with stress tensor:

$$\sigma_2^{yy} = -p_0 - 2ik\frac{F'(0)}{R}\epsilon\rho_2 ae^{ik\xi}, \qquad (p_0, \sigma_2^{xy}) = \epsilon\rho_2(P, T)ae^{ik\xi}, \tag{3.290}$$

and p_0 is a normal pressure. For P we can also write equivalent expression:

$$P = V'F - (V - c)F' - \frac{i}{kR}\left[F''' - k^2F' + (V''' - kV'')e^{-k\eta}\right]|_{\eta=0}. \tag{3.291}$$

Two other boundary conditions for Orr-Sommerfeld equation (3.285) result from finiteness of perturbation:

$$F, F'|_{\eta\to\infty} \to 0. \tag{3.292}$$

Equation (3.285) together with boundary conditions (3.287, 3.292) and equations (5.50, 3.289) form the closed system of equation for determination of complex eigenvalue c, which is complex phase speed of surface waves.

3.17.1. Solution of eigenvalue problem for special form of wind profile for large boundary layer thickness. It is convenient to approximate the parallel shear flow $V(y)$ by function

$$V(y) = V_0(1 - e^{-x/h}) \equiv 1 - e^{-x}. \tag{3.293}$$

This velocity profile allows us to find the explicit solution of Reley equation:

$$(V - c)(F'' - k^2 F) - U'' F = 0, \tag{3.294}$$

which is inviscid part of the Orr-Sommerfeld equation (3.285). It is very important that the curvature of this velocity profile $V''(0) \neq 0$ is nonzero for $\eta = 0$. This means that function (3.293) is a general case and that value of complex velocity c slightly changes for small variation of velocity profile (3.293). Early Benjamin [**?**] used other forms of $V(y)$ for which also it is possible to find explicit solutions of Orr-Sommerfeld equation, but it was very special cases of zero curvature $V''(0) = 0$, because asymptotic solutions of Orr-Sommerfeld equation (3.285) for $kR \to \infty$ strongly depend on $V''(0)$ [**?**] and if this curvature $V''(0)$ vanishes then these solutions qualitatively change.

Returning to profile (3.293) and making the substitution

$$z = 1 - e^{-(\eta-\eta_c)},\ \phi = (1 - z)^{-k} F, \tag{3.295}$$

where $V(\eta_c) = 0$, $\eta_c = -ln(1 - c)$, we reduce the Reley equation (3.294) to the hypergeometric one:

$$(1 - z)z\phi'' - (2k + 1)z\phi' + \phi = 0. \tag{3.296}$$

For $\eta \to \infty,\ z \to 1$ the boundary conditions (3.292) allow to write the solution of (3.296) in the neighborhood of $z = 1$ through the standard hypergeometric function:

$$\phi = \alpha F(k + \sqrt{k^2 + 1}, k - \sqrt{k^2 + 1}, 2k + 1, 1 - z), \qquad \alpha = const. \tag{3.297}$$

The function (3.297) is equal to the linear combination of two independent hypergeometric functions of variables z. One of which is regular one and the other one has logarithmic singularity for $\eta \to \eta_c,\ z \to 0$ [**?**]. Thus we may express the solution of Reley equation (3.294) in the neighborhood $\eta \to \eta_c$ satisfying to boundary conditions (3.292), in the form:

$$F_{reley} = \frac{\alpha\Gamma(1 + 2k)}{\Gamma(k + \sqrt{k^2 + 1})\Gamma(k - \sqrt{k^2 + 1})} \Big\{ \Big[\psi(k + \sqrt{k^2 + 1}) + \psi(k - \sqrt{k^2 + 1}) - \psi(2)\Big] (\eta - \eta_c) \\ -1 + (\eta - \eta_c)\ln(\eta - \eta_c) + (\gamma - k)(\eta - \eta_c)\Big\}, \tag{3.298}$$

where $\gamma = 0.5772\ldots$ is Euler constant.

The Reley equation (3.294) is a good approximation of the solutions of the Orr-Sommerfeld equation (3.285) for all η except thin viscous sublayers, where viscosity of upper fluid is significant one. The first sublayer, which is the so-called coincidence layer, occurs in a neighborhood of critical point $\eta = \eta_c,\ V(\eta) = c$, where resonance takes place due to the coincidence of local wind velocity $V(\eta_c)$ with phase speed c of surface waves. The inviscid Reley equation has logarithmic singularity at this point and the right viscous part of the Orr-Sommerfeld equation (3.285) becomes significant one. The thickness of coincidence layer has scaling $\mu = \frac{1}{(kR)^{1/3}}$ and the solutions of (3.285) expand by parameter μ in neighborhood of $\eta = \eta_c$. In a general case the viscous sublayer with thickness $1/(kR)^{1/2}$ also exists very close to the interface $\eta = 0$. But near to the threshold of Kelvin-Helmholtz instability due to $Re(c) \to 0, \eta_c \to 0$ this sublayer is merged with coincidence layer and solutions depend only on parameter μ [**?**]. At zero order by μ a general solution of the Orr-Sommerfeld equation in coincidence layer can be written as follows:

$$F = \beta_1 + \beta_2\theta + \beta_3\chi_3(\theta), \tag{3.299}$$

where $\theta = \frac{\eta-\eta_c}{\mu}$, $\quad \beta_{1,2,3} = const$, $\quad \chi_3(\theta) = \int\limits_{\infty}^{\theta} d\theta' \int\limits_{\infty}^{\theta'} d\theta'' \sqrt{\theta''} H^{(1)}_{1/3}\left[\frac{2}{3}(i\theta'')^{3/2}\right]$. In the range $\theta \to \infty$ it is necessary to consider the terms up to the first order by μ. Because $\chi_3(\theta)|_{\theta\to\infty} \to 0$, we have:

$$F = \beta_1 + \beta_2\theta + \beta_1\mu\frac{V''(\eta_c)}{V'(\eta_c)}\theta\ln\theta, \tag{3.300}$$

where we may put $\frac{V''(\eta_c)}{V'(\eta_c)} \simeq \frac{V''(0)}{V'(0)} = -1$ due to the small value of $\eta_c \to 0$. From matching procedure of outer (3.298) and inner (3.300) solutions and from boundary conditions (3.287) for $\eta = 0$ it is possible to find the unknown coefficients $\beta_1, \beta_2, \beta_3, \alpha$. Then it follows from (3.291) in the limit $\eta_c \to 0,\ \mu \to 0,\ |c| \ll \mu:$

$$P = \frac{F'''(0)}{ikR} = \frac{-\Gamma(k+\sqrt{k^2+1})\Gamma(k-\sqrt{k^2+1})}{\Gamma(1+2k)\left[\psi(k+\sqrt{k^2+1}) + \psi(k-\sqrt{k^2+1}) - \psi(2) + \gamma - k + \ln\mu + \frac{\chi_3'(0)}{\mu\chi_3(0)}\right]}. \tag{3.301}$$

Under the additional condition $k \sim 1$ and due to $\mu \ll 1$ the equation (3.301) significantly simplifies:

$$P = -\frac{\Gamma(k+\sqrt{k^2+1})\Gamma(k-\sqrt{k^2+1})}{\Gamma(1+2k)}\mu\frac{\chi_3(0)}{\chi_3'(0)}. \tag{3.302}$$

It is clear from here that for nonsmall values of k the viscous terms dominate. This leads to the conclusion that our basic assumption $\eta_c \to 0$ breaks and that the Miles instability dominates in this case. From physical point of view this is very natural result because if the boundary layer thickness is of the order of perturbation wavelength $k \sim 1$ then instability growth rate of surface waves strongly depends on the boundary layer structure and this layer certainly cannot be approximated by tangential discontinuity as it does in the Kelvin-Helmholtz theory. Moreover, it is evident from equation (3.301) that even in the absence of viscosity of the upper fluid the instability growth rate differs from Kelvin-Helmholtz one.

We note that the possibility of reduction of the Reley equation to the hypergeometric one for velocity profile (3.293) allows to find value of c without the additional restriction $|c| \ll 1$, when coincidence layer and viscous sublayer are asymptoticaly separated. But this case is not the subject of the present work.

3.17.2. Limit of thin boundary layer. If boundary layer thickness h is small in comparison with wave length, i.e. in dimensionless variables $k \ll 1$, but still $kR \gg 1$, then equations (3.298), (3.301) are significantly simplified. We obtain with accuracy up to k^2 :

$$F_{reley} = \alpha\big[k + (\eta - \eta_c) - k(\eta - \eta_c)\ln(\eta - \eta_c)\big]$$

$$P = -\frac{k}{1 - \frac{\chi_3'(0)}{\chi_3(0)}\frac{k}{\mu}}, \qquad \frac{\chi_3'(0)}{\chi_3(0)} = -1.1153 - i\,0.6440\ldots \tag{3.303}$$

Similarly, it follows from (3.289) that tangential stress

$$T = k^2\mu\frac{\chi_3''(0)}{\chi_3(0)}, \qquad \frac{\chi_3''(0)}{\chi_3(0)} = 0.6858 + i\,1.1880\ldots, \tag{3.304}$$

is proportional to $k^2\mu$ and thus T is negligible in comparison with P.

The Kelvin-Helmholtz theory corresponds to $P + iT = -kV_0^2 \equiv -k$, thus for $k/\mu \sim 1$ the Miles instability dominates. If $k/\mu \ll 1$ then from equations (3.303, 3.304, 5.50) we obtain the expression for the complex phase speed c for $k \simeq k_0 = \sqrt{g/\alpha}$:

$$k^2(c + 2ik\nu_1)^2 = \epsilon k^2\Big(-\delta + \frac{\chi_3'(0)}{\chi_3(0)}\frac{k}{\mu}\Big), \tag{3.305}$$

where $\delta = (V_0^2 - V_{cr}^2)/V_{cr}^2$ is supercriticity and critical wind velocity V_{cr} is given by (3.280). According to (3.305) the Kelvin-Helmholtz instability prevails Miles instability under the conditions:

$$\frac{k}{\mu} \ll 1, \qquad \frac{k}{\mu} \ll \delta, \qquad \mu \ll 1. \tag{3.306}$$

If $k/\mu \sim \delta$ then Miles instability dominates.

It can be noted that in a case $k \ll 1$ the similar to (3.303) equation can be obtained from the approximate Heisenberg solutions of the Reley equation (3.294) for arbitrary velocity profile $V(\eta)$ [**?**].

3.17.3. Limit of extremely thin boundary layer. If the boundary layer thickness is so small that kR is of the order of unity then the validity of above asymptotic by μ solutions breaks. But under the additional condition $kR \ll 1$ it is possible to obtain the analytic expressions for P, T. In the range $\eta \gg 1$ the Orr-Sommerfeld equation (3.285) is reduced to the uniform ordinary differential equation with constant coefficients. The general solution of this equation, vanishing at infinity, is given by the sum of two exponents:

$$F = \alpha_1 e^{-k\eta} + \alpha_2 e^{-\lambda\,\eta}, \qquad \alpha_{1,2} = const, \qquad \lambda = \sqrt{k^2 + ikR(1-c)} \tag{3.307}$$

Applicability of (3.307) breaks in thin layer with typical thickness $h \equiv 1$ near to the interface. But solution (3.307) changes at larger scales k^{-1}, $(kR)^{-1/2}$, at which velocity profile $V(\eta)$ is effectively approximated by velocity jump from $V(0) = 0$ to $V(0+0) = V_0$ and, as it will be shown below, the values P, T have weak dependence from form of velocity profile $V(\eta)$. In the limit $kR \ll 1$ for $\lambda\eta \ll 1$ the right side of the Orr-Sommerfeld equation dominates and this equation is reduced to the nonuniform ordinary differential equation with constant coefficients:

$$\frac{1}{ikR}\left[F^{IV} - 2k^2F'' + k^4F + (U^{IV} - 2kU''')e^{-k\eta}\right] = 0, \tag{3.308}$$

It is possible to check by direct substitution that its the general solution is given by:

$$F = (C_1 + C_2\eta)e^{k\eta} + (C_3 + C_4\eta)e^{-k\eta} + e^{k\eta}\int_\eta^\infty e^{-2k\eta}V'(\eta)d\eta, \quad \lambda\eta \ll 1,$$

where $C_{1,2,3,4}$ are arbitrary numbers. Expanding this solution for $k\eta \to 0$ in power series up the first order of η and satisfying to the boundary conditions (3.287), we obtain (here and below we again neglect $c \to 0$):

$$F = -(1 + k\eta)\int_0^\infty e^{-2k\eta}V'(\eta)d\eta + e^{k\eta}\int_\eta^\infty e^{-2k\eta}V'(\eta)d\eta. \tag{3.309}$$

It is convenient to rewrite the first integral in this expression in the form:

$$-\int_0^\infty e^{-2k\eta}V'(\eta)d\eta = -1 + kB, \qquad B = -2\int_0^\infty e^{-2k\eta}(V(\eta) - 1)d\eta, \tag{3.310}$$

where $B = O(1)$ under the condition that $V'(\eta)$ decreases more fast than $1/\eta^2$ for $\eta \to \infty$, what we suppose to be satisfied below. Solutions (3.307) and (3.309) can be matched in the intermediate range $\eta \gg 1$, $\lambda\eta \ll 1$, $k\eta \ll 1$, in which $B \to 0$. Expanding (3.307) in a power series up to the first order by η, we find from (3.307) and (3.309) the uniformly applicable for all η solution of the Orr-Sommerfeld equation (3.285) :

$$F = \alpha_1 e^{-k\eta} + \alpha_2 e^{-\lambda\eta} + e^{k\eta}\int_\eta^\infty e^{-2k\eta}V'(\eta)d\eta, \tag{3.311}$$

where

$$\alpha_1 = -1 + kB + \frac{2k(-1 + kB)}{-k + \lambda}, \quad \alpha_2 = -1 + kB - \alpha_2, \tag{3.312}$$

and constant B is given by (3.310). By direct substitution of (3.311) into (3.285) it is possible to establish that the corrections to (3.311) have the order k^2R. To estimate different terms in (3.311) we can note that due to the proportionality of the Reynolds number R to the boundary layer thickness h it is convenient to introduce also the value

$$\frac{R}{k} = \frac{V_0}{k\nu_2}, \tag{3.313}$$

which does not depend on h. We suppose that $R/k \gg 1$ (e.g. for air and water $R/k \simeq R/k_0 = 1200$). Then the equation (3.312) is simplified:

$$\alpha_1 = -1 - \frac{2k}{\lambda} + i\frac{2k}{R} + kB + O(\frac{k^3}{\lambda^3}), \quad \alpha_2 = \frac{2k}{\lambda} - i\frac{2k}{R} + O(\frac{k^3}{\lambda^3}). \tag{3.314}$$

From (3.289), (3.311), (3.314) we find the corrections to the pressure $P_0 = -k$ of Kelvin-Helmholtz theory:

$$\begin{gathered} P = -k(1 + \tfrac{2k}{\lambda}) + O(\tfrac{k^2}{\lambda^2}), \\ T = -\tfrac{i2k^2}{\lambda} + O(\tfrac{k^2}{\lambda^2}). \end{gathered} \tag{3.315}$$

But viscous part of normal stress (3.290) $2ikF'(0)/R = -2ikV'(0)/R$ prevails corrections (3.315) due to $R/k \gg 1$ under the additional condition that $V'(0)$ is of the order of V_0/h. For simplicity let us put $V'(0) = V_0/h \equiv 1$ (this corresponds e.g. to (3.293). Then right side of equation (5.50) can be written as follows:

$$P + 2ik\frac{F'(0)}{R} + iT = -k[1 + \frac{2i}{R}] + O(\frac{k^2}{\lambda^2}). \tag{3.316}$$

Similarly to the consideration of equation (3.305) in §4 we obtain from eqs. (3.316), (5.50) the conditions of the Kelvin-Helmholtz instability dominance:

$$k \ll 1, \qquad kR \ll 1, \qquad 2/R \ll \delta. \tag{3.317}$$

We note that because R is proportional to h then conditions (3.317) are satisfied for not too small boundary layer thickness.

In particular case of air blowing over water, when $k \simeq k_0 = \sqrt{g/\alpha} = 3.664 sm^{-1}$, $\nu_2 = 0.15 sm^2/sec$, $V_{cr} = 660 sm/sec$, the results of previous sections about the range of Kelvin-Helmholtz instability dominance can be written in the form:

a)for thin boundary layer it follows from (3.306) that $0.008sm \ll h \ll 0.27sm, \quad \delta \gg \left(\frac{h}{6.6\cdot 10^{-2}sm}\right)^{5/3}$;

b)for very thin boundary layer we obtain from (3.317) that $h \ll 0.008sm, \quad \delta \gg \frac{4.5\cdot 10^{-4}sm}{h}$.

In the intermediate range between cases a) and b), where $k \ll 1, \quad kR \sim 1$, the boundary layer thickness h has the same order as coincidence layer thickness. So it is not possible to use any asymptotic method for solving of the Orr-Sommerfeld equation. For air blowing over water in this intermediate range the dependence $c(k)$ is found from numerical calculations for $k \sim k_0$. For each value of c the boundary problem (3.285, 3.287, 3.292) is solved by orthonormalization procedure [**?**, **?**], then the roots of equation (5.50) are found by Newton's method. In particular it is established that for thickness $h \simeq 0.02/k_0 = 0.0054sm$ of boundary layer the Kelvin-Helmholtz instability dominates for the smallest supercriticity $\delta = (V_0^2 - V_{cr}^2)/V_{cr}^2$. For this thickness $Re(c) \ll Im(c)$, i.e. Kelvin-Helmholtz theory dominates approximately for $\delta > 0.1$. Note also that analytical results of §§3,4,5 are strongly supported by numerical calculations.

Thus the minimal necessary supercriticity above $V_{cr} = 660\ sm/sec$ for Kelvin-Helmholtz instability dominance is of the order of 10%. But developed in [**?**] nonlinear theory of Kelvin-Helmholtz instability used the condition $\delta \ll 1$. Thus the applicability of this theory to the case of air blowing over water is connected at least with rather severe restriction of the boundary layer thickness in the air near to air-water interface. Nevertheless the conditions (3.306) shows that Kelvin-Helmholtz theory can have wide range of applicability for the instability development of the interface between two fluids if the upper fluid has significantly lesser kinematic viscosity than air or if the substantial change of other parameters (which are surface tension α, the acceleration of gravity g, the ratio of fluids densities $\epsilon = \rho_2/\rho_1$) takes place.

CHAPTER 4

Surface Dynamics in Conformal Variables

4.1. Non-Canonical Hamiltonian Structure and Poisson Bracket for 2D Hydrodynamics with Free Surface

We study two-dimensional potential motion of ideal incompressible fluid with free surface of infinite depth. Fluid occupies the infinite region $-\infty < x < \infty$ in the horizontal direction x and extends down to $y \to -\infty$ in the vertical direction y as schematically shown on the left panel of Fig. 1. The time-dependent fluid free surface is represented in the parametric form as

$$x = x(u,t), \;\; y = y(u,t) \tag{4.1}$$

with the parameter u spanning the range $-\infty < u < \infty$ such that

$$x(u,t) \to \pm\infty \text{ and } y(u,t) \to 0 \text{ as } u \to \pm\infty. \tag{4.2}$$

We assume that the free surface does not have self-intersection, i.e. $\mathbf{r}(u_1,t) \neq \mathbf{r}(u_2,t)$ for any $u_1 \neq u_2$. In other words, the free surface is the simple plane curve. Here $\mathbf{r}(u,t) \equiv (x(u,t), y(u,t))$.

In the particular case when the free surface can be represented by a single-valued function of x,

$$y = \eta(x,t), \tag{4.3}$$

one can also represent domain occupied by the fluid as $-\infty < y \leq \eta$ and $-\infty < x < \infty$. Such single-valued case has been widely considered (see e.g. Ref. [**Sto57**]). We however do not restrict to that particular case which is recovered by choosing $u = x$.

Potential motion implies that a velocity $\mathbf{v}$ of fluid is determined by a velocity potential $\Phi(\mathbf{r},t)$ as $\mathbf{v} = \nabla\Phi$ with $\nabla \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. The incompressibility condition $\nabla \cdot \mathbf{v} = 0$ results in the Laplace equation

$$\nabla^2\Phi = 0 \tag{4.4}$$

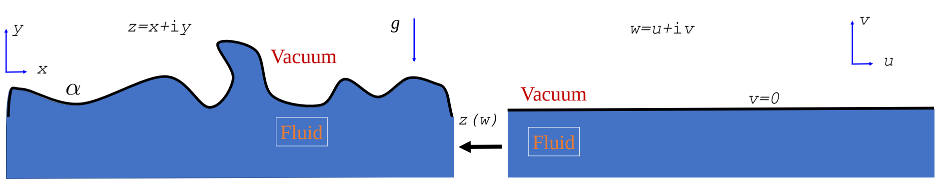

FIGURE 1. Dark area represents the domain occupied by fluid in the physical plane $z = x + \mathrm{i}y$ (left) and the same domain in $w = u + \mathrm{i}v$ plane (right). Thick solid lines correspond to the fluid's free surface in both planes.

inside fluid. Eq. (4.4) is supplemented with decaying boundary condition (BC) at infinity in the horizontal direction,

$$\nabla\Phi \to 0 \text{ for } |x| \to \infty, \tag{4.5}$$

and a vanishing of the normal velocity the fluid's bottom,

$$\left.\frac{\partial\Phi}{\partial n}\right|_{y\to-\infty} = 0. \tag{4.6}$$

Without loss of generality BCs (4.5) and (4.6) can be replaced by Dirichlet BC

$$\Phi \to 0 \text{ at } |\mathbf{r}| \to \infty. \tag{4.7}$$

BCs at the free surface are time-dependent and consist of kinematic and dynamic BCs. Kinematic BC ensures that free surface moves with the normal velocity component v_n of fluid particles at the free surface. Motion of the free surface is determined by time derivatives of the parameterization (4.1) and kinematic BC is given by a projection into normal directions as

$$\mathbf{n}\cdot(x_t, y_t) = v_n \equiv \mathbf{n}\cdot\nabla\Phi|_{x=x(u,t),\ y=y(u,t)}, \tag{4.8}$$

where

$$\mathbf{n} = \frac{(-y_u, x_u)}{(x_u^2 + y_u^2)^{1/2}} \tag{4.9}$$

is the outward unit normal vector to the free surface and subscripts here and below means partial derivatives, $x_t \equiv \frac{\partial x(u,t)}{\partial t}$ etc.

Eqs. (4.8) and (4.9) result in a compact expression

$$y_t x_u - x_t y_u = [x_u\Phi_y - y_u\Phi_x]|_{x=x(u,t),\ y=y(u,t)} \tag{4.10}$$

for the kinematic BC.

Tangential component of the vector $\mathbf{r}_t = (x_t, y_t)$ is not fixed by kinematic BC (4.10) but can be chosen at our convenience. E.g., one can define u to be the Lagrangian coordinate of fluid particles at the free surface (fluid particles once on the free surface never leave it). Then tangential component of $\mathbf{r}_t$ would coincide with the tangential component of $\nabla\Phi|_{x=x(u,t),\ y=y(u,t)}$. Another possible choice is to choose u to be the arclength along the free surface. However, we use neither Lagrangian or arclength formulation below. Instead, throughout the paper we use the conformal variables for the free surface parameterization as described below in Section 4.1.1. Another particular form of (4.1) is given by Eq. (4.3), which corresponds to choosing $u = x$ (as mentioned above, it is possible only if $\eta(x,t)$ is the single-valued function of x). In that case Eq. (4.9) is reduced to $\mathbf{n} = (-\eta_x, 1)(1+\eta_x^2)^{-1/2}$ and kinematic BC Eq. (4.10) is given by

$$\eta_t = (1+\eta_x^2)^{1/2} v_n = (-\eta_x\Phi_x + \Phi_y)|_{\ y=\eta(x,t)}\,. \tag{4.11}$$

This form of kinematic BC has been widely used (see e.g. Ref. [**Sto57**]).

A dynamic BC, which is the time-dependent Bernoulli equation (see e.g. [**LL89a**]) at the free surface, is given by

$$\left.\left(\Phi_t + \frac{1}{2}(\nabla\Phi)^2 + gy\right)\right|_{x=x(u,t),\ y=y(u,t)} = -P_\alpha, \tag{4.12}$$

where g is the acceleration due to gravity and

$$P_\alpha = -\frac{\alpha(x_u y_{uu} - x_{uu} y_u)}{(x_u^2 + y_u)^{3/2}} \tag{4.13}$$

is the pressure jump at the free surface due to the surface tension coefficient α. Here without loss of generality we assumed that pressure is zero above the free surface (i.e. in vacuum). All results below apply both to the surface gravity wave case ($g > 0$) and the Rayleigh-Taylor problem ($g < 0$). Below we also consider a particular case $g = 0$ when inertia forces well exceed gravity force. For the case of single-valued parameterization (4.3), Eq. (4.13) is reduced to the well-known expression (see e.g. Ref. [**Zak68**])

$$P_\alpha = -\alpha \frac{\partial}{\partial x}[\eta_x (1 + \eta_x^2)^{-1/2}] = -\alpha \eta_{xx} (1 + \eta_x^2)^{-1/2}. \tag{4.14}$$

Eqs. (4.12) and (4.13), together with decaying BCs (4.2) and (4.5), imply that a Bernoulli constant (generally located at right hand side (r.h.s) of Eq. (4.12)) is zero.

Eqs. (4.1),(4.2),(4.4)-(4.9),(4.12) and (4.13) form a closed set of equations which is equivalent to Euler equations for dynamics of ideal fluid with free surface for any chosen free surface parameterization (4.1). Here at each moment of time t, Laplace Eq. (4.4) has to be solved with Dirichlet BC

$$\psi(x,t) \equiv \Phi(\mathbf{r},t)|_{x=x(u,t),\ y=y(u,t)} \tag{4.15}$$

and BCs (4.5), (4.6). That boundary value problem has the unique solution. The knowledge of $\Phi(\mathbf{r},t)$ allows to find the normal velocity v_n at the free surface as in Eq. (4.10). It can be interpreted as finding the Dirichlet-Neumann operator for the Laplace Eq. (4.4) [**CS93**]. Then one can advance in time to find new value of $\psi(x,t)$ from Eqs. (4.10) and (4.12) using that

$$\psi_t = [\Phi_t + x_t \Phi_x + y_t \Phi_y]\,|_{x=x(u,t),\ y=y(u,t)} \tag{4.16}$$

as well as evolve a parameterization (4.1) and so on. Here Eq. (4.16) results from the definition (4.15).

The set (4.1),(4.2),(4.4),(4.7),(4.9) and (4.12) preserves the total energy

$$H = K + P, \tag{4.17}$$

where

$$K = \frac{1}{2}\int\limits_\Omega (\nabla\Phi)^2 \mathrm{d}x\mathrm{d}y \tag{4.18}$$

is the kinetic energy and

$$P = g\int\limits_\Omega y\,\mathrm{d}x\mathrm{d}y - g\int\limits_{y\le 0} y\,\mathrm{d}x\mathrm{d}y + \alpha\int\limits_{-\infty}^{\infty}\left(\sqrt{x_u^2 + y_u^2} - x_u\right)\mathrm{d}u \tag{4.19}$$

is the potential energy. Here $\mathrm{d}x\mathrm{d}y$ is the element of fluid volume (more precisely it is the fluid's area because we restricted to 2D fluid motion with the third spatial dimension being trivial), Ω is the area occupied by the fluid which extends down to $y \to -\infty$ in the vertical direction. The term $g\int\limits_{y\le 0} y\,\mathrm{d}x\mathrm{d}y$ corresponds to the gravitational energy of unperturbed fluid (flat free surface) and it is subtracted from the integral over Ω to ensure that the total contribution of the gravitational energy, $g\int_\Omega y\,\mathrm{d}x\mathrm{d}y - g\int\limits_{y\le 0} y\,\mathrm{d}x\mathrm{d}y$, is finite. In other words, one can understand these two terms as the limit $h\to\infty$ and $L\to\infty$, where h is the fluid depth with the bottom at $y=-h$ and L is the horizontal extend of the fluid. Then $g\int\limits_{y\le 0} y\,\mathrm{d}x\mathrm{d}y = -\frac{gh^2L}{2}$, where using this expression below we assume taking the limits $h\to\infty$ and $L\to\infty$. The surface tension energy $\alpha\int_{-\infty}^{\infty}\left(\sqrt{x_u^2+y_u^2}-x_u\right)\mathrm{d}u$ in Eq. (4.19) is determined by the arclength of free surface with $-x_u$ term added to ensure that the surface energy is zero for unperturbed fluid with $y\equiv 0$.

If we introduce the vector field $\mathbf{F}=\hat{y}y^2/2$ with $\hat{y}$ being the unit vector in positive y direction, then the gravitational energy in Eq. (4.19) takes the following form $g\int_\Omega \nabla\cdot\mathbf{F}\,\mathrm{d}x\mathrm{d}y - \frac{gh^2L}{2}|_{h,L\to\infty}$. By the divergence theorem of vector analysis (in our 2D case it can be also reduced to the Green's theorem) this gravitational energy is converted into the surface integral $g\int_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s+\frac{gh^2L}{2}|_{h,L\to\infty}$ (line integral in 2D over arclength $\mathrm{d}s=\sqrt{x_u^2+y_u^2}\mathrm{d}u$ with $\partial\Omega$ being the boundary of Ω) which together with Eq. (4.9) results in

$$P=\frac{g}{2}\int\limits_{-\infty}^{\infty} y^2\,x_u\mathrm{d}u+\alpha\int\limits_{-\infty}^{\infty}\left(\sqrt{x_u^2+y_u^2}-x_u\right)\mathrm{d}u. \tag{4.20}$$

In the simplest case of the single-valued surface parametrization Eq. (4.3), Eqs. (4.18) and (4.20) take the simpler forms

$$K=\frac{1}{2}\int\limits_{-\infty}^{\infty}\mathrm{d}x\int\limits_{-\infty}^{\eta}(\nabla\Phi)^2\mathrm{d}y \tag{4.21}$$

and

$$P=\frac{g}{2}\int\limits_{-\infty}^{\infty}\eta^2\mathrm{d}x+\alpha\int\limits_{-\infty}^{\infty}\left(\sqrt{1+\eta_x^2}-1\right)dx, \tag{4.22}$$

respectively.

It was proved in Ref. [**Zak68**] that ψ and η for the single-valued surface parametrization (4.3) satisfy the canonical Hamiltonian system

$$\frac{\partial\eta}{\partial t}=\frac{\delta H}{\delta\psi},\quad \frac{\partial\psi}{\partial t}=-\frac{\delta H}{\delta\eta} \tag{4.23}$$

with H given by Eqs. (4.17), (4.21) and (4.22). The Hamiltonian formalism of Ref. [**Zak68**] has been widely used for water waves, see e.g. Refs. [**ZLF92, KP03**] for review as well as it was generalized to the dynamics of the interface between two fluids [**KL95**]. In this paper we show that for the general "multivalued" case of the parametrization (4.1), the system of dynamical Eqs. (4.10) and (4.12) for $x(u,t)$, $y(u,t)$ and $\psi(u,t)$ also has a Hamiltonian structure if we additionally assume that $x(u,t)$ and $y(u,t)$ are defined from the conformal map of Section 4.1.1. However, that structure is non-canonical with the non-canonical Poisson bracket and depends on the choice of the parametrization of the surface.

Apparently, the system (4.10) has infinite number of degrees of freedom. The most important feature of integrable systems is the existence of "additional" constants of motion which are different from "natural" motion constants (integrals) (see Refs. [**GGKM67, ZS72, Arn89, ZF71, NMPZ84**]). For system (4.23) the natural integrals are the energy H (4.17), the total mass of fluid,

$$M = \int_{-\infty}^{\infty} \eta(x,t)\mathrm{d}x, \tag{4.24}$$

and the horizontal component of the momentum,

$$P_x = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\eta} \frac{\partial \Phi}{\partial x}\mathrm{d}y. \tag{4.25}$$

Φ is the harmonic function inside fluid because it satisfies the Laplace Eq. (4.4). The harmonic conjugate of Φ is a stream function Θ defined by

$$\Theta_x = -\Phi_y \text{ and } \Theta_y = \Phi_x. \tag{4.26}$$

Similar to Eq. (4.7), we set without loss of generality zero Dirichlet BC for Θ as

$$\Theta \to 0 \text{ at } |\mathbf{r}| \to \infty. \tag{4.27}$$

We define a complex velocity potential $\Pi(z,t)$ as

$$\Pi = \Phi + \mathrm{i}\Theta, \tag{4.28}$$

where

$$z = x + \mathrm{i}y \tag{4.29}$$

is the complex coordinate. Then Eqs. (4.26) turn into Cauchy-Riemann equations ensuring the analyticity of $\Pi(z,t)$ in the domain of z plane occupied by the fluid (with the free fluid's boundary defined by Eqs. (4.1) and (4.2)). A physical velocity with the components v_x and v_y (in x and y directions, respectively) is recovered from Π as $\frac{d\Pi}{dz} = v_x - \mathrm{i}v_y$.

Using $\Theta_y = \Phi_x$ from Eq. (4.26), we immediately convert the horizontal momentum (4.25) into $P_x = \int_{-\infty}^{\infty} \Theta\, \mathrm{d}x$ through integration by parts and Eq. (4.27) which results in

$$P_x = \int_{-\infty}^{\infty} \Theta(x(u,t), y(u,t), t) x_u(u,t)\mathrm{d}u. \tag{4.30}$$

Eq. (4.30) is also valid for the general multi-valued case (contrary to Eq. (4.25) which requires the particular parametrization (4.3)). To check that we replace Eq. (4.25) by $P_x = \int_\Omega \Theta_y \mathrm{d}x\mathrm{d}y = \int_\Omega \nabla \cdot \mathbf{F}\, \mathrm{d}x\mathrm{d}y$ with $\mathbf{F} = \hat{y}\Theta$ and, similar to the derivation of Eq. (4.20), we then obtain Eq. (4.30) from the divergence theorem and Eq. (4.9).

One can use Eqs. (4.26) to obtain the equivalent form of P_x as $P_x = \int_\Omega \Phi_x \mathrm{d}x\mathrm{d}y = \int_\Omega \nabla \cdot \mathbf{F}\, \mathrm{d}x\mathrm{d}y$ with $\mathbf{F} = \hat{x}\Phi$. Then the divergence theorem together with Eq. (4.9) results in

$$P_x = -\int_{-\infty}^{\infty} \psi(x(u,t), t) y_u(u,t)\mathrm{d}u. \tag{4.31}$$

In a similar way, a vertical component of momentum is given by

$$P_y = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\eta} \frac{\partial \Phi}{\partial y} \mathrm{d}y = \int_{-\infty}^{\infty} \psi \, \mathrm{d}x, \tag{4.32}$$

where we used integration by part and Eqs. (4.7) and (4.15). P_y is the integral of motion only for the zero gravity case, $g = 0$. A change of integration variable in Eq. (4.32) results in

$$P_y = \int_{-\infty}^{\infty} \psi \, x_u \mathrm{d}u. \tag{4.33}$$

Eq. (4.33) is also valid for the general multi-valued case. To check that we define in the general case that $P_y = \int_\Omega \Phi_y \, \mathrm{d}x\mathrm{d}y = \int_\Omega \nabla \cdot \mathbf{F} \, \mathrm{d}x\mathrm{d}y$ with $\mathbf{F} = \hat{y}\Phi$ and, similar to the derivation of Eq. (4.20), we obtain Eq. (4.33) from the divergence theorem and Eq. (4.9).

One can use Eqs. (4.26) to obtain the equivalent form of P_y as $P_y = -\int_\Omega \Theta_x \mathrm{d}x\mathrm{d}y = \int_\Omega \nabla \cdot \mathbf{F} \, \mathrm{d}x\mathrm{d}y$ with $\mathbf{F} = -\hat{x}\Theta$. Then the divergence theorem together with Eq. (4.9) results in

$$P_x = \int_{-\infty}^{\infty} \Theta(x(u,t), y(u,t), t) y_u(u,t) \mathrm{d}u. \tag{4.34}$$

For the parametrization (4.1), Eq. (4.24) is replaced by $M = \int_\Omega \mathrm{d}x\mathrm{d}y - \int_{y\le 0} \mathrm{d}x\mathrm{d}y = \int_\Omega \nabla \cdot \mathbf{F} \mathrm{d}x\mathrm{d}y - hL|_{h,L\to\infty}$ with $\mathbf{F} = \hat{y}y$. Similar to derivation of Eq. (4.20) we then use the divergence theorem and Eq. (4.9) to obtain that

$$M = \int_{-\infty}^{\infty} y(u,t) x_u(u,t) \mathrm{d}u. \tag{4.35}$$

In this paper we develop a Hamiltonian formalizm for the general multi-valued case compare with single-valued case established in Ref. [**Zak68**]. Plan of the paper is the following. In Section 4.1.1 we introduce the conformal variables as the particular case of the general parametrization (4.1). In Section 4.1.2 we introduce the Hamiltonian formalism for system (4.1),(4.2),(4.4)-(4.9) and (4.12) with the nonlocal non-canonical symplectic form and the corresponding Poisson bracket. Section 4.2 provides the explicit expression for the Hamiltonian equations resolved with respect to time derivatives. Section 4.2.1 rewrites these dynamic equations in the complex form and introduce another complex unknowns R and V. Section 4.2.3 introduce a generalization of the Hamiltonian of Euler equations with free surface to include additional physical effects such as the interaction of dielectric fluids with electric field and two fluid hydrodynamics of superfluid Helium with a free surface. It is shown that these equations allows very powerful reductions which suggests a complete integrability. Section **??** provides a summary of obtained results and discussion on future directions.

4.1.1. Conformal mapping. To choose a convenient version of the general parametrization (4.1), we consider the time-dependent conformal mapping

$$z(w,t) = x(u,v,t) + \mathrm{i}y(u,v,t) \tag{4.36}$$

of the lower complex half-plane $\mathbb{C}^-$ of the auxiliary complex variable

$$w \equiv u + \mathrm{i}v, \quad -\infty < u < \infty, \tag{4.37}$$

into the area in (x, y) plane occupied by the fluid. Here the real line $v = 0$ is mapped into the fluid free surface (see Fig. 1) and $\mathbb{C}^-$ is defined by the condition $-\infty < v < 0$. The function $z(w, t)$ is the analytic function of $w \in \mathbb{C}^-$. The conformal mapping (4.36) at $v = 0$ provides a particular form of the free surface parameterization (4.1) for the parameter u.

The conformal mapping (4.36) ensures that the function $\Pi(z, t)$ (4.28) transforms into $\Pi(w, t)$ which is analytic function of w for $w \in \mathbb{C}^-$ (in the bulk of fluid). Here and below we abuse the notation and use the same symbols for functions of either w or z (in other words, we assume that e.g. $\tilde{\Pi}(w, t) = \Pi(z(w, t), t)$ and remove $\tilde{}$ sign). The conformal transformation (4.36) also ensures Cauchy-Riemann equations $\Theta_u = -\Phi_v, \quad \Theta_v = \Phi_u$ in w plane.

The idea of using time-dependent conformal transformation like (4.36) to address systems equivalent/similar to Eqs. (4.1),(4.2),(4.4)-(4.9) and (4.12) was exploited by several authors including [**Ovs73, MOI81, Tan91, Tan93, DKSZ96b, CS98, CS05, Cha16, ZDV02**]. We follow [**DKSZ96b**] to recast the system (4.1),(4.2),(4.4)-(4.9) and (4.12) into the equivalent form for $x(u, t)$, $y(u, t)$ and $\psi(u, t)$ at the real line $w = u$ of the complex plane w using the conformal transformation (4.36). We show that the kinematical BC takes the form

$$y_t x_u - x_t y_u = -\hat{\mathcal{H}}\psi_u, \tag{4.38}$$

where

$$\hat{\mathcal{H}} f(u) = \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{+\infty} \frac{f(u')}{u' - u}\mathrm{d}u' \tag{4.39}$$

is the Hilbert transform ([**Hil05**]) with p.v. denoting a Cauchy principal value of integral. The dynamic BC takes the form

$$\psi_t y_u - \psi_u y_t + gy y_u = -\hat{\mathcal{H}}\left(\psi_t x_u - \psi_u x_t + gy x_u\right) - \alpha\frac{\partial}{\partial u}\frac{x_u}{|z_u|} + \alpha\hat{\mathcal{H}}\frac{\partial}{\partial u}\frac{y_u}{|z_u|}, \tag{4.40}$$

where $x(u, t)$ is expressed through $y(u, t)$ as follows

$$\tilde{x} \equiv x - u = -\hat{\mathcal{H}} y \tag{4.41}$$

(see Eq. (4.153) of Appendix 4.2.4 for the justification of Eq. (4.41) as well as the complimentary expression $\hat{\mathcal{H}}\tilde{x} = y$).

Eq. (4.41) exemplifies the general relation between the harmonically conjugated functions in $\mathbb{C}^-$ as was first obtained by David Hilbert ([**Hil05**]). The particular case of Eq. (4.41) results from the analyticity of $z(w, t)$ for $w \in \mathbb{C}^-$ which implies that $\tilde{x}$ and y are harmonically conjugated functions for $w \in \mathbb{C}^-$. Similarly, $\Pi(w, t)$ (4.28) is also analytic function for $w \in \mathbb{C}^-$ which results in

$$\Theta|_{w=u} = \hat{\mathcal{H}}\psi, \quad \psi = -\hat{\mathcal{H}}\Theta|_{w=u} \text{ for } w = u. \tag{4.42}$$

We notice that left hand side (l.h.s.) of Eq. (4.38) is the same as l.h.s of Eq. (4.10) multiplied by $(x_u^2 + y_u^2)^{1/2} = |z_u|$. R.h.s. of Eq. (4.10) multiplied by $(x_u^2 + y_u^2)^{1/2}$ is given by $\Phi_v|_{v=0} = -\Theta_u|_{v=0}$ (which is the normal velocity v_n to the surface in w plane multiplied by the Jacobian $x_u^2 + y_u^2$ of the conformal transformation (4.36), see e.g. Refs. [**DKSZ96b, DLK16**]). Then using Eqs. (4.15) and (4.42), we obtain Eq. (4.38).

Eq. (4.40) can be also obtained from Eqs. (4.4)-(4.9),(4.12),(4.13) and (4.15) by the change of variables (4.36). We do not provide it here to avoid somewhat bulky calculations. Instead, we derive both Eqs. (4.38) and (4.40) from Hamiltonian formalism in Section 4.1.2. See also Appendix A.2 of Ref. [**DLK16**] for detailed derivation of similar Eqs. for a case of the periodic BCs along x instead of decaying BCs (4.2) and (4.5).

We now transform the kinetic energy (4.18) into the integral over the real line $w = u$. The Laplace Eq. (4.4) implies that we can apply the Green's formula to Eq. (4.18) as $K = \frac{1}{2}\int_\Omega \nabla\cdot(\Phi\nabla\Phi)\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_{\partial\Omega}\psi v_n \mathrm{d}s = \frac{1}{2}\int_{\partial\Omega}\psi v_n\sqrt{x_u^2+y_u^2}\mathrm{d}u$. Using Eqs. (4.9), (4.15), (4.42) and rewriting v_n in conformal variable w (see e.g. Appendix A.1 of Ref. [**DLK16**] for the explicit expressions on the respective derivatives) one obtains that [**DKSZ96b**]

$$K = -\frac{1}{2}\int\limits_{-\infty}^{\infty}\psi\hat{\mathcal{H}}\psi_u\mathrm{d}u. \tag{4.43}$$

4.1.2. Hamiltonian formalism. Conformal mapping makes possible an extension of the Hamiltonian formalism of Eqs. (4.23) for single-valued function η of x into a general multi-valued case, i.e. to the parametrization (4.1). For that we notice that the Hamiltonian Eqs. (4.23) can be obtained from the minimization of the action functional

$$S = \int L\mathrm{d}t \tag{4.44}$$

with the Lagrangian

$$L = \int\limits_{-\infty}^{\infty}\psi\eta_t\mathrm{d}x - H. \tag{4.45}$$

We now generalize the Lagrangian (4.45) into multi-valued η through the parametrization (4.1) as

$$L = \int\limits_{-\infty}^{\infty}\psi(y_t x_u - x_t y_u)\mathrm{d}u - H \tag{4.46}$$

with the Hamiltonian

$$H = -\frac{1}{2}\int\limits_{-\infty}^{\infty}\psi\hat{\mathcal{H}}\psi_u\mathrm{d}u + \frac{g}{2}\int\limits_{-\infty}^{\infty}y^2\,x_u\mathrm{d}u + \alpha\int\limits_{-\infty}^{\infty}\left(\sqrt{x_u^2+y_u^2} - x_u\right)\mathrm{d}u. \tag{4.47}$$

as follows from Eqs. (4.17), (4.20) and (4.43). Here we used the change of variables in $\eta_t\mathrm{d}x$ of Eq. (4.45) from (x,t) into (u,t) which results in $\eta_t dx = (y_t x_u - x_t y_u)\mathrm{d}u$ (see also Appendix A.2 of Ref. [**DLK16**] for more details).

Using Eq. (4.41) to explicitly express $x(u,t)$ as the functional of $y(u,t)$, one can rewrite the Hamiltonian H (4.47) as follows

$$H = -\frac{1}{2}\int\limits_{-\infty}^{\infty}\psi\hat{\mathcal{H}}\psi_u\mathrm{d}u + \frac{g}{2}\int\limits_{-\infty}^{\infty}y^2\,(1-\hat{\mathcal{H}}y_u)\mathrm{d}u + \alpha\int\limits_{-\infty}^{\infty}\left(\sqrt{(1-\hat{\mathcal{H}}y_u)^2+y_u^2} - 1 + \hat{\mathcal{H}}y_u\right)\mathrm{d}u. \tag{4.48}$$

We can use either Eq. (4.47) or (4.48) at our convenience for finding the dynamic equations.

Vanishing of a variation $\delta S = 0$ of Eq. (4.44) over ψ together with Eq. (4.46) results in

$$y_t x_u - x_t y_u = -\hat{\mathcal{H}}\psi_u = \frac{\delta H}{\delta \psi} \tag{4.49}$$

which gives kinematic BC (4.38).

Variations over x and y must satisfy the condition (4.41). To ensure that condition we introduce the modification $\tilde{L}$ of the Lagrangian (4.46) and the modified action $\tilde{S}$ by adding the term with the Lagrange multiplier $f(u,t)$ as

$$\tilde{L} = L + \int\limits_{-\infty}^{\infty} f[y - \hat{\mathcal{H}}(x-u)]\mathrm{d}u, \quad \tilde{S} = \int \tilde{L}\mathrm{d}t, \tag{4.50}$$

which does not change Eq. (4.49).

To ensure the most compact derivation of the dynamical equations from the variation of $\tilde{S}$, we use the Hamiltonian (4.48) (which does not contain x) while we keep x (not expressing it as a functional of y) in the remaining terms of the modified action $\tilde{S}$ beyond H. Then a vanishing of a variation $\delta\tilde{S} = 0$ over x and y, together with (4.46), (4.47) and (4.50), result in Eqs.

$$y_u\psi_t - y_t\psi_u + \hat{\mathcal{H}}f = 0 \tag{4.51}$$

and

$$-x_u\psi_t + x_t\psi_u + f = \frac{\delta H}{\delta y} = gyx_u - g\hat{\mathcal{H}}(yy_u) - \alpha\hat{\mathcal{H}}\frac{\partial}{\partial u}\frac{x_u}{|z_u|} - \alpha\frac{\partial}{\partial u}\frac{y_u}{|z_u|}, \tag{4.52}$$

respectively. Here we used that

$$\frac{\delta F(x-u)}{\delta y} = \hat{\mathcal{H}}\frac{\delta F(x-u)}{\delta x} \tag{4.53}$$

for any functional F of $x(u) - u = -\hat{\mathcal{H}}y$.

Excluding the Lagrange multiplier f from Eqs. (4.51) and (4.52) by applying $\hat{\mathcal{H}}$ to Eq. (4.52) and subtracting the result from Eq. (4.51) we recover Eq. (4.40).

We note that there are two alternatives to using Eqs. (4.51) and (4.52). First one is to keep x in the Hamiltonian H (4.17), (4.20), (4.43) (instead of replacing it by $u - \hat{\mathcal{H}}(x-u)$ as was done in Eq. (4.48)). Then vanishing variations of $\tilde{S}$ (4.50) over x or y results in modification of Eqs. (4.51) and (4.52). Excluding f from these modified Eqs. still results in Eq. (4.40) as was obtained in Ref. [**DKSZ96b**]. Second alternative is to replace x by $u - \hat{\mathcal{H}}(x-u)$ in Eqs. (4.44), (4.46) and use the Hamiltonian (4.48). Then a vanishing variation of S (4.44) over y results in

$$\hat{\mathcal{H}}(\psi_t y_u - \psi_u y_t) - \psi_t x_u + \psi_u x_t = \frac{\delta H}{\delta y} = gyx_u - g\hat{\mathcal{H}}(yy_u) - \alpha\hat{\mathcal{H}}\frac{\partial}{\partial u}\frac{x_u}{|z_u|} - \alpha\frac{\partial}{\partial u}\frac{y_u}{|z_u|}. \tag{4.54}$$

Applying $-\hat{\mathcal{H}}$ to Eq. (4.54) we again recover Eq. (4.40). A variational derivative of the Hamiltonian over ψ in all cases is given by Eq. (4.49).

The second alternative allows to obtain Eq. (4.40) without the use of the Lagrange multiplier f. Below we use Eqs. (4.51) and (4.52) because they allow to significantly simplify subsequent transformations.

Applying $-\hat{\mathcal{H}}$ to Eq. (4.51) and adding it to Eq. (4.52) recovers Eq. (4.54). We use Eqs. (4.49) and (4.54) to rewrite Eqs. (4.38) and (4.40) in the "symplectic" Hamiltonian form [**ZD12**]

$$\hat{\Omega}\mathbf{Q}_t = \frac{\delta H}{\delta \mathbf{Q}}, \qquad \mathbf{Q} \equiv \begin{pmatrix} y \\ \psi \end{pmatrix}, \tag{4.55}$$

where the symplectic operator $\hat{\Omega}$ is given by

$$\hat{\Omega} = \begin{pmatrix} \hat{\Omega}_{11} & \hat{\Omega}_{12} \\ \hat{\Omega}_{21} & 0 \end{pmatrix}, \tag{4.56}$$

which is 2×2 skew-symmetric matrix operator with

$$\hat{\Omega}_{21}^{\dagger} = -\hat{\Omega}_{12}, \tag{4.57}$$

Here $\hat{\Omega}_{21}^{\dagger}$is the adjoint operator, $\langle f, \hat{\Omega}_{ij} g\rangle \equiv \langle \hat{\Omega}_{ij}^{\dagger} f, g\rangle$, $i,j = 1,2$, with respect to the scalar product $\langle f, g\rangle = \int_{-\infty}^{\infty} f(u)g(u)du$. Also $\hat{\Omega}_{11}$ is the skew-symmetric operator

$$\hat{\Omega}_{11}^{\dagger} = -\hat{\Omega}_{11}. \tag{4.58}$$

Eqs. (4.55) and (4.56) expressed in components are given by

$$\begin{aligned} \hat{\Omega}_{11} y_t + \hat{\Omega}_{12}\psi_t &= \frac{\delta H}{\delta y}, \\ -\hat{\Omega}_{12}^{\dagger} y_t &= \frac{\delta H}{\delta \psi}. \end{aligned} \tag{4.59}$$

Using Eqs. (4.49) and (4.54) we obtain that

$$\hat{\Omega}_{21} q = x_u q + y_u \hat{\mathcal{H}} q = (1 - \hat{\mathcal{H}} y_u) q + y_u \hat{\mathcal{H}} q \tag{4.60}$$

for any function $q = q(u)$. Using Eqs. (4.41) and (4.54) we obtain that

$$\hat{\Omega}_{11} q = -\hat{\mathcal{H}}(\psi_u q) - \psi_u \hat{\mathcal{H}} q, \quad \Omega_{12} q = -x_u q + \hat{\mathcal{H}}(y_u q) = -(1 - \hat{\mathcal{H}} y_u) q + \hat{\mathcal{H}}(y_u q). \tag{4.61}$$

Using integration by parts and definition (4.39) in Eqs. (4.60), (4.61) ensures a validity of Eqs. (4.57) and (4.58). We note that Eqs. (4.55)-(4.61) are valid for any Hamiltonian, not only for the Hamiltonian (4.48) provided we derive them from the variation of action (4.50). Because Eqs. (4.49) and (4.54) are obtained directly from the variation principle, the symplectic form, corresponding to the symplectic operator $\hat{\Omega}$ (4.56), is closed and nondegenerate (see Ref. [**Arn89**]).

Eqs. (4.49) and (4.54) are not resolved with respect to the time derivatives y_t and ψ_t. It is remarkable that the symplectic operator $\hat{\Omega}$ (4.56) can be explicitly inverted. We first find the explicit expression for y_t using Eq. (4.49) rewritten in the complex form

$$z_t \bar{z}_u - \bar{z}_t z_u = -2\mathrm{i}\hat{\mathcal{H}}\psi_u = 2\mathrm{i}\frac{\delta H}{\delta \psi}, \tag{4.62}$$

where $\bar{f}(w)$ means a complex conjugate of a function $f(w)$. Note that the complex conjugation $\bar{f}(w)$ of $f(w)$ in this paper is understood as applied with the assumption that $f(w)$ is the complex-valued function of the real argument w even if w takes the complex values so that

$$\bar{f}(w) \equiv \overline{f(\bar{w})}. \tag{4.63}$$

That definition ensures the analytical continuation of $f(w)$ from the real axis $w = u$ into the complex plane of $w \in \mathbb{C}$.

We use the Jacobian

$$J = x_u^2 + y_u^2 = z_u \bar{z}_u = |z_u|^2 \tag{4.64}$$

which is nonzero for $w \in \mathbb{C}^-$ because $z = z(w,t)$ is the conformal mapping there. Dividing Eq. (4.62) by J we obtain that

$$\frac{z_t}{z_u} - \frac{\bar{z}_t}{\bar{z}_u} = -\frac{2\mathrm{i}}{J}\hat{\mathcal{H}}\psi_u = \frac{2\mathrm{i}}{J}\frac{\delta H}{\delta \psi}. \tag{4.65}$$

Here $\frac{z_t}{z_u}$ is analytic in $\mathbb{C}^-$ and $\frac{\bar{z}_t}{\bar{z}_u}$ is analytic in $\mathbb{C}^+$.

It is convenient to introduce the operators

$$\hat{P}^- = \frac{1}{2}(1 + \mathrm{i}\hat{\mathcal{H}}) \quad \text{and} \quad \hat{P}^+ = \frac{1}{2}(1 - \mathrm{i}\hat{\mathcal{H}}) \tag{4.66}$$

which are the projector operators of a function $q(u)$ defined at the real line $w = u$ into functions $q^+(u)$ and $q^-(u)$ analytic in $w \in \mathbb{C}^-$ and $w \in \mathbb{C}^+$, respectively, such that

$$q = q^+ + q^-. \tag{4.67}$$

Here we assume that $q(u) \to 0$ for $u \to \pm\infty$. Eqs. (4.66) imply that

$$\hat{P}^+(q^+ + q^-) = q^+ \quad \text{and} \quad \hat{P}^-(q^+ + q^-) = q^-, \tag{4.68}$$

see more discussion of the operators (4.66) in Appendix 4.2.4. Also notice that Eqs. (4.66) result in the identities

$$\hat{\mathcal{H}}q = \mathrm{i}[q^+ - q^-] \tag{4.69}$$

and

$$\hat{P}^+ + \hat{P}^- = 1,\ (\hat{P}^+)^2 = \hat{P}^+,\ (\hat{P}^-)^2 = \hat{P}^-,\ \hat{P}^+\hat{P}^- = \hat{P}^-\hat{P}^+ = 0. \tag{4.70}$$

Applying $\hat{P}^-$ to Eq. (4.65) and multiplying by z_u after that we find that

$$z_t = -z_u\hat{P}^-\left[\frac{2\mathrm{i}}{J}\hat{\mathcal{H}}\psi_u\right] = z_u\hat{P}^-\left[\frac{2\mathrm{i}}{J}\frac{\delta H}{\delta \psi}\right] \tag{4.71}$$

which is explicit solution for time derivative in complex form. Taking the real and imaginary parts we obtain that

$$y_t = (y_u\hat{\mathcal{H}} - x_u)\left[\frac{1}{J}\hat{\mathcal{H}}\psi_u\right] = -(y_u\hat{\mathcal{H}} - x_u)\left[\frac{2}{J}\frac{\delta H}{\delta \psi}\right] \tag{4.72}$$

and

$$x_t = (x_u\hat{\mathcal{H}} + y_u)\left[\frac{1}{J}\hat{\mathcal{H}}\psi_u\right] = -(x_u\hat{\mathcal{H}} + y_u)\left[\frac{2}{J}\frac{\delta H}{\delta \psi}\right]. \tag{4.73}$$

We now multiply Eq. (4.51) by x_u and add to Eq. (4.52) multiplied by y_u to exclude ψ_t which results in

$$\psi_u(y_t x_u - y_u x_t) + y_u\frac{\delta H}{\delta y} = x_u\hat{\mathcal{H}}f + y_u f = -\mathrm{i}z_u\hat{P}^- f + \mathrm{i}\bar{z}_u\hat{P}^+ f. \tag{4.74}$$

We use Eq. (4.49) in l.h.s. of Eq. (4.74) to exclude time derivative and apply P^- to it to obtain Eq.

$$\hat{P}^- f = \frac{\mathrm{i}}{z_u}\hat{P}^-\left[y_u\frac{\delta H}{\delta y} - \psi_u\hat{\mathcal{H}}\psi_u\right] = \frac{\mathrm{i}}{z_u}\hat{P}^-\left[y_u\frac{\delta H}{\delta y} + \psi_u\frac{\delta H}{\delta \psi}\right], \tag{4.75}$$

which does not contain any time derivative. Taking a sum of Eq. (4.51) multiplied by i and Eq. (4.52) result in

$$\psi_t \bar{z}_u - \bar{z}_t \psi_u - 2\hat{P}^- f = -\frac{\delta H}{\delta y}. \tag{4.76}$$

Excluding $\hat{P}^- f$ and $\bar{z}_t$ in Eq. (4.76) through Eqs. (4.75) and (4.71) we obtain

$$\psi_t = -\psi_u \hat{P}^+ \left[\frac{2\mathrm{i}}{J}\frac{\delta H}{\delta \psi}\right] + \frac{2\mathrm{i}}{J}\hat{P}^- \left[y_u \frac{\delta H}{\delta y} + \psi_u \frac{\delta H}{\delta \psi}\right] - \frac{1}{\bar{z}_u}\frac{\delta H}{\delta y}. \tag{4.77}$$

Using Eq. (4.68) we transform Eq. (4.77) into

$$\psi_t = -\psi_u \hat{\mathcal{H}} \left[\frac{1}{J}\frac{\delta H}{\delta \psi}\right] - \frac{1}{J}\hat{\mathcal{H}} \left[\psi_u \frac{\delta H}{\delta \psi}\right] - \frac{x_u}{J}\frac{\delta H}{\delta y} - \frac{1}{J}\hat{\mathcal{H}} \left[y_u \frac{\delta H}{\delta y}\right]. \tag{4.78}$$

Eqs. (4.72) and (4.78) can be written in the general Hamiltonian form

$$\mathbf{Q}_t = \hat{\mathcal{R}}\frac{\delta H}{\delta \mathbf{Q}}, \quad \mathbf{Q} \equiv \begin{pmatrix} y \\ \psi \end{pmatrix}, \tag{4.79}$$

where

$$\hat{\mathcal{R}} = \hat{\Omega}^{-1} = \begin{pmatrix} 0 & \hat{\mathcal{R}}_{12} \\ \hat{\mathcal{R}}_{21} & \hat{\mathcal{R}}_{22} \end{pmatrix} \tag{4.80}$$

is 2×2 skew-symmetric matrix operator with the components

$$\begin{aligned}
&\hat{\mathcal{R}}_{11} q = 0, \\
&\hat{\mathcal{R}}_{12} q = \frac{x_u}{J} q - y_u \hat{\mathcal{H}}\left(\frac{q}{J}\right), \\
&\hat{\mathcal{R}}_{21} q = -\frac{x_u}{J} q - \frac{1}{J}\hat{\mathcal{H}}\left(y_u q\right), \quad \hat{\mathcal{R}}_{21}^\dagger = -\hat{\mathcal{R}}_{12}, \\
&\hat{\mathcal{R}}_{22} q = -\psi_u \hat{\mathcal{H}}\left(\frac{q}{J}\right) - \frac{1}{J}\hat{\mathcal{H}}\left(\psi_u q\right), \quad \hat{\mathcal{R}}_{11}^\dagger = -\hat{\mathcal{R}}_{11}.
\end{aligned} \tag{4.81}$$

We call $\hat{\mathcal{R}} = \hat{\Omega}^{-1}$ by the "implectic" operator (sometimes such type of inverse of the symplectic operator is also called by the co-symplectic operator, see e.g. Ref. [**Wei83, Mor98**]).

Writing Eq. (4.79) in components we also obtain that

$$\begin{aligned}
y_t &= \hat{\mathcal{R}}_{12}\frac{\delta H}{\delta \psi}, \\
\psi_t &= \hat{\mathcal{R}}_{21}\frac{\delta H}{\delta y} + \hat{\mathcal{R}}_{22}\frac{\delta H}{\delta \psi}.
\end{aligned} \tag{4.82}$$

Comparing Eqs. (4.55) and (4.79) we conclude that $\hat{\mathcal{R}} = \hat{\Omega}^{-1}$ which can be confirmed by the direct calculation that

$$\hat{\mathcal{R}}\hat{\Omega} = \hat{\Omega}\hat{\mathcal{R}} = I, \tag{4.83}$$

where I is the identity operator.

We use Eqs. (4.79) and (4.80) to define the Poisson bracket

$$\{F, G\} = \sum_{i,j=1}^{2} \int_{-\infty}^{\infty} \mathrm{d}u \left(\frac{\delta F}{\delta Q_i}\hat{\mathcal{R}}_{ij}\frac{\delta G}{\delta Q_j}\right) = \int_{-\infty}^{\infty} \mathrm{d}u \left(\frac{\delta F}{\delta y}\hat{\mathcal{R}}_{12}\frac{\delta G}{\delta \psi} + \frac{\delta F}{\delta \psi}\hat{\mathcal{R}}_{21}\frac{\delta G}{\delta y} + \frac{\delta F}{\delta \psi}\hat{\mathcal{R}}_{22}\frac{\delta G}{\delta \psi}\right) \tag{4.84}$$

between arbitrary functionals F and G of $\mathbf{Q}$. It is clear from that definition that any functionals ξ and η of y only commute to each other, i.e. $\{\xi,\eta\}=0$.

Eq. (4.84) allows to rewrite Eqs. (4.79) and (4.80) in the non-canonical Hamiltonian form corresponding to Poisson mechanics as follows

$$\mathbf{Q}_t=\{\mathbf{Q},H\}. \tag{4.85}$$

The Poisson bracket requires to satisfy a Jacobi identity

$$\{F,\{G,L\}\}+\{G,\{L,F\}\}+\{L,\{F,G\}\}=0 \tag{4.86}$$

for arbitrary functionals F, G and L of $\mathbf{Q}$. The Jacobi identity is ensured by our use of the variational principle for the action (4.50).

A functional F is the constant of motion of Eq. (4.85) provided $\{F,H\}=0$. It follows from Eq. (4.84) that any functionals F and G, which depend only on y, commute with each other, i.e. $\{F,G\}=0$. We note that the derivation of Eqs. (4.79)-(4.85) is valid for any Hamiltonian, not only for the Hamiltonian (4.48), because we derive these equations starting from the variation of action (4.50). It implies that Eq. (4.84) has no Casimir invariant (the constant of motion which does not depend on the particular choice of the Hamiltonian H, see e.g. Refs. [**Wei83, ZK97**]). Beyond our standard Hamiltonian (4.48), one can also apply Eqs. (4.84), (4.85) to more general cases as discussed in Section 4.2.3.

4.2. Dynamic equations for the Hamiltonian (4.48)

Eq. (4.72) provides the kinematic BC solved to y_t. Eq. (4.78) with the Hamiltonian (4.48) can be simplified as follows. We first notice that using Eq. (4.61), the gravity part of the variational derivative (4.54) can be represented as follows

$$\left.\frac{\delta H}{\delta y}\right|_{\alpha=0}=gyx_u-g\hat{\mathcal{H}}(yy_u)=-g\hat{\Omega}_{12}y. \tag{4.87}$$

Then the contribution of that gravity part into r.h.s. of Eq. (4.61) is given by

$$\hat{\mathcal{R}}_{21}\left.\frac{\delta H}{\delta y}\right|_{\alpha=0}=-g\hat{\mathcal{R}}_{21}\hat{\Omega}_{12}y=-gy, \tag{4.88}$$

where we use the definition (4.81) and Eq. (4.83).

Second step is to simplify the surface tension part

$$\left.\frac{\delta H}{\delta y}\right|_{g=0}=-\alpha\hat{\mathcal{H}}\frac{\partial}{\partial u}\frac{x_u}{|z_u|}-\alpha\frac{\partial}{\partial u}\frac{y_u}{|z_u|} \tag{4.89}$$

of the variational derivative (4.54). We also notice the identity

$$x_u\frac{\partial}{\partial u}\frac{x_u}{|z_u|}+y_u\frac{\partial}{\partial u}\frac{y_u}{|z_u|}=\frac{1}{2|z_u|}\frac{\partial}{\partial u}(x_u^2+y_u^2)+(x_u^2+y_u^2)\frac{\partial}{\partial u}\frac{1}{|z_u|}=0 \tag{4.90}$$

which is the particular case of the identity

$$\frac{\delta F}{\delta x}x_u+\frac{\delta F}{\delta y}y_u\equiv 0$$

for general parametrization invariant functionals $F((x(u),y(u))$, see e.g. Refs. [**Mor05, FMS18**]. Eq. (4.90) corresponds to $F=\int_{-\infty}^{\infty}(|z_u|-x_u)\mathrm{d}u$ which is the parametrization invariant functional because it represents the arclength of the surface (minus the arclength of unperturbed surface) and thus is independent on the particular surface parametrization $(x(u), y(u))$, see also Eq. (4.19) and discussion after it.

The contribution of the surface tension part into r.h.s. of Eq. (4.61) is given by

$$\hat{\mathcal{R}}_{21}\left.\frac{\delta H}{\delta y}\right|_{g=0} = \hat{\mathcal{R}}_{21}\left[-\alpha\hat{\mathcal{H}}\frac{\partial}{\partial u}\frac{x_u}{|z_u|} - \alpha\frac{\partial}{\partial u}\frac{y_u}{|z_u|}\right] = \frac{\alpha}{x_u}\frac{\partial}{\partial u}\frac{y_u}{|z_u|}, \tag{4.91}$$

where we used Eqs. (4.81), (4.83) and expressed $\frac{\partial}{\partial u}\frac{x_u}{|z_u|}$ through the identity (4.90). Eq. (4.91) has a removable singularity at $x_u = 0$. To explicitly remove that singularity we perform the explicit differentiation in r.h.s of this Eq. to obtain that

$$\hat{\mathcal{R}}_{21}\left.\frac{\delta H}{\delta y}\right|_{g=0} = \frac{\alpha(x_u y_{uu} - x_{uu}y_u)}{|z_u|^3}, \tag{4.92}$$

which provides the expression for the pressure jump (4.13). Using Eqs. (4.49),(4.88) and (4.92) we obtain a particular form of Eq. (4.78) for the Hamiltonian (4.48) as follows

$$\psi_t = \psi_u\hat{\mathcal{H}}\left[\frac{1}{|z_u|^2}\hat{\mathcal{H}}\psi_u\right] + \frac{1}{|z_u|^2}\hat{\mathcal{H}}\left[\psi_u\hat{\mathcal{H}}\psi_u\right] - gy + \frac{\alpha(x_u y_{uu} - x_{uu}y_u)}{|z_u|^3}. \tag{4.93}$$

Eqs. (4.41), (4.72) and (4.93) form a closed set of equations defined on the real line $w = u$. That system was first obtained in Ref. [**DKSZ96b**] with the surface tension term in the form (4.91). We notice that the same system can be obtained directly from Eqs. (4.1),(4.2),(4.4)-(4.9),(4.12),(4.13) and the definition of the conformal mapping (4.36) without any use of the variational principle of Section (4.1.2). However, such alternative derivation is significantly more cumbersome.

4.2.1. Dynamic equations in the complex form. Dynamical Eqs. (4.78) are defined on the real line $w = u$ with the analyticity of $z(w,t)$ and $\Pi(w,t)$ in $w \in \mathbb{C}^-$ taken into account through the Hilbert operator $\hat{\mathcal{H}}$. For the analysis of surface hydrodynamics, it is efficient to consider the analytical continuation of $z(w,t)$ and $\Pi(w,t)$ into $w \in \mathbb{C}^+$ with the time-dependent complex singularities of these functions fully determine their properties. The projector operators (4.66) are convenient tools for such analytical continuation with

$$\Pi = \psi + \mathrm{i}\hat{\mathcal{H}}\psi = 2\hat{P}^-\psi \tag{4.94}$$

and

$$z - u = -\hat{\mathcal{H}}y + \mathrm{i}y = 2\mathrm{i}\hat{P}^- y, \tag{4.95}$$

see Appendix 4.2.4 for more details. Analytical continuation of Eqs. (4.94) and (4.95) into complex plane $w \in \mathbb{C}$ amounts to a straightforward replacing u by w in Eq. (4.149) (as well as in Eqs. (4.156) and (4.157), see also Appendix 4.2.4) which is always allowed provided $w \in \mathbb{C}^+$ and $w \in \mathbb{C}^-$ for $\hat{P}^+q(w)$ and $\hat{P}^-q(w)$, respectively. This is possible because the pole singularity at $u' = u \pm \mathrm{i}0$ in the integrand of Eq. (4.149) does not cross the integration contour $-\infty < u' < \infty$ as w continuously changes from $w = u$ into the complex values. Analytical continuation in the opposite direction (i.e. into $w \in \mathbb{C}^+$ for $\hat{P}^-q(w)$ and $w \in \mathbb{C}^-$ for $\hat{P}^+q(w)$) however requires to move/deform the integration contour $-\infty < u' < \infty$ which is possible only so long as complex singularities are not reached. We also remind our definition (4.63) of complex conjugation which ensures how to define $\bar{f}(w)$ for $w \in \mathbb{C}$. Another convenient way of analytical continuation from the real line $w = u$

into $\mathbb{C}$ is to use Eqs. (4.155)-(4.157). However, such continuation into $w \in \mathbb{C}^+$ for $\hat{P}^- q(w)$ and $w \in \mathbb{C}^-$ for $\hat{P}^+ q(w))$ is limited by the convergence of integrals in Eqs. (4.156) and (4.157) which implies that $|Im(w)|$ cannot exceed the distance of a singularity closest to the real axis. We also note that if the function $q(w)$ is analytic in $\mathbb{C}^-$ then $\bar{q}(w)$ is analytic in $\mathbb{C}^+$ and vise versa.

We replace variations over y and ψ of Section 4.1.2 by variation over z, $\bar{z}$, Π and $\bar{\Pi}$ according to

$$\frac{\delta}{\delta y} = 2\mathrm{i}\hat{P}^+ \frac{\delta}{\delta z} - 2\mathrm{i}\hat{P}^- \frac{\delta}{\delta \bar{z}} \quad \text{and} \quad \frac{\delta}{\delta \psi} = 2\hat{P}^+ \frac{\delta}{\delta \Pi} + 2\hat{P}^- \frac{\delta}{\delta \bar{\Pi}} \tag{4.96}$$

as follows from Eqs. (4.94) and (4.95), see also Eq. (4.53). Here we used that

$$x = \frac{z+\bar{z}}{2}, \ y = \frac{z-\bar{z}}{2\mathrm{i}} \text{ and } \psi = \frac{\Pi + \bar{\Pi}}{2}, \tag{4.97}$$

as follows from Eqs. (4.36) and (4.28). In variational derivatives (4.96) we assume that z, $\bar{z}$, Π and $\bar{\Pi}$ are independent variables.

Applying $\hat{P}^-$ to Eqs. (4.71) and (4.77) together with Eqs. (4.96) and (4.97), we obtain the following dynamic equations

$$z_t = \mathrm{i}Uz_u, \tag{4.98}$$

$$\Pi_t = \mathrm{i}U\Pi_u - B - \mathcal{P}, \tag{4.99}$$

where

$$U = 4\hat{P}^- \left\{ \frac{1}{J} \left[\hat{P}^- \frac{\delta H}{\delta \bar{\Pi}} + \hat{P}^+ \frac{\delta H}{\delta \Pi} \right] \right\}, \tag{4.100}$$

is the complex transport velocity,

$$\mathcal{P} = -4\mathrm{i}\hat{P}^- \left\{ \frac{1}{J} \left[\hat{P}^- \left(\bar{z}_u \frac{\delta H}{\delta \bar{z}} \right) - \hat{P}^+ \left(z_u \frac{\delta H}{\delta z} \right) \right] \right\}, \tag{4.101}$$

and

$$B = -4\mathrm{i}\hat{P}^- \left\{ \frac{1}{J} \left[\hat{P}^- \left(\bar{\Pi}_u \frac{\delta H}{\delta \bar{\Pi}} \right) - \hat{P}^+ \left(\Pi_u \frac{\delta H}{\delta \Pi} \right) \right] \right\}. \tag{4.102}$$

Here we used that z and Π and analytic in $\mathbb{C}^-$ while $\bar{z}$ and $\bar{\Pi}$ are analytic in $\mathbb{C}^+$. Taking an imaginary part of Eq. (4.98) and a real part of Eq. (4.99) one can recover Eqs. (4.72) and (4.78).

Eqs. (4.98)-(4.102) are convenient for analytical study. A version of dynamic equations is obtained by the change of variables (suggested in Ref. [**Dya01**])

$$R = \frac{1}{z_u}, \tag{4.103}$$

$$V = \mathrm{i}\frac{\partial \Pi}{\partial z} = \mathrm{i}R\Pi_u. \tag{4.104}$$

Eqs. (4.98) and (4.99) in terms of variables (4.103) and (4.104) take the following form

$$\frac{\partial R}{\partial t} = \mathrm{i}\left(UR_u - RU_u\right), \tag{4.105}$$

$$\frac{\partial V}{\partial t} = \mathrm{i}\left[UV_u - R(B_u + \mathcal{P}_u)\right]. \tag{4.106}$$

These dynamic equations are valid for any Hamiltonian. They are also convenient for numerical simulations to avoid a numerical instability at small spatial scales, see e.g. Ref. [**ZDP06**] and related analysis of weakly nonlinear case in Ref. [**LZ05**]. Note that R and V include only a derivative of the conformal mapping (4.36) and the complex potential Π over w while $z(w,t)$ and $\Pi(w,t)$ are recovered from solution of these Eqs. as $z = \int \frac{1}{R} dw$ and $\Pi = -\mathrm{i} \int \frac{V}{R} dw$. Respectively, these relation can be used to recover the integrals of motion (4.30), (4.33) and (4.35) from R and V.

We now rewrite our standard Hamiltonian (4.48) in terms of variables z, $\bar{z}$, Π and $\bar{\Pi}$ which gives that

(4.107)
$$H = \int_{-\infty}^{\infty} du \left[\frac{\mathrm{i}}{8}(\Pi_u \bar{\Pi} - \Pi \bar{\Pi}_u) - \frac{g}{16}(z-\bar{z})^2(z_u + \bar{z}_u) + \alpha \left(\sqrt{z_u \bar{z}_u} - \frac{z_u + \bar{z}_u}{2} \right) \right].$$

Eqs. (4.100)-(4.102) and (4.107) results in

$$U = \mathrm{i}\hat{P}^- \left\{ \frac{1}{J} \left[\Pi_u - \bar{\Pi}_u \right] \right\} = \hat{P}^-(R\bar{V} + \bar{R}V), \tag{4.108}$$

$$\mathcal{P} = -\mathrm{i}g(z-w) - 2\mathrm{i}\alpha \hat{P}^-(Q_u \bar{Q} - Q\bar{Q}_u), \tag{4.109}$$

and

$$B = \hat{P}^- \left\{ \frac{|\Pi_u|^2}{|z_u|^2} \right\} = \hat{P}^-(|V|^2), \tag{4.110}$$

where

$$Q \equiv \frac{1}{\sqrt{z_u}} = \sqrt{R}. \tag{4.111}$$

Plugging in Eqs. (4.108)-(4.110) into Eqs. (4.105) and (4.106) we obtain

$$\frac{\partial R}{\partial t} = \mathrm{i}\left(U R_u - R U_u \right), \tag{4.112}$$

$$\frac{\partial V}{\partial t} = \mathrm{i}\left[U V_u - R B_u \right] + g(R-1) - 2\alpha R \hat{P}^- \frac{\partial}{\partial u}(Q_u \bar{Q} - Q\bar{Q}_u). \tag{4.113}$$

Other authors have referred to these equations as the “Dyachenko” equations ([**Dya01**]) which serve as a basis for numerical study of free surface hydrodynamics. They can be also immediately rewritten fully in terms of Q and V as follows

(4.114)
$$\frac{\partial Q}{\partial t} = \mathrm{i}\left(U Q_u - \frac{1}{2} Q U_u \right),$$

(4.115)
$$U = \hat{P}(Q^2 \bar{V} + \bar{Q}^2 V),$$

(4.116)
$$\frac{\partial V}{\partial t} = \mathrm{i}\left[U V_u - Q^2 \hat{P}^- \frac{\partial}{\partial u}(|V|^2) \right] + g(Q^2 - 1) - 2\alpha Q^2 \hat{P}^- \frac{\partial}{\partial u}(Q_u \bar{Q} - Q\bar{Q}_u).$$

4.2.2. Dynamic equations in complex form without nonlocal operators. Both Eqs. (4.112),(4.113) and (4.114)-(4.116) involves $\hat{P}^-$ which is the nonlocal operator. Sometimes for analytical study and looking for the explicit solutions one may need to avoid such nonlocal operator. To do that we use Eq. (4.62) with r.h.s. rewritten through Eq. (4.94) which gives

$$z_t\bar{z}_u - \bar{z}_t z_u = \bar{\Pi}_u - \Pi_u \tag{4.117}$$

for the kinematic BC in the complex form.

To satisfy the dynamic BC we use Eq. (4.93), where the term $\hat{\mathcal{H}}\left[\frac{1}{|z_u|^2}\hat{\mathcal{H}}\psi_u\right]$ is expressed through the complex conjugate of Eq. (4.71) and Eqs. (4.66) which results in

$$\hat{\mathcal{H}}\left[\frac{1}{|z_u|^2}\hat{\mathcal{H}}\psi_u\right] = \frac{\bar{z}_t}{\bar{z}_u} - \mathrm{i}\frac{1}{|z_u|^2}\hat{\mathcal{H}}\psi_u. \tag{4.118}$$

Plugging in Eq. (4.118) into Eq. (4.93) and using Eqs. (4.66) we obtain that

$$\psi_t = \psi_u\frac{\bar{z}_t}{\bar{z}_u} - \frac{1}{|z_u|^2}2\mathrm{i}\hat{P}^-\left[\psi_u\hat{\mathcal{H}}\psi_u\right] - gy + \frac{\alpha(x_u y_{uu} - x_{uu}y_u)}{|z_u|^3}. \tag{4.119}$$

We now note that using Eqs. (4.69) and (4.97) allows to write that $\hat{P}^-\left[\psi_u\hat{\mathcal{H}}\psi_u\right] = \frac{\mathrm{i}}{4}\hat{P}^-\left[\bar{\Pi}_u^2 - \Pi_u^2\right] = -\frac{\mathrm{i}}{4}\Pi_u^2$ thus reducing Eq. (4.119) to

$$\psi_t\bar{z}_u - \psi_u\bar{z}_t + \frac{\Pi_u^2}{2z_u} + \frac{g}{2\mathrm{i}}\bar{z}_u(z-\bar{z}) + \frac{\mathrm{i}\alpha\bar{z}_u}{2|z_u|}\left(\frac{z_{uu}}{z_u} - \frac{\bar{z}_{uu}}{\bar{z}_u}\right) = 0, \tag{4.120}$$

where we also expressed gravity and surface tension terms through z and $\bar{z}$ using Eqs. (4.97). Eq. (4.121) for the particular case $g = \alpha = 0$ was first derived in Ref. [**ZD12**] (except there are trivial misprints in Eq. 3.54 of that Ref.). Eq. (4.121) is the complex version of Bernouilli equation. Using Eqs. (4.97) one can also express ψ in Eq. (4.121) through Π and $\bar{\Pi}$ which gives a fully complex form of Bernouilli equation as follows

$$(\Pi_t + \bar{\Pi}_t)\bar{z}_u - (\Pi_u + \bar{\Pi}_u)\bar{z}_t + \frac{\Pi_u^2}{z_u} - \mathrm{i}g\bar{z}_u(z-\bar{z}) + \frac{\mathrm{i}\alpha\bar{z}_u}{|z_u|}\left(\frac{z_{uu}}{z_u} - \frac{\bar{z}_{uu}}{\bar{z}_u}\right) = 0. \tag{4.121}$$

Eqs. (4.117) and (4.121) are the dynamic equations in the complex form. They are not resolved with respect to the time derivative but they do not contain any nonlocal operator.

4.2.3. Generalized hydrodynamics and integrability. We notice that all expressions derived in Section 4.1.2 starting from Eq. (4.55) and in Section 4.2.1 before Eq. (4.107) are valid for arbitrary Hamiltonian H. In this Section we go beyond the standard Hamiltonian (4.107) to apply our Hamiltonian formalism for other physical systems beyond the Euler equations with free surface, gravity and surface tension. We call the corresponding dynamical equations by “generalized hydrodynamics”.

The new Hamiltonian is written as

$$H = H_{Eul} + \tilde{H}, \tag{4.122}$$

where H_{Eul} is the standard Hamiltonian (4.107) and

$$\tilde{H} = \frac{\mathrm{i}\beta}{8}\int(z_u + \bar{z}_u - 2)(z-\bar{z})\mathrm{d}u = \frac{\beta}{2}\int y\hat{\mathcal{H}}y_u\mathrm{d}u \tag{4.123}$$

is the "generalized" part which adds up to the potential energy. Here β is the real constant. Using FT (A.479), one can also rewrite Eq. (4.123) through Parseval's identity as

$$\tilde{H} = -\frac{\beta}{2}\int |k||y_k|^2 dk \tag{4.124}$$

which shows that $\tilde{H}$ is the sign-definite quantity. Here we also used that the Hilbert operator $\hat{\mathcal{H}}$ turns into a multiplication operator under FT as $(\hat{\mathcal{H}}_u f)_k = \mathrm{i}\,\mathrm{sign}\,(k)\, f_k$ which follows from Eqs. (4.66) and Appendix 4.2.4. Thus the additional potential energy $\tilde{H}$ is positive for $\beta < 0$ and negative for $\beta > 0$.

There are several physical interpretation of $\tilde{H}$. First case $\beta > 0$ corresponds e.g. to the dielectric fluid with a charged and ideally conducting free surface in the vertical electric field [**Zub00, Zub02, Zub08a**]. Such situation is realized on the charged free surface of a superfluid Helium [**CC69, Shi70**]. Then Eq. (4.123) is valid provided surface charges fully screen the electric field above the fluid free surface. This limit was first realized experimentally in Ref. [**Ede80**]. Negative sign of $\tilde{H}$ implies instability due to the presence of the electric field. Another application occurs for the quantum Kelvin-Helmholtz instability of counterflow of two components of superfluid Helium [**LZ18**]. Second case $\beta < 0$ corresponds e.g. to the dielectric fluid with a free surface in the horizontal electric field [**ZZ06, ZZ08, ZK14a**] and references therein. Positive sign of $\tilde{H}$ implies a stabilizing effect of the horizontal electric field. Similar effects can occur in magnetic fluids. See [**Zub08a, ZK14a, LZ18**] for more references on physical realizations of the generalized hydrodynamics.

We now consider the dynamics Eqs. (4.105), (4.106) for the Hamiltonian (4.122),(4.123). Then U is still given by Eq. (4.108) according to Eq. (4.100) because $\tilde{H}$ does not depend on Π. Eq. (4.101) results in

$$\mathcal{P} = -\mathrm{i}g(z-w) - 2\mathrm{i}\alpha\hat{P}^-(Q_u\bar{Q} - Q\bar{Q}_u) + \beta\hat{P}^-(R\bar{R}-1), \tag{4.125}$$

while B remain the same as in Eqs. (4.108) and (4.110) because the definitions (4.100) and (4.102) involve only variations over Π and $\bar{\Pi}$.

Eqs. (4.105),(4.106),(4.108),(4.110),(4.111) and (4.125) result in the generalization of Dyachenko Eqs. (4.112),(4.113) as follows

$$\frac{\partial R}{\partial t} = \mathrm{i}\left(UR_u - RU_u\right), \tag{4.126}$$

$$\frac{\partial V}{\partial t} = \mathrm{i}\left[UV_u - RB_u - \beta R\hat{P}^-\frac{\partial}{\partial u}(R\bar{R})\right] + g(R-1) - 2\alpha R\hat{P}^-\frac{\partial}{\partial u}(Q_u\bar{Q} - Q\bar{Q}_u). \tag{4.127}$$

As a particular example until the end of this section we consider Eqs. (4.126) and (4.127) for $g = \alpha = 0$. We define r as

$$r = R - 1 \tag{4.128}$$

and linearizes Eqs. (4.108),(4.110),(4.126) and (4.127) over small amplitude solutions in r and V which gives

$$\begin{aligned} r_t &= -\mathrm{i}V_u, \\ V_t &= -\mathrm{i}\beta r_u, \end{aligned} \tag{4.129}$$

where we used that r does not have zeroth Fourier harmonics implying $\hat{P}^{-}r = r$ and $\hat{P}^{-}\bar{r} = 0$. Excluding V from Eq. (4.129) results in

$$r_{tt} = -\beta r_{uu}. \tag{4.130}$$

If $\beta = -s^2 < 0$, $s > 0$, then Eq. (4.130) turns into a wave equation,

$$r_{tt} = s^2 r_{uu}, \tag{4.131}$$

while for $\beta = s^2 > 0$ we obtain an elliptic equation.

We now go beyond a linearization and consider fully nonlinear Eqs. (4.108),(4.110),(4.126) and (4.127) for $\beta = -s^2$. We assume a reduction

$$V = \mathrm{i}sr. \tag{4.132}$$

Then Eqs. (4.108) and (4.110) result in $B = s^2\hat{P}^{-}(|r|^2)$ and $U = \mathrm{i}sr$. Plugging in these expressions into Eqs. (4.126) and (4.127) results in a single equation

$$r_t = sr_u, \tag{4.133}$$

with a general solution

$$\begin{aligned} r &= f(u + st), \\ v &= \mathrm{i}sf(u + st) \end{aligned} \tag{4.134}$$

for the arbitrary function $f(u)$. This is a remarkable result because it is valid for arbitrary level of nonlinearity. In a similar way, a reduction

$$V = -\mathrm{i}sr \tag{4.135}$$

in Eqs. (4.108),(4.110),(4.126) and (4.127) results in a single equation

$$r_t = -sr_u, \tag{4.136}$$

with a general solution

$$\begin{aligned} r &= g(u - st), \\ v &= -\mathrm{i}sg(u - st) \end{aligned} \tag{4.137}$$

for the arbitrary function $g(u)$.

The existence of the general solutions (4.134) and (4.137) for the reductions (4.132) and (4.135), however, does not imply that one can obtain the explicit solution of the general Eqs. (4.126) and (4.127) because a linear superposition of solutions (4.134) and (4.137) is not generally a solution of Eqs. (4.126) and (4.127).

We now consider the second case $\beta = s^2 > 0$ and look at a reduction

$$V = sr. \tag{4.138}$$

Then Eqs. (4.108) and (4.110) result in $B = s^2\hat{P}^{-}(|r|^2)$ and $U = s[r + 2\hat{P}^{-}(|r|^2)]$. Plugging in these expressions into Eqs. (4.126) and (4.127) results in a single equation (both equations for r_t and V_t coincide)

$$r_t = \mathrm{i}s\left(r_u[-1 + 2\hat{P}^{-}(|r|^2)] - (1 + r)2\hat{P}^{-}(|r|^2)_u\right). \tag{4.139}$$

In a similar way, a reduction

$$V = -sr \tag{4.140}$$

in Eqs. (4.108),(4.110),(4.126) and (4.127) results in a single equation

$$r_t = -\mathrm{i}s\left(r_u[-1 + 2\hat{P}^{-}(|r|^2)] - (1 + r)2\hat{P}^{-}(|r|^2)_u\right). \tag{4.141}$$

Eqs. (4.139) and (4.141) interchange under a change of the sign of the time so it is sufficient to study one of them.

Infinite number of explicit solutions of Eqs. (4.139) and (4.141) can be constructed. We however do that indirectly by first considering the reduction (4.138) for variables z and Π instead of R and V. We use Eq. (4.98) and its complex conjugate $\bar{z}_t = -\mathrm{i}\bar{U}\bar{z}_u$ together with Eqs. (4.108) and (4.103) to obtain that

$$\mathrm{i}\left(\frac{\bar{z}_t}{\bar{z}_u} - \frac{z_t}{z_u}\right) = R\bar{V} + \bar{R}V = \frac{\bar{V}}{z_u} + \frac{V}{\bar{z}_u}. \tag{4.142}$$

Eq. (4.138) and its complex conjugate imply that

$$\begin{aligned} V &= s(R-1) = s\frac{1-z_u}{z_u}, \\ \bar{V} &= s(\bar{R}-1) = s\frac{1-\bar{z}_u}{\bar{z}_u} \end{aligned} \tag{4.143}$$

which allows to exclude V and $\bar{V}$ from Eq. (4.142) resulting in the closed equation for z as

$$\mathrm{i}(\bar{z}_t z_u - z_t \bar{z}_u) = s(2 - z_u - \bar{z}_u). \tag{4.144}$$

A change of variables $z = G - \mathrm{i}st$ in Eq. (4.144) results in the Laplace growth equation (LGE) given by [**Zub00, Zub02, Zub08a**]

$$\mathrm{Im}\left(\bar{G}_t G_u\right) = -s. \tag{4.145}$$

LGE is integrable in a sense of the existence of infinite number of integrals of motion and its relation to the dispersionless limit of the integrable Toda hierarchy [**MWWZ00**].

One can also mention that LGE was derived as the approximation of Hele-Shaw flow (the ideal fluid pushed through a viscous fluid in a narrow gap between two parallel plates), see Refs. [**PK45, Gal45, SB84, How86, BKL$^+$86, MWD94**]. Also Ref. [**Cro00b**] found that exact solutions for free-surface Euler flows with surface tension (such as Crapper's classic capillary water wave solutions [**Cra57**] and solutions of Refs. [**Tan96, Cro00a, Cro99**]) are related to steady solutions of Hele–Shaw flows (with non-zero surface tension).

The reduction (4.140) also results in LGE by the trivial change of sign in Eq. (4.264). Similar to the case $\beta = -s^2 < 0$ above, the existence of infinite number of solutions for the reductions (4.138) and (4.140) in the case $\beta = s^2 > 0$ does not imply that one can obtain the explicit solution of the general Eqs. (4.126) and (4.127) because a linear superposition of solutions of the corresponding LGEs is not generally a solution of Eqs. (4.126) and (4.127). Nevertheless, we make a conjecture that the full system Eqs. (4.108),(4.110),(4.126) and (4.127) is integrable both for $\beta < 0$ and $\beta > 0$.

We derived the non-canonical Hamiltonian system (4.79) which is equivalent to the Euler equation with a free surface for general multi-valued parameterization of surface by the conformal transformation (4.36). This generalizes the canonical Hamiltonian system (4.23) of Ref. [**Zak68**] which is valid only for single-valued surface parameterization. The Hamiltonian coincide with the total energy (kinetic plus potential energy) of the ideal fluid in the gravitational field with the surface tension. A non-canonical Hamiltonian system (4.79) can be written in terms of Poisson mechanics (4.85) with the non-degenerate Poisson bracket (4.84), i.e. it does not have any Casimir invariant. That bracket is identically zero between any two functionals of the canonical transformation (4.36). In future work we plan to focus on finding of integrals of motion which are functional of that conformal map only so they will commute with each other which might be a sign of the complete integrability of the Hamiltonian system (4.79). It was conjectured in Ref. [**DZ94b**] that the system (4.23) is completely integrable at least for the case of the zero surface tension. Since then the arguments *pro* and *contra* were presented, see e.g. Ref. [**DKZ13**]. Thus this question of possible integrability is still open and very important.

We also reformulated the Hamiltonian system (4.79) in the complex form which is convenient to analyze the dynamics in terms of analytical continuation of solutions into the upper complex half-plane. A full knowledge of such singularities would provide a complete description of the free surface hydrodynamics and corresponding Riemann surfaces as was e.g. demonstrated on the particular example of Stokes wave in Ref. [**Lus16**].

Additionally, we analyzed the generalized hydrodynamics with multiple applications ranging from dielectric fluid with free surface in the electric field to the two fluid hydrodynamics of superfluid Helium. In that case we identified powerful reductions which allowed to find general classes of particular solutions. We conjecture that the generalized hydrodynamics might be completely integrable.

Extension of $2D$ results of this paper into $3D$ is beyond the scope of this work. We only note that the Hamiltonian Eqs. (4.23) for single-valued parameterization are valid in $3D$ also [**Zak68**]. Also multi-valued parametrization can be extended into $3D$ provided the variation of waves is slow in the third dimension as shown in Ref. [**Rub05**].

4.2.4. Projectors to functions analytic in upper and lower complex half-planes. This appendix justifies the definitions (4.66) of the projector operators $\hat{P}^{\pm}$ as well as provides a derivation of Eqs. (4.41) and (4.42). The Sokhotskii-Plemelj theorem (see e.g. [**Gak66, PM08**]) results in

$$\int\limits_{-\infty}^{\infty} \frac{q(u')du'}{u'-u+\mathrm{i}0} = \text{p.v.} \int\limits_{-\infty}^{\infty} \frac{q(u')du'}{u'-u} - \mathrm{i}\pi q(u) = \pi\hat{\mathcal{H}}q - \mathrm{i}\pi q(u), \tag{4.146}$$

$$\int\limits_{-\infty}^{\infty} \frac{q(u')du'}{u'-u-\mathrm{i}0} = \text{p.v.} \int\limits_{-\infty}^{\infty} \frac{q(u')du'}{u'-u} + \mathrm{i}\pi q(u) = \pi\hat{\mathcal{H}}q + \mathrm{i}\pi q(u), \tag{4.147}$$

where we used the definition (4.39) and i0 means $\mathrm{i}\epsilon$, $\epsilon \to 0^+$. Here $q(u) \in \mathbb{C}$, $q(u) \to 0$ for $u \to \pm\infty$, as well as we assumed that $q(u)$ is Hölder continuous function, i.e. $|q(u) - q(u')| \le C|u-u'|^{\gamma}$ for any real u, u' and constants $C > 0$, $0 < \gamma \le 1$.

The non-zero limit $q(u) \to q_0 = const$ at $u \to \pm\infty$ can be also considered, where $q_0 \in \mathbb{C}$. To ensure a finite value of $\hat{\mathcal{H}}q$ in (4.147), we assume that a decay condition

$$|q(u) - q_0| \le A|u|^{-\gamma_1} \tag{4.148}$$

holds for $u \to \pm\infty$ with the constant values $\gamma_1 > 0$ and $A > 0$. However, the decaying boundary conditions (1.2) and (1.7) imply that $q_0 = 0$ in our case. The Hölder continuity requirement (see e.g. [**Tit48, Gak66, Pan96**]). E.g., instead of the Hölder continuity one can assume that $q \in L^p$ then $\hat{\mathcal{H}}q \in L^p$ for any $p \in (1, \infty)$ with $\|q\|_{L^p} \equiv \left(\int_{-\infty}^{\infty} |q(u)|^p \mathrm{d}u\right)^{1/p}$. The condition $q \in L^p$ is sufficient for the existence of the inverse of $\hat{\mathcal{H}}$ such that $\hat{\mathcal{H}}^2 q = -q$ almost everywhere. The Hilbert transform can be also considered for bounded almost everywhere functions $q \in L^\infty$ which implies that $\hat{\mathcal{H}}q$ belongs to the bounded mean oscillation (BMO) classes of functions [**Fef71, FS72**]. However, Hölder continuity requirement and the decay condition (4.148) are typically sufficient for our purposes as well as they ensures that $\hat{\mathcal{H}}^2 q = -q$ pointwise. E.g. a singularity of a limiting Stokes wave $\propto u^{2/3}$ [**Sto80**] corresponds to $\gamma = 2/3$. The limiting standing wave is expected to have a singularity with $\gamma = 1/2$ [**PP52, Gra73, Wil11**]. Generally in this paper, $q(u)$ is formed from functions analytic at the real line $w = u$ and their complex conjugates. It implies that typically $\gamma = 1$. Only in exceptional cases, complex singularities reach $w = u$ from $w \in \mathbb{C}$ implying that $\gamma < 1$ as for the limiting Stokes wave and limiting standing wave.

Using Eqs. (4.146) and (4.147), we rewrite Eq. (4.66) as follows

$$\hat{P}^{\pm} q = \frac{1}{2}(1 \mp \mathrm{i}\hat{\mathcal{H}})q = \pm\frac{1}{2\pi \mathrm{i}}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{q(u')du'}{u' - u} + \frac{1}{2}q(u) = \pm\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{q(u')du'}{u' - u \mp \mathrm{i}0}. \tag{4.149}$$

Extending u into the complex plane of w in Eqs. (4.149) either in $\mathbb{C}^+$ or in $\mathbb{C}^-$ (one can also interpret that as closing complex integration contours in $\mathbb{C}^+$ or in $\mathbb{C}^-$) we obtain that

$$q^+ \equiv \hat{P}^+ q \tag{4.150}$$

is analytic in $\mathbb{C}^+$ and

$$q^- \equiv \hat{P}^- q \tag{4.151}$$

is analytic in $\mathbb{C}^-$ such that $q^{\pm}(u) \to 0$ for $u \to \pm\infty$. Using Eqs. (4.149)-(4.151) we obtain that

$$q = q^+ + q^-. \tag{4.152}$$

Eqs. (4.150)-(4.152) justify the definition (4.66) of $\hat{P}^{\pm}$ as the projector operators as well as Eq. (4.67) if we keep in mind that $q_0 = 0$ for all functions of interest because of the decaying boundary conditions (4.2) and (4.7). We note that Eqs. (4.68) can be also immediately obtained by plugging Eq. (4.152) into Eqs. (4.149) and moving integration contour from the real line $u = w$ either upwards into $\mathbb{C}^+$ or downwards into $\mathbb{C}^-$.

Assume that $q(w)$ is the analytic function for $w \in \mathbb{C}^-$, i.e. $q^- \equiv 0$ in Eq. (4.152). Moving the integration contour in Eq. (4.147) from the real line $u = w$ downwards into $\mathbb{C}^-$ implies the zero value of the integral. Then taking the real and imaginary parts of r.h.s. of Eq. (4.147), i.e. setting $\hat{\mathcal{H}}q + \mathrm{i}\pi q(u) = 0$, results in the relations between real and imaginary parts of q at the real line $w = u$ as follows ([**Hil05**])

$$\hat{\mathcal{H}}Re(q) = Im(q), \quad \hat{\mathcal{H}}Im(q) = -Re(q). \tag{4.153}$$

We also notice that Eqs. (4.41) and (4.42) are obtained from Eqs. (4.153) if we set either $q(w,t) = z(w,t) - w$ or $q = \Pi(w,t)$ which ensures that $q(w,t)$ is analytic for $w \in \mathbb{C}^-$.

Another view of the projector operators $\hat{P}^{\pm}$ can be obtained if we use the Fourier transform (FT)

$$q_k \equiv \frac{1}{(2\pi)^{1/2}} \int\limits_{-\infty}^{\infty} q(u) \exp(-\mathrm{i}ku)\, \mathrm{d}u \tag{4.154}$$

and introduce the splitting of $q(u)$ as

$$q(u) = q^+(u) + q^-(u), \tag{4.155}$$

where

$$q^+(w) = \frac{1}{(2\pi)^{1/2}} \int\limits_{0}^{\infty} q_k \exp(\mathrm{i}kw)\, \mathrm{d}k \tag{4.156}$$

is the analytical (holomorphic) function in $\mathbb{C}^+$ and

$$q^-(w) = \frac{1}{(2\pi)^{1/2}} \int\limits_{-\infty}^{0} q_k \exp(\mathrm{i}kw)\, \mathrm{d}k \tag{4.157}$$

is the analytical function in $\mathbb{C}^-$. Here we assume that the inverse FT,

$$\mathcal{F}^{-1}[q_k](u) \equiv \frac{1}{(2\pi)^{1/2}} \int\limits_{-\infty}^{\infty} q_k \exp(\mathrm{i}ku)\, \mathrm{d}k,$$

equals almost everywhere to $q(u)$ for real values of u. This is valid e.g. if $q(u)$ belongs to both L^1 (absolutely integrable) and L^2 (square integrable) classes (see e.g. Ref. [**Rud86**]). If the function $q(w)$ is analytic in $\mathbb{C}^-$ then $\bar{q}(w)$ is analytic in $\mathbb{C}^+$ as also seen from equations (4.155)-(4.157).

(*====================================*)

4.3. Short branch cut approximation and square root singularity solutions

In this section we derive the dynamical equations of the short branch cut approximation and establish their integrability in characteristics in subsection 4.3.1 as well as provide particular solutions in subsection 4.3.2.

4.3.1. Short branch cut approximation. Consider the branch cut γ connecting branch points at $w = a(t) \in \mathbb{C}^+$ and $w = b(t) \in \mathbb{C}^+$. The branch cut is called short one if its distance to the real axis, $\min(|Im(a)|, |Im(b)|)$, is large compared with $|a-b|$. It allows to define a small parameter ϵ as follows

$$\epsilon \equiv |a-b|/\min(|Im(a)|, |Im(b)|) \ll 1. \tag{4.158}$$

We neglect other singularities/branch cuts in R and V by assuming that they either identically zero or give small contribution at the real axis $w = u$. Then we define

$$\begin{aligned} R(w,t) - 1 &= \int\limits_a^b \frac{\tilde{R}(w',t)\mathrm{d}w'}{w-w'}, \\ V(w,t) &= \int\limits_a^b \frac{\tilde{V}(w',t)\mathrm{d}w'}{w-w'}, \end{aligned} \tag{4.159}$$

where $\tilde{R}(w',t)$ and $\tilde{V}(w',t)$ are densities along branch cut such that the jump of R across branch cut at $w = w'$ is $2\pi\mathrm{i}\tilde{R}(w',t)$ and similar the jump for V is $2\pi\mathrm{i}\tilde{V}(w',t)$ as follows from the Sokhotskii-Plemelj theorem (see e.g. [**Gak66, PM08**]). Integration in Eqs. (4.159) is taken over any contour which is a simple arc in $\mathbb{C}^+$ connecting $w = a$ and $w = b$. This contour defines a branch cut. There is a freedom in choice of that branch cut connecting two branch points $w = a$ and $w = b$. We however assume that the arclength of the branch cut is of the same order of magnitude as $|a-b|$, i.e. that arclength is not very much different from the length of the segment of the straight line connecting $w = a$ and $w = b$. Also $\tilde{R}(w',t)$ and $\tilde{V}(w',t)$ are assumed to be the continuous functions of w'. Also $\tilde{R}(w',t)$ and $\tilde{V}(w',t)$ can be zero at some parts of the contour. The functions $\bar{R}$ and $\bar{V}$ are given by

$$\begin{aligned} \bar{R}(w,t) - 1 &= \int\limits_{\bar{a}}^{\bar{b}} \frac{\bar{\tilde{R}}(\bar{w}',t)\mathrm{d}\bar{w}'}{w-\bar{w}'}, \\ \bar{V}(w,t) &= \int\limits_{\bar{a}}^{\bar{b}} \frac{\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}\bar{w}'}{w-\bar{w}'}, \end{aligned} \tag{4.160}$$

with the contour $\bar{\gamma}$ connecting $w = \bar{a}$ and $w = \bar{b}$ being the reflection of the contour of Eq. (4.159) with respect to the real axis $w = Re(w)$.

Functions $U(w,t)$ and $B(w,t)$ can be rewritten as

$$\begin{aligned} U &= R\bar{V} + \bar{R}V - \hat{P}^+(R\bar{V} + \bar{R}V), \\ B &= V\bar{V} - \hat{P}^+(V\bar{V}), \end{aligned} \tag{4.161}$$

where we used the definition (4.66) to represent $\hat{P}^-$ as $\hat{P}^- = 1 - \hat{P}^+$. Because $\hat{P}^+ f$ is analytic for $w \in \mathbb{C}^+$ for any function f, as well as both $\bar{R}$ and $\bar{V}$ are analytic for $w \in \mathbb{C}^+$ according to the definition (4.63), we conclude from Eq. (4.161) that both U and B have a branch cut γ connecting $w = a$ and $w = b$ inherited from branch cut of R and V. Then similar to Eqs. (4.159), we represent $U(w,t)$ and $B(w,t)$

through the integrals of the densities $\tilde{U}(w',t)$ and $\tilde{B}(w',t)$ along the branch cut as

$$U(w,t) = \int_a^b \frac{\tilde{U}(w',t)\mathrm{d}w'}{w-w'},$$
$$B(w,t) = \int_a^b \frac{\tilde{B}(w',t)\mathrm{d}w'}{w-w'}. \tag{4.162}$$

Using Eqs. (4.159) and (4.160), a calculation of the projectors in the definitions (4.242) is performed through the partial fractions as follows

$$\hat{P}^-\left[(R-1)\bar{V}\right] = \hat{P}^- \int_a^b \int_{\bar{a}}^{\bar{b}} \frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'}{(w-w'')(w-\bar{w}')}$$
$$= \hat{P}^- \int_a^b \int_{\bar{a}}^{\bar{b}} \frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'}{w''-\bar{w}'}\left(\frac{1}{w-w''} - \frac{1}{w-\bar{w}'}\right)$$
$$= \int_a^b \int_{\bar{a}}^{\bar{b}} \frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'}{w''-\bar{w}'}\frac{1}{w-w''} = \int_a^b \frac{\tilde{R}(w'',t)\bar{V}(w'',t)\mathrm{d}w''}{w-w''}, \tag{4.163}$$

where at the last line we used the definition (4.160). Similar to Eq. (4.163), one obtains that

$$\hat{P}^-\left[(\bar{R}-1)V\right] = \int_a^b \frac{\tilde{V}(w'',t)[\bar{R}(w'',t)-1]\mathrm{d}w''}{w-w''} \tag{4.164}$$

and

$$B(w,t) = \hat{P}^-\left[V\bar{V}\right] = \int_a^b \frac{\tilde{V}(w'',t)\bar{V}(w'',t)\mathrm{d}w''}{w-w''}. \tag{4.165}$$

Eqs. (4.242),(4.162)-(4.165) result in

$$\tilde{U}(w,t) = \tilde{V}(w,t)\bar{R}(w,t) + \tilde{R}(w,t)\bar{V}(w,t) \tag{4.166}$$

and

$$\tilde{B}(w,t) = \tilde{V}(w,t)\bar{V}(w,t), \tag{4.167}$$

where $w \in \gamma$.

The functions $\bar{R}(w,t)$ and $\bar{V}(w,t)$ are analytic for $w \notin \bar{\gamma}$ including $w \in \mathbb{C}^+$ and they are represented by the convergent Taylor series in the open disk $|w-w_0| < r_d$ with $w_0 \in \gamma$. The radius of convergence r_d is given by distance from w_0 to $\bar{\gamma}$. For the short branch cut $r_d \simeq 2|a| \gg |b-a|$. Without the loss of generality we assume that the center of branch cut is located at the imaginary axis, i.e. $Re(a+b) = 0$ and choose $w_0 \in \gamma$ to be also at the imaginary axis, $Re(w_0) = 0$. E.g. for the simplest choice of branch cut γ to be the segment of straight line connecting $w = a$ and $w = b$, we then obtain that

$$w_0 = (a+b)/2. \tag{4.168}$$

In the short branch cut approximation (4.158) we keep only zeroth order terms in Taylor series for $\bar{R}(w,t)$ and $\bar{V}(w,t)$ and denote

$$R_c(t) \equiv \bar{R}(w_0(t),t) \ \text{ and } V_c(t) \equiv \bar{V}(w_0(t),t). \tag{4.169}$$

Using Eqs. (4.162),(4.166)-(4.169) we then obtain in that approximation that

$$U = RV_c + R_cV - V_c, \quad B = V_cV. \tag{4.170}$$

More accurate approximation for U and B can be obtained from Eqs. (4.162),(4.166)-(4.169) by taking into account more terms in Taylor series of $\bar{R}(w,t)$ and $\bar{V}(w,t)$ at $w = w_0$ beyond Eq. (4.169). For instance, by keeping linear terms,

$$\begin{aligned} \bar{R}(w,t) &\simeq R_c(t) + (w - w_0(t))R'_c, \quad R'_c \equiv \frac{\partial}{\partial w}R(w,t)\bigg|_{w=w_0}, \\ \bar{V}(w,t) &\simeq V_c(t) + (w - w_0(t))V'_c, \quad V'_c \equiv \frac{\partial}{\partial w}V(w,t)\bigg|_{w=w_0}, \end{aligned} \tag{4.171}$$

we obtain a modification of Eq. (4.170) as $U \to U + \Delta U$ and $B \to B + \Delta B$, where

$$\begin{aligned} \Delta U &= -\langle\tilde{R}\rangle V'_c + \langle\tilde{V}\rangle R'_c + (w - w_0)[V(w,t)R'_c + (R(\omega,t) - 1)V'_c]. \\ \Delta B &= -\langle\tilde{V}\rangle V'_c + (w - w_0)V(w,t)V'_c. \end{aligned} \tag{4.172}$$

Here $\langle\tilde{R}\rangle \equiv \int_a^b \tilde{R}(w)\mathrm{d}w$ and $\langle\tilde{V}\rangle \equiv \int_a^b \tilde{V}(w)\mathrm{d}w$. The short branch cut approximation requires that both

$$|\Delta U| \ll |U| \quad \text{and} \quad |\Delta B| \ll |B|. \tag{4.173}$$

Qualitatively it implies that singularities in R and V must not be too strong. E.g., if a singularity in R is stronger than in V, as studied in [**DDLZ19**], then these conditions require that $|Im(a)V'_c\tilde{R}| \ll |R_c\tilde{V}|$. We note that the limit of infinitely short branch cut recovers pole solutions of [**DDLZ19**].

Any approximation of $\bar{R}(w,t)$ and $\bar{V}(w,t)$ in Eqs. (4.166),(4.167) by polynomials in powers of $w - w_0$ turns Dyachenko Eqs. (4.241)-(4.243) into hyperbolic-type PDE with variable coefficients both in t and w. In the simplest case of zeroth order polynomials, Eqs. (4.241)-(4.243), (4.170) and conditions (4.173) result in the dynamical equations of the short branch cut approximation,

$$\begin{aligned} R_t + \mathrm{i}V_cR_u &= \mathrm{i}R_c(VR_u - V_uR), \\ V_t + \mathrm{i}V_cV_u &= \mathrm{i}R_cVV_u + g(R-1), \end{aligned} \tag{4.174}$$

which have variable coefficients $R_c(t)$ and $V_c(t)$ in t only. A more general case of the higher order polynomials, i.e. going beyond the short branch cut approximation implying variable coefficients in w (as exemplified in Eqs. (4.171)), will be considered in the separate paper.

In the complex moving frame,

$$\chi = w - \mathrm{i}\int_0^t V_c(t')\mathrm{d}t', \tag{4.175}$$

we obtain from Eqs. (4.174), that

$$\begin{aligned} R_t &= \mathrm{i}R_c(VR_\chi - V_\chi R), \\ V_t &= \mathrm{i}R_cVV_\chi + g(R-1), \end{aligned} \tag{4.176}$$

where the space derivative is over a new independent variable χ.

We now neglect the term with g in Eq. (4.176), resulting in

$$R_t = \mathrm{i}R_c(VR_\chi - V_\chi R), \tag{4.177}$$

$$V_t = \mathrm{i}R_c V V_\chi, \tag{4.178}$$

which is justified either if $g = 0$ or $|R-1| \ll 1$. This second condition implies that the free surface is initially nearly flat (this approximation applies only for small enough time while the condition $|R-1| \ll 1$ remains valid).

Eq. (4.178) is decoupled from Eq. (4.177) and turns into the complex Hopf equation

$$V_\tau = V V_\chi \tag{4.179}$$

under the transformation to the new complex time

$$\tau(t) = \mathrm{i}\int\limits_0^t R_c(t')\mathrm{d}t' \tag{4.180}$$

and the respectively redefined Eq. (4.175) as

$$\chi = w - \int\limits_0^\tau \frac{V_c(t(\tau'))}{R_c(t(\tau'))}\mathrm{d}\tau'. \tag{4.181}$$

Under the same transformation (4.180) and (4.181), Eq. (4.177) turns into

$$R_\tau = VR_\chi - V_\chi R, \tag{4.182}$$

which is convenient to transform back from R (4.103) to z which gives

$$z_\tau = Vz_\chi + c(\tau). \tag{4.183}$$

Eq. (4.183) ensures that Eq. (4.182) is valid for the arbitrary function $c(\tau)$ of τ. To fix that freedom in the choice of $c(\tau)$, we have, similar to the discussion after Eq. (**??**) in Section **??**, to take into account the decaying BCs (**??**) and (**??**). Using the definitions (4.180) and (4.181), we obtain that a change of independent variables from (χ, τ) to (w, t) in Eq. (4.186) results in

$$\frac{z_t}{\mathrm{i}R_c} + \frac{V_c}{R_c}z_w = Vz_w + c(\tau(t)). \tag{4.184}$$

Taking the limit $w = u$, $u \to \pm\infty$, one obtains from Eq. (4.184) and BCs (**??**), (**??**) that

$$c(\tau) = \frac{V_c}{R_c}, \tag{4.185}$$

Respectively, Eq. (4.183) is reduced to

$$z_\tau = Vz_\chi + \frac{V_c}{R_c}. \tag{4.186}$$

Eqs. (4.179) and (4.186) are easily integrable. Assume that $F(w)$ and $G(w)$ are arbitrary functions analytic for $w \in \mathbb{C}^-$ such that $F(w) \to 0$ as $w \to \infty$ and

$G(w) \to w$ as $w \to \infty$. Then a general solution of system (4.179) and (4.186) is given by

$$V = F(\chi_0), \tag{4.187}$$

$$z = G(\chi_0) + \int_0^\tau c(\tau')\mathrm{d}\tau', \tag{4.188}$$

where the function $\chi_0(\chi, \tau)$ is determined by the solution of the implicit equation

$$\chi = \chi_0 - F(\chi_0)\tau \tag{4.189}$$

and

$$\int_0^\tau c(\tau')\mathrm{d}\tau' = \int_0^\tau \frac{V_c(\tau')}{R_c(\tau')}\mathrm{d}\tau' \tag{4.190}$$

as follows from Eq. (4.185).

Eqs. (4.187) and (4.189) define a parametric representation of a Riemann surface $\Gamma_V(w)$. If $F(\chi_0)$ is the rational function then $\Gamma_V(w)$ has genus zero at the initial time $t = 0$ (see e.g. Ref. [**DFN85**] for definition of genus of surface). For $t > 0$, branch points emerge in $\Gamma_V(w)$ thus making genus nonzero. Branch points on the surface Γ_V are zeros of the derivative $\frac{d\chi}{d\chi_0} = 1 - F'(\chi_0)\tau$. Generally, these zeros are simple. Assume such zero to be located at $\chi_0 = \chi_c$. Then one can write that $\chi = (\chi_0 - \chi_c)^2 h(\chi_0)$ implying a square root branch point on Γ_V (one can solve that implicit equation for $\chi_0(\chi)$ to see that). Here $h(\chi_0)$ is the analytic function of χ_0 at $\chi_0 = \chi_c$ such that $h(\chi_c) \neq 0$. A number of such branch points (and, respectively, the number of sheets of $\Gamma_V(w)$) can be arbitrary large depending on the rational function $F(\chi_0)$.

A pair of Eqs. (4.187) and (4.188) give a parametric representation of the "physical" Riemann surface $G(z)$. This surface is not changing with time meaning that a velocity field of the fantom fluid defined in Ref. [**DDLZ19**] is time-independent. This fact additionally shows that the short branch cut approximation has only a limited range of applicability.

4.3.2. Particular solutions. According to Refs. [**DDLZ19, LZ20**], Eq. (4.187) does not allow decaying at $w \to \infty$ solution in terms of rational functions for $t > 0$ because any Nth order pole in V immediately results in the $2N + 1$ order pole term in the right-hand side (r.h.s.) of Eq. (4.187) which cannot be balanced by the maximum $N + 1$ pole order term in l.h.s. of Eq (4.187). Assume that

$$F(w) = -\frac{A}{w - a_0} = V|_{\tau=0}, \tag{4.191}$$

where A and a_0 are the complex constants such that $a_0 \in \mathbb{C}^+$. This initial condition has a pole at $w = a_0$. Then solving Eqs. (4.189) and (4.191) for χ_0, we obtain that

$$\chi_0 = \frac{\chi + a_0}{2} \pm \sqrt{\frac{(\chi - a_0)^2}{4} - A\tau} \tag{4.192}$$

which has two square root branch points at

$$\chi = a_0 \pm \sqrt{4A\tau}. \tag{4.193}$$

We choose a branch cut to be the straight line segment of length $|2\sqrt{4A\tau}|$ connecting two branch points (4.193).

Eqs. (4.180),(4.181),(4.187),(4.189)-(4.192) result in

$$(\chi_0)_\chi = \frac{1}{2} + \frac{\chi - a_0}{4\sqrt{\frac{(\chi-a_0)^2}{4} - A\tau}} \tag{4.194}$$

and

$$V = \frac{-2A}{\chi - a_0 + \sqrt{(\chi-a_0)^2 - 4A\tau}} = -\frac{\chi - a_0 - \sqrt{(\chi-a_0)^2 - 4A\tau}}{2\tau}, \tag{4.195}$$

where the branch of the square root $\sqrt{\ldots}$ is chosen to have $\sqrt{(\chi-a_0)^2} = \chi - a_0$ thus satisfying the initial condition (4.191).

The length of the branch cut according to (4.193) is increasing with time as $2\sqrt{4A\tau}$ and the solution (4.195) remains valid while the short cut approximation (4.158) is valid, i.e.

$$|2\sqrt{4A\tau}| \ll |Im(a_0)|. \tag{4.196}$$

That condition can be generalized by taking into account Eqs. (4.180) and (4.181).

Eq. (4.188) for z depends on the arbitrary function $G(\chi_0)$ so we can immediately construct the infinite set of solutions for z. E.g., choosing

$$G(\xi) = \xi \text{ for any } \xi \in \mathbb{C}, \tag{4.197}$$

we obtain from Eq. (4.188) and (4.192) that

$$z = \frac{\chi + a_0}{2} + \sqrt{\frac{(\chi-a_0)^2}{4} - A\tau} + \int\limits_0^\tau \frac{V_c(\tau')}{R_c(\tau')} \mathrm{d}\tau' \tag{4.198}$$

with the same choice of the branch of square root as in Eq. (4.195). Below in this section we always assume the same choice of the root. Using the definition (4.103) we obtain from Eq. (4.192) that

$$R = \frac{1}{(\chi_0)_\chi G_{\chi_0}(\chi_0)}. \tag{4.199}$$

Eqs. (4.197)-(4.199) result in

$$R = \frac{2\sqrt{(\chi-a_0)^2 - 4A\tau}}{\chi - a_0 + \sqrt{(\chi-a_0)^2 - 4A\tau}} = \frac{\sqrt{(\chi-a_0)^2 - 4A\tau}\left(\chi - a_0 - \sqrt{(\chi-a_0)^2 - 4A\tau}\right)}{2A\tau}. \tag{4.200}$$

This case corresponds to $R|_{t=0} \equiv 1$, which is the initially flat free surface evolving from the initial velocity distribution (4.191). A solution with such initial condition was first studied in Ref. [**KSZ93, KSZ94**] in the approximation of weak nonlinearity. It follows from Eq. (4.193) that one of two branch points reaches the real line $w = Re(w)$ in a finite time for a general complex value of the complex constant A (the only exception is $A > 0$ when both branch cuts move horizontally parallel to the real line). It means a formation of singularity on the free surface. However, well before that the condition (4.158) of the applicability of the short branch cut approximation is violated as the lower branch point approaches the real line. In Section 4.3.4 we discuss such type of solution in details for the periodic boundary conditions and compare it with the full numerical solution of Euler equations indicating that the singularity in full equations does reach the real line in a finite time.

We now convert the solution (4.193) for the location of branch points into w plane and the physical time t. The location of $w_0(t)$ (4.168) is determined by taking the midpoint

$$\chi_{mid} \equiv a_0 \tag{4.201}$$

between the two branch points (4.193) and after that using the definitions (4.180) and (4.181) to shift χ by $\int\limits_0^\tau \frac{V_c(t(\tau'))}{R_c(t(\tau'))}\mathrm{d}\tau'$ to return from the independent variable χ to w. It gives that

$$w_0(t) = a_0 + \int\limits_0^\tau \frac{V_c(t(\tau'))}{R_c(t(\tau'))}\mathrm{d}\tau'. \tag{4.202}$$

For z we use the initial condition (4.197) so that

$$R(w,t)|_{t=0} \equiv 1. \tag{4.203}$$

In the simplest approximation of Eqs. (4.180) and (4.181), we set

$$\chi \simeq w - \mathrm{i}V_c(0)t = w + \mathrm{i}\frac{\bar{A}t}{a_0 - \bar{a}_0} \tag{4.204}$$

and

$$\tau \simeq \mathrm{i}R_c(0)t = \mathrm{i}\,t. \tag{4.205}$$

where we used Eqs. (4.180),(4.191),(4.193),(4.202) and (4.203).

Using Eqs. (4.169),(4.193),(4.204) and (4.205), we obtain the approximate positions of branch points in w as follows

$$w = w_\pm \simeq a_0 \pm \sqrt{4A\mathrm{i}\,t} - \mathrm{i}\frac{\bar{A}t}{a_0 - \bar{a}_0}. \tag{4.206}$$

It is shown in Section 4.3.4 that the periodic boundary conditions version of equation (4.206) is accurate for the values of t well below the applicability condition (4.196) of the short branch cut approximation. Thus it might be sufficient in many practical calculations to use Eq. (4.206) instead of more accurate evaluations of integrals in Eqs. (4.180) and (4.181).

As another particular initial shape of surface we choose that

$$G(\xi) = \xi + B\log[\xi - C] \text{ for any } \xi \in \mathbb{C} \text{ with } \quad C \neq a_0 \text{ and } Im(C) > 0, \tag{4.207}$$

where B and C are complex constants. We note that Eq. (4.207) does not satisfy BC (**??**). That asymptotic deficiency can be fixed if we add the extra term $-B\log[\xi - C_1]$, $Im(C_1) > 0$ in r.h.s. of Eq. (4.207) which is however beyond the scope of that paper. Respective, by ignoring such fix we also neglect the last term in r.h.s. of Eq. (4.188). Then Eqs. (4.188) and (4.207) imply that at any moment of time τ,

$$z = \chi_0(\chi,\tau) + B\log[\chi_0(\chi,\tau) - C], \tag{4.208}$$

where $\chi_0(\chi,\tau)$ is given by Eq. (4.192).

Respectively, using Eq. (4.199), we obtain from Eq. (4.208) that

$$R = \frac{1}{(\chi_0)_\chi} \frac{\chi_0 - C}{\chi_0 - C + B} = \frac{1}{(\chi_0)_\chi}\left(1 - \frac{B}{\chi_0 - C + B}\right) \tag{4.209}$$

and

$$z_\chi = (\chi_0)_\chi \left(1 + \frac{B}{\chi_0 - C}\right). \tag{4.210}$$

We note that Eq. (4.207) has the branch cut which extends to the complex infinity. However, the corresponding R at $t = 0$ has only the pose singularity, see Eq.(4.209). Thus the short branch cut approximation remains valid for the initial condition (4.207) at least for small enough t.

If $C \neq a_0$ then it follows from Eq. (4.207) that the function z_χ has a simple pole at $\chi_0 = C$. Using Eq. (4.192), we then obtain that a trajectory of motion of that pole in χ plane is given by

$$\chi = C - \frac{A\tau}{a_0 - C}. \tag{4.211}$$

It follows from Eq. (4.210) that the residue of z_χ at that point is the integral of motion in χ plane, which is exactly equal to the constant B.

In a similar way, the function R in Eq. (4.209) has a simple pole at $\chi_0 = C - B$ provided $C - B \neq a_0$. A trajectory of motion of that pole in χ plane is given by

$$\chi = C - B - \frac{A\tau}{a_0 - C + B}. \tag{4.212}$$

However, the residue of that pole is not a constant of motion. We notice V is regular at $\chi_0 = C - B$ (because we assumed $C - B \neq a_0$) at least for small enough time. Such local solution (with the pole in R and no pole in V) is compatible with the analysis of [**DDLZ19, LZ20**] of the system (4.241)-(4.243), where solutions with the pole in both R and V was excluded while a solution with the pole in R only was allowed.

Another particular case is to set

$$R(w, t = 0) = \frac{B_1}{w - a_0} = R|_{\tau=0}, \tag{4.213}$$

where B_1 is the complex constant. This initial condition has a pole at the same $w = a_0$ as the initial pole in V defined in Eq. (4.191). Then Eqs. (4.197)-(4.199) result in

$$\begin{aligned} R &= \frac{4B_1\sqrt{(\chi - a_0)^2 - 4A\tau}}{(\chi - a_0 + \sqrt{(\chi - a_0)^2 - 4A\tau})^2} \\ &= \frac{B_1\sqrt{(\chi - a_0)^2 - 4A\tau}\left(\chi - a_0 - \sqrt{(\chi - a_0)^2 - 4A\tau}\right)^2}{4A^2\tau^2}. \end{aligned} \tag{4.214}$$

The particular solution (4.195),(4.214) recovers the asymptotic result of Ref. [**Tan93**] (Case (a) of Section 4 of that Ref.) obtained in that Ref. by the matched asymptotic expansions at $t \to 0$.

Eq. (4.214) makes sense locally near the pole position but cannot be valid globally because R must approach 1 as $w \to \infty$. So we provided that case only for the exact comparison with Ref. [**Tan93**]. We however can easily fix that deficiency through the replacement of Eq. (4.213) by

$$R(w,t=0)=1+\frac{B_1}{w-a_0}=R|_{\tau=0}. \tag{4.215}$$

Using Eq. (4.103) we then obtain that

$$z(w,t=0)=w-B_1\log(w-a_0+B_1) \tag{4.216}$$

and, using Eqs. (4.188),(4.192), that

$$z=\frac{\chi+a_0}{2}+\sqrt{\frac{(\chi-a_0)^2}{4}-A\tau}-B_1\log\left(\frac{\chi-a_0}{2}+\sqrt{\frac{(\chi-a_0)^2}{4}-A\tau}+B_1\right). \tag{4.217}$$

Differentiating Eq. (4.217) over w and using Eq. (4.103) results in

$$\begin{aligned} R&=\frac{2\sqrt{(\chi-a_0)^2-4A\tau}(2B_1+\chi-a_0+\sqrt{(\chi-a_0)^2-4A\tau})}{(\chi-a_0+\sqrt{(\chi-a_0)^2-4A\tau})^2}\\ &=\frac{\sqrt{(\chi-a_0)^2-4A\tau}(2B_1+\chi-a_0+\sqrt{(\chi-a_0)^2-4A\tau})\left(\chi-a_0-\sqrt{(\chi-a_0)^2-4A\tau}\right)^2}{8A^2\tau^2}. \end{aligned} \tag{4.218}$$

Eq. (4.214) is recovered from Eq. (4.218) in the limit $B_1\to\infty$.

We note that in all particular cases (4.200), (4.214) and (4.218), the series expansion at any of two branch points (4.193) shows that $R=0$ at these points, which is in the perfect agreement with the analytical results of Ref. [**LZ20**]. We also remind that all these particular cases share the same V from Eq. (4.195).

To express V and R in all these cases in terms of w and t requires to find expression of χ and τ through w and t using Eqs. (4.180) and (4.181). For that one can use definitions (4.169) with χ_0 determined in terms of χ through Eq. (4.193). After that a general condition (4.158) can be also verified. We note that all particular examples above correspond to the moving branch points according to Eq. (4.193). It implies that the condition (4.158) is violated at large times so the short branch cut approximation is valid in all these particular cases only for a finite duration of time.

The second sheet of Riemann surface Γ_V corresponds to the opposite choice of sign in Eq. (4.192). It means that we have to change the sign in front of each square root in Eqs. (4.195),(4.200),(4.214) and Eq. (4.218). It immediately implies that $V\to\infty$ and $R\to\infty$ as $\chi\to\infty$ in all these equations for the second (nonphysical) sheet of Riemann surface.

In all particular examples of this Section, the functions V and R are analytic functions of $\sqrt{(\chi-a_0)^2-4A\tau}$, i.e. they are analytic in two sheets of Riemann surface of w. This fact is the result of the approximation (4.174) effectively assuming that both $\tilde{V}$ and $\tilde{R}$ are constant on the branch cut. Going beyond that short cut approximation, we expect that V and R can be analytically continued into a much more complicated Riemann surfaces $\Gamma_V(w)$ and $\Gamma_R(w)$ with the unknown total number of sheets. Our experience with the Stokes wave in Ref. [**Lus16**] suggests that generally the number of sheets is infinite. Some exceptional cases like found in Refs. [**KZ14, ZK18**] have only a finite number of sheets of Riemann surface (these solutions however have diverging values of V and R at $w\to\infty$). We suggest that the detailed study of such many- and infinite-sheet Riemann surfaces is one of the most important goal in free surface hydrodynamics. This topic is however beyond the scope of this paper. We also notice that even in the simplest considered case (4.191), the function R can have the arbitrary number of additional poles and branch points depending on the choice of the function $G(\chi_0)$ in Eq. (4.187) instead of particular cases considered in this Section.

4.3.3. Short branch cut approximation and square root singularity solutions for the periodic case. In this section we extend the results of Section 4.3 into the 2π-periodic boundary conditions (**??**) instead of the decaying boundary conditions (**??**) used in Section 4.3. In that periodic case, instead of a single branch cut connecting branch points at $w = a(t) \in \mathbb{C}^+$ and $w = b(t) \in \mathbb{C}^+$ of Section 4.3, we consider the periodic sum of branch cuts and use the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+a} = \pi \cot \pi a. \tag{4.219}$$

Then taking the sum over branch cuts amounts to replacing $w - w'$ by $(1/2)\tan\left[(w-w')/2\right]$ in the denominators of Eq. (4.159) and all other similar expressions. In particular, Eq. (4.159) is replaced by

$$\begin{aligned} R(w,t) - 1 &= \frac{1}{2}\int\limits_a^b \frac{\tilde{R}(w',t)\mathrm{d}w'}{\tan\frac{w-w'}{2}}, \\ V(w,t) &= \frac{1}{2}\int\limits_a^b \frac{\tilde{V}(w',t)\mathrm{d}w'}{\tan\frac{w-w'}{2}}, \end{aligned} \tag{4.220}$$

Eq. (4.160) is replaced by

$$\begin{aligned} \bar{R}(w,t) - 1 &= \frac{1}{2}\int\limits_{\bar{a}}^{\bar{b}} \frac{\bar{\tilde{R}}(\bar{w}',t)\mathrm{d}\bar{w}'}{\tan\frac{w-\bar{w}'}{2}}, \\ \bar{V}(w,t) &= \frac{1}{2}\int\limits_{\bar{a}}^{\bar{b}} \frac{\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}\bar{w}'}{\tan\frac{w-\bar{w}'}{2}}. \end{aligned} \tag{4.221}$$

and Eq. (4.162) is replaced by

$$\begin{aligned} U(w,t) &= \frac{1}{2}\int\limits_a^b \frac{\tilde{U}(w',t)\mathrm{d}w'}{\tan\frac{w-w'}{2}}, \\ B(w,t) &= \frac{1}{2}\int\limits_a^b \frac{\tilde{B}(w',t)\mathrm{d}w'}{\tan\frac{w-w'}{2}}. \end{aligned} \tag{4.222}$$

Instead of the partial fractions used in Eq. (4.163), it is more convenient to use the integral representation of the projector $\hat{P}^-$ (4.66) for the periodic functions (see e.g. Ref. [**LDS17**]) which follows from Eq. (4.39) and the Sokhotskii-Plemelj theorem (see e.g. [**Gak66, PM08**]) giving that

$$\hat{P}^{-}f=-\frac{1}{2\pi\mathrm{i}}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi}\frac{f(u')du'}{u'-u+\mathrm{i}0+2\pi n}=-\frac{1}{4\pi\mathrm{i}}\int_{-\pi}^{\pi}\frac{f(u')du'}{\tan\frac{u'-u+\mathrm{i}0}{2}}, \tag{4.223}$$

where i0 means $\mathrm{i}\epsilon$, $\epsilon\to 0^{+}$ and we used the identity (4.219).

Using Eqs. (4.220), (4.221) and (4.222), a calculation of the projectors in the definitions (4.242) is performed through moving the integration contour from $(-\pi,\pi)$ to $(-\pi-\mathrm{i}\infty,\pi-\mathrm{i}\infty)$ together with the identity $\tan(-\mathrm{i}\infty)=-\mathrm{i}$ which give that

$$\begin{aligned}
\hat{P}^{-}\left[(R-1)\bar{V}\right] &= \frac{1}{4}\hat{P}^{-}\int_{a}^{b}\int_{\bar{a}}^{\bar{b}}\frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'}{\tan\frac{w-w''}{2}\tan\frac{w-\bar{w}'}{2}}\\
&= -\frac{1}{16\pi\mathrm{i}}\int_{-\pi}^{\pi}\int_{a}^{b}\int_{\bar{a}}^{\bar{b}}\frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'\mathrm{d}u'}{\tan\frac{u'-w''}{2}\tan\frac{u'-\bar{w}'}{2}\tan\frac{u'-w+\mathrm{i}0}{2}}\\
&= -\frac{1}{4}\int_{a}^{b}\int_{\bar{a}}^{\bar{b}}\frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'}{1}\left(\frac{-1}{2}-\frac{1}{\tan\frac{w-w''}{2}\tan\frac{w-\bar{w}'}{2}}+\frac{1}{\tan\frac{\bar{w}'-w''}{2}\tan\frac{w-\bar{w}'}{2}}\right)\\
&= -\frac{1}{4}\int_{a}^{b}\int_{\bar{a}}^{\bar{b}}\frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'}{1}\left(\frac{-1}{2}-\frac{1}{\tan\frac{w-\bar{w}'}{2}}\left[\frac{\tan\frac{\bar{w}'-w''}{2}-\tan\frac{w-w''}{2}}{\tan\frac{w-w''}{2}\tan\frac{\bar{w}'-w''}{2}}\right]\right)\\
&= -\frac{1}{4}\int_{a}^{b}\int_{\bar{a}}^{\bar{b}}\frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'}{1}\left(\frac{1}{2}+\left[\frac{1}{\tan\frac{w-w''}{2}\tan\frac{\bar{w}'-w''}{2}}\right]\right)\\
&= -\frac{1}{8}\int_{a}^{b}\int_{\bar{a}}^{\bar{b}}\frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'}{1}+\frac{1}{4}\int_{a}^{b}\int_{\bar{a}}^{\bar{b}}\frac{\tilde{R}(w'',t)\bar{\tilde{V}}(\bar{w}',t)\mathrm{d}w''\mathrm{d}\bar{w}'}{\tan\frac{w-w''}{2}\tan\frac{w''-\bar{w}'}{2}},
\end{aligned} \tag{4.224}$$

where we used the following trigonometric identity

$$\cot(a-b)=\frac{1+\tan a\tan b}{\tan a-\tan b}$$

with $a=\frac{\bar{w}'-w''}{2}$, $b=\frac{w-w''}{2}$ and $a-b=\frac{\bar{w}'-w}{2}$.

Now using the definitions (4.220) and (4.221) in Eq. (4.224), we obtain that

$$\hat{P}^{-}\left[(R-1)\bar{V}\right]=-\frac{1}{2}[R(-\mathrm{i}\infty,t)-1]\bar{V}(\mathrm{i}\infty,t)+\frac{1}{2}\int_{a}^{b}\frac{\tilde{R}(w'',t)\bar{V}(w'',t)\mathrm{d}w''}{\tan\frac{w-w''}{2}}, \tag{4.225}$$

where at the first term in r.h.s. of Eq. (4.224) we take appropriate limits to use the analyticity of R and $\bar{V}$ as follows: $\tan\frac{w-w'}{2}\to-\mathrm{i}\infty$ as $w\to-\mathrm{i}\infty$ in the first Eq. (4.220) $\tan\frac{w-\bar{w}'}{2}\to\mathrm{i}\infty$ as $w\to\mathrm{i}\infty$ in the second Eq. (4.221).

Similar to Eq. (4.225), one obtains that

(4.226)
$$\hat{P}^{-}\left[(\bar{R}-1)V\right] = -\frac{1}{2}[\bar{R}(\mathrm{i}\infty,t)-1]V(-\mathrm{i}\infty,t) + \frac{1}{2}\int_a^b \frac{\tilde{V}(w'',t)[\bar{R}(w'',t)-1]\mathrm{d}w''}{\tan\frac{w-w''}{2}}$$

and

(4.227)
$$\hat{B}(w,t) = \hat{P}^{-}\left[V\bar{V}\right] = -\frac{1}{2}[\bar{V}(\mathrm{i}\infty,t)]V(-\mathrm{i}\infty,t) + \frac{1}{2}\int_a^b \frac{\tilde{V}(w'',t)\bar{V}(w'',t)\mathrm{d}w''}{\tan\frac{w-w''}{2}}.$$

We obtain from Eqs. (4.63) and (**??**) that

$$R(-\mathrm{i}\infty,t)-1 = \bar{R}(\mathrm{i}\infty,t)-1 = \bar{V}(\mathrm{i}\infty,t)] = V(-\mathrm{i}\infty,t) = 0. \tag{4.228}$$

Then Eqs. (4.242),(4.222)-(4.165) and (4.228) result in the same Eqs. (4.166) and (4.167) as for the decaying BCs case (**??**) considered in Section 4.3.

Similar to Section 4.3, we consider the short branch cut approximation for the periodic case recovering exactly the same Eqs. as (4.169)-(4.184). The only difference in addressing these equations in comparison with Section 4.3 is to use the periodic BC (**??**). Respectively, instead of Eq. (4.185), we have to use the conditions (**??**) and (4.35) to determine $c(\tau)$.

As a particular example, we assume a periodic initial condition

$$F(w) = -\frac{A}{2\tan\frac{w-a_0}{2}} + \frac{\mathrm{i}A}{2} = V|_{\tau=0}, \tag{4.229}$$

where A and a_0 are complex constants such that $a_0 \in \mathbb{C}^+$. This initial condition is the periodic analog of Eq. (4.191) with the extra constant term $\frac{\mathrm{i}A}{2}$ added to make sure that $V \to 0$ at $Im(w) \to -\mathrm{i}\infty$, i.e. the decay of the velocity deep inside fluid. This initial condition has poles at $w = a_0 + 2\pi n,\ n = 0, \pm1, \pm2, \ldots$. In contrast with Eqs. (4.189) and (4.191), Eqs. (4.189) and (4.229) cannot be explicitly solved for χ_0. Thus Eqs. (4.187)-(4.189) provide only the implicit form of the solution for the initial condition (4.229). Note that $c(\tau)$ in this section is generally not given by Eq. (4.185) but is determined the conditions (**??**) and (4.35).

We can still explicitly obtain that the locations of square root branch points if we differentiate Eq. (4.189) over χ resulting in

$$1 = (\chi_0)_\chi \left[1 + \frac{dF(\chi_0)}{d\chi_0}\tau\right] \tag{4.230}$$

and notice $(\chi_0)_\chi$ is singular at the square root branch points (see e.g. Eq. (4.194) in the non-periodic case). It implies from Eq. (4.230) that

$$1 + \frac{dF(\chi_0)}{d\chi_0}\tau = 0 \tag{4.231}$$

at each branch point. Solving Eq. (4.231) we obtain the location of branch point in χ_0 variable as follows

$$\chi_0 = \chi_{0,\pm} \equiv a_0 \pm 2\arcsin\left(\frac{\sqrt{A\tau}}{2}\right) + 2\pi n,\ n = 0, \pm1, \pm2, \ldots. \tag{4.232}$$

Then using Eqs. (4.229) ,(4.231) and (4.232) we obtain that the square root branch points are located at
(4.233)
$$\chi=\chi_\pm=a_0-\frac{\mathrm{i}A\tau}{2}\pm\left[\frac{1}{2}\sqrt{A\tau}\sqrt{4-A\tau}+2\arcsin\left(\frac{\sqrt{A\tau}}{2}\right)\right]+2\pi n,\ n=0,\pm1,\pm2,\ldots.$$

Eq. (4.193) is recovered from Eq. (4.233) at the leading order $O(\tau^{1/2})$ for $A\tau\to 0$ and $n=0$. Similar to Eq. (4.193), we choose a branch cut to be the straight line segment of length $|\sqrt{A\tau}\sqrt{4-A\tau}+4\arcsin\left(\frac{\sqrt{A\tau}}{2}\right)|$ connecting the two branch points (4.233). The location of $w_0(t)$ (4.168) is determined by taking the midpoint

(4.234) $$\chi_{mid}\equiv a_0-\frac{\mathrm{i}A\tau}{2}$$

between the two branch points (4.233) and after that using the definitions (4.180) and (4.181) to shift χ by $\int\limits_0^\tau \frac{V_c(t(\tau'))}{R_c(t(\tau'))}\mathrm{d}\tau'$ to return from the independent variable χ to w. It gives that

(4.235) $$w_0(t)=a_0-\frac{\mathrm{i}A\tau}{2}+\int\limits_0^\tau\frac{V_c(t(\tau'))}{R_c(t(\tau'))}\mathrm{d}\tau'.$$

For z we use the initial condition (4.197) so that

(4.236) $$R(w,t)|_{t=0}\equiv 1.$$

The length of the branch cut is increasing with time as $\propto\sqrt{\tau}$ at $\sqrt{A\tau}$ according to (4.233) and the solution (4.195) remains valid at least while the short cut approximation (4.158) is valid, i.e.

(4.237) $$|2\sqrt{4A\tau}|\ll|Im(a_0)|.$$

For comparison with simulations one has to return from the independent variables τ and χ to the original variables t and w using Eqs. (4.180) and (4.181). In the simplest approximation of Eqs. (4.180) and (4.181), we set
(4.238)
$$\chi\simeq w-\mathrm{i}V_c(0)t=w-\mathrm{i}\left[-\frac{\bar{A}}{2\tan\frac{w-\bar{a}_0}{2}}-\frac{\mathrm{i}\bar{A}}{2}\right]\Bigg|_{w=a_0,t=0}\times t=w-\mathrm{i}\left[-\frac{\bar{A}}{2\tan\frac{a_0-\bar{a}_0}{2}}-\frac{\mathrm{i}\bar{A}}{2}\right]t$$

and

(4.239) $$\tau\simeq\mathrm{i}R_c(0)t=\mathrm{i}\,t.$$

where we used Eqs. (4.180),(4.193),(4.229),(4.235) and (4.236).

Using Eqs. (4.169),(4.233),(4.238) and (4.239), we obtain the approximate positions of branch points in w as follows

(4.240)
$$w=w_\pm\simeq a_0+\mathrm{i}\left[-\frac{\bar{A}}{2\tan\frac{a_0-\bar{a}_0}{2}}-\frac{\mathrm{i}\bar{A}}{2}\right]t+\frac{At}{2}$$
$$\pm\left[\frac{1}{2}\sqrt{A\mathrm{i}\,t}\sqrt{4-A\mathrm{i}\,t}+2\arcsin\left(\frac{\sqrt{A\mathrm{i}\,t}}{2}\right)\right]+2\pi n,\ n=0,\pm1,\pm2,\ldots.$$

For the precise location of branch points instead of the approximation (4.240), we have to find the dependence of τ on t and χ on w and t using Eqs. (4.180) and (4.181) together with the definitions (4.169) and Eq. (4.235). These equations are implicit ones. Also one has to find $\chi_0(\chi, \tau)$ from the implicit Eqs. (4.189) and (4.229) to be able to use Eqs. (4.187) and (4.188) for finding V_c and R_c from the definitions (4.169).

Similar to the solutions of Section 4.3, the particular solutions considered in this section also have the moving branch points according to Eq. (4.233). It follows from Eq. (4.193) that one of two branch points reaches the real line $w = Re(w)$ in a finite time for a general complex value of the complex constant A. It means a formation of singularity on the free surface. However, well before that the condition (4.158) of the applicability of the short branch cut approximation is violated as the lower branch point approaches the real line.

Similar to the discussion in Section 4.3, one can find a wide range of particular solutions for the periodic case of this section based on the general solutions (4.187) and (4.188).

4.3.4. Comparison of short branch cut approximation with full numerical solution. In this section we compare the short branch cut approximation described in the Section 4.3.3 with the full numerical solution of the system

$$\frac{\partial R}{\partial t} = \mathrm{i}\,(UR_u - RU_u)\,, \tag{4.241}$$

$$U = \hat{P}^-(R\bar{V} + \bar{R}V), \quad B = \hat{P}^-(V\bar{V}), \tag{4.242}$$

$$\frac{\partial V}{\partial t} = \mathrm{i}\,[UV_u - RB_u] + g(R-1), \tag{4.243}$$

satisfying the initial conditions (4.229) and (4.236). We assume that there is no gravity ($g = 0$). Both functions $z(w, t)$ and $\Pi(w, t)$ are recovered from the variables R and V by means of the relations (4.103) and (4.104), where we assume zero mean fluid (4.35) is at zero.

These initial conditions result in a pair of branch points that move according to Eq. (4.233). The direction of motion depends on the argument of the complex constant A. In the simulations we chose three values $A = 1$, $A = \mathrm{i}$ and $A = -\mathrm{i}$. The initial pole of the complex velocity, V, is located at $a_0 = \mathrm{i}$. Generally A can be the arbitrary complex number and a_0 can be the arbitrary complex number from $\mathbb{C}^+$. The Figures 2–5 show the spatial profiles (right panel), and the location of branch points (left panel) for both the branch cut approximation of the Section 4.3.3, and the full numerical solutions. The branch points are located at $w = w_\pm(t) \equiv w_{\pm,r}(t) + \mathrm{i}w_{\pm,i}(t)$, where $w_{\pm,r}(t)$ and $w_{\pm,i}(t)$ are real–valued. At each time step the location is recovered from the numerical simulations by the rational approximation procedure outlined in Section **??**. Additionally, $v_c = Im(w_{-,r}(t))$ can be also determined from the asymptotic of the exponential decay rate of the Fourier coefficients $\hat{z}_k \sim e^{-v_c|k|}$ of $z(w)$ for $|k| \to \infty$, see e.g. Ref. [**DLK16, LDS17**] for more details of that Fourier technique. Eq. (4.233) provides the analytic formula for the location of branch points in terms of the τ and χ in the branch cut approximation. However, the dependencies of $\tau(t)$ and $\chi(w, t)$ are given by an implicit relation that follows from the Eqs. (4.229). For the sake of convenience we use the approximate Eqs. (4.238) and (4.239) which result in the explicit expression (4.240) for branch point locations in terms of w and t.

We solve the implicit Eqs. (4.187) and (4.189) for $\chi_0(w,\tau)$ at every instant of time τ to determine the shape of the free surface $x(u,t)+\mathrm{i}y(u,t)$. The extra conditions (**??**) and (4.35) are used to find $c(\tau)$, and a subsequent substitution in the Eqs. (4.187)–(4.188) and (4.229) gives the shape of the surface $z(u,t)$.

The summary of a comparison of the short branch cut approximation and the numerical solutions is given below:

(a) For $A=\mathrm{i}$, both branch points move along the imaginary axis as follows from Eq. (4.233). The lower branch point, $w_- = \mathrm{i}w_{-,i}(t)$ moves downward from $w=a_0$, and the upper branch point $w_+ = \mathrm{i}w_{+,i}(t)$ is moves upward from $w=a_0$. Fig. 2a illustrates a dependence of the vertical coordinate of $w_{-,i}(t)$ on time, as determined from the Eq. (4.233) and the numerical simulations. The positions of branch points are recovered from numerical simulations by the procedure based on a least-squares rational approximation of complex functions and is described in details in Refs. [**DLK16, LDS17, DDLZ19**]. The vertical coordinate of the lower branch point is also estimated from the asymptotics of the decay rate of the Fourier spectrum giving the same result. Fig. 2b shows the spatial profiles of the free surface and a comparison of the short branch cut approximation and full numerics. It is seen that the spatial profile has a form of jet. Also Fig. 2b shows the time dependence of the maximum error in the surface elevation $y(x,t)$ between the numerical solution and short branch cut approximation.

As discussed in all particular solutions of Sections 4.3 and 4.3.3, one of the branch points of the analytical solution in the short branch cut reaches the real line $w = Re(w)$ in a finite time meaning a formation of the singularity of the free surface in a finite time. This is exactly what is seen in Fig. 2a. However, we also see in Fig. 2a that the full numerical solution seems does not produce a finite time singularity. Instead, the singularity appears to occurs at the infinite time $t\to\infty$. To quantify that statement we performed a fit to the stretcher exponential law $v = ae^{-kt^b}$, where a, b and k are three real fitting constants. We find that $b = 1.333 \simeq 4/3$ provides the best fit as seen in Fig. 3. Purely exponential fit $b=1$ is also shown providing not as good fit. Another not as good fit is e.g. $b=2$, i.e. the Gaussian exponent (not shown in Fig. (3). The detail discussion of the topic of finite time singularity is beyond the scope of this paper.

(b) For $A=-\mathrm{i}$, both branch points start to move in the horizontal direction, but unlike the problem on infinite line $-\infty < x < \infty$, the branch points in periodic problem develop vertical speed and approach the real axis. At later times branch cut recovered from numerics is not short thus violating b the short branch cut approximation. However, the positions of branch points recovered from short branch cut approximation agree semi-quantitavely with numerical simulations even at late times. Figure 4b shows the spatial profiles of the free surface at different times.

(c) For $A=1$, both branch points start moving in the complex plane from the initial position at $w=\mathrm{i}a$ as illustrated in the Figure 5a. Contrary to the other two cases, the positions of the branch points are not symmetric with respect to the imaginary axis. The Figure 5b shows how the shape of the free surface moves in time with increasing of steepness thus promoting overturning of the wave in a finite time.

We may conclude that the short branch cut approximation gives excellent results up to the values of small parameter $\epsilon \gtrsim 0.9$, well–outside of the applicability region for the short branch cut approximation (4.158).

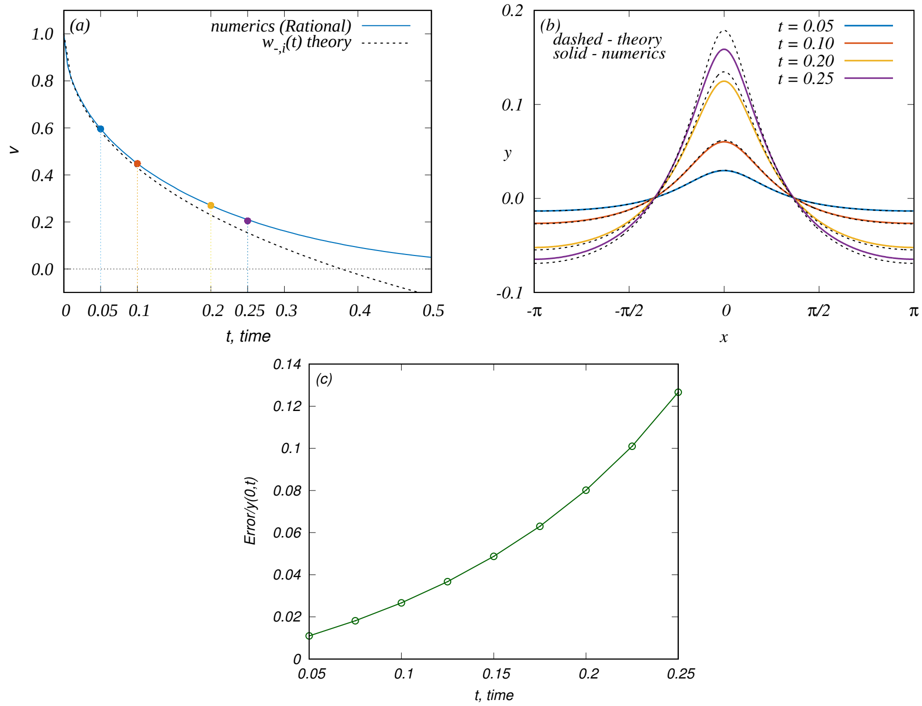

FIGURE 2. (a) The vertical position $v = Im(w_-(t)) = w_{-,i}(t)$ of the lower branch point vs. time t in the simulation with initial conditions (4.229) and (4.236) with $A = \mathrm{i}$ and $a_0 = \mathrm{i}$. The lower branch point location $w_-(t)$ is recovered from the full numerical simulations of Eqs. (4.241)–(4.243) by means of a numerical rational approximation (solid line) compared with its location from the short branch cut approximation (4.233) (dashed line). See Section **??** about the rational approximation. The relative error of the theoretic prediction vs. numerics is 1.93% at short time $t = 0.05$, about 3.93% at $t = 0.1$, and 15.2% at $t = 0.2$. At $t = 0.05$ and $t = 0.2$, the value of the parameter ϵ introduced in Eq. (4.158) is 0.61 and 0.89 respectively. These values are well outside the asymptotic condition $\epsilon \ll 1$ when the short branch cut theory is guaranteed to be applicable. (b) The spatial profiles of the fluid surface at different times: the result of numerical simulation (solid lines), and the short branch cut approximation (dashed lines). (c) The time dependence of the maximum of the error for the surface elevation $y(x,t)$ between the numerical solution and the short branch cut approximation. The maximum of error occurs at $x = 0$ as seen from the surface profiles in (b). The error is normalized to the values of $y(0,x)$ from the numerical solution.

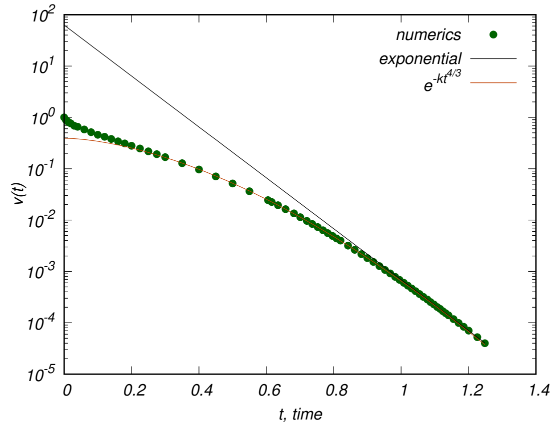

FIGURE 3. A fit of $v = w_{-,i}(t)$ from Fig. 2a (shown by dots) into the stretched exponent $v = ae^{-kt^b}$ with $b = 1$ (black solid line) and $b = 4/3$ (red solid line). It is seen that $b = 4/3$ is much better fit that $b = 1$. Here $a = 0.39795 \pm 0.01406$ and $k = 6.40096 \pm 0.03348$ are the fitting constants.

4.4. Nonlinear development of Kelvin-Helmholtz instability for counterflow of superfluid and normal components of Helium II

Kelvin-Helmholtz instability (KHI) is perhaps the most important hydrodynamic instability which commonly occurs either at the interface between two fluids moving with different velocities or in the presence of the tangential velocity jump/shear flow in the same fluid [**LL89a**]. Recently KHI attracted significant experimental and theoretical attention in superfluids. KHI was studied either for interface between different phases of ^{3}He [**Vol02, BEE$^+$02, Vol03, FEH$^+$06, Vol15**], which has many similarities with KHI in classical fluids, or KHI from relative motion of components of ^{4}He [**HB14, RLMD16, BLP$^+$16, GGL$^+$16, GVGV17**] which has no classical analog thus we refer to it as quantum KHI. We focus on the second case, i.e. on quantum KHI of the free surface of ^{4}He in the superfluid phase (He-II state) in the presence of counterflow of superfluid and normal fluid components [**Kor91, Kor02**]. Principle difference here from KHI of classical fluids is that relative fluid motion in quantum KHI occurs not from different sides of interface but from the same side of the He-II free surface with fluids components coexisting in the same volume which is purely quantum effect. A counterflow is achieved in experiment by the action of a stationary heat flow within the liquid in the direction tangent to the free surface as shown in Fig. 6.

Linear analysis of both classical KHI and quantum KHI results in the exponential growth of surface perturbations [**LL89a, Kor91, Kor02**]. As these initial perturbations reach amplitudes comparable with their wavelength, nonlinear effects must be considered. Weak nonlinearity approximation takes into account the leading order nonlinear correction over the small parameter which is the typical slope of surface. Weakly nonlinear equations for development of KHI of classical fluids results in a finite time singularity [**KL95**] which means that solutions become strongly nonlinear beyond the perturbation theory. Two-dimensional (2D)

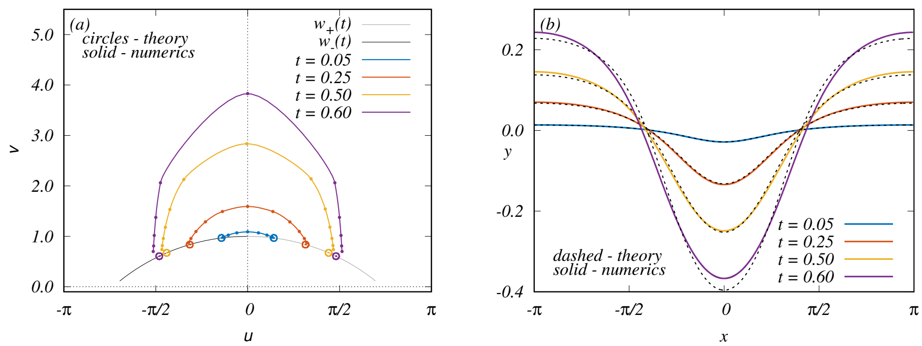

FIGURE 4. (a) The location of the branch cut in the analytical continuation of the complex velocity V (4.104) in the complex plane at several instants of time. The initial conditions are given by (4.229) and (4.236) with $A = -\mathrm{i}$ and $a_0 = \mathrm{i}$. The branch cuts recovered from the numerical simulations of the equations (4.241)–(4.243). The filled circles show the positions of poles of rational approximation, and the open circles correspond to the branch point locations given by the analytic formula for $w_{\pm,i}(t)$ from Eq. (4.233) at the respective time. The solid black lines are the trajectories of $w_{\pm,i}(t)$ from the short cut approximation. The difference in the position of the branch points estimated from the numerical simulation and the short branch cut theory is 2.89% at time $t = 0.05$ ($\epsilon = 0.92$), and is 6.45% at time $t = 0.50$ ($\epsilon = 4.12$). The $w_{\pm,i}(t)$ from Eq. (4.233) give an excellent estimate for the branch points even for $\epsilon > 1$. (b) The spatial profiles of the fluid surface from numerical simulation (solid lines), and short cut approximation (dashed lines).

dynamics of interface between two fluids in weak nonlinearity approximation can be reduced to the motion of complex singularities through the analytical continuation into complex plane from the interface [**KSZ93, ZK14b**]. Approach of singularity to the interface always means a formation of its geometric singularity. Other examples of analysis of weakly nonlinear 2D dynamics through motion of singularities include the interface between ideal fluid and light highly viscous fluid [**Lus04**], and vortex sheet in ideal fluid [**Moo79**]. Extending weakly nonlinear solutions into strongly nonlinear solution is challenging and mostly was done for the particular case of free surface hydrodynamics (i.e. the density of the second fluid turns into zero) [**Tan91, Tan93, CBT99, DLK13, KZ14, Lus16**] including drops pinchoff [**EFLS07, TLZ09**]. Another exception is the ideal fluid pushed through viscous fluid in a narrow gap between two parallel plates (Hele–Shaw flow) which can be approximately reduced to the Laplace growth equation (LGE) admitting an infinite set of exact solutions [**PK45, Gal45, SB84, How86, BKL$^+$86, MWD94**].

We use a key property of quantum KHI that both fluid components share the same volume. It allows to find the exact strongly nonlinear solutions and moreover, general integrability of growing solutions. This is achieved through the exact reduction of quantum KHI dynamics to LGE for arbitrary level of nonlinearity.

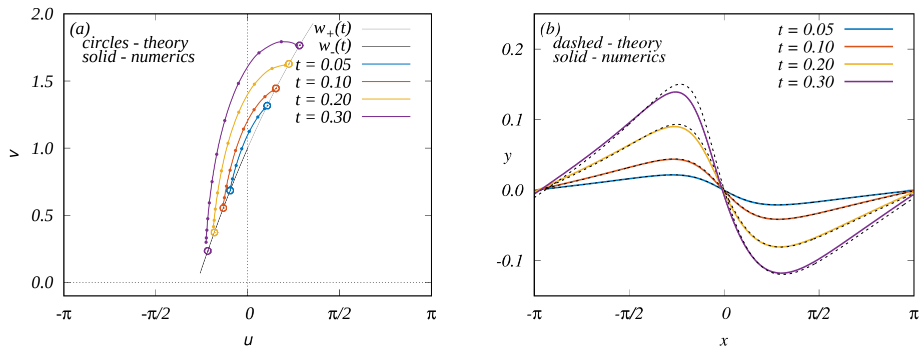

FIGURE 5. (a) The location of the branch cut in the analytical continuation of the complex velocity V (4.104) at different moments of time in the complex plane for initial conditions given by (4.229) and (4.236) with $A = 1$ and $a_0 = \mathrm{i}$. The branch cuts recovered from full numerical simulations of Eqs. (4.241)–(4.243) is given by solid line. The filled circles represent the poles of the rational approximation of the branch cut, and the open circles correspond to the branch point locations $w_{\pm,i}(t)$ from the equation (4.233). The gray line passing through $w = \mathrm{i}$ is the trajectory of $w_{\pm,i}(t)$ as obtained from Eq. (4.233). It is observed that for small time $w_{\pm,i}(t)$ from Eq. (4.233) approximates the branch points to 3.31% (relative error) at $t = 0.05$ with $\epsilon = 0.68$, and 7.69% (relative error) at $t = 0.25$ with $\epsilon = 1.18$. (b) The shape of free surface of the fluid at different times: numerical simulation (solid lines), and the short branch cut approximation (dashed lines).

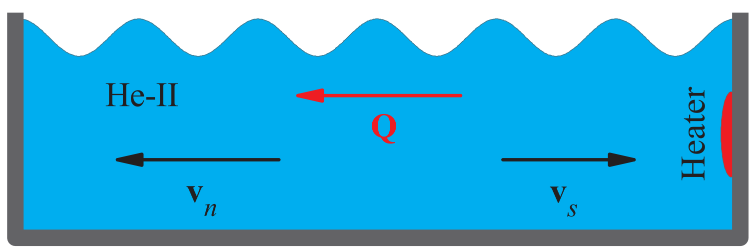

FIGURE 6. A schematic of counterflow in superfluid ^{4}He. Heater results in the flux of heat $\mathbf{Q}$ which is carried by the normal fluid component with velocity $\mathbf{v}_n$ while superfluid component moves in the opposite direction with the velocity $\mathbf{v}_s$. Both components coexist in the same volume of fluid and share the same free surface.

These new solutions, in particular, describe the formation of cusps (dimples) on the He-II free surface in a finite time with both a surface curvature and velocities of components of He-II diverging at singular points. We expect that these singularities will be possible to observe in He-II experiment which is different from weaker singularities of the Moore's type (were identified from approximate analysis in Refs.

[**Moo79, KSZ93, ZK14b**]) and predicts smooth surface with jump only in second derivative. LGE is integrable in a sense of the existence of infinite number of integrals of motion and relation to the dispersionless limit of the integrable Toda hierarchy [**MWWZ00**]. We suggest that the obtained reduction of quantum KHI to LGE is important to the general problem of integrability of surface dynamics [**DZ94a**]. It provides a very rare example of integrable physical system.

Superfluid component of He-II necessary has quantized vortices if counterflow velocity exceeds several mm/s with their density growing with that velocity [**?**]. Here we consider the dynamics of He-II at macroscopic scale where we can average over vortices. We neglect average vorticity from such averaging as well as ignore vorticity of normal component similar to Ref. [**Kor91, Kor02**] which refers to that approximation as non-dissipative two-fluid description. In that approximation the dynamics of both fluid components is potential one, i.e. $\mathbf{v}_s = \nabla\Phi_s$ and $\mathbf{v}_n = \nabla\Phi_n$, where $\mathbf{v}_s$, $\mathbf{v}_n$ are velocities of superfluid and normal components with Φ_s and Φ_n being the corresponding velocity potentials. We assume that both components are incompressible with densities $\rho_s \equiv \text{const}$, $\rho_n \equiv \text{const}$ and the total density $\rho \equiv \rho_s + \rho_n$. Incompressibility implies Laplace equation for each component, $\nabla^2\Phi_{n,s} = 0$. We focus on 2D flow $\mathbf{r} \equiv (x, y)$, where x and y are horizontal and vertical coordinates, respectively. We assume that both fluids occupy the region $-\infty < y \leq \eta(x,t)$, where $y = \eta(x,t)$ is the free surface elevation with the unperturbed surface given by $\eta(x,t) \equiv 0$. The flow of both components deep inside He-II ($y \to -\infty$) as well as at $|x| \to \infty$ is assumed to be uniform following x direction, which implies $\Phi_{n,s} \to V_{n,s}\, x$, where $V_{n,s}$ are the corresponding horizontal velocities. We use the reference frame of the center of mass such that $\rho_n V_n + \rho_s V_s = 0$ and introduce the relative velocity $V = V_s - V_n > 0$ between fluid components meaning that $V_{n,s} = \mp\rho_{s,n}V/\rho$.

The dynamic boundary condition (BC) at the free surface ($y = \eta$) follows from the generalization of Bernoulli Eq. into two fluid components, see e.g. Chap. 140 of Ref. [**LL89a**] and Refs. [**Kor91, Kor02**])

$$\begin{aligned} &\rho_n\left(\frac{\partial\Phi_n}{\partial t} + \frac{(\nabla\Phi_n)^2}{2}\right) + \rho_s\left(\frac{\partial\Phi_s}{\partial t} + \frac{(\nabla\Phi_s)^2}{2}\right)\Bigg|_{y=\eta} \\ &\qquad = \Gamma - P_\alpha - P_g, \end{aligned} \tag{4.244}$$

where $P_\alpha = -\alpha\frac{\partial}{\partial x}[\eta_x(1+\eta_x^2)^{-1/2}]$ is the pressure jump at the free surface due to the surface tension α (the pressure is zero outside the fluid assuming that there is a vacuum there), $\eta_x \equiv \partial\eta/\partial x$, $P_g = \rho g\eta$ is the gravity pressure (the contribution of the acceleration due to gravity g) and $\Gamma = \rho_n\rho_s V^2/(2\rho)$ is the Bernoulli constant which ensures that Eq. (4.244) is satisfied at $|x| \to \infty$.

The kinematic BCs at the free surface are given by

$$\eta_t(1+\eta_x^2)^{-1/2} = \partial_n\Phi_n|_{y=\eta} = \partial_n\Phi_s|_{y=\eta}, \tag{4.245}$$

where $\eta_t \equiv \partial\eta/\partial t$, $\partial_n \equiv \mathbf{n}\cdot\nabla$ is the outward normal derivative to the free surface with $\mathbf{n} = (-\eta_x, 1)(1+\eta_x^2)^{-1/2}$. Eqs. (4.244) and (4.245) together with $\nabla^2\Phi_n = \nabla^2\Phi_s = 0$ and BC at infinity form a closed set of equations of two-fluid hydrodynamics for KHI problem.

We introduce the average velocity $\mathbf{v} = (\rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s)/\rho$ and the auxiliary potentials $\Phi = (\rho_n\Phi_n + \rho_s\Phi_s)/\rho$, $\phi = \sqrt{\rho_n\rho_s}(\Phi_n - \Phi_s)/\rho$ which are linear combinations of Φ_n and Φ_s thus satisfying Laplace equation together with $\nabla\Phi = \mathbf{v}$. BCs at either

$y \to -\infty$ or $|x| \to \infty$ are reduced to

$$\Phi \to 0 \text{ and } \phi \to -Vx\sqrt{\rho_n\rho_s}/\rho. \tag{4.246}$$

Eq. (4.244) turns into

$$\left.\frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \frac{(\nabla\phi)^2}{2}\right|_{y=\eta} = \frac{c^2}{2} - \frac{P_\alpha + P_g}{\rho}, \tag{4.247}$$

where $c = \sqrt{2\Gamma/\rho}$ is the constant which has the dimension of velocity. Eqs. (4.245) are reduced to

$$\eta_t(1+\eta_x^2)^{-1/2} = \partial_n\Phi|_{y=\eta} \tag{4.248}$$

and

$$\partial_n\phi|_{y=\eta} = 0. \tag{4.249}$$

We replace ϕ by its harmonic conjugate ψ such that Cauchy-Riemann equations $\phi_x = \psi_y$ and $\phi_y = -\psi_x$ are valid. BC (4.249) for Laplace Eq.

$$\nabla^2\psi = 0 \tag{4.250}$$

at the free surface reduces to vanishing of tangential derivatives $\partial_\tau\psi|_{y=\eta} = 0$ because $\partial_\tau\psi|_{y=\eta} = -\partial_n\phi|_{y=\eta}$. Without the loss of generality we set

$$\psi|_{y=\eta} = 0. \tag{4.251}$$

BC at either $y \to -\infty$ or $|x| \to \infty$ are reduced to

$$\psi \to -Vy\sqrt{\rho_n\rho_s}/\rho = -cy. \tag{4.252}$$

If we introduce the stream functions $\Psi_{n,s}$ for the components of He-II (they satisfy Cauchy-Riemann equations $\partial_x\Phi_{n,s} = \partial_y\Psi_{n,s}$ and $\partial_y\Phi_{n,s} = -\partial_x\Psi_{n,s}$), then $\psi = (\Psi_n - \Psi_s)\sqrt{\rho_n\rho_s}/\rho$. ψ is fully determined by $\eta(x,t)$ from (4.251) and (4.252) while being independent on Φ. Dynamic BC (4.247) in terms of Φ and ψ is given by

$$\left.\frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \frac{(\nabla\psi)^2}{2}\right|_{y=\eta} = \frac{c^2}{2} - \frac{P_\alpha + P_g}{\rho}. \tag{4.253}$$

Eqs. (4.248), (4.251) and (4.253) together with $\nabla^2\Phi = \nabla^2\psi = 0$ and BCs (4.246) and (4.252) at infinity form a closed set of equations equivalent (through harmonic conjugation) to solving two-fluid He-II hydrodynamics for KHI problem. It is remarkable that this set is equivalent (up to trivial change of constants) to the problem of 2D dynamics of charged surface of ideal fluid in the limit when surface charges fully screen the electric field above the fluid free surface. This limit was realized experimentally for the He-II (with negligible ρ_n) free surface charged by electrons [**Ede80**]. In that case Φ has the meaning of the only (ideal) fluid component and ψ represents (up to the multiplication on constant) the electrostatic potential in the ideal fluid. The term $\propto (\nabla\psi)^2$ in Eq. (4.253) corresponds to the electrostatic pressure.

Refs. [**Zub00, Zub02**] found exact time-dependent solutions for this problem of the dynamics of charged surface of superfluid He-II in the limit of zero surface tension and gravity as well as for the limit of zero temperature (i.e. neglecting the normal component of He-II). We apply that approach for full (nonlinear) KHI problem with finite temperature. We set $\alpha = g = 0$ in right-hand side of Eq. (4.253). Below we provide estimates of the applicability of such neglect of surface tension and gravity for two-component dynamics of He-II.

Our goal is to reduce Eqs. (4.248), (4.251) and (4.253) together with $\nabla^2\Phi = \nabla^2\psi = 0$ and BCs (4.246) and (4.252) to solving LGE. Differentiation of Eq. (4.251) over t and x results in

$$\eta_t = -\psi_t/\psi_y|_{y=\eta}, \qquad \eta_x = -\psi_x/\psi_y|_{y=\eta}$$

respectively. Using these expressions in kinematic BC (4.248) rewritten in the equivalent form $\eta_t = \Phi_y - \eta_x\Phi_x|_{y=\eta}$, allows to obtain that

$$\psi_t + \nabla\psi\cdot\nabla\Phi|_{y=\eta} = 0. \tag{4.254}$$

The sum and difference of Eqs. (4.253) and (4.254) (with $P_\alpha = P_g = 0$) result in

$$F_t^{(\pm)} \mp cF_y^{(\pm)} + (\nabla F^{(\pm)})^2\Big|_{y=\eta} = 0, \tag{4.255}$$

where we introduced the harmonics potentials

$$F^{(\pm)} = (\Phi \pm \psi \pm cy)/2, \tag{4.256}$$

which satisfy the Laplace Eqs.

$$\nabla^2 F^{(\pm)} = 0,\ F^{(\pm)} \to 0 \ \text{ for } y\to-\infty \text{ or } |x|\to\infty. \tag{4.257}$$

According to Eqs. (4.251) and (4.256), the motion of the free surface is determined by the implicit expression

$$c\eta = F^{(+)} - F^{(-)}|_{y=\eta}. \tag{4.258}$$

Returning to physical $\Phi_{n,s}$ and $\Psi_{n,s}$, we find that

$$2\rho F^{(\pm)} = \rho_n\Phi_n + \rho_s\Phi_s \pm \sqrt{\rho_n\rho_s}\left(\Psi_n - \Psi_s + Vy\right). \tag{4.259}$$

Eqs. (4.257) together with (4.255), and (4.258) is equivalent to the KHI problem. It is crucial that nonlinear Eqs. (4.255) decouple into separate Eqs. for $F^{(+)}$ and $F^{(-)}$. We note that such decoupling does not occur for the classical KHI problem (the interface between two fluids) where the velocity potentials and stream functions of each of two fluids are defined in physically distinct regions ($y < \eta$ and $y > \eta$) thus making impossible a superposition of the type (4.259). (Decoupling is however possible by other method in small angle approximation with leading quadratic nonlinearity in perturbation series for classical KHI [**ZK14b**].)

The full set of equations (4.255), (4.257)-(4.258) is still generally coupled through Eq. (4.258). But an additional assumption (reduction) that either

$$F^{(+)} = 0 \text{ or } F^{(-)} = 0 \tag{4.260}$$

ensures the closed Eqs. which have a wide family of exact nontrivial solutions described below. That assumption remains valid as time evolves. It follows from Eqs. (4.256) that (4.260) ensure the relations between $\Phi_{n,s}$ and $\Psi_{n,s}$ as

$$\mp\sqrt{\rho_n\rho_s}\left(\Psi_n - \Psi_s + Vy\right) = \rho_n\Phi_n + \rho_s\Phi_s.$$

We look at the physical meaning of our reductions (4.260), based on the particular limit of small amplitude surface waves. We neglect the nonlinear term in Eqs. (4.255) resulting in the linear system which we solve in the form of plane waves

$$\begin{aligned} F^{(\pm)} &= a^{(\pm)}\exp(\mathrm{i}kx + ky - \mathrm{i}\omega^{(\pm)}t),\ \eta = b^{(+)} \\ &\times \exp(\mathrm{i}kx - \mathrm{i}\omega^{(+)}t) + b^{(-)}\exp(\mathrm{i}kx - \mathrm{i}\omega^{(-)}t), \end{aligned} \tag{4.261}$$

where $a^{(\pm)}$ and $b^{(\pm)}$ are small constants, $\omega^{(\pm)}$ are frequencies and k is the wavenumber. First Eq. in (4.261) ensures the exact solution of Eqs. (4.257) with decaying

BCs at $y \to -\infty$. Substitution of (4.261) into Eq. (4.258) and the linearization of (4.255) results in the relations

$$\omega^{(\pm)} = \pm \mathrm{i}ck, \qquad cb^{(\pm)} = \pm a^{(\pm)}. \tag{4.262}$$

which are two branches of the dispersion relation of KHI with $g = \alpha = 0$ [**LL89a, Kor91, Kor02**]. Superscripts "+" and "−" correspond to exponentially growing and decaying perturbations of the flat free surface, respectively. Eqs. (4.260) choose one of these two branches. Thus Eqs. (4.255), (4.257)-(4.258) together with (4.260) represent the fully nonlinear stage of such separation into two branches.

The generic initial conditions include both unstable and stable part (4.261) with the unstable part dominates as time evolves. Also it was shown in Refs. [**Zub02, Zub08b**] that small perturbation of $F^{(-)}$ on the background of large $F^{(+)}$ decays to zero. Thus the choice of the reduction $F^{(-)} = 0$ (which is assumed below) in Eq. (4.260) is the natural one to address the nonlinear stage of KHI. Then Eqs. (4.256) imply that $F^{(+)} = \Phi = \psi + cy$, i.e. Φ is determined by ψ. The boundary value problem (BVP) (4.250)-(4.252) solves for ψ at each t. The motion of the free surface is determined by Eq. (4.248) as

$$(\eta_t - c)(1 + \eta_x^2)^{-1/2} = \partial_n \psi|_{y=\eta}\,. \tag{4.263}$$

To solve BVP (4.250)-(4.252) we consider the conformal map $z = z(w,t)$ [**DKSZ96a**] from the lower complex half-plane $-\infty < v \le 0$, $-\infty < u < +\infty$ of the complex variable $w = u + \mathrm{i}v$ into the area $-\infty < y \le \eta(x,t)$ occupied by the fluid in the physical plane $z = x + \mathrm{i}y$ with the real line $v = 0$ mapped into fluid free surface. Then the free surface is given in the parametric form $y = Y(u,t) \equiv \operatorname{Im} z(u,t)$ and $x = X(u,t) \equiv \operatorname{Re} z(u,t)$. Solutions of both BVP (4.250)-(4.252) and the harmonically conjugated BVP $\nabla^2\phi = 0$, (4.246), (4.249) in (u,v) variables are given by $\phi + \mathrm{i}\psi = -c(u + \mathrm{i}v)$. It means that the conformal variables u and v have a simple physical meaning: $u = -\phi/c$ and $v = -\psi/c$ corresponding (up to multipication to the constant $-1/c$) to the harmonically conjugated potentials ϕ and ψ.

We consider w as independent variable while $z(w,t)$ as the unknown function. Eq. (4.263) is given by $Y_tX_u - Y_uX_t = cX_u - c$, which can be rewritten as

$$\operatorname{Im}\left(\bar{G}_tG_u\right) = c. \tag{4.264}$$

where $G(u,t) = z(u,t) - \mathrm{i}ct$. Eq. (4.264) has the exact form of LGE which has the infinite number of exact solutions often involving logarithms (see e.g. Refs.[**SB84, How86, BKL$^+$86, MWD94**]). We look at a periodic solution [**BKL$^+$86**] with the wavenumber k,

$$z = w - \mathrm{i}kA^2(t)/2 - \mathrm{i}A(t)\exp[-\mathrm{i}kw], \tag{4.265}$$

where $A(t)$ is the amplitude of the free surface surface perturbation satisfying a nonlinear ordinary differential equation $dA/dt = ckA\left(1 - k^2A^2\right)^{-1}$which develops a finite-time singularity in dA/dt at the time $t = t_c$ with $A(t_c) = 1/k$. As t approaches t_c, a leading order solution is given by $A = 1/k - \sqrt{c\tau/k} + O(\tau)$, where $\tau = t_c - t$. Singularities of the conformal map (4.265) are determined by a condition $z_w = 0$ implying that they approach the real line $v = 0$ from above with the increase of t. That line is reached at $\tau = 0$ and $u = 2\pi n/k$, $n = 0, \pm 1, \pm 2, \ldots$ In particular, choosing $n = 0$, expanding at $u = 0$ and assuming $\tau \to 0$ we obtain that

$$\begin{aligned} X &= u\sqrt{ck\tau} + k^2u^3/6 + O(u\tau + u^3\tau^{1/2}), \\ Y &= -3/2k + 2\sqrt{c\tau/k} + ku^2/2 + O(\tau + u^2\tau^{1/2}). \end{aligned} \tag{4.266}$$

Fig. 7 shows an example of such solution at different t. It follows from Eq. (4.266) that a cusp pointing downward (a dimple) $y + 3/2k \propto |x|^{2/3}$ is formed at the free surface at $t = t_c$ (i.e. $\tau = 0$) with the vertical velocity diverging as $\tau^{-1/2}$ at the tip of the cusp [**SB84, BKL$^+$86**].

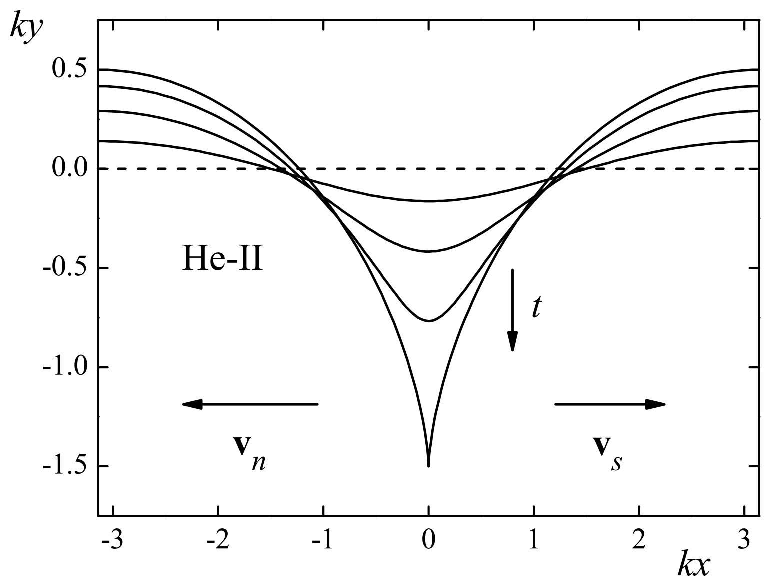

FIGURE 7. Evolution of an initial periodic perturbation of the free surface $y(x)$ for Eq. (4.265) with $kA(0) \approx 0.15$. The surface shape is shown over one spatial period for the times $ckt = 0$, 0.8, 1.2, 1.4 until the cusp singularity is formed. The dashed line shows the unperturbed free surface, $y \equiv 0$.

Near the singularity (the tip of the cusp) one has to take into account the surface tension and the finite viscosity of the normal component to regularize the singularity. Surface tension near the singularity is given by $P_\alpha \approx \alpha/r$, where r is the radius of curvature of the free surface. It follows from Eq. (4.266) that $r \approx c\tau$, which implies that $P_\alpha \approx \alpha/c\tau$. The dynamic pressure P_v, which determines the development of KHI in LGE reduction, is given by $P_v = \rho v^2/2 \equiv \rho\left[(\nabla\Phi)^2) + (\nabla\psi)^2\right]/2$, where v is the typical velocity. Near singularity $v \simeq \sqrt{2c/k\tau}$ and $P_v = \rho c/k\tau$. Thus both P_v and $P_\alpha \propto \tau^{-1}$. Surface tension effect is small if the Weber number $\mathrm{We} = P_v/P_\alpha$, the ratio of dynamic and surface tension pressures, is well above 1. Using $\mathrm{We} \approx \rho c^2/\alpha k = \rho_n\rho_s V^2/(\rho\alpha k)$, and assuming $\mathrm{We} \gtrsim 1$ for applicability of LGE regime, we obtain the condition for the wavelength $\lambda = 2\pi/k \gtrsim 2\pi\rho\alpha/(\rho_n\rho_s V^2)$. He-II at the temperature 1.5 K has $\rho_n = 0.016$ g/cm^3, $\rho_s = 0.129$ g/cm^3 and $\alpha = 0.332$ dyn/cm [**DB98**]. E.g. if $V = 15$ cm/s then $\lambda \gtrsim 0.64$ cm.

The relative strength of inertial and viscous forces near the singularity is determined by the Reynolds number $\mathrm{Re} = vr/\nu$, where ν is kinematic viscosity of He-II. Using that $v \approx \sqrt{2c/k\tau}$ and $r \approx c\tau$, implies that $\mathrm{Re} \approx c\nu^{-1}\sqrt{2r/k}$, i.e. Re turns small for $r \to 0$ and viscosity has to be taken into account. A typical scale r_ν below which the flow of the normal component cannot be considered as potential one is estimated by setting $\mathrm{Re} \approx 1$ which gives $r_\nu \approx k\nu^2/2c^2$. For the temperature 1.5 K, we use $\nu = 9.27 \cdot 10^{-5}$ cm^2/s [**DB98**]. Then $r_\nu \approx 1.8 \cdot 10^{-10}$ cm, i.e. $r_\nu \ll \lambda$ thus the viscous effect is much less than the surface tension. The influence of gravity, which is determined by the Froude number $\mathrm{Fr} = P_v/P_g$, is small near the singularity because the gravity pressure $P_g \simeq \rho g y$ is finite while P_v diverges as τ^{-1} implying divergence of Fr.

Thus we reduced fully nonlinear quantum KHI dynamics to LGE which has the infinite set of exact solutions with the generic formation of cusps at the free surface in a finite time. The key is the exact transform from two-fluid description into the effective single-fluid description of Eq. (4.253). It suggests a roadmap for efficient use of conformal map to include gravity and capillarity into dynamics. Adding capillarity would ensure singularity regularization at small spatial scales. Conformal map can be used for electro-hydrodynamic instability [**Ede80, Zub08b**] and Faraday waves [**LMDP17**] of He-II. Viscosity can be taken into account through conformal map in Stokes flow regime of normal component which would go beyond weakly nonlinear result [**Lus04**].

Free surface represents vortex sheet which results in the additional generation of quantized vortices at the nonlinear stage of KHI. It is expected to push quantum turbulence states T1 towards T2/T3 states [**?**].

CHAPTER 5

Solitons in multidimensions and soliton stability

5.1. Solitons in multidimensions

Consider the soliton solution

$$\psi(\mathbf{r},t) = R(\mathbf{r})e^{i\lambda^2 t} \tag{5.1}$$

for the cubic focusing NLSE

$$i\psi_t + \nabla^2\psi + |\psi|^2\psi = 0 \tag{5.2}$$

in the spatial dimension $D < 4$. Equations (5.1) and (5.2) result in the time-independent equation

$$-\lambda^2 R + \nabla^2 R + R^3 = 0 \tag{5.3}$$

for the real-valued soliton amplitude $R(\mathbf{r}) \in \mathbb{R}$.

Multiplying equation (5.3) by R and integrating over $\mathbf{r}$ we obtain, using integration by parts for the decaying boundary conditions $R(\mathbf{r}) \to 0$ for $|\mathbf{r}| \to \infty$ that

$$-\lambda^2 N_s - X_s + 2Y_s = 0, \tag{5.4}$$

where

$$N_s := \int R^2(\mathbf{r})d\mathbf{r} \tag{5.5}$$

is the number of particles N for the soliton solution (5.1). Similar,

$$X_s := \int |\nabla R|^2 d\mathbf{r} \tag{5.6}$$

and

$$Y_s := \frac{1}{2}\int R^4 d\mathbf{r} \tag{5.7}$$

are the kinetic energy and the absolute value of the potential energy for the soliton solution (5.1), respectively.

Multiplying equation (5.3) by $\sum\limits_{j=1}^{D} x_j\partial_{x_j} R$ and integrating over $\mathbf{r}$ we obtain that

$$\frac{D}{2}\lambda^2 N_s + \frac{D-2}{2}X_s - \frac{D}{2}Y_s = 0. \tag{5.8}$$

Equations (5.4) and (5.8) are sometimes referred to as ???? *Pokhozhaev's identities*. Using equations (5.4) and (5.8) we obtain that

$$
\begin{aligned}
X_s &= \frac{D}{4-D}\lambda^2 N_s, \\
Y_s &= \frac{2}{4-D}\lambda^2 N_s \\
H_s &= \frac{D-2}{4-D}\lambda^2 N_s.
\end{aligned}
\tag{5.9}
$$

Soliton solution (5.1) of NLSE (5.2) allows a scaling scaling transformation

$$R(\mathbf{r}) = \lambda R_0(\lambda \mathbf{r}), \tag{5.10}$$

where $R_0(\mathbf{r})$ is the soliton solution (5.1) with $\lambda = 1$, i.e.

$$-R_0 + \nabla^2 R_0 + R_0^3 = 0. \tag{5.11}$$

5.2. The Kadomtzev - Petviashwili equation

In this chapter we will derive the universal equation describing long, quasi-one-dimensional waves of small amplitude in almost all versions of Hydrodynamics with dispersion. We will study the solutions of this system which are close to simple waves of small amplitude. Thus we will assume that the density variation is small,

$$\rho = \rho_0 + \delta\rho \tag{5.12}$$

$$\frac{\delta\rho}{\rho_0} << 1 \tag{5.13}$$

and expand the simple waves' velocity function $s(\rho)$ in powers of $\frac{\delta\rho}{\rho_0}$:

$$s(\rho) = c\left(1 + \lambda\frac{\delta\rho}{\rho_0} + ...\right) \tag{5.14}$$

Here, λ is some dimensionless constant.

In the case of a polytropic gas, λ can be found from (1.6), $\lambda = \rho_0 \frac{\partial s}{\partial \rho}\big|_{\rho=\rho_0}$:

$$\lambda = \frac{\gamma+1}{2} \tag{5.15}$$

In all realistic models of a continuous medium, $\lambda > 0$. In this approximation, equation (1.2) reads

$$\frac{\partial\delta\rho}{\partial t} + c\left(1 + \lambda\frac{\delta\rho}{\rho_0}\right)\frac{\partial\delta\rho}{\partial x} = 0 \tag{5.16}$$

This equation must be accomplished by ???? high-order terms. To do this, we consider the dispersion relation (3.15). By taking the square root and assuming that $\varepsilon L^2 k^2 << 1$, one gets

$$\omega = c|k|\left(1 + \frac{1}{2}\varepsilon L^2 k^2\right) \tag{5.17}$$

Next we assume that the wave vector k has longitudinal and orthogonal components k_x and $k_\perp$, and $|k_\perp| < k_z$. Then,

$$|k| = \sqrt{k_x^2 + k_\perp^2} \approx k_x + \frac{1}{2}\frac{k_\perp^2}{k_x} \tag{5.18}$$

Finally, we can simplify expression (6.4) up to the following form:

$$\omega = c\left(k_x + \frac{1}{2}\frac{k_\perp^2}{k_x} + \frac{1}{2}\varepsilon L^2 k_x^3\right) \tag{5.19}$$

Now we should modify equation (1.3) so that, in the linear approximation, it has a solution

$$\delta\rho \approx e^{-i\omega(k)t + i\vec{k}\cdot\vec{r}} \tag{5.20}$$

There is only one such equation,

$$\frac{1}{c}\frac{\partial\delta\rho}{\partial t} + \left(1 + \lambda\frac{\delta\rho}{\rho_0}\right)\frac{\partial\delta\rho}{\partial x} - \frac{1}{2}\varepsilon L^2\frac{\partial^3\delta\rho}{\partial x^3} + \frac{1}{2}\partial^{-1}\Delta_\perp\delta\rho = 0 \tag{5.21}$$

Now we introduce the dimensionless variable $\frac{\lambda}{c}\delta\rho = 6u$ and rescale the time and spatial coordinates, $x \to L\sqrt{\frac{|\varepsilon|}{2}}x$, $r_\perp \to L\sqrt{|\varepsilon|}r_\perp$, and $t \to \frac{L}{c}\sqrt{\frac{|\varepsilon|}{2}}t$.

Then we can shift to the moving frame, $x \to x - ct$, which gives

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}\right) + \nabla_\perp u = 0 \tag{5.22}$$

This is the KP-1 equation.

If $\varepsilon < 0$, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + \nabla_\perp u = 0 \tag{5.23}$$

This is the KP-2 equation.

In the absence of dependence on perpendicular coordinates, the KP equations reduce to the Korteweg-de Vries equations,

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0 \tag{5.24}$$

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{5.25}$$

In fact, equations (6.8) and (6.9) are equivalent. Equation (6.9) turns into (6.8) with a simple transformation, $u \to -u$, $t \to -t$. This same transformation turns the KP-1 equation into the following:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}\right) - \nabla_\perp u = 0 \tag{5.26}$$

Equations (6.7) and (6.10) constitute a Hamiltonian system. They can be rewritten in the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{\delta H}{\delta u} = 0 \tag{5.27}$$

Where, for KP1,

$$H = \int\left(u^3 + \frac{1}{2}u_x^2 + \frac{1}{2}(\nabla_\perp\partial^{-1}u)^2\right)dx d\vec{r}_\perp \tag{5.28}$$

and for KP2,

$$H = \int\left(u^3 + \frac{1}{2}u_x^2 - \frac{1}{2}(\nabla_\perp\partial^{-1}u)^2\right)dx d\vec{r}_\perp \tag{5.29}$$

The full three-dimensional KP equations have numerous applications in nonlinear wave dynamics. But they are not integrable systems; only the two-dimensional equations are integrable. They are:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)-\frac{\partial^2 u}{\partial y^2}=0 \tag{5.30}$$

(KP1) and

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+\frac{\partial^2 u}{\partial y^2}=0 \tag{5.31}$$

(KP2). In the absence of dependence on y, both equations (6.14) and (6.15) lead to the same KdV equation, (6.9). This equation has the solitonic solution

$$u=\frac{2k^2}{cosh^2k[(x-x+0)-4k^2t]} \tag{5.32}$$

Here, $k>0$ and x_0 $(-\infty<x_0<\infty)$ are arbitrary real constants.

All of the solutions of the KdV equations are independent of y. One can construct a solution depending only on the variable $\chi=x-\alpha y$. Now we obtain the KdV equation in a moving frame,

$$\frac{\partial u}{\partial t}\pm\alpha^2\frac{\partial u}{\partial x}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0 \tag{5.33}$$

The sign before α^2 depends on the type of KP equation. Equation (6.17) describes the following oblique solitions:

$$u=\frac{2k^2}{cosh^2k[x-x_0-\alpha y-(4k^2\pm\alpha^2)t]} \tag{5.34}$$

It is interesting to study stationary waves in the frameworks of the KP1 and KP2 equations. To get equations of those waves, one must set $u=u(x+at,y+bt)$. Now,

$$\frac{\partial u}{\partial t}=a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}$$

and we can end up with the following family of equations:

$$\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial x}\left(6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)\pm\frac{\partial^2 u}{\partial y^2}=0 \tag{5.35}$$

Hereafter, we will only consider the special cases $b=0$, $a=\pm1$. By choosing different combinations of signs of constants a and α, we will end up with four canonical euqations known as the Boussinesq equations:

$$\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}+3\frac{\partial^2}{\partial x^2}u^2=0 \tag{5.36}$$

$$\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^4 u}{\partial x^4}+3\frac{\partial^2}{\partial x^2}u^2=0 \tag{5.37}$$

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}+3\frac{\partial^2}{\partial x^2}u^2=0 \tag{5.38}$$

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}-\frac{\partial^4 u}{\partial x^4}+3\frac{\partial^2}{\partial x^2}u^2=0 \tag{5.39}$$

If we linearize these equations and set $u \approx e^{iqy+ipx}$, the we get the dispersion relations

$$q^2 = p^2 + p^4 \tag{5.40}$$

$$q^2 = p^2 - p^4 \tag{5.41}$$

$$q^2 = -p^2 + p^4 \tag{5.42}$$

$$q^2 = -p^2 - p^4 \tag{5.43}$$

One can see that only equation (6.20) has a linearly stable basic solution, $u_0 = 0$. In all other cases, this ground state is unstable. Equations (6.21) and (6.23) are badly ill-posed, while in (6.22) the instability takes place only for long enough waves, $p^2 < 1$.

5.3. Soliton Stability

5.4. Transverse modulational instability of counterpropagating quasi-phase-matched beams in a quadratically nonlinear medium

Pattern formation is an active topic in nonlinear optics. Early theoretical investigations demonstrated that cross phase modulation between counterpropagating beams in media with cubic nonlinearity leads to a cooperative, absolute instability[**?**, **?**, **?**], and the formation of spatial patterns[**?**]. The instability was first observed using counterpropagating beams in a bulk medium[**?**]. Concurrent work revealed similar phenomena in optical cavities containing a passive nonlinear medium[**?**, **?**, **?**]. The cavity geometry is intrinsically more complex due to the interplay of linear and nonlinear resonances which leads to new features, such as pattern formation in the presence of nonlinear loss[**?**].

In the last few years the scenario of pattern formation in parametric $\chi^{(2)}$ media with a nonlinear polarization that is a quadratic function of the optical fields has been investigated extensively[**?**, **?**, **?**, **?**]. Convective modulational instability has been observed in a forward propagating traveling wave interaction[**?**]. In order to generate patterns it is necessary to provide feedback such that the system exhibits an absolute instability. One way of introducing feedback is to allow the interacting beams to counterpropagate. Although backwards parametric interactions were proposed in the sixties[**?**] there is not sufficient birefringence in available materials to phasematch a fundamental wave E_1 at frequency ω_1 to a counterpropagating second harmonic E_2 at frequency $\omega_2 = 2\omega_1$. The studies to date of pattern formation in quadratic media have therefore been based on mean field analysis of an intracavity geometry where the cavity provides the feedback necessary for an absolute instability[**?**, **?**, **?**, **?**].

In this letter we study transverse instability in a bulk quadratic medium of length L without a cavity, in the geometry shown in Fig. 1. The absence of cavity effects allows the transverse instability to be studied in a more basic form. In order to provide the necessary coupling between counterpropagating beams we consider a backwards quasi phase matched interaction [**?**, **?**] in a periodically poled material with a nonlinear susceptibility of the form $\chi^{(2)} = 2\epsilon_0 d_m \cos k_m z$, where ϵ_0 is the vacuum permittivity and d_m is the effective value of the quadratic susceptibility tensor. When $k_m \simeq k_2 + 2k_1$, where $k_i = \omega_i n_i / c$ (n_i is the refractive index of field E_i, c is the speed of light in vacuum) a forward propagating beam at ω_1 couples to a backwards propagating beam at ω_2. As was suggested in Ref. [**?**] transverse instabilities may be excited in this geometry. When the counterpropagating pump beams have equal intensities we find a simple dispersion relation that describes the presence of an absolute instability for nonzero phase mismatch ($\Delta k = 2k_1 + k_2 - k_m \neq 0$). Quasi phase matching relaxes restrictions on the choice of wavelength, and allows the largest components of the electro-optic tensor in a given material to be accessed. As we show below, threshold powers for observation of the transverse instability using, for example, periodically poled LiNbO_3 are in the range of a few MW, and thus accessible with pulsed lasers.

Counterpropagation of a fundamental field $E_1 = (\mathcal{E}_1/2)\exp[i(k_1 z - \omega_1 t)] + c.c.$ and its second harmonic $E_2 = (\mathcal{E}_2/2)\exp[i(-k_2 z - 2\omega_1 t)] + c.c.$ is described by the set

$$
\begin{aligned}
\left(\frac{n_1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} - \frac{i}{2k_1}\nabla_\perp^2\right)\mathcal{E}_1 &= i\frac{\omega_1}{cn_1}\frac{d_m}{2}e^{-\imath\Delta k z}\mathcal{E}_1{}^*\mathcal{E}_2, \\
\left(\frac{n_2}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial z} - \frac{i}{2k_2}\nabla_\perp^2\right)\mathcal{E}_2 &= i\frac{\omega_1}{cn_2}\frac{d_m}{2}e^{+\imath\Delta k z}\mathcal{E}_1^2,
\end{aligned}
\tag{5.44}
$$

where $\nabla_\perp^2$ operates on the transverse coordinates $\mathbf{r} = (x, y)$. Before proceeding it is convenient to rewrite Eqs. (5.44) in the scaled dimensionless form

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z} - i\nabla_\perp^2\right)A_1 &= A_1{}^*A_2, \\
\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z} - \frac{i}{2}\nabla_\perp^2\right)A_2 &= +i\beta A_2 - A_1{}^2,
\end{aligned}
\tag{5.45}
$$

where we have made the substitutions $\mathcal{E}_1 = (2cn_1/\omega_1 d_m L)A_1$, $\mathcal{E}_2 = -i(2cn_1/\omega_1 d_m L)A_2 \exp(i\Delta k z)$, $t \to (n_1 L/c)t$, $z \to Lz$, $\mathbf{r} \to \sqrt{L/2k_1}\mathbf{r}$, $\beta = \Delta k L$, and $n_2 \approx n_1$.

Equations (5.45) have stationary plane wave solutions

$$
\begin{aligned}
\bar{A}_1 &= a_1 \exp[ia_2 z], \\
\bar{A}_2 &= ia_2 \exp[2ia_2 z],
\end{aligned}
\tag{5.46}
$$

with a_1, a_2 real amplitudes, and a_1 positive. Solutions (5.46) exist provided the phase mismatch takes the value $\beta = -\left(2a_2 + \frac{a_1^2}{a_2}\right)$.

We then look for modulational instability using the ansatz

$$
\begin{aligned}
A_1 &= \bar{A}_1\left(1 + f_+(z)e^{\imath\mathbf{k}_\perp\cdot\mathbf{r}} + f_-(z)e^{-\imath\mathbf{k}_\perp\cdot\mathbf{r}}\right)e^{\nu t}, \\
A_2 &= \bar{A}_2\left(1 + b_+(z)e^{\imath\mathbf{k}_\perp\cdot\mathbf{r}} + b_-(z)e^{-\imath\mathbf{k}_\perp\cdot\mathbf{r}}\right)e^{\nu t},
\end{aligned}
\tag{5.47}
$$

where $\pm\mathbf{k}_\perp$ is the transverse wavevector of the sidebands in scaled dimensionless form. Linearization of Eqs.(5.45) in the amplitudes of the sidebands $f_\pm, b_\pm$ gives the set

$$
\begin{aligned}
(\frac{d}{dz} + \nu + ik_d)f_+ &= ia_2(-f_+ + f_-^* + b_+), \\
(\frac{d}{dz} + \nu - ik_d)f_-^* &= -ia_2(f_+ - f_-^* + b_-^*), \\
(\frac{d}{dz} - \nu - i\frac{k_d}{2})b_+ &= -i\frac{a_1^2}{a_2}(2f_+ - b_+), \\
(\frac{d}{dz} - \nu + i\frac{k_d}{2})b_-^* &= i\frac{a_1^2}{a_2}(2f_-^* - b_-^*),
\end{aligned} \tag{5.48}
$$

where $k_d = k_\perp^2$. Equations (5.48) together with the boundary conditions

$$
f_\pm(0) = b_\pm(1) = 0 \tag{5.49}
$$

form a well posed boundary value problem for the eigenvalue ν which is the instability growth rate of the sidebands. The solvability condition for the boundary value problem defined by Eqs.(5.48,5.49) subject to the requirement $\mathrm{Re}(\nu) = 0$ gives an equation for the instability threshold. For general values of the parameters of the problem the resulting expression is cumbersome. In the restricted case of ground state amplitudes of equal moduli $a_1 = \pm a_2$, and assuming further that at threshold the sidebands are not frequency shifted with respect to the ground state ($\mathrm{Im}(\nu) = 0$) we find a compact form for the threshold condition:

$$
\begin{aligned}
&2(2a_2 - k_d)\cos w_p \cos w_m + \\
&\frac{12a_2^3 - 4a_2^2 k_d + a_2 k_d^2 + k_d^3}{w_p w_m} \sin w_p \sin w_m + \\
&\frac{40a_2^3 - 36a_2^2 k_d + 6a_2 k_d^2 + 9k_d^3}{8a_2^2} = 0,
\end{aligned} \tag{5.50}
$$

where

$$
\begin{aligned}
w_{p,m} &= \sqrt{\frac{-12a_2^2 + 12a_2 k_d + 5k_d^2 \pm w}{8}}, \\
w &= \sqrt{144a_2^4 - 32a_2^3 k_d - 40a_2^2 k_d^2 + 24a_2 k_d^3 + 9k_d^4}.
\end{aligned}
$$

The dashed line in Fig. 2 shows a_2 as a function of k_d found from solving Eq. (5.50) for the lowest branch of the transverse instability threshold. The minimum threshold $a_2 \simeq 1.94$ occurs for $k_d \simeq 2.79$. Note that in addition to the threshold curve shown in Fig. 2 other solutions of (5.50) exist corresponding to higher lying threshold curves. For a_2 negative Eq.(5.50) has no solutions. It may be noted that in the cascading limit of large β Eq. (5.45b) reduces to $A_2 = -(i/\beta)A_1^2$ which allows Eqs. (5.45) to be rewritten as a single equation for A_1 with an effective cubic nonlinearity that takes the form $-(i/\beta)|A_1|^2 A_1$. The nonlinearity is self-focusing for a_2 positive and self-defocusing for a_2 negative. We can thus state that modulational instability without frequency shifts occurs only under conditions corresponding to a self-focusing nonlinearity.

In order to verify that the analytic solution obtained for $\mathrm{Im}(\nu) = 0$ corresponds to the lowest threshold Eqs. (5.48,5.49) were solved numerically for $a_2 = \pm a_1$ and arbitrary complex ν. It turns out that for $a_2 > 0$ and $k_d < 4.61$ the solution found from solving Eq. (5.50) gives the lowest instability threshold. At $a_2 \simeq 3.26$, $k_d \simeq 4.61$ a bifurcation occurs and for larger k_d the lowest instability threshold is obtained in the presence of frequency detuning ($\mathrm{Im}(\nu) \neq 0$), as shown by the solid and dotted curves in Fig. 2.

For $a_2 < 0$ all solutions were found numerically from Eqs. (5.48,5.49), and they are accompanied by frequency shifts as shown in Fig. 2. The lowest instability threshold occurs on the solution branch shown in Fig. 2 at $a_2 \simeq -1.68$ and $k_d \simeq 2.28$. Additional branches with higher thresholds are not shown in the figure, although they cross the depicted branch at several places for $k_d \lesssim 0.7$. Note that for both signs of a_2 the solution branches are duplicated since the sign of $\mathrm{Im}(\nu)$ is arbitrary.

It is of interest to estimate the pump power necessary for experimental observation of the transverse instability. The pump beam irradiance is given by $I = (2\epsilon_0 c^3 n_1^3/\omega_1^2 d_m^2 L^2)a_1^2$ in units of W/m^2. The characteristic transverse spatial scale of the instability is $\Lambda = 2\pi\sqrt{L/2k_1 k_d}$. Assuming a gaussian type pump beam and defining the ratio of the gaussian beam diameter d_g to the spatial scale Λ by $m = d_g/\Lambda$, we find an expression for the required pump power $P = (\pi^3 m^2 cL/4n_1\omega_1 k_d)I$. Note that m cannot be too small since that would imply that the generated sidebands do not overlap spatially with the pump beams over the length L of the crystal. Simple geometrical arguments lead to the approximate requirement $m > (2/\pi)k_d$. Although the minimum instability threshold found in Fig. 2. occurs at $|a_2| \simeq 1.7$ it is accompanied by frequency shifts that may complicate experimental observations. We will therefore estimate on the basis of the slightly higher threshold at $k_d \simeq 2.7$, $a_1 = a_2 \simeq 1.9$. Assuming $m = 5$, a pump wavelength of 1.06 μm, a crystal length of 1 cm, $n_1 = 2.2$, and $d_m \sim 30$ pm/V which corresponds to $LiNbO_3$, we find $I \sim 6.4$ MW/cm^2, $\Lambda \sim 75$ μm, and $P \sim 3.5$ kW. This level of pump power is readily available with a nanosecond pulsed Nd:YAG laser. Note that in the scheme considered here the same pump intensity must be provided at both the fundamental and second harmonic frequencies.

Currently available poling techniques cannot, however, meet the requirement of $\Lambda_m = 2\pi/k_m \sim 120$ nm. An alternative is to quasi phase match to a high order of a longer period poling. This technique was used recently in an experimental demonstration of backwards second harmonic generation[**?**]. For square-shaped modulation of the nonlinear coefficient the effective nonlinearity is $d_{m,eff} = (4/\pi)d_m/p$ where p is the order of the phase matching. A realizable poling period of 3.5 μm gives $p = 29$, $P \sim 1.8$ MW, and a peak irradiance $I \sim 3.3$ GW/cm^2, that is still well below damage thresholds for $LiNbO_3$.

Finally it should be mentioned that at these irradiance levels two-photon absorption of the second harmonic beam may increase the threshold for observation of the instability. For a second harmonic beam at 0.53 μm recent measurements of the two-photon absorption coefficient in bulk $LiNbO_3$ indicate a value of $\beta_a = 2.5 \times 10^{-12}$ m/W [**?**]. Thus at the suggested irradiance of $I = 3.3$ GW/cm^2 the irradiance reduction experienced in a crystal of length $L = 1$ cm is about $\exp(-L\beta_a I) = 0.43$ or 57%. Note that the absorption occurs on the second harmonic beam but not on the counterpropagating fundamental. We expect that the power requirements for experimental observations will be increased by not more than a factor of 2-3.

Thus have demonstrated the presence of transverse modulational instability of counterpropagating beams in a bulk $\chi^{(2)}$ medium without cavity feedback. By analogy with similar absolute instabilities in media with a $\chi^{(3)}$ nonlinearity we expect that the nonlinear stage of the instability will result in the formation of spatial patterns. Numerical estimates indicate that the effect should be observable using currently available quasi phase matched media.

5.5. Wave Collapses

The Vlasov-Petrishchev-Talanov criterion [**?**] of a wave collapse within the framework of two-dimensional nonlinear Schrodinger equation, found in 1971, is a cornerstone in the theory of wave collapses. This was the first rigorous result for nonlinear systems with dispersion, which showed the possibility of the formation of a wave-field singularity in a finite time, despite the presence of the linear dispersion of waves, the effect impeding the formation of point singularities (focii) in the linear optics.

The nonlinear Schrodinger equation (NLSE) is written for the wave function ψ and, in terms of dimensionless variables, has the form

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + |\psi|^2\psi = 0. \tag{5.51}$$

Hereafter, the subscript t denotes a partial derivative over time. The NLSE describes the motion of a quantum-mechanical particle in a self-consistent potential with attraction: $U = -|\psi|^2$. Just the attraction is a cause of the singularity formation. From the quantum-mechanical viewpoint, the collapse within the framework of the nonlinear Schrodinger equation can be interpreted as the fall off a particle to an attracting center in a self-consistent potential [**ZK86**].

Equation (5.51) is often called the Gross-Pitaevsky equation [**?**]. In particular, this equation fairly accurately describes long-wavelength oscillations of the condensate of a weakly imperfect Bose gas with negative scattering length. At present, Eq. (5.51) is the basic model for studying the nonlinear dynamics of Bose condensates (see, e.g., [**?, ?**]). In nonlinear optics, two-dimensional Eq. (5.51) describes the stationary self-focusing of light in a medium with Kerr nonlinearity. In this case, the wave function is the envelope of the field of a quasi-monochromatic electromagnetic wave, the time t has the meaning of the coordinate along the direction of light-beam propagation, and the second term in Eq. (5.51) describes diffraction of the beam in the transverse direction.

The Vlasov-Petrishchev-Talanov criterion follows from th e following relationship for the second time derivative of the mean square size $\langle r^2\rangle = N^{-1}\int r^2|\psi|^2 d\mathbf{r}$ of the distribution:

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 d\mathbf{r} = 8H, \tag{5.52}$$

where

$$N = \int |\psi|^2 d\mathbf{r}$$

is the total number of quasi-particles and the Hamiltonian H is given by

$$H = \int |\nabla\psi|^2 d\mathbf{r} - \frac{1}{2}\int |\psi|^4 d\mathbf{r} \equiv X - Y. \tag{5.53}$$

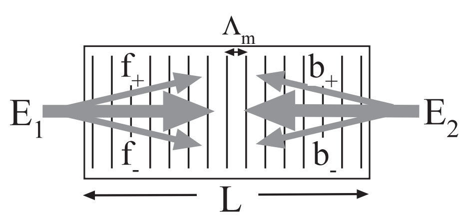

FIGURE 1. Interaction of counterpropagating fundamental E_1, and second harmonic E_2, beams in a $\chi^{(2)}$ medium poled with period Λ_m. Spatial sidebands of amplitude $f_\pm, b_\pm$ are generated inside the medium.

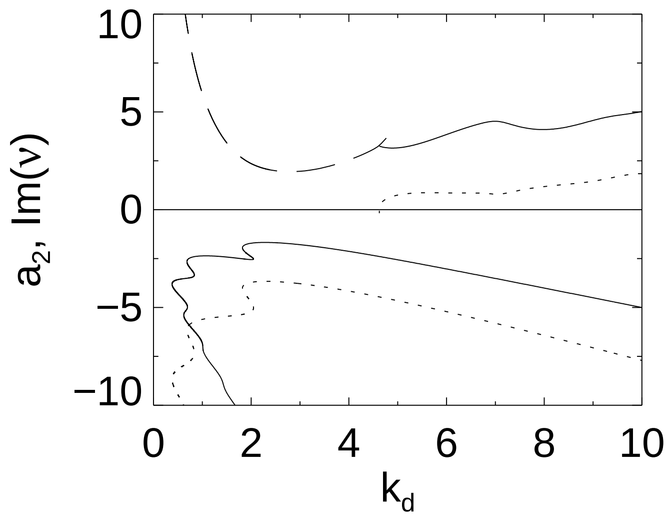

FIGURE 2. Pump amplitude a_2 and frequency shifts $\mathrm{Im}(\nu)$ at the threshold for transverse instability. The dashed line is a_2 found from Eq. (5.50), while the solid and dotted lines are a_2 and $\mathrm{Im}(\nu)$ respectively, found from numerical solution of Eqs. (5.48,5.49).

Equality (5.52) is verified by the direct calculation. This relationship is often called the virial theorem since the quantity $N\langle r^2\rangle$ can be considered the inertia moment. In the classical mechanics, the simplest way of deriving the virial theorem, i.e., the relationship between the average kinetic and potential energies, consists namely in calculating the second time derivative of the total inertia moment of the whole system.

Since H is a conserved quantity, Eq. (5.52) can be integrated two times. This yields the relationship

$$\int r^2|\psi|^2 d\mathbf{r} = 4Ht^2 + C_1 t + C_2, \tag{5.54}$$

where $C_{1,2}$ are the additional integrals of motion. The existence of these integrals is explained by two Noether symmetries: the lens transform (this fact was established by V.I. Talanov [**?**] in 1970) and the scaling transforms [**?**] (see also [**?, ?**]).

The Vlasov-Petrishchev-Talanov criterion follows immediately from Eq. (5.54). The mean square size $< r^2 >$ of any field distribution with negative Hamiltonian

$$H < 0, \tag{5.55}$$

and arbitrary C1 and C2, becomes zero in a finite time, which, with allowance for the conservation of N, is indicative of the formation of a singularity of the field ψ.

In the three-dimensional case ($D = 3$), Eq. (5.52) is replaced by

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 d\mathbf{r} = 4(2H - Y), \tag{5.56}$$

and equality (5.54), by the inequality

$$N\langle r^2\rangle < 4Ht^2 + C_1 t + C_2\,, \tag{5.57}$$

where $C_{1,2}$ are the integration constants determined by the initial conditions. The last relationship yields the same sufficient criterion of the collapse as for $D = 2$: $H < 0$ [?].

Being one of the most striking, the Vlasov-Petrishchev-Talanov criterion became well known mostly in the physical literature, while mathematicians recognized the paper [?] later. In 1974, Levine [?] proposed to search sufficient criteria of collapses using majorizing second-order differential inequalities for a positively defined quantity R, which have the form (see also [?])

$$R_{tt}R - (1+\alpha)R_t^2 \geq f(R)\,, \tag{5.58}$$

where the number $\alpha > 0$ and the function $f(R)$ are specified based on the model being studied. The collapse corresponds to tending of the quantity R to infinity in a finite time.

The simple ansatz (see [?])

$$R = A^{-1/\alpha}, \tag{5.59}$$

reduces Eq. (5.58) to

$$A_{tt} \leq -\frac{\partial V}{\partial A}, \tag{5.60}$$

where

$$V(A) = \int^{A} \alpha A^{1+2/\alpha} f(A^{-1/\alpha})\,dA.$$

There is a simple mechanical interpretation of this inequality. The quantities A and $V(A)$ play the role of the coordinate and potential energy of a particle, respectively. In the particular case of NLSE (5.51), the potential is a linear function $V(A) = -8HA$ and $A = N\langle r^2\rangle$.

The time of collapse in this system corresponds to the instant when the particle reaches the coordinate origin $A = 0$. Hence one can easily obtain the necessary conditions for the potential form $V(A)$ and the initial conditions for which the particle can fall on the center, and estimate t_0, the time of collapse. To ensure that the particle reaches the coordinate origin $A = 0$, it is sufficient that the potential $V(A)$ increases monotonically with increasing A. Then, in the case of leftward motion ($A_t < 0$), inequality (5.149) can be integrated once:

$$E(t) = \frac{A_T^2}{2} + V(A) \geq E(0). \tag{5.61}$$

Here, E(0) is the particle energy at the initial time $t = 0$ and $E(t)$ is the current value of the particle energy. The sign $\geq$ in the inequality (5.61) means that the particle energy increases in the course of leftward motion, and the upper limit of the quantity t_0 is given by the integral

$$t_0 \leq \int_0^{A(0)} \frac{dA}{2\sqrt{E(0) - V(A)}}. \tag{5.62}$$

Thus, the negative initial velocity

$$A_t(0) < 0$$

is the sufficient criterion of collapse for monotonic potentials. If the initial velocity is positive, then, generally, it is possible that the particle is not rejected from the reflecting point at which its velocity is zero. However, if the derivative $\partial V/\partial A$ is bounded from below by a certain (positive) value B, then the particle will eventually reach the coordinate origin for an arbitrary initial velocity. This follows from the estimate

$$A \leq -\int_0^t dt_1 \int_0^{t_1} \frac{\partial V}{\partial A} dt_2 + C_1 t + C_2 \leq -\frac{1}{2} B t^2 + C_1 t + C_2.$$

obtained by integrating Eq.(5.149) twice over t.

In 1978, Kalantarov and Ladyzhenskaya [**?**] applied the Levine's approach for deriving the criterion of collapse for the equation of a nonlinear string, which is often called the Boussinesq equation. The problem of collapse for an integrable equation, e.g., the Boussinesq equation (see [15]), was surprising at that time and stimulated great debates. Later on, this issue was clarified in [**?, ?**] within the framework of the inverse scattering transform.

The later works aimed at obtaining integral criteria of collapses (see [**?**]-[**?**]), showed that all such criteria are somehow based on using majorizing inequalities similar to Eq. (5.58). In this review, based on the concrete examples, we show how the positively defined integral quantity R is constructed and derive the corresponding majorizing inequalities. Taking into account the form of the expression for R and the temporal behavior of this quantity, one can infer which parameters of a wave field (e.g., the field amplitude or its spatial derivatives) tend to infinity and estimate the time of collapse.

The method itself, with which inequalities similar to Eq. (5.58) can be constructed, is given a significant place in this review. This approach is based on various integral estimates. First of all, we mean inequalities of the Sobolev type, which follow from the famous embedding theorems by S. L. Sobolev, in particular, for L_p and spaces W_2^1 with the norms

$$\|u\|_p = \left[\int |u|^p dx\right]^{1/p}, \ (p > 0), \ \ \|u\|_{W_2^1} = \left[\int (|u|^2 + |\nabla u|^2) dx\right]^{1/2},$$

respectively. These equations were used for the first time to prove the stability of ion-acoustic solitons in a magnetized plasma [**?**]. The fact that the Hamiltonian is bounded was laid in the basis of this proof. This approach was widely used later to ascertain the stability of solitons of various types see, e.g., the review [**?**]). Weinstein [**?**] applied these inequalities to the collapse problem (see also [**?**]). These estimates are closely related to soliton solutions which play the role of separatrices and delimit collapsing states and the distributions spreading due to the linear effects, i.e., diffraction and dispersion of waves. In particular, this is the case for the three-dimensional NLSE.

5.6. Criterion of collapse in the three-dimensional NLSE

Equation (5.56) shows that the criterion of collapse, given by Eq. (5.55), is not sufficiently accurate for $D = 3$ since this criterion was derived using the simplest estimate for the quantity Y . In this section, we show how the estimate (5.55) can be improved and demonstrate the possibility of a collapse for a positive Hamiltonian whose threshold value is equal to the value of H corresponding to the basic soliton

(s) solution of Eq. (5.51):

$$\psi(r,t) = \lambda\phi_0(\lambda r)e^{i\lambda^2 t}. \tag{5.63}$$

Here, the normalized wave function ?0 is spherically symmetric, does not have nodes, and obeys the equation

$$-\phi_0 + \Delta\phi_0 + \mid \phi_0 \mid^2 \phi_0 = 0. \tag{5.64}$$

This implies that the total number of particles corresponding to a soliton solution is inversely proportional to λ:

$$N_s = N_0/\lambda, \tag{5.65}$$

where, according to [28], $N_0 = 18.94$. It can easily be seen that solution (5.63) is a stationary point of the Hamiltonian H for a fixed particle number N:

$$\delta(H + \lambda^2 N) = 0.$$

A number of conclusions on the possible stability or instability of soliton solutions can be drawn already from this relationship. Following [**?**], we consider scaling transforms of the wave function ψ, for which the total number of particles is conserved ($N = \text{inv}$):

$$\psi(r) \to a^{-3/2}\psi_s(r/a).$$

In this case, the Hamiltonian H becomes a function of the scaling parameter a of the transform :

$$H(a) = \frac{X_s}{a^2} - \frac{Y_s}{a^3}.$$

A soliton corresponds to the maximum of this function, which is reached at $a = 1$. This means that such a soliton can be unstable, and this instability actually exists. According to Eq. (5.65), the Kolokolov-Vakhitov linear criterion of stability [**?**]

$$\frac{\partial N_s}{\partial \lambda^2} > 0, \tag{5.66}$$

is not satisfied, which means instability of the soliton. It is also important to note that for $a \to 0$, due to the nonlinear interaction, the Hamiltonian $H(a)$ turns out to be a function unbounded from below, which implies the possibility of a collapse as the typical size of the distribution decreases. At the same time, this function tends to zero for $a \to \infty$ due to the dispersion term in $H(a)$. Therefore, a soliton solution can be treated as a separatrix delimiting collapsing and non-collapsing solutions. In what follows we use namely this fact to find a more rigorous criterion of collapse.

Between the integrals X and Y at the maximum point of the Hamiltonian, corresponding to the soliton solution, we have the relation: $2X_s = 3Y_s$. Another relationship between these integrals can be obtained by multiplying Eq. (5.64) by ψ and integrating over $\mathbf{r}$. As a result, two obtained relationships yield

$$X_s = 3N_0^2/N_s, \quad Y_s = 2N_0^2/N_s, \quad H_s = N_0^2/N_s. \tag{5.67}$$

To find a more precise estimate of the right-hand side of Eq. (5.56), we use the multiplicative variant of the Sobolev inequality [**?**, **?**]:

$$Y \leq CN^{1/2}X^{3/2}, \tag{5.68}$$

where C is the constant. The best value of C is found by minimizing the functional J:

$$C_{best} = \min\, J[\psi] = \min\left(\frac{N^{1/2}X^{3/2}}{Y}\right). \tag{5.69}$$

It is not difficult to verify that the corresponding Lagrange-Euler equation for the extremum of the functional J is identical to Eq.(5.64 for the soliton solutions. The quantity J attains its minimal value at the spherically symmetric soliton without nodes (the ground state). According to Eq. (5.67), the constant C_{best} turns out to be equal to (see, e.g., [**?**, **?**]):

$$C_{best} = \frac{2}{3\sqrt{3}}\frac{1}{N_0}. \tag{5.70}$$

As a result, the inequality (5.68) becomes

$$Y \leq \frac{2}{3\sqrt{3}}\frac{N^{1/2}}{N_0}X^{3/2}. \tag{5.71}$$

Next, consider the function

$$F = X - C_{best}N^{1/2}X^{3/2}. \tag{5.72}$$

By construction, F is the lower boundary of the Hamiltonian H for fixed N and, as a function of X, has the maximum $F = H_N = N_0^2/N$ at $X = X_N = 3N_0^2/N$. Here, H_N is the value of the Hamiltonian for the ground-state soliton. Thus, the plane (H, X) can be divided into three regions:

I. $0 < H < H_N$ for $0 < X < X_N$.

II. $H_N > H > -\infty$ for $X_N \leq X < \infty$ (this region includes the half-plane H ¡0).

III. $H > H_N$ for $0 < X < \infty$.

In the first region, for arbitrary values of H the quantity X does not exceed its maximal value $X_N = 3N_0^2/N$. Hence, according to the uncertainty relation,

$$\langle r^2 \rangle \geq \frac{9}{4}\frac{N}{X}, \tag{5.73}$$

the average $\langle r^2 \rangle$ occurs bounded from below. As a result, the collapse of the distribution as a whole turns out to be impossible. Moreover, since X is bounded from above, one can show that the formation of a singularity of the wave function ψ is also impossible for any $t > 0$ [**?**]..

For the second region we can write the following estimate

$$N\frac{d^2}{dt^2}\langle r^2 \rangle = 4(2H - Y) = 4(3H - X) \leq 4(3H - X_N) = 12(H - H_N)\,, \tag{5.74}$$

Integration of Eq. (5.74), yields the inequality

$$N\langle r^2 \rangle \leq 6(H - H_N)t^2 + C_1 t + C_2, \tag{5.75}$$

from which a more sharp criterion for three-dimensional collapse is obtained:

$$H < H_N = N_0^2/N\,. \tag{5.76}$$

Here, H_N is the value of the Hamiltonian for the ground-state soliton. The criterion (5.76), as well as the Vlasov- Petrishchev-Talanov criterion, is only sufficient.

In the third region, where $H > H_N$, the collapse of a distribution as a whole is possible only for certain initial conditions. By appropriately choosing the constants $C_{1,2}$ one can ensure that the average $\langle r^2 \rangle$ can vanish. For example, collapse takes place for $C_1 \neq 0$, which corresponds to an initially prefocused distribution, for which the value of X_0 exceeds X_N under the following constrain (see [13]):

$$C_1 \leq -\sqrt{24(H - H_N)C_2}.$$

5.7. Criterion of collapse for the nonlinear Klein-Gordon equation and the Ginzburg-Landau equation

Consider the Lorentz-invariant Klein-Gordon equation with cubic nonlinear term:

$$\eta_{tt} = \gamma\eta + \Delta\eta + |\eta|^2\eta \ . \tag{5.77}$$

The equation of such type was derived in [**KL95**] for description of the over-threshold (the positive growth rate $\gamma > 0$) behavior of surface gravity-capillary waves excited by wind due to the Kelvin-Helmholtz instability. In this case, η has the meaning of the envelope of a wave with the carrier wavevector $\mathbf{k_0} = \mathbf{n}\sqrt{g/\alpha}$, where g is the gravity acceleration, α is the surface tension coefficient, $\mathbf{n}$ is the unit vector aligned with the wind velocity, and Δ is the two-dimensional Laplace operator. An important feature of Eq. (27) is the attraction, i.e, the plus sign of the nonlinear term $|\eta|^2\eta$. The latter means that the nonlinearity does not stabilize instability of spatially homogeneous states for $\gamma > 0$, but, on the contrary, amplifies this instability and leads to an explosive growth of the amplitude η in a finite time. Consider the question on the criteria of collapse for spatially localized distributions $\eta(r,t)$. Let us introduce the positively de.defined quantity $R = \int |\eta|^2 d\mathbf{r} > 0$ and consider its temporal evolution. Equation (5.77) makes it possible to find that

$$\frac{d^2R}{dt^2} = -4H + \int [6|\eta_t|^2 + 2|\nabla\eta|^2 + 2|\gamma||\eta^2]d\mathbf{r}. \tag{5.78}$$

Here, H is the Hamiltonian of the system (5.77), which is equal to

$$H = \int (|\eta_t|^2 - \gamma|\eta|^2 + |\nabla\eta|^2 - \frac{1}{2}|\eta|^4)d\mathbf{r}.$$

Then, multiplying Eq.(5.78) by R and using the Cauchy-Bunyakowsky inequality, we arrive at the inequality similar to (5.58):

$$R_{tt}R - \frac{3}{2}R_t^2 \geq -4HR + 2\gamma R^2. \tag{5.79}$$

Applying the ansatz $R = A^{-2}$ (see Eq. (5.59)), we obtain inequality (5.149) where the potential is given by the formula

$$V(A) = -\frac{HA^4}{2} + \gamma\frac{A^2}{2}.$$

If the norm R becomes infinite in a finite time, then the solution of Eq. (5.77) will no longer be smooth, and a singularity occurs in the solution no later than the quantity R becomes infinite. The appearance of a singularity for the quantity A means that the particle reaches the coordinate origin $A = 0$ in a finite time. If the velocity A_t is negative, then we arrive at the inequality (5.61):

$$E(t) = \frac{A_t^2}{2} + V(A) \geq E(0) \ . \tag{5.80}$$

The sign of the inequality means that the particle gains energy while moving toward the center. Using this relationship, we can easily find all cases where the particle can reach the point $A = 0$. The collapse takes place in the following cases:
1) for $H < 0$ and $\gamma < 0$ if $E(0) > 0$ and $A_t < 0$;
2) for $H < 0$ and $\gamma > 0$ if $A_t(0) < 0$;
3) for $H > 0$ and $\gamma > 0$ if $A_t(0) < 0$, $A^2(0) < \gamma/(2H)$, and $E(0) < \gamma^2/(8H)$ or if $A_t(0) < 0$ and $E(0) > \gamma^2/(8H)$.
In all these cases, the upper estimate for the time of collapse t_0 is obtained using integral (5.62).

Since the particle has a certain velocity near the coordinate origin $A = 0$, the following estimate holds for the norm $R = \int |\eta|^2 d\mathbf{r}$ near the singularity:

$$\int |\eta|^2 d\mathbf{r} \geq \frac{C}{(t_0 - t)^2} \ . \tag{5.81}$$

It is important to note that estimate (5.81) does not depend on the dimension of space. This estimate corresponds to an amplitude increasing as $\eta \sim (t_0 - t)^{-1}$ and to an almost unchanged or even expanding region of the collapsing solution.

A similar case is realized for a collapse described by the Ginzburg-Landau equation of the form

$$\frac{\partial \psi}{\partial t} = \gamma\psi + \Delta\psi + 2|\psi|^2\psi. \tag{5.82}$$

This modification of the Ginzburg-Landau equation is valid near the critical point in the case of hard excitation. For example, this equation describes generation of pulses in a ring laser with a nonlinear absorbing cell bleaching with increasing intensity [**?**] and also holds near the convection threshold in binary liquids [**?**], [**?**].

If ψ is real-valued, then Eq. (5.82) reduces to the nonlinear heat transfer equation (see the classical paper [**?**] by Piskunov, Petrovsky, and Kolmogorov concerning the application of this equation to the problem of combustion).

In the case of spatially homogeneous solutions, the fact of the singularity formation for ψ can easily be verified. If $\gamma > 0$, then ψ firstly grows for small initial data and then, at the nonlinear stage, "explodes" in a finite time. If $\gamma < 0$, then the explosive regime can only be realized if $2|\psi_0|^2 > |\gamma|$.

Let us now obtain the criterion of collapse for spatially localized distributions ($\psi \to 0$ for $r \to \infty$).

The equation (5.82) belongs to the class of gradient systems, i.e., can be written as

$$\frac{\partial \psi}{\partial t} = -\frac{\delta F}{\delta \psi *}. \tag{5.83}$$

Here

$$F = \int (|\nabla\psi|^2 - \gamma|\psi|^2 - |\psi|^4) d\mathbf{r} \equiv X - \gamma N - Y \tag{5.84}$$

is the Lyapunov functional, or the free energy, since

$$\frac{dF}{dt} = -2\int |\psi_t|^2 d\mathbf{r} \leq 0. \tag{5.85}$$

Thus, F is a decreasing function of time. In particular, if $F < 0$ at the initial time instant, then the free energy F remains negative for all $t > 0$. In this case, according to Eq. (5.82) the quantity N obeys the equation

$$\frac{dN}{dt} = 2\gamma N - 2X + 4Y = -2F + 2Y, \tag{5.86}$$

which shows that the power N increases with time if F is negative. Let the ratio $-F/N$ be chosen as the functional R. If the initial conditions correspond to a negative value of F, then R is positive definite at any time. The temporal derivative of R is given by the expression

$$\frac{dR}{dt} = \frac{2\int |\psi_t|^2 d\mathbf{r}}{N} + \frac{FN_t}{N^2}. \tag{5.87}$$

The first term in this formula is estimated using the Cauchy-Bunyakowsky inequality:

$$\frac{dN}{dt} = 2\int |\psi||\psi_t| d\mathbf{r} \leq 2N^{1/2}(\int \psi_t|^2 d\mathbf{r})^{1/2}.$$

Substituting the estimate for the integral $\int \psi_t|^2 d\mathbf{r}$ into Eq. (5.87) and taking into account definition (5.84) of F, we obtain the desired majorizing inequality:

$$\frac{dR}{dt} \geq \frac{N_t}{N^2}\left[\frac{N_t}{2} + F\right] \geq 2\frac{R}{N}(-F + Y + N) \geq 2R^2. \tag{5.88}$$

This first-order differential inequality is the simplest version of the general inequality (5.58). The inequality (5.88) is easily integrated, which yields the estimate for R [**?**] (see also [**?**]):

$$2R \geq \frac{1}{t_0 - t},$$

where the time t_0 is expressed in terms of the initial value $R|_{t=0} = R_0$:

$$t_0 = \frac{1}{2R_0}.$$

This estimate can be made more stringent for $\gamma < 0$ in a similar way as was used to derive criterion (5.76). In this case, the sufficient collapse criterion yields the upper estimate for the initial value of F [**?**] :

$$F|_{t=0} < F_s,$$

where F_s is the free energy of ground-state soliton solution (5.63) of the NLSE.

5.8. Collapses in the hydrodynamic type systems

Let us begin this section with the simplest hydrodynamical model, i.e., the Euler equations for dust with negligible pressure,

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\mathbf{v} = 0, \tag{5.89}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = 0. \tag{5.90}$$

Here, we assume that the velocity $\mathbf{v}$ sufficiently rapidly tends to zero at the infinity to ensure that all the norms considered below are finite.

Equations (5.89) and (5.90) can easily be solved within the framework of the Lagrangian description. In terms of the Lagrangian variables, Eq. (5.90) describes the motion of free particles:

$$\mathbf{r} = \mathbf{a} + \mathbf{v}_0(\mathbf{a})t, \tag{5.91}$$

where $\mathbf{v}_0(\mathbf{a})$ is the initial velocity. The free motion of Lagrangian particles is the reason for breaking in the dust hydrodynamics, which corresponds to intersection of the trajectories of fluid particles. In terms of the mapping describing the transformation from the Eulerian to Lagrangian description, the breaking means that the Jacobian J becomes zero. This process represents a collapse for Eqs. (5.89) and (5.90), which results in an infinite velocity gradient and density ρ at points where the Jacobian is zero. That is why the breaking is sometimes called the gradient catastrophe.

It is worth mentioning that Eqs. (5.89) and (5.90) have important astrophysical applications. According to Ya.B. Zeldovich [**?**], the breaking is the reason for the formation of protogalaxies from the stellar dust.

The condition of breaking and the time of breaking are locally determined from the initial velocity $\mathbf{v}_0(\mathbf{a})$. In this section, we do not consider this issue in detail (see, e.g., [**?**]) and restrict ourselves to obtaining the sufficient integral criterion of breaking.

We begin with one-dimensional equation (5.90):

$$v_t + vv_x = 0\ . \tag{5.92}$$

and introduce the integrals n,

$$I_n = \int_{-\infty}^{\infty} v_x^n\ dx,$$

in which n is an integer. It can easily be obtained from E q. (5.92) that the evolution of these integrals is determined by the following recurrence relationships:

$$\frac{dI_n}{dt} = -(n-1)\ I_{n+1}. \tag{5.93}$$

In particular, for $n = 2$ and $n = 3$ we have

$$\frac{d}{dt}\int v_x^2\ dx = -\int v_x^3\ dx\ , \tag{5.94}$$

$$\frac{d}{dt}\int v_x^3\ dx = -2\int v_x^4\ dx\ . \tag{5.95}$$

Applying the Cauchy-Bunyakowsky inequality to the right-hand part of Eq. (5.94), we find

$$\frac{dI_2}{dt} \le \left(\int v_x^4 dx\right)^{1/2} I_2^{1/2}. \tag{5.96}$$

Next, substituting

$$I_4 = \int v_x^4\ dx = \frac{1}{2}\frac{d^2 I_2}{dt^2}$$

into Eq. (5.96), we arrive at the closed inequality for the integral I_2:

$$I_2 \cdot \frac{d^2 I_2}{dt^2} - 2\left(\frac{dI_2}{dt}\right)^2 \ge 0, \tag{5.97}$$

which is the particular case of Eq. (5.58) with $f(R) = 0$. Respectively, the change $I_2 = A^{-1}$ transforms Eq. (5.96) into the inequality

$$A_{tt} \le 0. \tag{5.98}$$

which implies that the criterion of collapse is given by the constraint on the initial value of the derivative dI_2/dt for $t = 0$ [**?**]:

$$I_{2t}(0) > 0$$

or

$$I_3(0) = \int v_{0x}^3 dx < 0.$$

A similar criterion for a multi-dimensional problem can also be obtained in the case where the matrix

$$U_{ij} = \frac{\partial v_j}{\partial x_i}$$

has real eigenvalues for $t = 0$. This can be realized if the following condition is valid:

$$2(\text{tr } S)^2 - \frac{2}{D}\text{tr}(S^2) \geq \omega^2, \tag{5.99}$$

Here, tr denotes the trace of a matrix, $S = 1/2(U + U^T)$ is the symmetric part of the velocity deformation matrix U, $\omega = \text{rot}\,\mathbf{v}$ is the vorticity, and the subscript T denotes transposition.

In the multi-dimensional case, the majorizing inequality is written for the quantity

$$I = \int (\det U)^2 d\mathbf{r}$$

and has the form (see the details of derivation in [**?**]):

$$I\frac{d^2 I}{dt^2} - \left(1 + \frac{1}{D}\right)\left(\frac{dI}{dt}\right)^2 \geq 0, \tag{5.100}$$

where D is the space dimension. Inequality (5.100) is reduced to Eq. (48) by means of the ansatz

$$A = \frac{1}{I^{1/D}}.$$

The criterion of collapse, as before, consists in the requirement that the initial particle velocity be negative:

$$A_t(0) < 0 \text{ or } I_t(0) > 0\ .$$

The corresponding time of collapse is estimated as follows:

$$t_0 < \frac{DI(0)}{I_t(0)} = -\frac{1}{\langle\lambda(0)\rangle},$$

where $\langle\lambda\rangle$ is the average eigenvalue of the matrix $U_{t=0}$:

$$\langle\lambda\rangle = \frac{1}{D}\sum_i \bar{\lambda}_i = \frac{1}{DI}\int \text{ tr } U \ (\det\ U)^2 d\mathbf{r}.$$

The next example of a system of hydrodynamical type is the Boussinesq equation

$$U_{tt} - U_{xx} + U_{xxxx} + (U^2)_{xx} = 0. \tag{5.101}$$

This equation yields the following dispersion relation for small-amplitude waves:

$$\omega^2 = k^2 + k^4,$$

which is identical to the Bogoliubov spectrum for oscillations of the condensate of a weakly imperfect Bose gas. The nonlinearity in Eq. (5.101) is hydrodynamical.

In the absence of dispersion, i.e., when $\omega^2 \approx k^2$, Eq. (5.101) reduces to the nonlinear acoustic equation which describes, in particular, breaking of a finite-amplitude acoustic wave (the simple Riemann wave). In this case, the function U can be considered the density fluctuation for which the natural constraint

$$\int_{-\infty}^{\infty} U dx = 0, \tag{5.102}$$

expressing the conservation of mass, should hold.

Hence, in contrast to Eqs. (5.89) and (5.90), both factors, i.e., the nonlinearity and the dispersion, are taken into account in Eq. (5.101). Solutions in the form of stationary waves dependent on $x - ct$, which are identical to the cnoidal wave solutions for the Korteweg - de Vries equation, can easily be found for Eq. (5.101). In particular, the soliton solutions of both equations are the same. Moreover, both of these equations admit the Lax representation, i.e., can be integrated by means of the inverse scattering transform [**?**, **?**]. Despite its "integrability," Eq. (5.101) has solutions of the collapsing type. This fact was demonstrated for the first time by Kalantarov and Ladyzhenskaya [**?**] in 1978. The existence of collapse in Eq. (5.101) was proved by constructing a majorizing inequality similar to Eq. (5.58) for the positive definite quantity

$$R = \int W^2 dx,$$

where $U = W_x$ (in accordance with Eq. (5.102) W is assumed to tend to zero at the infinity).

Equation (5.101) is a Hamiltonian equation, and its Hamiltonian is given by

$$H = \int [(\Phi_x)^2/2 + (W_x)^2/2 + (W_{xx})^2/2 - (W_x)^3/3] dx \equiv I_1 + I_2 + I_3 - I_4, \tag{5.103}$$

where Φ has the meaning of a velocity potential which is defined from the equations

$$W_t = \Phi_x \,, \quad \Phi_t = W_x - W_{xxx} - W_x^2. \tag{5.104}$$

To find the corresponding inequality, let us firstly find the time derivative of the quantity R:

$$R_t = -2 \int W \Phi_x dx \leq 2R^{1/2}(2I_1)^{1/2}. \tag{5.105}$$

The second derivative, according to (5.104), is equal to

$$R_{tt} = 4I_1 - 4I_2 - 4I_3 + 6I_4. \tag{5.106}$$

Then, substituting R and its derivatives (5.105) and (5.106) into the expression $R_{tt}R - (1+\alpha)R_t^2$, performing simple algebra, and taking into account definition (5.103) of H, we obtain

$$R_{tt}R - (1+\alpha)R_t^2 \geq R[-6H + 2I_2 + 2I_3 + 2(1-4\alpha)I_1]. \tag{5.107}$$

Hence it is seen that, since the integrals $I_{2,3}$ are positive, this inequality for $\alpha = 1/4$ can be reduced to the form of Eq. (5.58):

$$R_{tt}R - (1+1/4)R_t^2 \geq -6HR. \tag{5.108}$$

Applying the change

$$R = A^{-1/4},$$

we arrive at the inequality (5.149) in which the potential

$$V(A) = -\frac{1}{4}HA^6.$$

This shows that the particle can reach the coordinate origin if the initial velocity is negative and the initial energy $E(0)$ is positive. If $H > 0$, then the collapse, i.e., tending of the integral R to in.nity in a finite time, is possible provided the condition

$$A_t(0) < 0 \text{ or } \int W\Phi_x dx\Big|_{t=0} > 0.$$

is satisfied. If $H > 0$, then the collapse is also possible, but only under two additional conditions:

$$A_t(0) < 0\,,\;\; E(0) > 0,$$

which, in terms of the variables W and Φ, are written as follows:

$$R_t(0) > 0\,,\; 32(R_t(0))^2 - \frac{H}{R^{14}(0)} > 0.$$

As it is shown in [**?**], these criteria for Eq. (5.101) can be generalized to the case of periodic boundary conditions.

The majorizing inequality can also be obtained for the "improved" Boussinesq equation with negative dispersion [**?**]:

$$U_{tt} - U_{xx} - U_{xxtt} + (U^2)_{xx} = 0. \tag{5.109}$$

In contrast to the classical integrable Boussinesq equation with negative dispersion, Eq. (5.109) corresponds to the stable dispersion relation

$$\omega^2 = \frac{k^2}{1+k^2}.$$

The majorizing inequality for Eq. (5.109) is constructed for the quantity [**?**]

$$R = \int (g^2 + g_x^2)dx > 0,$$

where $g_x = U$ and $0 < \alpha \leq 1/4$.

5.9. Generalizing the Vlasov-Petrishchev-Talanov criterion

In this section, we consider how criteria similar to the Vlasov-Petrishchev-Talanov criterion can be used for studying collapses in three-dimensional wave systems. Two examples will be discussed: the generalized Kadomtsev-Petviashvili equation

$$\frac{\partial}{\partial z}\left(U_t + U_{zzz} + (n+2)(n+1)U^{n+2}U_z\right) = \Delta_\perp U \tag{5.110}$$

and the so-called hyperbolic nonlinear Schrodinger equation

$$i\frac{\partial \psi}{\partial t} + \Delta_\perp \psi - \psi_{zz} + |\psi|^2\psi = 0. \tag{5.111}$$

In these two equations $\Delta_\perp = \partial_{xx} + \partial_{yy}$ is the Laplace operator in the plane perpendicular to the z axis corresponding to the main direction of wave propagation.

5.9.1. Collapse in KP-type equations. Consider collapse in Kadomtsev-Petviashvili (KP)-type systems. We begin with the classic KP equation with the positive dispersion

$$\frac{\partial}{\partial x}\left(u_t + u_{xxx} + 6uu_x\right) = 3\Delta_\perp u, \tag{5.112}$$

which describes the propagation of a beam of weakly nonlinear acoustic waves in a medium with weak dispersion as discussed in Section ???. The second term in the left-hand side of Eq (5.112) is responsible for dispersion effects, the third term describes the effects of sound wave steepening, and the term in the right-hand side describes the diffraction

The case $n = 3$ in Eq. (5.112) corresponds to the classical Kadomtsev-Petviashvili equation. The Kadomtsev-Petviashvili equation with $n = 4$ appears in a description of waves of the acoustic type in antiferromagnets for certain angles of propagation [**?**]..

The hyperbolic NLSE (5.111) appears in a description of the evolution of quasi-monochromatic waves with negative dispersion, which corresponds to the minus sign of the term ψ_{zz} (see [**Zak68, ?, ?**]). The classical examples of such waves are the surface gravity waves on deep water and electromagnetic waves in dielectrics with normal dispersion.

We begin our analysis from the Kadomtsev-Petviashvili equation (5.112) assuming $n = 4$. Consider the quantity

$$\langle r_\perp^2 \rangle = \frac{1}{N}\int r_\perp^2 U^2 d\mathbf{r},$$

which, owing to the conservation of $N = \int U^2 d\mathbf{r}$, has the meaning of the average square radius of the beam. It was shown for the first time in [**?**] that an analog of the virial theorem (5.56) exists for the quantity $\langle r_\perp^2 \rangle$:

$$\frac{d^2}{dt^2}\int r_\perp^2 U^2 d\mathbf{r} = 48H - 8\int U_z^2 d\mathbf{r}, \tag{5.113}$$

where H is the conserved Hamiltonian, equal to

$$H = \int \left[\frac{U_z^2}{2} + \frac{\nabla_\perp W^2}{2} - U^4\right] d\mathbf{r}.$$

for $n = 4$. This immediately yields the estimate

$$\frac{d^2}{dt^2}\int r_\perp^2 U^2 d\mathbf{r} < 48H. \tag{5.114}$$

Hence, the collapse, i.e., shrinking (formally to a zero size), of a beam takes place for $H < 0$ [**?**]. It can be shown that the criterion of collapse, $H < 0$, holds for an arbitrary integer $n > 4$.

Nevertheless, the criterion of collapse for the classical Kadomtsev-Petviashvili equation, which is most important from the viewpoint of various applications, remains an open question, although the fact of unboundedness of the Hamiltonian H from below for fixed $N = \int U^2 d\mathbf{r}$ and the collapse realized in numerical simulations [**?, ?**] are indicative of the possibility of a collapse.

Concerning the hyperbolic nonlinear Schrodinger equation (5.111) in the three-dimensional case, we show how an analysis of inequalities of the virial type yields the conclusion that the collapse of a wave packet as a whole is impossible at the stage when a pulse compresses in all three directions. We should note that, strictly speaking, the proof outlined below does not exclude the possibility of singularity formation for initial conditions different from those considered in this section.

The equation (5.111) with hyperbolic second-order operator describes different behavior of a wave packet in the longitudinal direction (along the z axis) and in the direction transverse to this axis. The quasiparticles are attracted to each other in the transverse direction and tend to shrink the beam across the z axis. The masses of quasi-particles in the longitudinal direction are negative, so the nonlinearity tends to increase the packet size along the z axis. In this respect, the main question is whether there is enough time for the singularity formation due to the transverse shrinking, despite the longitudinal expansion of the quasi-particles.

The equation (5.111) belongs to the Hamiltonian one with

$$H = \int |\nabla_\perp \psi|^2 d\mathbf{r} - \int |\psi_z|^2 d\mathbf{r} - \frac{1}{2}|\psi|^4 d\mathbf{r} \equiv I_\perp - I_z - Y. \tag{5.115}$$

Consider variations in the wave-packet average square sizes $\langle r_\perp^2 \rangle$ and $\langle z^2 \rangle$ along and across the z axis, respectively. Calculations similar to (5.52) give

$$N\frac{d^2}{dt^2}\langle r_\perp^2\rangle = 4\left[2\int |\nabla_\perp\psi|^2 d\mathbf{r} - \int |\psi|^4 d\mathbf{r}\right], \tag{5.116}$$

$$N\frac{d^2}{dt^2}\langle z^2\rangle = 8\int |\psi_z|^2 d\mathbf{r} + 2\int |\psi|^4 d\mathbf{r}. \tag{5.117}$$

The quantities $\langle r_\perp^2\rangle$ and $I_\perp$ from (5.116), as well as $\langle z^2\rangle$ and I_z from Eq. (5.117), obey the uncertainty relations

$$I_\perp\langle r_\perp^2\rangle \geq N,\ \ I_z\langle z^2\rangle \geq 1/4. \tag{5.118}$$

With the help of these relations and using the definition (5.115) of H, one can estimate the right-hand sides of Eqs. (5.116) and (5.117):

$$N\frac{d^2}{dt^2}\langle r_\perp^2\rangle = 8H + 8I_z \geq -4H + 2\frac{N}{\langle z^2\rangle}, \tag{5.119}$$

$$N\frac{d^2}{dt^2}\langle z^2\rangle = -4H + .4I_z + 4I_\perp > -4H + 4\frac{N}{\langle r_\perp^2\rangle} \tag{5.120}$$

Consider now the regime of shrinking in all directions when

$$\frac{d}{dt}\langle r_\perp^2\rangle < 0,\ \ \frac{d}{dt}\langle z^2\rangle < 0,$$

which is most favorable from the collapse viewpoint, and show that a collapse, treated as a decrease in the average square transverse and/or longitudinal sizes to zero ($\langle r_\perp^2\rangle \to 0$, $\langle z^2\rangle \to 0$), is impossible in this case.

At first, we prove that the average longitudinal square size z^2 of the wave packet cannot vanish if $\frac{d}{dt}\langle z^2\rangle < 0$. Consider Eq. (5.117), from which, with account of Eq. (5.118), the closed inequality for $\langle z^2\rangle$ can be obtained:

$$N\frac{d^2}{dt^2}\langle z^2\rangle \geq 8I_z \geq 2\frac{N}{\langle z^2\rangle}. \tag{5.121}$$

This relationship can be integrated once over time if $\frac{d}{dt}\langle z^2\rangle < 0$, $d_z 2_/ dt < 0$:

$$\mathcal{E}(t) = \frac{1}{2}\left(\frac{d\langle z^2\rangle}{dt}\right)^2 - 2\ln\langle z^2\rangle \leq \mathcal{E}(0), \tag{5.122}$$

where $\mathcal{E}(0)$ is the initial value of $\mathcal{E}(t)$. If $\langle z^2\rangle \to 0$, then the left-hand side of this inequality increases to infinity due to the logarithmic term, that contradicts the inequality (5.122). Thus, compression $\langle z^2\rangle \to 0$ is impossible.

Let us now show that the collapse to zero in the transverse direction is also impossible. For this aim we multiply the inequality (5.119) by $\frac{d}{dt}\langle z^2\rangle < 0$, and the inequality (5.120) by $\frac{d}{dt}\langle r_\perp^2\rangle < 0$, sum the results, and then integrate the obtained expression over time from zero to t. As a result, we get

$$E(t) = N\frac{d\langle r_\perp^2\rangle}{dt}\frac{d\langle z^2\rangle}{dt} - 8H\langle z^2\rangle + 4H\langle r_\perp^2\rangle - 2N\ln\langle z^2\rangle - 4N\ln\langle r_\perp^2\rangle \leq E(0), \tag{5.123}$$

where $E(0)$ is the value of $E(t)$ at the initial instant of time. It follows immediately from this inequality that collapse at the shrinking stage is impossible since the first term on the right-hand side of Eq.(5.123) is positive by definition, the terms proportional to H are finite, and the logarithmic term turns out to be infinitely large for $\langle r_\perp^2\rangle \to 0$. This does not conform to the fact that the function $E(t)$ is bounded from above by the initial value $E(0)$. Hence, the collapse of a three-dimensional wave-field distribution as a whole is impossible at the most "dangerous" stage of shrinking in all directions [**?**]. Does this mean that collapse in such a system is entirely impossible? Strictly speaking, not, because, firstly, criteria similar to the Vlasov-Petrishchev-Talanov criterion are sufficient and, secondly, the above analysis shows that collapse, if possible, should be searched for regimes corresponding to the outflow of the gas of quasi-particles in the longitudinal direction, which stipulates the increase in the longitudinal size of the wave packet. However, despite significant interest in this problem (see, e.g., recent publications [**?, ?, ?**] concerning this issue), this questions remains open at present.

5.10. Critical Collapse of NLSE

The catastrophic collapse (self-focusing) of a high power laser beam has been routinely observed in nonlinear Kerr media since the advent of lasers [**Ask62, CGT64, Boy08, SS99**]. The propagation of a laser beam through the Kerr media is described by the nonlinear Schrödinger equation (NLSE) in dimensionless form,

$$i\partial_z\psi + \nabla^2\psi + |\psi|^2\psi = 0, \tag{5.124}$$

where the beam is directed along z-axis, $\mathbf{r} \equiv (x, y)$ are the transverse coordinates, $\psi(\mathbf{r}, z)$ is the envelope of the electric field, and $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. NLSE (5.124) also describes the dynamics of attractive Bose-Einstein condensate (BEC) [**PS03**] (z is replaced by the time variable in that case). In addition, NLSE emerges in numerous optical, hydrodynamic, and plasma problems, and describes the propagation of nonlinear waves in general nonlinear systems with cubic nonlinearity.

Equation (5.124) can be rewritten in the Hamiltonian form

$$i\psi_t = \frac{\delta H}{\delta\psi^*} \tag{5.125}$$

with the Hamiltonian

$$H = \int \left(|\nabla\psi|^2 - \frac{1}{2}|\psi|^4\right)d\mathbf{r}. \tag{5.126}$$

Another conserved quantity, $N \equiv \int |\psi|^2 d\mathbf{r}$, has the meaning of the optical power (or the number of particles in the BEC). The sufficient condition for the collapse is $H < 0$, while the necessary condition is $N > N_c$, where N_c is the critical power defined below.

While the large power $N \gg N_c$ typically produces multiple collapses (multiple filamentation of the laser beam [**BSL$^+$04**]) with strong turbulence behavior [**DNPZ92, LV10**], the dynamics of each collapsing filament is universal and can be considered independently. Each collapsing filament carries the power N only moderately above N_c. We consider a single collapsing filament (laser beam) centered at $\mathbf{r} = 0$. For $z \to z_c$ the collapsing solution of NLSE quickly approaches the cylindrically symmetric solution, which is convenient to represent through the following change of variables [**SS99**]:

$$\psi(r, z) = \frac{1}{L} V(\rho, \tau) e^{i\tau + iLL_z\rho^2/4}, \quad |\mathbf{r}| \equiv r, \tag{5.127}$$

Here, $L(z)$ is the z-dependent beam width, and

$$\rho = \frac{r}{L}, \quad \tau = \int_0^z \frac{dz'}{L^2(z')} \tag{5.128}$$

are blow up variables such that $\tau \to \infty$ as $z \to z_c$. Transformation (5.127) was inspired by the discovery of the additional conformal symmetry of NLSE which is called the "lens transform" [**Tal70, KT85, FP99**].

It follows from (5.124), (5.127) and (5.128) that $V(\rho, \tau)$ satisfies

$$i\partial_\tau V + \nabla_\rho^2 V - V + |V|^2 V + \frac{\beta}{4}\rho^2 V = 0, \tag{5.129}$$

where

$$\beta = -L^3 L_{zz} \quad \text{and} \quad \nabla_\rho^2 \equiv \partial_\rho^2 + \rho^{-1}\partial_\rho. \tag{5.130}$$

As $z \to z_c$, β approaches zero adiabatically slowly and $V(\rho)$ approaches the ground state soliton $R(\rho)$ [**FP99**]. The ground state soliton is the radially symmetric, z-independent solution of NLSE, $-R + \nabla_\rho^2 R + R^3 = 0$. It is positive definite, i.e., $R > 0$, with asymptotic $R(\rho) = e^{-\rho}[A_R\rho^{-1/2} + O(\rho^{-3/2})]$, $\rho \to \infty$, $A_R \equiv 3.518062\ldots$ [**FP99**]. Also R defines the critical power

$$N_c \equiv 2\pi \int R^2 \rho d\rho = 11.7008965\ldots \tag{5.131}$$

The limiting behavior in $V \to R$ as $z \to z_c$ implies that the $\partial_\tau V$ term in (5.129) is a small correction compare to the other terms. Also β can be interpreted as quantity proportional to the excess of particles above critical, $N - N_c$, in the collapsing region [**Mal93, FP99**].

(*===*)

Consider the nonlinear Schrödinger equation in the spatial dimension D with a power law nonlinearity

$$i\psi_t + \nabla^2\psi + |\psi|^{2\sigma}\psi = 0 \tag{5.132}$$

and the decaying boundary condition $\psi(\mathbf{r}, t) \to 0$ for $|\mathbf{r}| \to \infty$. Here $\sigma > 0$. Similar to cubic NLSE (2.173), NLSE (5.132) is the Hamiltonian system

$$\psi_t = -i\frac{\delta H}{\delta \bar{\psi}} \tag{5.133}$$

with the Hamiltonian

$$H = \int \left(|\nabla\psi|^2 - \frac{1}{\sigma+1}|\psi|^{2(\sigma+1)} \right) d\mathbf{r}. \tag{5.134}$$

Consider the second spatial moment of $|\psi|^2$ defined as

$$A(t) = \int |\mathbf{r}|^2 |\psi|^2 d\mathbf{r}. \tag{5.135}$$

Using (5.132), integrating by parts, and taking into account vanishing boundary conditions at infinity one obtains

$$A_t = \int 2ix_j(\psi\partial_{x_j}\bar{\psi} - \bar{\psi}\partial_{x_j}\psi)d^3\mathbf{r}, \tag{5.136}$$

where $\partial_{x_j} \equiv \frac{\partial}{\partial x_j}$ and repeated index j means summation over all space coordinates, $j = 1, \ldots, D$. After a second differentiation over t, one obtains that

$$A_{tt} = 8H - \frac{4(\sigma D - 2)}{\sigma + 1} \int |\psi|^{2(\sigma+1)} d\mathbf{r} \tag{5.137}$$

which is called by a *virial theorem.*

It follows from equation (5.137) that

$$A_{tt} \le 8H \tag{5.138}$$

if $\sigma D - 2 \ge 0$. Integrating the differential inequality (5.138) one obtains that

$$A(t) \le 4Ht^2 + A_t(0)t + A(0). \tag{5.139}$$

If $H < 0$ then it follows from the inequality (5.139) that $A(t) < 0$ for large enough t. It contradicts the positive definiteness of A which follows from the definition (5.135). Thus qualitatively we expect that the mean square width

$$\langle r^2 \rangle := \frac{\int |\mathbf{r}|^2 |\psi|^2 d\mathbf{r}}{N} \tag{5.140}$$

vanishes in the finite time. Then the conservation of N requires that $max(|\psi|) \to \infty$. To quantify that statement we use the following chain of inequalities

(*===*)

5.11. Collapse in NLSE with the nonlocal interactions and dipolar Bose-Einstein condensate

The dynamics of Bose-Einstein condensate (BEC) with short-range s-wave interaction have been the subject of extensive research in recent years [**DGPS99, DCC$^+$01, PS03**]. Condensates with a positive scattering length have a repulsive (defocusing) nonlinearity which stabilizes the condensate with the help of external trap. Condensates with a negative scattering length have an attractive (focusing) nonlinearity which formally admits solitons. However, without trap these solitons are unstable and their perturbation leads either to collapse of condensate [**FK93, DGPS99, DCC$^+$01, PS03**] or condensate expansion. External trap prevents expansion of condensate and makes solitons metastable for a sufficiently small number of atoms. Otherwise, for larger number of atoms, the focusing nonlinearity results in collapse of solitons. The effect of a long-range dipolar interaction on BEC was first studied theoretically [**YY00, GRmcdzP00, SSZL00, Lus02, CMKF09**] and more recently observed experimentally [**GWH$^+$05, LKF$^+$07, KLM$^+$08**] (see also [**Bar08, LMS$^+$09**] for review). In particular, collapse of BEC with dominant dipole-dipole forces predicted based on approximate variational estimate [**SSZL00**] and obtained based on exact analysis [**Lus02**] was recently observed in experiment [**LMF$^+$08**].

Here we look for possibility of collapse of BEC due to a long-range attraction vs. formation of stable self-trapped condensate for a general type of long-range interaction

$$V(\mathbf{r}) = \frac{f(\mathbf{n})}{r^b}, \quad b > 0, \quad \mathbf{n} \equiv \frac{\mathbf{r}}{r}, \quad r \equiv |\mathbf{r}|, \tag{5.141}$$

where $f(\mathbf{n})$ is an arbitrary bounded function $|f(\mathbf{n})| < \infty$ and $\mathbf{r} = (x_1, x_2, x_3)$. We do not require $f(\mathbf{n})$ to be sign-definite. By attractive interaction we mean that $f(\mathbf{n})$ is negative at least for some nonzero range of angles so that one can choose a wave function to provide negative contribution to energy functional.

Possible experimental realization of (5.141) are numerous. E.g., recent experimental advances allow to study interaction of ultracold Rydberg atoms with principle quantum number about 100 (see e.g. [**AVG98, HRB$^+$08, JUH$^+$08, SWM09, PMB$^+$10**]). These interactions between atoms in highly excited Rydberg levels are long-range and dominated by dipole-dipole-type forces. Strength of interaction between Rb atoms is about 10^{12} times stronger (at typical distance $\sim 10\mu m$) than interaction between Rb atoms in a ground state (see e.g. [**SWM09**] for review). Strength and angular dependence of interaction between Rydberg atoms can be tuned in a wide range [**BMZ07, SWM09**]. E.g., spatial dependence for Rb with principle quantum number $n \simeq 100$ can be $\propto 1/r^3$ for $r \lesssim 9.5\mu m$ and $\propto 1/r^6$ (van der Waals character) for $r \gtrsim 9.5\mu m$ [**SWM09**]. Another alternative is to admix Rydberg atoms with the ground state atoms creating effective long-range interaction potential [**PMB$^+$10**]. Short-range s-wave scattering interaction is limited to much smaller distance $\sim$ few nm so that the range of dominance of long-range interaction potential is quite high. Radiative lifetime of Rydberg atoms scales as n^3 and for large n that time is in ms range [**SWM09**]. If we compare that time scale with the collapse time 0.1ms of BEC with dipole-dipole interaction [**LMF$^+$08**] one can conclude that observation of BEC collapse with Rydberg atoms is feasible. Another possible form of long-range attractive interaction is gravity-like $1/r$ potential which is proposed to be realized in a system of atoms with laser induced dipoles such that an arrangement of several laser fields causes cancelation of anisotropic terms [**OGKA00**]. Terms $\propto 1/r^2$ are also possible [**OGKA00**].

5.11.1. Virial theorem for the nonlocal Gross-Pitaevskii equation. The mean-field BEC dynamics is governed by a nonlocal Gross-Pitaevskii equation (NGPE)

$$\begin{aligned} i\hbar\frac{\partial\Psi(\mathbf{r})}{\partial t} &= \Big[-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2(\gamma_1^2x_1^2 + \gamma_2^2x_2^2 + \gamma_3^2x_3^2) \\ &\quad + g|\Psi(\mathbf{r})|^2 + \int d^3\mathbf{r}'\, V(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2\Big]\Psi(\mathbf{r}), \end{aligned} \tag{5.142}$$

where Ψ is the condensate wave function, the contact interaction is $\propto g = 4\pi\hbar^2 a/m$, a is the $s-$wave scattering length, m is the atomic mass, ω_0 is the external trap frequency, γ_1, γ_2, γ_3 are the anisotropy factors of the trap, and the wavefunction is normalized to the number of atoms, $\int |\Psi|^2 d^3\mathbf{r} = N$. Contact interaction term can be also included into the potential $V(\mathbf{r})$ as $\frac{g}{2}\delta(\mathbf{r})$ but we have not done that because we focus here on effect of the long-rage potential (5.141). If $V(\mathbf{r}) \equiv 0$ then a standard Gross-Pitaevski equation (GPE) [**DGPS99**] is recovered.

NGPE (5.142) can be written through variation $i\hbar\frac{\partial\Psi}{\partial t} = \frac{\delta E}{\delta \Psi^*}$ of the energy functional

$$E = E_K + E_P + E_{NL} + E_R, \tag{5.143}$$

which is an integral of motion: $\frac{dE}{dt} = 0$, and

$$\begin{aligned} E_K &= \int \frac{\hbar^2}{2m}|\nabla\Psi|^2 d^3\mathbf{r}, \quad E_{NL} = \frac{g}{2}\int |\Psi|^4 d^3\mathbf{r}, \\ E_P &= \int \frac{1}{2}m\omega_0^2(\gamma_1^2 x_1^2 + \gamma_2^2 x_2^2 + \gamma_3^2 x_3^2)|\Psi|^2 d^3\mathbf{r}, \\ E_R &= \frac{1}{2}\int |\Psi(\mathbf{r})|^2 V(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2 d^3\mathbf{r} d^3\mathbf{r}'. \end{aligned} \tag{5.144}$$

Consider time evolution of the mean square radius of the wave function, $\langle r^2\rangle \equiv \int r^2|\Psi|^2 d^3\mathbf{r}/N$. Using (5.142), integrating by parts, and taking into account vanishing boundary conditions at infinity one obtains

$$\partial_t\langle r^2\rangle = \frac{\hbar}{2mN}\int 2ix_j(\Psi\partial_{x_j}\Psi^* - \Psi^*\partial_{x_j}\Psi)d^3\mathbf{r}, \tag{5.145}$$

where $\partial_t \equiv \frac{\partial}{\partial t}$, $\partial_{x_j} \equiv \frac{\partial}{\partial x_j}$ and repeated index j means summation over all space coordinates, $j = 1, \ldots, 3$. After a second differentiation over t, one gets [**Lus02**]

$$\begin{aligned} \partial_t^2\langle r^2\rangle = \frac{1}{2mN}\Big[&8E_K - 8E_P + 12E_{NL} \\ &-2\int |\Psi(\mathbf{r})|^2|\Psi(\mathbf{r}')|^2(x_j\partial_{x_j} + x_j'\partial_{x_j'})V(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}\Big], \end{aligned} \tag{5.146}$$

which is called by a virial theorem [**Lus02**] similar to GPE [**VPT71, Zak72, Lus95, Pit96, Ber98, LS00**].

It follows from (5.141) that $(x_j\partial_{x_j} + x_j'\partial_{x_j'})V(\mathbf{r}-\mathbf{r}') = -bV(\mathbf{r}-\mathbf{r}')$ and using (5.143) we rewrite (5.146) as follows

$$\begin{aligned} \partial_t^2\langle r^2\rangle = \frac{1}{2mN}\Big[&4bE + (8-4b)E_K - (8+4b)E_P \\ &+(12-4b)E_{NL}\Big]. \end{aligned} \tag{5.147}$$

Here the nonlocal nonlinear term E_R was absorbed into E in comparison with (5.146). Catastrophic collapse of BEC in terms of NGPE means a singularity formation, $\max|\Psi| \to \infty$, in a finite time. Because of conservation of N, the typical size of atomic cloud near singularity must vanish. The virial theorem (5.147) describes collapse when the positive-definite quantity $\langle r^2\rangle$ becomes negative in finite time implying $\max|\Psi| \to \infty$ before $\langle r^2\rangle$ turns negative. The kinetic energy E_K diverges to infinity at collapse time. Then the potential energy must also diverge to ensure conservation of the energy functional E. But the divergence of the potential energy implies that $\max|\Psi| \to \infty$ because of the conservation of N. Another way to see divergence to infinity of E_K is from the uncertainty relation $E_K \geq \frac{\hbar^2}{2m}(9/4)N/\langle r^2\rangle$ (see [**Lus95, LS00**] as well as (5.148) below) for $\langle r^2\rangle \to 0$. Generally $\langle r^2\rangle$ may not vanish at collapse (e.g. if there are nonzero values of $|\Psi|$ away from collapse center) but E_K diverges to infinity at collapse time for sure because of $\max|\Psi| \to \infty$. We use below divergence to infinity of E_K as necessary and sufficient condition of collapse formation while vanishing of $\langle r^2\rangle$ is only sufficient condition for collapse.

NGPE is not applicable near singularity and another physical mechanisms are important such as inelastic two- and three-body collisions which can cause a loss of atoms from the condensate [**DGPS99**]. In addition, multipole expansion used for derivation of the dipole-dipole-type potential is not applicable on a very short distances (about a few Bohr radii). However, as explained above, NGPE with the potential (5.141) is a good approximation for a wide range of typical interatomic distances ranging from a fraction of nm to $\sim 10\mu m$.

5.11.2. Sufficient collapse conditions for $2 \le b \le 3$**.** Consider case $2 \le b \le 3$. Then one immediately obtains from equation (5.147) for $g \le 0$ that $\partial_t^2\langle r^2\rangle \le \frac{2bE}{mN}$. Integrating that differential inequality over time we get that $\langle r^2\rangle \le \frac{bE}{mN}t^2 + \partial_t\langle r^2\rangle|_{t=0}t + \langle r^2\rangle|_{t=0}$. If $E<0$ we conclude that $\langle r^2\rangle \to 0$ for large enough t which provides a sufficient criterion of collapse of BEC. Condition $E<0$ is sufficient but not necessary for collapse. We now use generalized uncertainty relations between $E_K,\ N, \langle r^2\rangle,\ \partial_t\langle r^2\rangle$ [**Lus95, LS00**] to obtain much stricter condition of collapse. For the reader's convenience we repeat the derivation of Refs. [**Lus95, Lus02**] to show that these uncertainty relations result from the Cauchy-Schwarz inequality and the equation (5.145) with use of integration by parts ($\Psi \equiv Re^{i\phi}$, $R = |\Psi|$) as follows

$$\begin{aligned} &E_K = \frac{\hbar^2}{2m}\int \Big[(\nabla R)^2 + (\nabla\phi)^2R^2\Big]d^3\mathbf{r},\\ &\frac{2mN}{\hbar}\left|\partial_t\langle r^2\rangle\right| = 4\left|\int x_j\partial_{x_j}\phi R^2 d^3\mathbf{r}\right|\\ &\le 4\Big(N\langle r^2\rangle\int(\nabla\phi)^2R^2d^3\mathbf{r}\Big)^{1/2},\\ &N = -\frac{2}{3}\int x_jR\partial_{x_j}Rd^3\mathbf{r} \le \frac{2}{3}\Big(N\langle r^2\rangle\int(\nabla R)^2d^3\mathbf{r}\Big)^{1/2}. \end{aligned} \tag{5.148}$$

Using the equations. (5.147), (5.148) one can obtain a basic differential inequality:

$$\begin{aligned} \partial_t^2\langle r^2\rangle \le \frac{1}{2mN}\Big[4bE - (b-2)\frac{\hbar^2}{2m}\Big(\frac{9N}{\langle r^2\rangle} + \frac{m^2N(\partial_t\langle r^2\rangle)^2}{\hbar^2\langle r^2\rangle}\Big)\\ -(4+2b)m\omega_0^2NF(\gamma)\langle r^2\rangle\Big], \end{aligned} \tag{5.149}$$

where $F(\gamma) \equiv \min(\gamma_1^2,\gamma_2^2,\gamma_3^2)$ results from the estimate of upper bound of the term $\propto E_P$ in the equation (5.147). Change of variable $\langle r^2\rangle = B^{4/(b+2)}/N$ gives the following differential inequality:

$$\begin{aligned} \partial_t^2B \le \frac{b+2}{2m}\Big[bEB^{\frac{b-2}{b+2}} - (b-2)\frac{\hbar^2}{8m}\frac{9N^2}{B^{\frac{6-b}{b+2}}}\\ - \frac{b+2}{2}m\omega_0^2F(\gamma)B\Big], \end{aligned} \tag{5.150}$$

which can be rewritten as

$$B_{tt} = -\frac{\partial U(B)}{\partial B} - q^2(t), \tag{5.151}$$

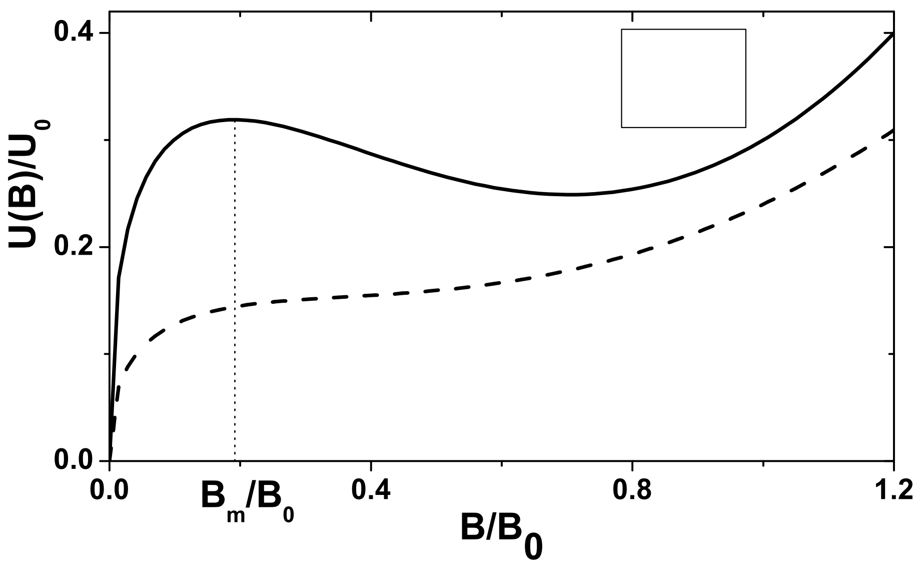

FIGURE 3. Typical behaviour of the potential $U(B)$ from the equation (5.152) for $E \leq E_{critical}$ (curve 1) $E > E_{critical}$ (curve 2). $U_0 = (N\hbar/m)^{\frac{b+2}{2}}\omega_0^{\frac{2-b}{2}}, \quad B_0 = (N\hbar/m\omega_0)^{\frac{b+2}{4}}$.

where

$$U(B) = -\frac{(b+2)^2}{4m}EB^{\frac{2b}{b+2}} + \frac{\hbar^2 9(b+2)^2N^2}{32m^2}B^{\frac{2b-4}{b+2}} + \frac{(b+2)^2}{8}\omega_0^2F(\gamma)B^2, \tag{5.152}$$

and $q^2(t)$ is some unknown nonnegative function of time. Equation (5.151) has a simple mechanical analogy [**Lus95**] with the motion of a "particle" with coordinate B under the influence of the potential force $-\frac{\partial U(B)}{\partial B}$ in addition to the force $-q^2(t)$. Due to the influence of the nonpotential force $-q^2(t)$ the total energy $\mathcal{E}$ of the "particle" is time dependent: $\mathcal{E}(t) = \frac{B_t^2}{2} + U(B)$. Collapse certainly occurs if the "particle" reaches the origin $B = 0$. It is clear that if the particle were to reach the origin without the influence of the force $-q^2(t)$ then it would reach the origin even faster under the additional influence of this nonpositive force. Therefore one can consider below the particle dynamics without the influence of the nonconservative force $-q^2(t)$ to prove sufficient collapse conditions.

It follows from the equation (5.152) that the potential $U(B)$ is a monotonic function for $E \leq 3\hbar\omega_0 N[(b^2-4)F(\gamma)]^{1/2}/(2b) \equiv E_{critical}$ (see curve 1 in Fig. 3) while for $E > E_{critical}$ the potential $U(B)$ has a barrier at $B_m^{4/(b+2)} = b\Big(E - \big[E^2 -$

$E^2_{critical}\big]^{1/2}\Big)/\big[(b+2)m\omega_0^2 F(\gamma)\big]$ with particle energy $\mathcal{E}_m = U(B_m)$ at the top (see curve 2 in Fig. 3). One can separate sufficient collapse condition into three different cases:

(a) for $E \le E_{critical}$ the particle reaches the origin in a finite time irrespective of the initial value of $B\big|_{t=0}$;

(b) for $E > E_{critical}$ and $\mathcal{E}(0) > \mathcal{E}_m$, the particle is able to overcome the barrier thus it always falls to the origin in a finite time irrespective of the initial value of $B\big|_{t=0}$;

(c) for $E > E_{critical}$ and $\mathcal{E}(0) < \mathcal{E}_m$, the particle is not able to overcome the barrier thus it falls to the origin in a finite time only if $B\big|_{t=0} < B_m$.

It is important to stress that we have proven here analytically only sufficient collapse conditions. Generally even if none of conditions a,b,c are satisfied one can not exclude collapse formation for some particular values of the initial conditions of the equation (5.142). Generally it is determined by the nonpotential force $-q^2(t)$. The inequality (5.149) reduces to equality for $\gamma_1^2 = \gamma_2^2 = \gamma_3^2$, $g = 0$ and a Gaussian initial condition with $\psi|_{t=0} = \frac{N^{1/2}}{\pi^{3/4} r_0^{3/2}} e^{-r^2/(2r_0^2)}$. For that initial condition

$$E = \frac{3\hbar^2}{4m}\frac{N}{r_0^2} + \pi^{-1/2} f N^2 r_0^{-b}\Gamma(3/2 - b/2) + \frac{3}{4} m\omega_0^2 r_0^2 N \tag{5.153}$$

provided $f(\mathbf{n}) = Const \equiv f$.

Assume that the trap contribution to E is negligible (i.e. we set $\omega_0 \to 0$) then $E < 0$ in (5.153) either if the constant r_0 is small and $b > 2$ or if $b = 2$ and $N > N_c^{(var)}$, where

$$N_c^{(var)} = -3\hbar^2/(4mf). \tag{5.154}$$

It means that for $2 \le b \le 3$ we can easily have the simplest sufficient collapse condition $E < 0$ satisfied for the long-distance potential alone (for $g = 0$), i.e. that potential alone can result in collapse of BEC. $N_c^{(var)}$ in (5.154) is the variational estimate for the critical number of particles N_c for $b = 2$. If $N < N_c$ then collapse is impossible for any initial conditions (and for any trap) as shown below. N_c is independent on the trap and it is an analog of the critical particle number for the standard two-dimensional (2D) GPE with contact interactions only.

For $2 < b < 3$ we can also introduce another critical value of particles, $N_{c,trap}$ from the condition that for $N > N_{c,trap}$ the energy E does not have minimum as a function of system parameters for fixed N. We find from the equation (5.153) that E does not have minimum for any r_0 if $N > N_{c,trap}^{(var)}$, where

$$N_{c,trap}^{(var)} = -\frac{\hbar^{5/2}}{m^{3/2}\omega_0^{1/2} f}\,\frac{6(b-2)^{\frac{b-2}{4}}\sqrt{\pi}}{b(b+2)^{\frac{b+2}{4}}\Gamma(3/2 - b/2)} \tag{5.155}$$

is the variational estimate for $N_{c,trap}$. It means that any soliton-type solution is impossible for $N > N_{c,trap}$ and collapse inevitably occurs. The critical number of particles $N_{c,trap}$ is defined for $2 < b < 3$ and is determined by the trap (without trap particles could spread unboundedly preventing collapse for the wide class of initial conditions while the trap blocks that scenario and eventually results in collapse for $N > N_{c,trap}$). $N_{c,trap}$ is the analog of the critical number of particles $N_{cr,GPE} \equiv \kappa\frac{a_{ho}}{|a|}$, $\kappa \simeq 0.5$, $a_{ho} = (\hbar/m\omega_0)^{1/2}$ in the standard three-dimensional (3D) GPE [**RHBE95, RCC⁺01**]. Both $N_{c,trap}^{(var)}$ and $N_{cr,GPE}$ are undefined without trap because formally both NGPE for $2 < b < 3$ and standard 3D GPE can have collapse for arbitrary small number of particles for appropriately chosen initial conditions (of course these equations are based on the mean-field approximation so for small $N \sim 1$ these equations will not be applicable).

For $2 < b < 3$ and $N < N_c^{(var)}$ the energy E (5.153) has a local minimum for a finite value of r_0 while $E \to -\infty$ for $r_0 \to 0$. It means that depending on initial conditions (i.e. the initial value of r_0) BEC either collapses in a finite time or stabilizes on soliton solution. That soliton solution is however metastable because of finite probability of tunneling of condensate to small values of r_0.

$b = 3$ is the special case because convergence of integral E_R at small distances requires that angular integration (integration over a sphere of radius one) of $f(\mathbf{n})$ gives zero: $\int f(\mathbf{n}) d\mathbf{n} = 0$. A particular example for $b = 3$ was considered in Ref. [**Lus02**] for the case of the dipole-dipole interaction potential with all dipoles oriented in one fixed direction. In that case indeed $\int f(\mathbf{n}) d\mathbf{n} = 0$. Also in that case E_{NL} vanishes from (5.147) which allows collapse even for $g > 0$.

If either $b = 3$ and $\int f(\mathbf{n}) d\mathbf{n} \neq 0$ or $b > 3$ then it is necessary to introduce a cutoff at a small distance r_c (typically at few Bohr radii) and the potential would loose a general form (5.141). Also for $b > 3$ the integral $\int_{|\mathbf{r}|>r_c} V(\mathbf{r}) d^3\mathbf{r}$ is finite so generally we have a very similar situation to a standard δ-correlated potential [**DGPS99**]. Thus $b = 3$ is a border between short-range potentials (for $b > 3$) and long-range potentials (for $b \leq 3$) in 3D.

5.11.3. Nonexistence of collapse for $b < 2$. Now we prove that for $b < 2$ collapse is impossible for $g = 0$ because singularity of (5.141) is not strong enough. We use the inequality $\int \frac{|\Psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|^2} d^3\mathbf{r} \leq 4 \int |\nabla \Psi(\mathbf{r})|^2 d^3\mathbf{r}$ [**Lad69**], which holds for any $\mathbf{r}'$. We generalize that inequality using the Hölder's inequality (assuming $b < 2$) as follows

$$
\begin{aligned}
&\int \frac{|\Psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|^b} d^3\mathbf{r} = \int |\Psi(\mathbf{r})|^{2-b} \frac{|\Psi(\mathbf{r})|^b}{|\mathbf{r}-\mathbf{r}'|^b} d^3\mathbf{r} \\
&\leq \left[\int \left(|\Psi(\mathbf{r})|^{2-b} \right)^{\frac{1}{1-b/2}} d^3\mathbf{r} \right]^{1-\frac{b}{2}} \\
&\times \left[\int \left(\frac{|\Psi(\mathbf{r})|^b}{|\mathbf{r}-\mathbf{r}'|^b} \right)^{\frac{2}{b}} \right]^{\frac{b}{2}} d^3\mathbf{r} \leq 2^b N^{1-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K \right)^{\frac{b}{2}} .
\end{aligned} \tag{5.156}
$$

Using now boundness of $f : f(\mathbf{n}) \leq f_m \equiv \max_{\mathbf{n}} |f(\mathbf{n})|$ in (5.141) and inequality (5.156) we obtain a bound for E_R in (5.144)

$$
E_R \geq -f_m \, 2^{b-1} N^{2-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K \right)^{\frac{b}{2}} , \tag{5.157}
$$

which gives a respective bound of E in (5.144) (recall that we assume $g = 0$):

$$
E \geq E_K - f_m \, 2^{b-1} N^{2-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K \right)^{\frac{b}{2}} \equiv P(E_K). \tag{5.158}
$$

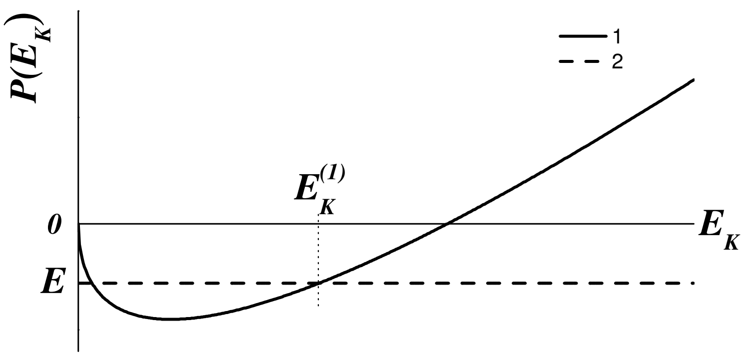

FIGURE 4. Schematic of the function $P(E_K)$ defined in (5.158) (solid curve). It is seen that the equation $P(E_K) = E$ (E is shown by dashed line) has either one of two roots for $E_K > 0$ depending on sign of E. $E_K^{(1)}$ designates the largest of these roots.

A function $P(E_K)$ in (5.158) has a minimum for $E_K = E_K^{(0)} \equiv 2^{-2}[f_m\, b]^{2/(2-b)} N^{\frac{4-b}{2-b}} (2m/\hbar^2)^{b/(2-b)}$ resulting in a lower bound

$$E \geq -\frac{2-b}{b} 2^{-2}[f_m\, b]^{\frac{2}{2-b}} N^{\frac{4-b}{2-b}} \left(\frac{2m}{\hbar^2}\right)^{\frac{b}{2-b}}. \tag{5.159}$$

Boundness of the energy functional E from below ensures that collapse is impossible for $b < 2$. To prove that we show boundness of E_K while collapse requires $E_K \to \infty$. We choose any value of E which satisfy (5.159). Fig. 4 shows schematically the function $P(E_K)$ from (5.158). Inequality (5.158) requires that $E_K \le E_K^{(1)}(E)$, where $E_K^{(1)}(E)$ is the largest root or equation $P(E_K) = E$. It proves that E_K is bounded for fixed N which completes the proof of absence of collapse for $b < 2$. Particular version of that result for $b = 1$ and $f(\mathbf{n}) = const$ was first obtained in Ref. [**Tur85**]. Nonexistence of collapse for the nonsingular potential $V(\mathbf{r})$ was shown previously based on approximate analysis in Ref. [**PGKGR00**]. Proof of nonexistence of collapse for particular example of nonsingular potentials with positive-definite bounded Fourier transform was given in Ref. [**BKWR02**]. These results can be easily generalized for any bounded potential similar to above analysis. Thus collapse can occur for singular potential only and singularity should be strong enough, i.e. $b \ge 2$.

5.12. Nonlinear stability of ground state soliton for $b < 2$

We now look for a soliton solution of NGPE (5.142) as $\Psi(\mathbf{r}, t) = A(\mathbf{r})e^{-i\mu t/\hbar}$, where μ is the chemical potential. In that case NGPE (5.142) reduces to a time-independent equation

$$\begin{aligned}&\Big[-\mu - \frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2(\gamma_1^2x_1^2 + \gamma_2^2x_2^2 + \gamma_3^2x_3^2)\\ &\quad + \int d^3\mathbf{r}'\, V(\mathbf{r} - \mathbf{r}')A(\mathbf{r}')^2\Big] A(\mathbf{r}) = 0,\end{aligned} \tag{5.160}$$

where we again assume $g = 0$ although generalization to $g \neq 0$ case is straightforward. Equation (5.160) is the stationary point of the energy functional E for a fixed number of particles: $\delta(E - \mu N) = 0$. Multiplying equation (5.160) by A and $x_j\partial_{x_j}A$ and integrating by parts one obtains using (5.141) and (5.143) that

$$\begin{aligned}&E_{K,s} = -\mu N_s \frac{b}{4-b} + E_{P,s}, \quad E_{R,s} = \mu N_s \frac{2}{4-b},\\ &E_s = -\mu N_s \frac{b-2}{4-b} + 2E_{P,s},\end{aligned} \tag{5.161}$$

where subscript "s" means values of all integrals are taken on the soliton solution. Especially simple and interesting is the case of self-trapping ($\omega_0 = 0$) when condensate is in steady state without any external trap. All integrals in that case depend on the number of particles N_s only.

Assume radial symmetry $f(\mathbf{n}) = Const < 0$ in (5.141). A ground state soliton is determined from a condition that $A(\mathbf{r})$ never crosses zero [**ZK74, KRZ86**]. To prove the ground soliton stability we show that it realizes a minimum of the Hamiltonian for a fixed N_s. One can make inequality (5.156) sharper by minimizing a functional $\mathcal{F}(\Psi) \equiv N^{1-\frac{b}{2}}\left(\frac{2m}{\hbar^2}E_K\right)^{\frac{b}{2}} / \int \frac{|\Psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|^b}d^3\mathbf{r}$. That minimum is achieved at one of stationary points $\frac{\delta\mathcal{F}}{\delta\Psi^*} = 0$ and after simple rescaling one can see that these points correspond to soliton solutions of the time-independent NGPE (5.160). Among these stationary points the minimum is achieved at the ground state soliton $\Psi_{s,ground}$. It gives a bound $\mathcal{F}(\Psi) \ge \min\mathcal{F}(\Psi) = \mathcal{F}(\Psi_{s,ground})$ which is sharper than the inequality (5.156). Following an analysis similar to equations (5.157)-(5.159) we obtain that for any Ψ

$$E \geq \min E = E_{s,ground}, \tag{5.162}$$

i.e. the ground state soliton solution attains the minimum of E for fixed N. It proves exactly the stability of soliton for $f(\mathbf{n}) = Const$. Similar ideas were used in a nonlinear Schrödinger equation (NLS) which is GPE with $\omega_0 = 0$ [**ZK74, KRZ86**]. The ground state soliton was also found numerically for $b = 1$ [**CFMW08**].

For more general $f(\mathbf{n}) \neq Const$ minimum of E is still negative if $f(\mathbf{n})$ is negative for a nonzero range of values of $\mathbf{n}$. So in that case we expect that the ground state soliton solution attains that minimum and, respectively, is stable. If $f(\mathbf{n}) > 0$ for any $\mathbf{n}$ then $\min E = 0$. It corresponds to the unbounded spatial spreading of NGPE solution for any initial conditions. Self-trapping is impossible in that case and solitons are possible for $\omega_0 \neq 0$ only.

Case $b = 2$ is on the boundary between bounded and unbounded energy functional as can be seen from inequalities (5.158) and (5.162). If $N > N_{s,ground}$ then E is unbounded. If $N < N_{s,ground}$ then $E \geq E_{s,ground} = 0$ as follows from (5.161) for $b = 2$. Thus $N_{s,ground}$ is the critical number of particles for collapse: $N_c = N_{s,ground}$. This is the exact result compare with the variational estimate (5.154). That critical particle number N_c is similar to to the critical particle number for the collapse of the standard 2D GPE (as well as similar to the critical power in nonlinear optics) [**VPT71**]. As we discussed above, it is important to distinguish N_c from the critical number of particles of 3D GPE with $\omega_0 \neq 0$ [**DGPS99, DCC$^+$01**].

5.12.1. Weak and strong regimes of collapse. To qualitatively distinguish different regimes of collapse and solitons one can consider, in addition to the exact analysis above, a scaling transformations $\Psi(\mathbf{r}) \to L^{-3/2}\Psi(\mathbf{r}/L)$ [**ZK86**] which conserves the number of particles. Under this transformation the energy functional E (for $\omega_0 = 0$) depends on the parameter L as follows

$$E(L) = L^{-2}E_K + L^{-b}E_R. \tag{5.163}$$

The virial theorem (5.147) and the relations (5.161), (5.163) have striking similarities with GPE if we replace b by the spatial dimension D in GPE. That analogy suggests to refer the case $b = 2$ as the critical NGPE and $b > 2$ as the supercritical NGPE because cases $D = 2$ and $D > 2$ are called by critical and supercritical ones, respectively, for standard GPE (NLS) [**ZK86**]. similarly, we refer to collapse for $b = 2$ as a critical collapse and for $b > 2$ as a supercritical collapse. Fig. 5 shows a typical dependence of (5.163) on L for $b > 2$, $b = 2$ and $b < 2$ assuming $E_R < 0$. For $b > 2$ there is a maximum of E (curve 1 in Fig. 5) corresponding to unstable soliton. Solution of NGPE either collapses or expands. For $b = 2$ there is no extremum and collapse is impossible for $N < N_{s,ground}$ (curve 2 in Fig. 5) while condensate can collapse for $N > N_{s,ground}$ (curve 3 in Fig. 5). The ground state soliton corresponds to $N = N_{s,ground}$ and $E = 0$ locating exactly at the boundary between collapsing and noncollapsing regimes. For $b < 2$ there is a minimum which corresponds to the stable ground state soliton (curve 4 in Fig. 5).

Solutions of both GPE and NGPE with $\omega_0 = 0$ near collapse typically consist of background of nearly linear waves and a central collapsing self-similar nonlinear core. The scaling (5.163) describes the dynamics of the core with time-dependent $L(t)$ such that $L(t) \to 0$ near collapse. Waves have negligible potential energy

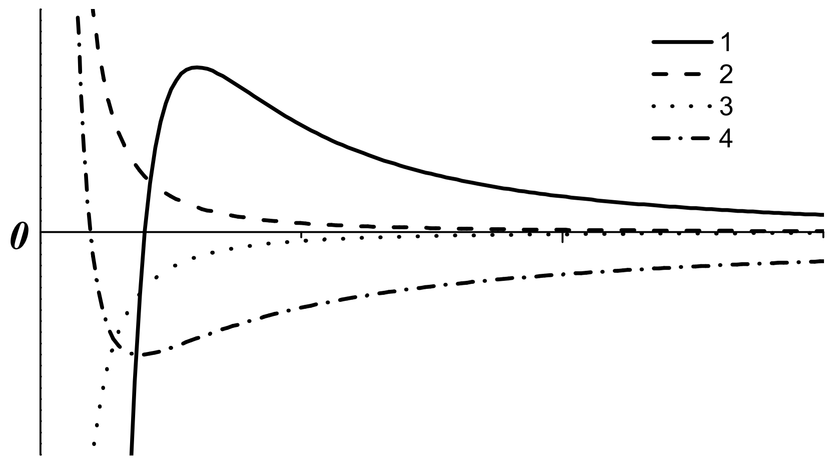

FIGURE 5. Schematic of $E(L)$ from (5.163) for $b > 2$ (curve 1), $b = 2$ and $N < N_{s,ground}$ (curve 2), $b = 2$ and $N > N_{s,ground}$ (curve 3) and $b < 2$ (curve 4).

but carry a positive kinetic energy $E_{waves} \simeq E_{K,waves}$. The total energy $E = E_{collapse} + E_{waves}$ is constant, where $E_{collapse}$ is the core energy.

It follows from (5.163) that for $b = 2$ one can simultaneously allow conservation of N and $E_{collapse}$ so that negligible number of particles are emitted from the core. This scenario is called a strong collapse with the self-similar collapsing core centered at $\mathbf{r} = \mathbf{0}$ and approximated as

$$|\psi_{c,strong}(\mathbf{r}, t)| \simeq \frac{1}{L(t)^{3/2}} \chi\left(\frac{\mathbf{r}}{L(t)}\right), \; L(t) \to 0 \text{ for } t \to t_0, \tag{5.164}$$

where the function $\chi(\xi)$ with $\xi \equiv \mathbf{r}/L(t)$ describes the spatial structure of the collapsing solution and t_0 is the collapse time.

The equation (5.164) is applicable for $|\xi| < \xi_c$, where $\xi_c \gtrsim 1$. The number of particles $N_{collapse,strong}$ in the collapsing solution is nearly constant provided ξ_c is nearly constant: $N_{collapse,strong} \simeq \int_{|\mathbf{r}|<\xi_c L(t)} |\psi_{c,strong}(\mathbf{r}, t)|^2 d^3\mathbf{r} = \int_{|\xi|<\xi_c} \chi^2(\xi) d^3\xi \sim 1$. Thus the critical collapse is always strong one with $N_{collapse,strong} \simeq N_c$. Substitution of (5.164) into NGPE allows to conclude that all terms are of the same order in powers of L (except the trapping potential E_P which is not important near collapse, as well as we assume $g = 0$) if $L(t) \propto (t_0 - t)^{1/2}$. By analogy with 2D GPE which has a critical collapse [**Fra85, LPSS88, DNPZ92**] we also expect to observe logarithmic corrections to $(t_0 - t)^{1/2}$ which is a typical feature of critical collapses in many systems (see e.g. [**Lus10**]). If $N \gg N_c$ then multiple collapses will occur each capturing about N_c particles which is the analog of multiple filamentation turbulence and beam spray in nonlinear optics and laser-plasma ineractions[**LV10, LR06**]. The universality in the number of particles captured in each collapse $N_{collapse,strong} \simeq N_c$ holds for the critical collapse only and does not hold for the supercritical case.

In supercritical case $2 < b \leq 3$ the term $\propto L^{-b}$ in (5.163) dominates with $E_{collapse} \to -\infty$ as $L(t) \to 0$. Then the only way to ensure a conservation of E is to assume a strong emission of linear waves (particles) from the collapsing core. Near the collapse time t_0 only a vanishing number of particles remains in the core (of course all that is true until NGPE losses its applicability) which is called a weak collapse [**ZK86**]. Instead of the self-similar solution (5.164), the weak collapse is described by another type of self-similar solution:

$$|\psi_{c,weak}(\mathbf{r},t)| \simeq \frac{1}{L(t)^{\alpha}}\eta\left(\frac{\mathbf{r}}{L(t)}\right),\ L(t) \to 0 \text{ for } t \to t_0, \\ \alpha = \frac{5-b}{2}, \tag{5.165}$$

where the function $\eta(\xi)$ with $\xi \equiv \mathbf{r}/L(t)$ describes the spatial structure of the collapsing solution and $\alpha = \frac{5-b}{2}$ is chosen from the condition that a substitution of (5.165) into NGPE allows the same leading order in powers of L for linear (the kinetic energy) and nonlinear (the potential energy from $V(\mathbf{r})$) terms. Here we again neglect the trapping potential and assume $g = 0$. Assuming now that left-hand-side of NGPE is of the same order in power of $L(t)$ we obtain that $L(t) \propto (t_0 - t)^{1/2}$. Similar to (5.164), we assume that the equation (5.165) is applicable for $|\xi| < \xi_c$, where $\xi_c \gtrsim 1$. Number of particles $N_{collapse,weak}$ in the collapsing solution approaches to zero near collapse: $N_{collapse,weak} \simeq \int_{|\mathbf{r}|<\xi_c L(t)} |\psi_{c,weak}(\mathbf{r},t)|^2 d^3\mathbf{r} = L(t)^{b-2}\int_{|\xi|<\xi_c}\eta^2(\xi)d^3\xi \sim L(t)^{b-2} \to 0$ for $t \to t_0$.

Solution of NGPE in the form of the strong collapse (5.164) can be also considered for $2 < b \leq 3$ which results in the dominance of the nonlinear interaction and time-dependent terms in NGPE over the kinetic energy term. It was shown that a similar solution for supercritical GPE [**ZK86**] is unstable and we expect that it might be also unstable for NGPE. Thus supercritical collapse can be either weak or strong ones but weak one appears to be more probable.

5.13. From multiple collapses to collapse turbulence: Non-Gaussian Statistics of Multiple Filamentation

The self-focusing and multiple filamentation of an intense laser beam propagating through a Kerr media has been the subject of intense research since the advent of lasers [**?, SS99**]. Multiple filamentation has many applications ranging from laser fusion [**?, LR04, LR06**] to the propagation of ultrashort pulses in the atmosphere [**?, ?**]. Here we study the statistics of multiple filamentation, which can be viewed as an example of strong optical turbulence with intermittency [**DNPZ92**], i.e. strong non-Gaussian fluctuations of the amplitude of the laser field [**Fri95**].

Long non-Gaussian tails of the PDF of light amplitude fluctuations have been previously observed in filamentation experiments [**?**] and optical rogue waves [**?**]. Long tails were obtained in solutions of the complex Ginzburg-Landau equation with a quintic nonlinearity [**?**]. The analytical form of a long tail of PDF for velocity gradient dominated by near-singular shocks was obtained in solutions of the forced Burgers equation [**?**].

Here we describe the propagation of a laser beam through the amplified Kerr media by the regularized nonlinear Schrödinger equation (RNLS) in dimensionless form,

$$i\partial_z\psi + (1 - ia\epsilon)\nabla^2\psi + (1 + ic\epsilon)|\psi|^2\psi = i\epsilon b\psi, \tag{5.166}$$

where the beam is directed along z-axis, $\mathbf{r} \equiv (x, y)$ are the transverse coordinates, $\psi(\mathbf{r}, z)$ is the envelope of the electric field, and $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. The term with a describes a wavenumber-dependent linear absorption, the term with c corresponds to two-photon absorption, and b is the linear gain coefficient. It is assumed that the wavenumber-independent part of linear absorption is included in b so that the a-term corresponds to the expansion of the general absorption coefficient near the carrier wavenumber of the laser beam in the Fourier domain. Here, $\epsilon \ll 1$ and we generally assume a, c, $b \sim 1$. RNLS (5.166) is also called the complex Ginzburg-Landau equation.

RNLS (5.166) can be realized experimentally in numerous systems, including e.g. the propagation of a laser beam in a ring cavity with a thin slab of Kerr media and amplification. In this case the nonlinear phase shift of the laser beam at each round trip is small so we can obtain (5.166) in a mean-field approximation [**?, LS00**] with z corresponding to the number of round trips in the cavity. RNLS (5.166) also describes the multiple filamentation of an intense, ultrashort laser beam in a Kerr media if we average over the temporal extent of the pulse [**?**]. E.g., multiple filamentation experiment [**?**] with two-photon-dominated absorption corresponds to $c\epsilon \simeq 0.025$.

Neglecting dissipation and amplification, we recover the nonlinear Schrödinger equation (NLS)

$$i\partial_z\psi + \nabla^2\psi + |\psi|^2\psi = 0. \tag{5.167}$$

NLS describes a catastrophic collapse (also called wave collapse) of filaments, $\max_{\mathbf{r}} |\psi| \equiv |\psi|_{max} \to \infty$, in a finite distance along z, if the optical power $N = \int |\psi|^2 d\mathbf{r}$ is above the critical power $N_c \simeq 11.701$ [**SS99**].

The optical power is not conserved in RNLS (5.166) for $\epsilon \neq 0$. If $b > 0$, the amplification term on the right hand side of (5.166) results in an increase of N. If $b = 0$ we assume that $N \gg N_c$ for $z = 0$ (e.g. $N/N_c \sim 10^4$ in [**?**]). In both cases the modulational instability [**SS99**] leads to the growth of perturbations of the beam and seeds multiple collapsing filaments. These two cases are called forced and decaying turbulence, respectively, in reference to turbulence in the Navier-Stokes equations [**Fri95**]. The statistical properties of these two cases are similar, provided in the decaying case we consider distances along z at which a relative cumulative decay of N is small.

In this Letter we focus on a forced case in which a dynamic balance is achieved between the pumping of optical power into the laser beam and dissipation. Figure 6(a) shows the evolution of $N(z)$ obtained from a numerical solution of RNLS (5.166). The optical power grows until reaching a statistical steady-state corresponding to fully developed optical turbulence with $N \simeq 1200$. In this regime the amplitude $|\psi|$ is characterized by the random distribution of filaments in $\mathbf{r}$ and z, as seen in the snapshot of $|\psi|$ for a fixed z in Figure 7. Dissipation is important only when the amplitude of each collapsing filament is near to its maximum (see Figure 6(b)) and as well as for large wavenumbers k. When $|\psi|_{max}(z)$ goes through a maximum, N experiences a fast decay due to dissipation. The influence of periodic boundary conditions on the statistical properties of optical turbulence can be

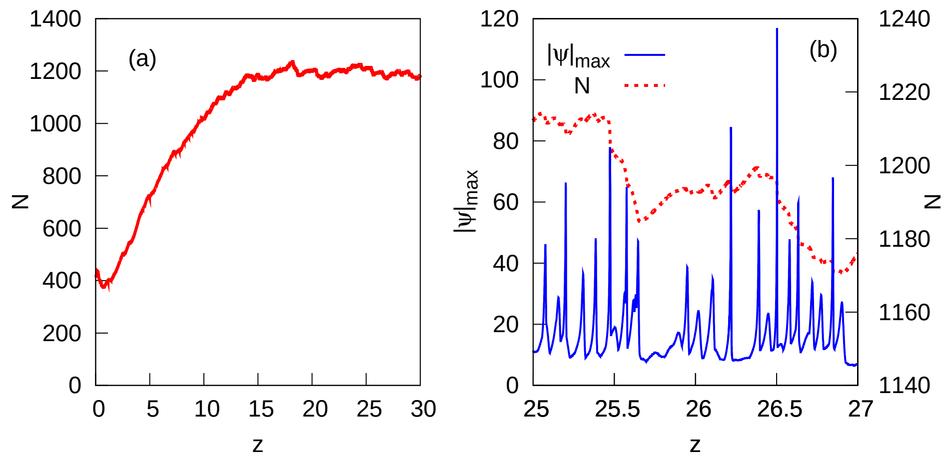

FIGURE 6. (a) $N(z)$ from simulation of RNLS (5.166) with $b = 20$, $a = c = 1$, $\epsilon = 0.01$. (b) The zoom of $N(z)$ in a smaller interval in z (solid curve, scale on the right) superimposed the $|\psi|_{max}(z)$ (dashed curve, scale on the left). All simulations used the fourth-order pseudo-spectral split-step algorithm on $-12.8 \leq x, y \leq 12.8$ with periodic boundary conditions at resolution 4096×4096 grid points. Initial conditions were a superposition of 100 randomly placed real-valued Gaussians with amplitudes and radii on $[-2, 2]$ and $[1, 2]$.

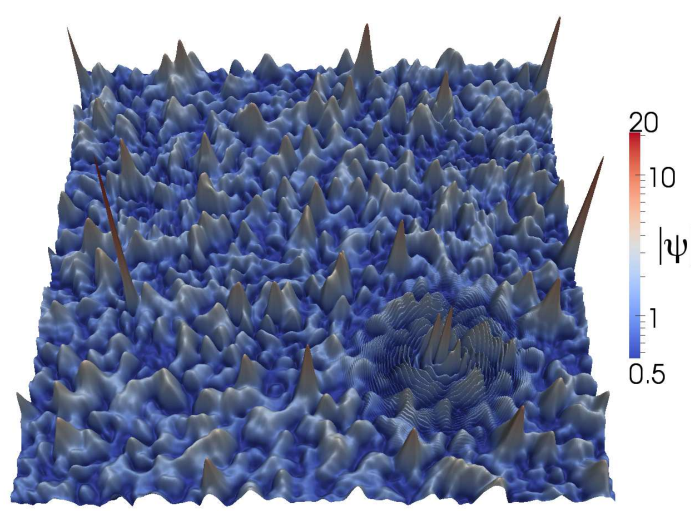

FIGURE 7. Snapshot of $|\psi|$ (vertical axis) vs. spatial coordinates (x, y) for simulation of Figure 6.

neglected if the simulation domain is large enough, so that $N \gg N_c$ (in Figure 6, $N \sim 100N_c$).

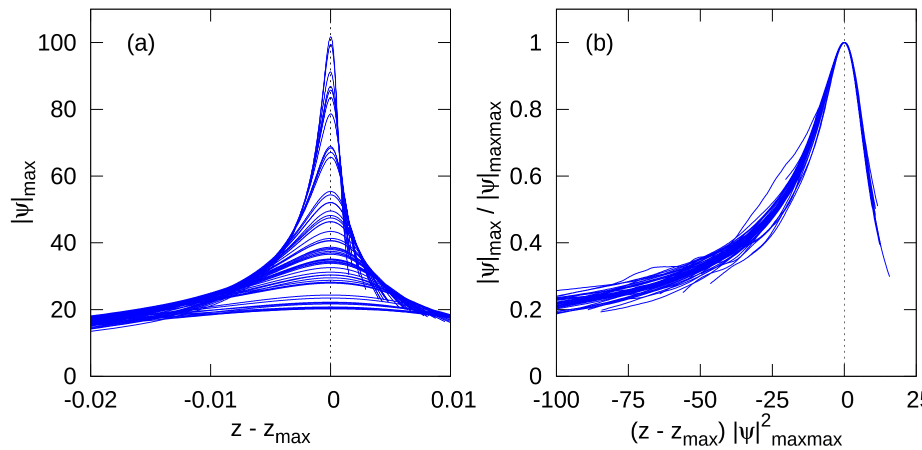

FIGURE 8. Dependence of $|\psi|_{max}(z)$ for multiple individual collapsing filaments (a) in the non-rescaled units; (b) the rescaled units (see the text for the description of rescaling). Individual filaments are extracted from simulation of RNLS (5.166) with parameters of Figure 6.

The evolution of each collapsing filament is well approximated for large $|\psi|$ by a self-similar radially-symmetric solution [**SS99, FP99**]:

$$|\psi(\mathbf{r}, z)| = \frac{1}{L(z)} R_0(\rho), \quad \rho = \frac{r}{L(z)}, \quad r \equiv |\mathbf{r}|, \tag{5.168}$$

where $L(z)$ is the transverse spatial scale of a filament and R_0 is the ground state soliton solution of NLS (5.167), given by $\nabla^2 R_0 - R_0 + R_0^3 = 0$, and corresponding to the critical power, $N_c = \int R_0(r)^2 d\mathbf{r}$ [**SS99**].

If $\epsilon = 0$ then $L(z) \simeq (2\pi)^{1/2}(z_0 - z)^{1/2}/(\ln|\ln(z_0 - z)|)^{1/2}$ describes a singularity (catastrophic collapse of a filament) as $z \to z_0$ [**SS99, FP99**]. For $\epsilon \neq 0$ the collapse is regularized and $|\psi|_{max}(z)$ achieves a maximum $|\psi|_{maxmax}$ at some $z = z_{max}$. A function

$$\gamma \equiv L\frac{dL}{dz}, \tag{5.169}$$

changes slowly with z compared to L at $z \lesssim z_{max}$. In the vicinity of a collapse, the forcing term in the right hand side of RNLS (5.166) can be neglected; the resulting equation can be written in rescaled units $z|\psi|^2_{maxmax}$, $\mathbf{r}|\psi|_{maxmax}$, and $\psi/|\psi|_{maxmax}$. (Here, we have also shifted z_{max} to $z = 0$.) As shown in Figure 8, $|\psi|_{max}(z)$ rescaled in these units exhibits a universal behavior for all near-singular filaments, — even for $\epsilon \neq 0$, and independent of the complicated structure of optical turbulence. This universality is a characteristic feature of two-photon absorption term in RNLS (5.166), but may not hold for other types of absorption.

Once the amplitude of a filament reaches its maximum, the amplitude decreases and subsequently the filament decays into outgoing cylindrical waves as seen in Figure 7. Superposition of these almost linear waves forms a nearly random Gaussian field and seeds new filaments. Figure 9 shows the probability $\mathcal{P}(h)$ for the amplitude

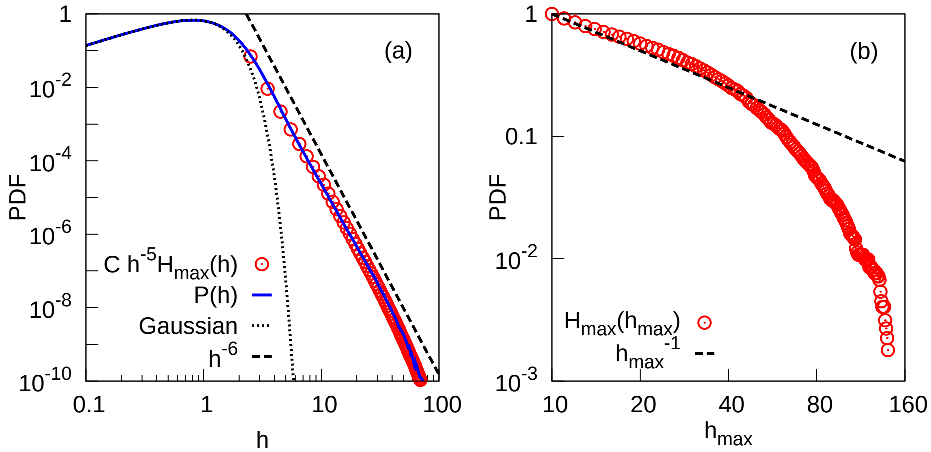

FIGURE 9. (a) $\mathcal{P}(h)$ for $|\psi| = h$ (solid curve) for the same simulation as in Figure 6. Dashed line shows fit to the Gaussian distribution and dotted line shows h^{-6} power law. Circles correspond to the solution of (5.172). (b) $H_{max}(h_{max})$ for $|\psi|_{maxmax}$. 2231 collapse events with $|\psi|_{maxmax} > 10$ are included in simulations in log-log scale (red dots). Dotted line shows h^{-1} power law. Scattering of data points for $h \gtrsim 110$ is due to lack of statistical ensemble for large collapses and is reduced for larger simulation times.

$|\psi|$ to have a value h, determined from simulations as

$$\mathcal{P}(h) = \frac{\int \delta(|\psi(\mathbf{r}, z)| - h) d\mathbf{r} dz}{\int d\mathbf{r} dz}. \tag{5.170}$$

Here, the integrals are taken over all values of $\mathbf{r}$ and all values of z after the turbulence has reached the statistically steady state. We observe that the fit to the Gaussian distribution works very well for $|\psi| \lesssim 2$ which corresponds to almost linear waves, while for $|\psi| \gtrsim 3$ the PDF has a power law-like dependence indicating intermittency [**?**].

We now show that the power-like tail of $\mathcal{P}(h)$ results from the near-singular filaments. This approach dates back to the idea of describing strong turbulence in the Navier-Stokes equations through singularities of the Euler equations [**Fri95**]. Unfortunately, this hydrodynamic problem remains unsolved. The forced Burgers equation remains the only example of an analytical description of strong turbulence in which the tail of the PDF for negative gradients follows a well established $(-7/2)$ power law [**?**], dominated by the dynamics of near-singular shocks.

First we calculate the contribution to the PDF from individual collapsing filaments. As shown in Figure 8, the filament amplitude $|\psi|_{max}$ reaches the maximum $|\psi|_{maxmax} \equiv h_{max}$ at $z = z_{max}$, and rapidly decays for $z > z_{max}$. While neglecting the contribution to $\mathcal{P}(h)$ from $z \gtrsim z_{max}$, we calculate the contribution of an individual filament to $\mathcal{P}(h)$ through the conditional probability $\mathcal{P}(h|h_{max})$ using

(5.168), (5.170), and (5.169) as follows

$$\begin{aligned}
\mathcal{P}\left(h|h_{max}\right) &\propto \int^{z_{max}} dz \int d\mathbf{r}\delta\left(h-\frac{1}{L(z)}R_0\left(\frac{r}{L(z)}\right)\right)\\
&\propto \int d\rho\,\rho \int^{L(z_{max})} \frac{dL\,L^3}{\gamma}\delta\left(h-\frac{1}{L(z)}R_0(\rho)\right)\\
&\simeq \int \frac{d\rho\,\rho}{\langle\gamma\rangle h^5}[R_0(\rho)]^4\Theta\left(\frac{R_0(0)}{L(z_{max})}-h\right)\\
&= Const\ h^{-5}\ \Theta\left(h_{max}-h\right),
\end{aligned} \tag{5.171}$$

where $h_{max} = R_0(0)/L(z_{max})$, and $\Theta(x)$ is the Heaviside step function. Here, we have changed the integration variable from z to L and approximated $\gamma(z)$ under the integral by its average value $\langle\gamma\rangle$ as $\gamma(z) \simeq \langle\gamma\rangle \sim -0.5$. This approximation is valid for $z \lesssim z_{max}$ outside the neighborhood of $z = z_{max}$.

As a second step we calculate $\mathcal{P}(h)$ by integration over all values of h_{max} using equation (5.171) as follows

$$\begin{aligned}
\mathcal{P}(h) &= \int dh_{max}\mathcal{P}(h|h_{max})\mathcal{P}_{max}(h_{max})\\
&\simeq Const\ h^{-5}\int dh_{max}\Theta(h_{max}-h)\mathcal{P}_{max}(h_{max})\\
&= Const\ h^{-5}H_{max}(h),
\end{aligned} \tag{5.172}$$

where $\mathcal{P}_{max}(h_{max})$ is the PDF for $h_{max} = |\psi|_{maxmax}$ and $H_{max}(h) \equiv \int_h^\infty \mathcal{P}_{max}(h_{max})dh_{max}$ is the cumulative probability that $|\psi|_{maxmax} > h$.

Figure 9(b) shows $H_{max}(h_{max})$. Circles in Figure 9(a) show the prediction of equation (5.172) with $H_{max}(h_{max})$ from Figure 9(b). The constant in equation (5.172) was chosen to fit the circles and the solid curve in Figure 9(a). The very good agreement between these two curves, first, justifies the assumptions used in derivation of the equation (5.172), and second, shows that the intermittency of optical turbulence of RNLS (5.166) is due to collapse dynamics, which is the main result of this Letter. Figure 9 also shows that $H_{max}(h_{max})$ is not well approximated by h_{max}^{-1} (and is not universal because it depends on the parameters a, b, c and ϵ.) Consequently, h^{-6} is only a crude approximation for $\mathcal{P}(h)$.

(*==================================*)

CHAPTER 6

Inverse Scattering Transform and Integrability

6.1. Nonlinear PDEs as compatibility condition for system of linear PDEs

Consider a system of linear PDEs

$$\partial_x \psi = \mathbf{U}\psi, \tag{6.1a}$$

$$\partial_t \psi = \mathbf{V}\psi, \tag{6.1b}$$

for the unknown $\psi(x,t) \in \mathbb{C}^n$ and independent variables $x, t \in \mathbb{R}$. Here $\mathbf{U}, \mathbf{V} \in \mathbb{C}^{n\times n}$ are complex matrices which depend on of x, t and have the given explicit dependence on the additional parameter $\lambda \in \mathbb{C}$. Calculating partial derivatives $\partial_t\partial_x\psi$ and $\partial_x\partial_t\psi$ using equations (6.1) we obtain that

$$\partial_t\partial_x\psi = \partial_t\left(\mathbf{U}\psi\right) = \mathbf{U}_t\psi + \mathbf{U}\psi_t = \mathbf{U}_t\psi + \mathbf{U}\mathbf{V}\psi, \tag{6.2a}$$

$$\partial_x\partial_t\psi = \partial_x\left(\mathbf{V}\psi\right) = \mathbf{V}_x\psi + \mathbf{V}\psi_x = \mathbf{V}_x\psi + \mathbf{V}\mathbf{U}\psi. \tag{6.2b}$$

The compatibility condition $\partial_t\partial_x\psi = \partial_x\partial_t\psi$ must be satisfied for any ψ so that equating (6.2a) and (6.2b) we obtain that

$$\partial_t\mathbf{U} - \partial_x\mathbf{V} + \mathbf{U}\mathbf{V} - \mathbf{V}\mathbf{U} = \partial_t\mathbf{U} - \partial_x\mathbf{V} + [\mathbf{U},\mathbf{V}] = 0, \tag{6.3}$$

where $[\mathbf{U},\mathbf{V}] := \mathbf{U}\mathbf{V} - \mathbf{V}\mathbf{U}$ is the *commutator* of $\mathbf{U}$ and $\mathbf{V}$. The equation (6.3) is called the *zero-curvature condition.*

Remark. The name of zero-curvature condition originates from the following geometrical interpretation of equation (6.3). In terms of the differential geometry, equations (6.1) define a connection on a two-dimensional vector bundle over the (x,t)-plane. The first equation (6.1a) describes how to "parallel-translate" a vector ψ in the x-direction, and the second equation (6.1b) describes how to "parallel-translate" a vector ψ in the t-direction. The matrices $\mathbf{U}$ and $\mathbf{V}$ are then the connection coefficients. A connection is said to have zero curvature if parallel translation of a vector ψ along a path from a point (x_1,t_1) to another point (x_2,t_2) gives the same result independent of path connecting the points. This is the same thing as asserting the existence of a full two-dimensional basis of simultaneous solutions of the equations $\partial_x\psi = \mathbf{U}\psi$ and $\partial_t\psi = \mathbf{V}\psi$, which is the above zero-curvature condition that must be satisfied by the connection coefficients.

IWe choose the matrices $\mathbf{U}$ and $\mathbf{V}$ to have the following polynomial form over λ,

$$\mathbf{U} = \begin{pmatrix} i\lambda & iq \\ ir & -i\lambda \end{pmatrix} = i\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + i\lambda\sigma_3 = \mathbf{U}_0 + \lambda\mathbf{U}_1, \tag{6.4a}$$

$$\mathbf{V} = \begin{pmatrix} 2i\lambda^2 - iqr & 2i\lambda q + q_x \\ 2i\lambda r - r_x & -2i\lambda^2 + iqr \end{pmatrix} = \mathbf{V}_0 + \lambda\mathbf{V}_1 + \lambda^2\mathbf{V}_2, \tag{6.4b}$$

where

$$\begin{aligned} \mathbf{U}_0 &= i\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} = iq\sigma_+ + ir\sigma_-, \\ \mathbf{U}_1 &= i\sigma_3, \\ \mathbf{V}_0 &= \begin{pmatrix} -i\,qr & q_x \\ -r_x & i\,qr \end{pmatrix} = -iqr\sigma_3 + q_x\sigma_+ - r_x\sigma_-, \\ \mathbf{V}_1 &= 2i\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} = 2\mathbf{U}_0, \\ \mathbf{V}_2 &= 2i\sigma_3 \end{aligned} \tag{6.5}$$

and $r = r(x,t) \in \mathbb{C}$, $q = q(x,t) \in \mathbb{C}$. Here σ_i, $i = 1, 2, 3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.6}$$

which satisfy the following relations

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1\sigma_2 = i\sigma_3, \ \sigma_2\sigma_3 = i\sigma_1, \ \sigma_3\sigma_1 = i\sigma_2. \tag{6.7}$$

Also we use in equations (6.4) and (6.5) the following linear combinations of Pauli matrices

$$\sigma_+ = \frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6.8}$$

Collecting terms with different powers of λ in equation (6.3) we obtain that orders λ^3, λ^2 and λ^1 are satisfied identically. At order λ^0 we obtain that

$$\begin{aligned} i\partial_t q - \partial_x^2 q - 2q^2 r &= 0, \\ i\partial_t r + \partial_x^2 r + 2qr^2 &= 0. \end{aligned} \tag{6.9}$$

Taking

$$r = \bar{q} \tag{6.10}$$

we obtain the focusing NLSE

$$i\partial_t r + \partial_x^2 r + 2|r|^2 r = 0 \tag{6.11}$$

while for

$$r = -\bar{q} \tag{6.12}$$

we obtain the defocusing NLSE

$$i\partial_t r + \partial_x^2 r - 2|r|^2 r = 0. \tag{6.13}$$

We now take $\mathbf{V}$ as the following cubic polynomial in λ

$$\mathbf{V} = \sum_{j=0}^{3} \lambda^j \mathbf{V}_j = \begin{pmatrix} 4i\lambda^3 - 2i\lambda qr + qr_x - rq_x & 4i\lambda^2 q + 2\lambda q_x - iq_{xx} - 2iq^2 r \\ 4i\lambda^2 r - 2\lambda r_x - ir_{xx} - 2ir^2 q & -4i\lambda^3 + 2i\lambda qr - qr_x + rq_x \end{pmatrix}, \tag{6.14}$$

where

$$
\begin{aligned}
&\mathbf{V}_0 = \begin{pmatrix} qr_x - rq_x & -iq_{xx} - 2iq^2 r \\ -ir_{xx} - 2iqr^2 & -qr_x + rq_x \end{pmatrix}, \\
&\mathbf{V}_1 = -2iqr\sigma_3 + 2\begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix}, \\
&\mathbf{V}_2 = 4i\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \\
&\mathbf{V}_3 = 4i\sigma_3.
\end{aligned} \tag{6.15}
$$

Collect terms with different powers of λ in equation (6.3) together with $\mathbf{U}$ from (6.4a) and $\mathbf{V}$ from (6.14) we obtain that orders λ^4, λ^3, λ^2 and λ^1 are satisfied identically. At order λ^0 we find that

$$
\begin{aligned}
q_t + q_{xxx} + 6qrq_x &= 0, \\
r_t + r_{xxx} + 6qrr_x &= 0.
\end{aligned} \tag{6.16}
$$

Taking $r = 1$ in equation (6.16) we obtain KdV,

$$q_t + q_{xxx} + 6qq_x = 0. \tag{6.17a}$$

Taking $r = q$ in equation (6.16) we obtain mKdV

$$q_t + q_{xxx} + 6q^2 q_x = 0. \tag{6.18a}$$

More general choice $r = 1 + \alpha r$ in equation (6.16) results in mixed KdV and mKdV

$$q_t + q_{xxx} + 6qq_x + 6\alpha q^2 q_x = 0. \tag{6.19a}$$

We note that all these cases have the same $\mathbf{U}$ defined in equation (6.4a).

6.2. Zakharov-Shabat scattering problem for general case

Equation (6.1a) written in components is given by

$$
\begin{aligned}
\partial_x \varphi^{(1)} &= \mathrm{i}\lambda\varphi^{(1)} + \mathrm{i}q(x)\varphi^{(2)}, \\
\partial_x \varphi^{(2)} &= -\mathrm{i}\lambda\varphi^{(2)} + \mathrm{i}r(x)\varphi^{(1)}.
\end{aligned} \tag{6.20}
$$

6.3. Zakharov-Shabat scattering problem for KdV

We consider the Zakharov-Shabat scattering problem

$$
\begin{aligned}
&\partial_x \varphi^{(1)} = \mathrm{i}\lambda\varphi^{(1)} + \mathrm{i}q(x)\varphi^{(2)}, \\
&\partial_x \varphi^{(2)} = -\mathrm{i}\lambda\varphi^{(2)} - \mathrm{i}\varphi^{(1)}, \\
&x, \lambda \in \mathbb{R}, \quad \varphi^{(1)}(x,\lambda), \varphi^{(2)}(x,\lambda) \in \mathbb{C}, \\
&\int_{-\infty}^{\infty} (1 + |x|)|q| dx < \infty.
\end{aligned} \tag{6.21}
$$

The system (6.21) is convenient to reduce to the single second order ODE by excluding $\varphi^{(1)}$ and defining

$$f = \varphi^{(2)}. \tag{6.22}$$

6.3.1. Examples of explicit solutions of scattering problems. Assume that the potential $q(x)$ in equation (6.21) is the Dirac delta function

$$q(x) = \gamma\delta(x), \quad \gamma \in \mathbb{R}, \tag{6.23}$$

where γ is the constant.

6.4. Zakharov-Shabat scattering problem for NLSE

We consider the Zakharov-Shabat scattering problem

$$\begin{aligned}
&\partial_x\varphi^{(1)} = \mathrm{i}\lambda\varphi^{(1)} + \mathrm{i}s\bar{u}(x)\varphi^{(2)},\\
&\partial_x\varphi^{(2)} = -\mathrm{i}\lambda\varphi^{(2)} + \mathrm{i}u(x)\varphi^{(1)},\\
&x,\lambda \in \mathbb{R}, \quad \varphi^{(1)}(x,\lambda), \varphi^{(2)}(x,\lambda) \in \mathbb{C}, \quad s = \pm 1,\\
&\int\limits_{-\infty}^{\infty}(1+|x|)|u|dx < \infty
\end{aligned} \tag{6.24}$$

which corresponds to focusing NLSE for $s = 1$ and defocusing NLSE for $s = -1$ as follows from equations (6.10) and (6.12) of Section (6.1). Here we replaced $r(x)$ by $u(x)$ to reserve the letter r for the reflection coefficient as will be seen below. The convergence of the integral at the last line of equation (6.24) ensures that $|u| \to 0$ for $|x| \to \infty$. Here we use superscrips (i), $i = 1, 2$ to designate two components of the column vector

$$\varphi := \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \tag{6.25}$$

while subscripts below distinguish different column vectors.

The general solution of the system (6.24) is given by the linear combination of two independent solutions which form a basis. We choose two pairs of such independent solution. The first pair is defined by two explicit solutions of the system (6.24) at $x \to +\infty$ as follows

$$\begin{aligned}
\psi_1(x,\lambda) &= \begin{pmatrix} e^{\mathrm{i}\lambda x} \\ 0 \end{pmatrix} + o(1), \qquad x \to +\infty,\\
\psi_2(x,\lambda) &= \begin{pmatrix} 0 \\ e^{-\mathrm{i}\lambda x} \end{pmatrix} + o(1), \qquad x \to +\infty.
\end{aligned} \tag{6.26}$$

The second pair is defined by two explicit solutions of the system (6.24) at $x \to -\infty$ as follows

$$\begin{aligned}
\varphi_1(x,\lambda) &= \begin{pmatrix} e^{\mathrm{i}\lambda x} \\ 0 \end{pmatrix} + o(1), \qquad x \to -\infty,\\
\varphi_2(x,\lambda) &= \begin{pmatrix} 0 \\ e^{-\mathrm{i}\lambda x} \end{pmatrix} + o(1), \qquad x \to -\infty.
\end{aligned} \tag{6.27}$$

Asymptotics (6.26) and (6.27) define the initial conditions $\psi_{1,2}(x,\lambda)$ and $\varphi_{1,2}(x,\lambda)$. Solving Cauchy problems for the system (6.24) with these initial conditions $\psi_{1,2}(x,\lambda)$ and $\varphi_{1,2}(x,\lambda)$ on the entire real line $x \in \mathbb{R}$.

A Wronskian

$$W(\chi_1,\chi_2) := \begin{vmatrix} \chi_1^{(1)} & \chi_2^{(1)} \\ \chi_1^{(2)} & \chi_2^{(2)} \end{vmatrix} = \chi_1^{(1)}\chi_2^{(2)} - \chi_1^{(2)}\chi_2^{(1)} \tag{6.28}$$

between two solutions χ_1 and χ_2 of the system (6.24) is independent on x as follow from Abel's identity (but generally $W(\chi_1, \chi_2)$ depends on λ). Then it follows from equation (6.26) and (6.27) that $W(\varphi_1, \varphi_2) = W(\psi_1, \psi_2) = 1$ for any $x \in \mathbb{R}$ which ensures that each pair (φ_1, φ_2) and (ψ_1, ψ_2) form basis for solutions of the system (6.24). It implies that φ_1 and φ_2 can be represented as the linear combination of vectors of the basis (ψ_1, ψ_2) as follows

$$\varphi_i(x, \lambda) = \sum_{j=1}^{2} T_{ji}(\lambda)\psi_j(x, \lambda), \quad i = 1, 2, \tag{6.29}$$

where $T_{ji}(\lambda)$ are complex constants which are independent on x. The constant form the complex matrix $T(\lambda) \in \mathbb{C}^{2\times 2}$.

Equation (6.29) together with expressions (6.26) and (6.27) can be interpreted as the "scattering" on the complex potentials $p(x)$ and $q(x)$ of the plane wave $\psi_2 \propto e^{-\mathrm{i}\lambda x}$ at $x \to +\infty$ into the reflected plane wave $\psi_1 \propto e^{\mathrm{i}\lambda x}$ at $x \to +\infty$ and the transmitted wave $\varphi_i(x, \lambda)$ at $x \to -\infty$. Then the matrix $T(\lambda) \in \mathbb{C}^{2\times 2}$ is called the *scattering matrix*.

The system (6.24) is the reduction of more general system (6.20) which ensure the reductions (6.10) and (6.12). These reductions imply the additional symmetry relation as follows. If we apply the complex conjugation to the system (6.24) and interchange the first and second equations, we obtain that

$$\begin{aligned} \partial_x \bar{\varphi}^{(2)} &= \mathrm{i}\lambda \bar{\varphi}^{(2)} - \mathrm{i}\bar{u}(x)\bar{\varphi}^{(1)}, \\ \partial_x \bar{\varphi}^{(1)} &= -\mathrm{i}\lambda \bar{\varphi}^{(1)} - \mathrm{i}su(x)\bar{\varphi}^{(2)}, \end{aligned} \tag{6.30}$$

where we assume that $\lambda \in \mathbb{R}$. The by comparison of equations (6.24) and (6.30) we conclude that if

$$\varphi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \tag{6.31}$$

is the solution of (6.24) then the vector

$$\tilde{\varphi} = \begin{pmatrix} -s\bar{\varphi}^{(2)} \\ \bar{\varphi}^{(1)} \end{pmatrix} \tag{6.32}$$

is also the solution of (6.24). Sometimes the "tilde" operation defined by equations (6.31) and (6.32) is called by "involution" in analogy with the involution (or an involutory function) $f(y)$ defined by $f(f(y)) = x$, i.e. it is the function that is its own inverse. The twice application of tilde operation gives "almost" involution because using equations (6.31) and (6.32) twice we obtain that

$$\tilde{\tilde{\varphi}} = -s \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} = -s\varphi. \tag{6.33}$$

It means that strictly speaking only the case $s = -1$ corresponds to the involution while $s = 1$ produces the opposite sign, i.e. "anti-involution". Below we abuse that notion and simply call tilde operation by involution neglecting that sign difference.

We define

$$\psi(x, \lambda) := \psi_1(x, \lambda) \quad \text{and} \quad \varphi(x, \lambda) \equiv \varphi_2(x, \lambda). \tag{6.34}$$

Then the solutions of Cauchy problems (6.24), (6.26) and (6.27) ensure that

$$\tilde{\psi}(x, \lambda) := \psi_2(x, \lambda) \quad \text{and} \quad \tilde{\varphi}(x, \lambda) \equiv -s\varphi_1(x, \lambda) \tag{6.35}$$

for all $x, \lambda \in \mathbb{R}$. It means that instead working with two vector pairs (φ_1, φ_2) and (ψ_1, ψ_2) we can work with two vectors ψ and φ while their linearly independent counterparts $\tilde{\psi}$ and $\tilde{\varphi}$ are obtained by involution each time we need them.

We define that

$$\varphi(x,\lambda) = a(\lambda)\tilde{\psi}(x,\lambda) + b(\lambda)\psi(x,\lambda). \tag{6.36}$$

It we multiply equation by $1/a(\lambda)$ then the resulting equation

$$\frac{1}{a(\lambda)}\varphi(x,\lambda) = \tilde{\psi}(x,\lambda) + r(\lambda)\psi(x,\lambda), \qquad r(\lambda) := \frac{b(\lambda)}{a(\lambda)} \tag{6.37}$$

can be interpreted as the scattering of the plane wave $\tilde{\psi}(x,\lambda) \propto e^{-\mathrm{i}\lambda x}, x \to +\infty$ propagating from $x \to +\infty$ to the left thus scattering on the potential $q(x)$ according to equation (6.24). That scattering results in transmitted wave $\frac{1}{a(\lambda)}\varphi(x,\lambda) \propto \frac{1}{a(\lambda)}e^{-\mathrm{i}\lambda x}$ propagating to the left at $x \to +\infty$ and the reflected wave $\psi(x,\lambda) \propto e^{-\mathrm{i}\lambda x}$ propagating to the right at $x \to +\infty$. Respectively, $1/a(\lambda)$ is the transmission coefficient and $r(\lambda) = \frac{b(\lambda)}{a(\lambda)}$ is the reflection coefficient.

Using equation (6.36) and the definitions (6.34), (6.35), we obtain that

$$W(\psi,\varphi) = W(\psi,\varphi)|_{x\to+\infty} = \begin{vmatrix} e^{\mathrm{i}\lambda x} & b(\lambda)e^{\mathrm{i}\lambda x} \\ 0 & a(\lambda)e^{-\mathrm{i}\lambda x} \end{vmatrix} = a(\lambda). \tag{6.38}$$

In a similar way we obtain that

$$W(\psi,\tilde{\varphi}) = W(\psi,\tilde{\varphi})|_{x\to+\infty} = \begin{vmatrix} e^{\mathrm{i}\lambda x} & -s\,\bar{a}(\lambda)e^{\mathrm{i}\lambda x} \\ 0 & \bar{b}(\lambda)e^{-\mathrm{i}\lambda x} \end{vmatrix} = \bar{b}(\lambda), \tag{6.39}$$

$$W(\tilde{\psi},\varphi) = W(\tilde{\psi},\varphi)|_{x\to+\infty} = \begin{vmatrix} 0 & b(\lambda)e^{\mathrm{i}\lambda x} \\ e^{-\mathrm{i}\lambda x} & a(\lambda)e^{-\mathrm{i}\lambda x} \end{vmatrix} = -b(\lambda) \tag{6.40}$$

and

$$W(\tilde{\psi},\tilde{\varphi}) = W(\tilde{\psi},\tilde{\varphi})|_{x\to+\infty} = \begin{vmatrix} 0 & -s\,\bar{a}(\lambda)e^{\mathrm{i}\lambda x} \\ e^{-\mathrm{i}\lambda x} & \bar{b}(\lambda)e^{-\mathrm{i}\lambda x} \end{vmatrix} = s\,\bar{a}(\lambda). \tag{6.41}$$

6.4.1. *Example***. Zakharov-Shabat scattering problem for the rectangular well potential in NLSE case.** Consider the Zakharov-Shabat scattering problem

$$\begin{aligned} &\partial_x\varphi^{(1)} = \mathrm{i}\lambda\varphi^{(1)} + \mathrm{i}s\bar{u}(x)\varphi^{(2)}, \\ &\partial_x\varphi^{(2)} = -\mathrm{i}\lambda\varphi^{(2)} + \mathrm{i}u(x)\varphi^{(1)}, \\ &x,\lambda \in \mathbb{R}, \quad \varphi^{(1)}(x,\lambda), \varphi^{(2)}(x,\lambda) \in \mathbb{C}, \\ &\int\limits_{-\infty}^{\infty} (1+|x|)|u|dx < \infty \end{aligned} \tag{6.42}$$

for the rectangular well potential with the width x_0 and the amplitude u_0 such that

$$u(x) = \begin{cases} u_0, & |x| \le x_0/2, \\ 0, & |x| > x_0/2 > 0. \end{cases} \tag{6.43}$$

Choosing a transmitted wave at $x \to -\infty$ with amplitude one, an incoming wave with amplitude a and a reflected wave with amplitude b at $x \to \infty$ we obtain the solution of (6.42) in the following form

$$\begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\lambda x}, & x \le -x_0/2, \\ \begin{pmatrix} A^{(1)} \\ A^{(2)} \end{pmatrix} e^{-i\xi x} + \begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix} e^{i\xi x}, & |x| \le x_0/2, \\ a \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\lambda x} + b \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\lambda x}, & x \ge x_0/2, \end{cases} \tag{6.44}$$

where

$$\xi := (\lambda^2 + s|u_0|^2)^{1/2}. \tag{6.45}$$

The coefficients $A^{(1,2)}$, $B^{(1,2)}$ and equation (6.44) are not independent but have to be chosen to represent solution of (6.42). Plugging in equation (6.44) for $|x| \le x_0/2$ into the system (6.42) results in the following relations between these coefficients:

$$\begin{aligned} A^{(2)} &= -\frac{\xi + \lambda}{s\bar{u}_0} A^{(1)}, \\ B^{(2)} &= \frac{\xi - \lambda}{s\bar{u}_0} B^{(1)}. \end{aligned} \tag{6.46}$$

Equations (6.44)-(6.46) together with the continuity of $\varphi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}$ at $x = \pm x_0/2$ represent six conditions for six unknowns $a, b, A^{(1,2)}$, $B^{(1,2)}$ which results in

$$\begin{aligned} a(\lambda) &= e^{i\lambda x_0} \left[\cos(x_0\xi) - \frac{i\lambda}{\xi} \sin(x_0\xi) \right], \\ b(\lambda) &= \frac{is\bar{u}_0}{\xi} \sin(x_0\xi). \end{aligned} \tag{6.47}$$

Zeros of a corresponds to poles of the scattering coefficient $r = \frac{b}{a}$. Looking now into analytical continuation of a into the complex values of λ we conclude that depending on the sign s these zeros are located at... ???? We notice that the singularity at $\xi = 0$ (i.e. at $\lambda = \pm(-s)^{1/2}|u_0|^2 \pm$) is removable for both $a(\lambda)$ and $b(\lambda)$ in equation (6.47). Thus both $a(\lambda)$ and $b(\lambda)$ are the entire functions (i.e. functions which are analytic at all finite points over the whole complex plane $\lambda \in \mathbb{C}$). Also $a(\lambda) \to 1$ for $|\lambda| \to \infty$.

Limit $x_0 \to 0$ and $u_0 = \frac{\gamma}{x_0}$ produces the Dirac delta function potential $u(x) = \gamma\delta(x)$ ... ???

6.4.2. Analyticity of φ, ψ and $a(\lambda)$. We now consider the analytical properties of φ, ψ and $a(\lambda)$ in the complex plane $\lambda \in \mathbb{C}$. We will show that these functions are analytic in the upper complex half-pane $\lambda \in \mathbb{C}^+$ which corresponds to $Im(\lambda) > 0$.

The asymptotic of φ at $x \to -\infty$ is determined by equations (6.26) and (6.34) as follows

$$\varphi(x, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\lambda x} + o(1), \qquad x \to -\infty. \tag{6.48}$$

It implies in that limit that $\varphi(x,\lambda) \to 0$ for $\lambda_i := Im(\lambda) \to +\infty$ because $|e^{-\mathrm{i}\lambda x}| = e^{\lambda_i x} \to 0$. We transform the system (6.24) into the integral form by defining

6.5. Lax representation for Kadomtsev-Petviashvili equation

We consider the following overdetermined system of linear differential equations

$$\sigma \frac{\partial \Psi}{\partial y} + \hat{L}\Psi = 0, \;\; \sigma^2 = \pm 1, \tag{6.49a}$$

$$\frac{\partial \Psi}{\partial t} + \hat{A}\Psi = 0, \tag{6.49b}$$

where $\hat{L}$ and $\hat{A}$ are the linear differential operators over x defined by

$$\hat{L}\Psi = \frac{\partial^2 \Psi}{\partial x^2} + (U + \beta(y))\Psi \tag{6.50}$$

and

$$\hat{A}\Psi = 4\frac{\partial^3 \Psi}{\partial x^3} + V\frac{\partial \Psi}{\partial x} + W\Psi \tag{6.51}$$

with U, V and W being the functions of $x, y, t \in \mathbb{R}$ and β is the arbitrary smooth function of y (but independent on both x and t).

Similar to the analysis of the system (6.1), we calculate the cross derivatives using equation (6.49) which results in

(6.52a)

$$\frac{\partial^2 \Psi}{\partial t \partial y} = -\sigma^{-1}\frac{\partial}{\partial t}\left(\hat{L}\Psi\right) = -\sigma^{-1}\frac{\partial \hat{L}}{\partial t}\Psi - \sigma^{-1}\hat{L}\frac{\partial \Psi}{\partial t} = -\sigma^{-1}\frac{\partial \hat{L}}{\partial t}\Psi + \sigma^{-1}\hat{L}\hat{A}\Psi,$$

(6.52b)

$$\frac{\partial^2 \Psi}{\partial y \partial t} = -\frac{\partial}{\partial y}\left(\hat{A}\Psi\right) = -\frac{\partial \hat{A}}{\partial y}\Psi - \hat{A}\frac{\partial \Psi}{\partial y} = -\frac{\partial \hat{A}}{\partial y}\Psi + \sigma^{-1}\hat{A}\hat{L}\Psi.$$

A compatibility condition $\frac{\partial^2 \Psi}{\partial t \partial y} = \frac{\partial^2 \Psi}{\partial y \partial t}$ implies from equations (6.52a) and (6.52b) that

$$\left(\frac{\partial \hat{L}}{\partial t} - \sigma\frac{\partial \hat{A}}{\partial y} - [\hat{L}, \hat{A}]\right)\Psi = 0. \tag{6.53}$$

Equations (6.50) and (6.51) mean that $\frac{\partial \hat{L}}{\partial t}$ is the multiplication operator,

$$\frac{\partial \hat{L}}{\partial t}\Psi = \frac{\partial U}{\partial t}\Psi, \tag{6.54}$$

while $\frac{\partial \hat{A}}{\partial y}$ is the first order differential operator over x,

$$\frac{\partial \hat{A}}{\partial y}\Psi = \frac{\partial V}{\partial y}\frac{\partial \Psi}{\partial x} + \frac{\partial W}{\partial y}\Psi. \tag{6.55}$$

The commutator $[\hat{L}, \hat{A}]$ is given by

$$\begin{aligned}[\hat{L}, \hat{A}] = \hat{L}\hat{A} - \hat{A}\hat{L} = \left(2\frac{\partial V}{\partial x} - 12\frac{\partial U}{\partial x}\right)\frac{\partial^2}{\partial x^2}\\ + \left(\frac{\partial^2 V}{\partial x^2} - 12\frac{\partial^2 U}{\partial x^2} + 2\frac{\partial W}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\partial^2 W}{\partial x^2} - 4\frac{\partial^3 U}{\partial x^3} - V\frac{\partial U}{\partial x}.\end{aligned} \tag{6.56}$$

Hence combining equations (6.54)-(6.56), we conclude that the operator $\hat{T} := \frac{\partial \hat{L}}{\partial t} - \sigma \frac{\partial \hat{A}}{\partial y} - [\hat{L}, \hat{A}]$ is the second order differential operator over x, annihilating any sufficiently smooth function Ψ, i.e. $\hat{T}\Psi \equiv 0$. We can remove Ψ in equation (6.53) and require that

$$\hat{T} = \frac{\partial \hat{L}}{\partial t} - \sigma \frac{\partial \hat{A}}{\partial y} - [\hat{L}, \hat{A}] = 0. \tag{6.57}$$

We use equations (6.54)-(6.57) to obtain that a condition of vanishing of the terms in $\hat{T}$ which multiply $\frac{\partial^2}{\partial x^2}$ is given by

$$2\frac{\partial V}{\partial x} - 12\frac{\partial U}{\partial x} = 0, \tag{6.58}$$

a condition of vanishing of the terms in $\hat{T}$ which multiply $\frac{\partial}{\partial x}$ results in

$$-\sigma\frac{\partial V}{\partial y} = \frac{\partial^2 V}{\partial x^2} - 12\frac{\partial^2 U}{\partial x^2} + 2\frac{\partial W}{\partial x} \tag{6.59}$$

and a condition of vanishing of the terms in $\hat{T}$ without $\frac{\partial}{\partial x}$ give that

$$\frac{\partial U}{\partial t} - \sigma\frac{\partial W}{\partial y} = \frac{\partial^2 W}{\partial x^2} - 4\frac{\partial^3 U}{\partial x^3} - V\frac{\partial U}{\partial x}. \tag{6.60}$$

An integration of equation (6.58) results in

$$V = 6U + C, \tag{6.61}$$

where $C(y, t)$ can be any function of both t and y. Then equations (6.59) and (6.61) result in

$$-6\sigma\frac{\partial U}{\partial y} = \sigma\frac{\partial C}{\partial y} - 6\frac{\partial^2 U}{\partial x^2} + 2\frac{\partial W}{\partial x}. \tag{6.62}$$

We define a new function

$$q := -\frac{W}{3\sigma} + \frac{1}{\sigma}\frac{\partial U}{\partial x} - \frac{x}{6}\frac{\partial C}{\partial y} + \frac{C_2}{3\sigma} \tag{6.63}$$

such that

$$W = 3\frac{\partial U}{\partial x} - 3\sigma q - \frac{\sigma x}{2}\frac{\partial C}{\partial y} + C_2, \tag{6.64}$$

i.e. q is fully responsible for a deviation of r.h.s. of equation (6.62) from zero. Here C_2 is the arbitrary function of y and t. The definition (6.63) converts equations (6.60) and (6.62) without loss of generality into the system

$$\frac{\partial U}{\partial t} + C\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} + 6U\frac{\partial U}{\partial x} = -3\sigma^2\frac{\partial q}{\partial y} - \frac{\sigma^2 x}{2}\frac{\partial^2 C}{\partial y^2} + \sigma\frac{\partial C_2}{\partial y}, \tag{6.65a}$$

$$\frac{\partial U}{\partial y} = \frac{\partial q}{\partial x} \tag{6.65b}$$

We conclude from equations (6.50),(6.57) and (6.72) that the system (6.65) is a compatibility condition for the following pair of linear PDEs

(6.66a)

$$\sigma\frac{\partial\Psi}{\partial y}+\frac{\partial^2\Psi}{\partial x^2}+(U+\beta(y))\Psi=0,$$

(6.66b)

$$\frac{\partial\Psi}{\partial t}+4\frac{\partial^3\Psi}{\partial x^3}+(6U+C(y,t))\frac{\partial\Psi}{\partial x}+\left(3\frac{\partial U}{\partial x}-3\sigma q-\frac{\sigma x}{2}\frac{\partial C(y,t)}{\partial y}+C_2(y,t)\right)\Psi=0,$$

where q is subject to the condition (6.65b).

Differentiating (6.65a) over x and using equation (6.65b), we obtain that

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t}+C\frac{\partial U}{\partial x}+\frac{\partial^3 U}{\partial x^3}+6U\frac{\partial U}{\partial x}\right)=-3\sigma^2\frac{\partial^2 U}{\partial y^2}-\frac{\sigma^2}{2}\frac{\partial^2 C}{\partial y^2} \tag{6.67}$$

which recovers the KP equation provided $\frac{\partial^2 C}{\partial y^2}\equiv 0$, i.e.

$$C(y,t)=C_0(t)+yC_1(t), \tag{6.68}$$

where $C_0(t)$ and $C_1(t)$ are arbitrary functions of time.

Solving equation (6.65b) in this system for q and plugging into equation (6.65a) result in the KP equation with the additional terms in r.h.s. as follows,

$$\left(\frac{\partial U}{\partial t}+C\frac{\partial U}{\partial x}+\frac{\partial^3 U}{\partial x^3}+6U\frac{\partial U}{\partial x}\right)=-3\sigma^2\partial_x^{-1}\frac{\partial^2 U}{\partial y^2}-\frac{\sigma^2 x}{2}\frac{\partial^2 C}{\partial y^2}+\sigma\frac{\partial C_2}{\partial y}+\gamma, \tag{6.69}$$

where γ is the arbitrary function of y, t (γ can be also included into the modification of C_2) and ∂_x^{-1} is the inverse of the differentiation operator, i.e. the integration operator over x which is convenient for the decaying boundary conditions to define in the symmetric form as

$$\partial_x^{-1}f=\frac{1}{2}\left[\int\limits_{-\infty}^{x}f(x')dx'-\int\limits_{x}^{\infty}f(x')dx'\right]. \tag{6.70}$$

The arbitrary constant γ reflects that q is defined in equation (**??**) up to the addition of the arbitrary constant independent on x. Usually we set $\gamma=0$. The constant C can be removed from equation (6.69) by the Galilean transformation to the moving frame along the direction x with the speed C.

Equations (6.61) and (6.63) imply that equation (6.51) takes the following form

$$\hat{A}\Psi=4\frac{\partial^3\Psi}{\partial x^3}+(6U+C)\frac{\partial\Psi}{\partial x}+\left(3\frac{\partial U}{\partial x}-3\sigma q\right)\Psi \tag{6.71}$$

We conclude from equations (6.50),(6.57) and (6.72) that the KP equation (6.69) is a compatibility condition for the following pair of linear PDEs

$$\sigma\frac{\partial\Psi}{\partial y}+\frac{\partial^2\Psi}{\partial x^2}+(U+\beta)\Psi=0, \tag{6.72a}$$

$$\frac{\partial\Psi}{\partial t}+4\frac{\partial^3\Psi}{\partial x^3}+C\frac{\partial\Psi}{\partial x}+6U\frac{\partial\Psi}{\partial x}+\left(3\frac{\partial U}{\partial x}-3\sigma q\right)\Psi=0, \tag{6.72b}$$

where

$$q=\partial_x^{-1}\frac{\partial U}{\partial y}. \tag{6.73}$$

This overdetermined system is the "LAX Pair" for the KP equation. Replacing $\frac{\partial}{\partial y} \to -\frac{\partial}{\partial y}$ does not change anything. Let us consider the following Lax Pair:

$$\pm\frac{\partial \Psi}{\partial y} + \frac{\partial^2 \Psi}{\partial x^2} + U\Psi = 0 \tag{6.74}$$

$$\frac{\partial \Psi}{\partial t} + 4\frac{\partial^3 \Psi}{\partial x^3} + C\frac{\partial \Psi}{\partial x} + 6U\frac{\partial \Psi}{\partial x} + \left(3\frac{\partial U}{\partial x} + iq\right)\Psi = 0 \tag{6.75}$$

Here U and q are real functions. A compatibility condition for equation (7.14) is

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t} + C\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} + 6U\frac{\partial U}{\partial x}\right) = -3\sigma^2\frac{\partial^2 U}{\partial y^2} - \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial y^2} \tag{6.76}$$

which can be transformed to the KP-1 equation by the already-mentioned transform $t \to -t$, $U \to -U$. Note that both equations (7.12) and (7.15) have bump-type solitons.

If there is no dependence on y, one can set $\Psi \sim e^{-\lambda y}$ and system (7.1), (7.2) becomes

$$\hat{L}\Psi = \frac{\partial^2 \Psi}{\partial x^2} + U\Psi = \lambda\Psi \tag{6.77}$$

$$\frac{\partial \Psi}{\partial t} + \hat{A}\Psi = 0 \tag{6.78}$$

Now, $q = 0$ and

$$\hat{A}\Psi = 4\frac{\partial^3 \Psi}{\partial x^3} + 6U\frac{\partial \Psi}{\partial x} + 3\frac{\partial U}{\partial x}\Psi \tag{6.79}$$

We now have the classical Lax representation

$$\frac{\partial \hat{L}}{\partial t} = [\hat{L}, \hat{A}] \tag{6.80}$$

for the KdV equation,

$$\frac{\partial U}{\partial t} + 6U\frac{\partial U}{\partial x} + \frac{\partial^3 U}{\partial x^3} = 0$$

This representation can be easily extended to the matrix case. Suppose in (7.16) and (7.17) U is an $n \times x$ real-valued matrix. Then equation (7.18) leads to the equality

$$\frac{\partial U}{\partial t} + 3\left(U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial x}U\right) + \frac{\partial^3 U}{\partial x^3} = 0$$

or

$$\frac{\partial U}{\partial t} + 3\frac{\partial U^2}{\partial x} + \frac{\partial^3 U}{\partial x^3} = 0 \tag{6.81}$$

In particular, one can assume that U is symmetric, $U^T = U$. For instance,

$$U = \begin{bmatrix} p & r \\ r & q \end{bmatrix}$$

Then equation (7.19) composes the system

$$\frac{\partial p}{\partial t} + 3\frac{\partial}{\partial x}(p^2 + r^2) + \frac{\partial^3 p}{\partial x^3} = 0 \tag{6.82}$$

$$\frac{\partial q}{\partial t} + 3\frac{\partial}{\partial x}(q^2 + r^2) + \frac{\partial^3 q}{\partial x^3} = 0 \tag{6.83}$$

$$\frac{\partial r}{\partial t} + 3\frac{\partial}{\partial x}(p + q)r + \frac{\partial^3 r}{\partial x^3} = 0 \tag{6.84}$$

Notice that the entire procedure detailed above is purely algebraic. We can therefore treat all the variables, t, x, and U as complex numbers. Let us make the substitution $U \to aU$, $t \to at$, and $x \to ax$, where $a^{-2} = i\alpha$. We end up with the equation

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + i\alpha\frac{\partial^3 U}{\partial x^3} = 0 \tag{6.85}$$

Now, assuming that $U = \frac{\partial \Phi}{\partial x} + i\rho$, $a = i\alpha$. Equation (7.21) is then equivalent to the system

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho\frac{\partial \Phi}{\partial x} = -\alpha\frac{\partial^3 \Phi}{\partial x^3} \tag{6.86}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 - \frac{1}{2}\rho^2 = \alpha\frac{\partial^2 \rho}{\partial x^2} \tag{6.87}$$

This is a badly unstable system with Hamiltonian

$$H = \frac{1}{2}\int \rho\left(\frac{\partial \Phi}{\partial x}\right)^2 dx - \frac{1}{6}\int \rho^3 dx - \frac{\alpha}{2}\int \left(\frac{\partial^2 \Phi}{\partial x^2}\right)^2 dx + \frac{\alpha}{2}\int \left(\frac{\partial \rho}{\partial x}\right)^2 dx \tag{6.88}$$

Now we will obtain Lax representations for all four types of Boussinesq equations. First, we must set $\frac{\partial \hat{L}}{\partial t} = 0$ in equation (7.6). Then we set

$$\frac{\partial \hat{A}}{\partial t} = [\hat{A}, \hat{L}] \tag{6.89}$$

Now operators $\hat{A}$ and $\hat{L}$ have changed positions.

Looking at equations (6.20) through (6.23), we see that versions (6.21) and (6.22) come from the KP-2 equations. They are compatibility conditions of the following systems:

$$\hat{A}\Psi = 4\frac{\partial^3 \Psi}{\partial x^3} \pm \frac{\partial \Psi}{\partial x} + 6U\frac{\partial \Psi}{\partial x} + 3\left(\frac{\partial U}{\partial x} + q\right)\Psi = \lambda\Psi \tag{6.90}$$

$$q = -3\partial^{-1}\frac{\partial U}{\partial y}$$

$$\sqrt{3}\frac{\partial \Psi}{\partial y} = \frac{\partial^2 \Psi}{\partial x^2} + U\Psi \tag{6.91}$$

These systems generate equations (6.21) and (6.22).

Now the combination of equation (7.25) with the Schrödinger equation

$$i\sqrt{3}\frac{\partial \Psi}{\partial y} = \frac{\partial^2 \Psi}{\partial x^2} + U\Psi$$

generates equations (6.20) and (6.23).

6.6. Conservation laws for the KdV equation

The KdV equation,

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{6.92}$$

is the compatibility condition for equations

$$\psi_{xx} + (k^2 + u)\psi = 0 \tag{6.93}$$

$$\psi_t = 2(2k^2 - u)\psi_x + u_x\psi \tag{6.94}$$

If we define $\lambda = -k^2$, then $\psi_{xxx} = \lambda\psi_x - (u\psi)_x$.

Now we introduce

$$\psi = \chi e^{ikx+4ik^3t} \tag{6.95}$$

Equations (8.2) and (8.3) then take the form

$$\chi_x = -\frac{1}{2ik}(\chi_{xx} + u\chi) \tag{6.96}$$

$$\chi_{xx} = \frac{1}{2ik}(\chi_t + 2u\chi_x - u_x\chi) \tag{6.97}$$

¿From (8.5) and (8.6), one can get a very simple relation,

$$\chi_t + \chi_{xxx} + 3u\chi_x = 0 \tag{6.98}$$

Let us assume that $\chi \to 1$ as $k \to \infty$. Then we get the asymptotic expansion

$$\chi = 1 + \frac{\chi_1}{-2ik} + \frac{\chi_2}{(-2ik)^2} + ... \tag{6.99}$$

$$\chi_1 = \partial^{-1}u \tag{6.100}$$

$$\chi_2 = u_x + u\partial^{-1}u \tag{6.101}$$

By plugging (8.8) into (8.6), one obtains the KdV equation (8.1).

Further consideration ought to be made in slightly different terms. Let us denote

$$\chi = e^{\int_{-\infty}^{x} q(x,t)dx}$$

Then,

$$q = -\frac{1}{2ik}(q_x + q^2 + u) \tag{6.102}$$

$$q_t = \frac{\partial}{\partial x}(q_{xx} + 3q^2q_x + q^3 + 3u) \tag{6.103}$$

Hereafter, we will assume that $u \to 0$ at $x \to \pm\infty$. Then, $\chi \to 1$ at $x \to \pm\infty$ (one can choose χ according to this condition), and $q \to 0$ at $x \to \pm\infty$.

Let us denote

$$I(k) = \int_{-\infty}^{\infty} q(x,v,t)dx \tag{6.104}$$

In virtue of (8.11), $\frac{dI(k)}{dt} = 0$. $I(k)$ is a constant of the motion and depends on parameter k.

One can expand q in an asymptotic series in powers of $\frac{1}{-2ik}$at $k \to \infty$:

$$q = \sum_{n=1}^{\infty} \frac{q_n}{(-2ik)^n} \tag{6.105}$$

Now,

$$q_1 = u \tag{6.106}$$

$$q_2 = u_x \tag{6.107}$$

$$q_3 = u_{xx} + u^2 \tag{6.108}$$

$$q_4 = q_{3x} + 2q_1q_2 = u_{xxx} + \frac{3}{2}\frac{\partial u^2}{\partial x} \tag{6.109}$$

$$q_5 = q_{4x} + 2q_1q_3 + q_2^2 \tag{6.110}$$

$$= q_{4x} + 2u(u_{xx} + u^2) + u_x^2 \tag{6.111}$$

$$= u_{xxxx} + 4u_x^2 + 5uu_{xx} + 2u^3 \tag{6.112}$$

In general,

$$q_{n+1} = q_{nx} + \sum_{k=1}^{n-1} q_k q_{n-1-k} \tag{6.113}$$

¿From (8.14) one can see that

$$\int_{-\infty}^{\infty} q_2 dx = 0$$

$$\int_{-\infty}^{\infty} q_4 dx = 0$$

Later on, we will prove that

$$\int_{-\infty}^{\infty} q_{2k} dx = 0 \tag{6.114}$$

Now,

$$I_0 = \int_{-\infty}^{\infty} u dx \tag{6.115}$$

$$I_1 = \int_{-\infty}^{\infty} u^2 dx \tag{6.116}$$

$$I_2 = \int_{-\infty}^{\infty} (-u_x^2 + 2u^3) dx$$

All integrals

$$I_k = \int_{-\infty}^{\infty} q_{2k+1} dx$$

are constants of motion of polynomial type

$$I_k = \int P_k(u, u_x, ..., u^{(k-1)} dx \tag{6.117}$$

where P_k is some polynomial.

The next nontrivial integral,

$$I_3 = \int_{-\infty}^{\infty} q_7 dx = \int_{-\infty}^{\infty} (2q_1q_5 + 2q_2q_4 + q_3^2) dx = \int_{-\infty}^{\infty} (u_{xx}^2 - 10uu_x^2 + 5u^4) dx \tag{6.118}$$

Notice that the KdV equation (8.1) is a Hamiltonian system:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{\delta H}{\delta u} = 0 \tag{6.119}$$

$$H = \frac{1}{2} I_2 = \int_{-\infty}^{\infty} \left(-\frac{1}{2} u_x^2 + u^3 \right) dx \tag{6.120}$$

Integrals (8.19) are local; one can also study non-local integrals. For instance, one can expand q in powers of parameter $2ik$ in the vicinity of $k = 0$:

$$q = p_0 + 2ikp_1 + (2ik)^2 p_2 + ... \tag{6.121}$$

Then p_0 satisfies the Riccati equation,

$$p_{0x} + p_0^2 + u = 0 \tag{6.122}$$

All higher terms of expansion (8.22) are solutions of inhomogeneous linear equations

$$p_{1x} + 2p_0p_1 = p_0$$

and so on.

Let us study more carefully the KP equation,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) = \pm 3 \frac{\partial^2 u}{\partial y^2} \tag{6.123}$$

We will assume that $u \to 0$ at $x \to \pm\infty$ and $|u| < \infty$ on the entire xy-plane. After integrating (9.1) by x, we get

$$\frac{\partial^2 A}{\partial y^2} = 0 \tag{6.124}$$

$$A = \int_{-\infty}^{\infty} u dx \tag{6.125}$$

¿From (9.2) we can conclude that

$$A = A(t) \tag{6.126}$$

In chapter 7 we defined q as

$$q = 3\partial_x^{-1} \frac{\partial u}{\partial y} = -3 \frac{\partial}{\partial y} \int_{-\infty}^{x} u dx \tag{6.127}$$

By virtue of condition (9.4), $q(\infty) = 0$. Then equation (9.1) can be written

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u_{xx} + 3u^2) = \mp \frac{\partial q}{\partial y} \tag{6.128}$$

For the oblique solution (6.8), $q = -\alpha u$, $\frac{\partial q}{\partial y} = \alpha^2 \frac{\partial u}{\partial x}$, and $\int_{-\infty}^{\infty} \frac{\partial q}{\partial y} dx = 0$. Then,

$$A = \int_{-\infty}^{\infty} u dx = A_0 = const \tag{6.129}$$

Notice that the equality

$$\frac{\partial A}{\partial t} = 0 \tag{6.130}$$

is just a consequence of the nature of the Hamiltonian structure:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{\delta H}{\delta u} = 0 \tag{6.131}$$

Equation (9.7) holds for any Hamiltonian satisfying the condition $\frac{\delta H}{\delta u} \to 0$ at $x \to \pm\infty$. In the general case, $A = A(y)$, but for both KP equations is it just a constant.

Actually, $A(y)$ is not a "normal" motion constant, like H itself, but is rather a "Kazimir function" conserved due to the degeneracy of the Poisson structure (9.8). Notice that, in the presence of solitons, $H = \infty$, while $\frac{\delta H}{\delta u}$ is finite and well-defined.

Let us find other conservation laws of the KP equations. Hereafter, we will concentrate on the KP-2 equation. Transition to the KP-1 equation can be done by the standard change, $y \to iy$ and $q \to iq$.

$$\frac{\partial \psi}{\partial y} + \frac{\partial^2 \psi}{\partial x^2} + u\psi = 0 \tag{6.132}$$

$$\frac{\partial \psi}{\partial t} + 4\frac{\partial^3 \psi}{\partial x^3} + 6u\frac{\partial \psi}{\partial x} + (3u_x + q)\psi = 0 \tag{6.133}$$

where $q = -\frac{\partial}{\partial y}\partial^{-1}u$.

As before, we introduce the function χ

$$\psi = \chi e^{-kx-k^2+4k^3 t} \tag{6.134}$$

χ satisfies the equations

$$2k\chi_x = \chi_y + \chi_{xx} + u\chi \tag{6.135}$$

and

$$12k^2\chi_x - 16k\chi_{xx} - 6ku\chi + \chi_t + 4\chi_{xxx} + 6u\chi_x + (3u_x + q)\chi = 0 \tag{6.136}$$

After some simplifications, equation (9.13) must be reduced to one of the following two forms:

$$\chi_{xt} + \chi_{xxxx} + 3\chi_{yy} + 3(u\chi_x)_x - 3(\partial^{-1}u_y\chi)_x = 0 \tag{6.137}$$

$$\frac{\partial}{\partial t}\chi_x + \frac{\partial}{\partial x}(\chi_{xxx} + 3u\chi_x + q\chi)_x + 3\frac{\partial}{\partial y}(\chi_y + u\chi) = 0 \tag{6.138}$$

Equation (9.15) is the differential form of the set of conservation laws

$$\chi_{xt} + \chi_{xxxx} + 3\chi_{yy} + 3(u\chi_x)_x + 3(u\chi_y - \partial^{-1}u_y\chi_x) = 0 \tag{6.139}$$

Now we can denote $\varepsilon = 1/2k$ and expand χ in powers of ε:

$$\chi = 1 + \varepsilon\chi_1 + \varepsilon^2\chi_2 + ... \tag{6.140}$$

Equation (9.12) then becomes

$$\chi_x = \varepsilon(\chi_y + \chi_{xx} + u\chi)$$

Thus,

$$\chi_1 = \partial^{-1}u \tag{6.141}$$

$$\chi_2 = u_x + \partial^{-1}u_y = u_x - \frac{1}{3}q \tag{6.142}$$

Plugging (9.18) into (9.15) leads to the KP-2 equation. Substituting higher-order expansion terms from (9.16) into (9.15) leads to other nontrivial relations imposed on u and q. Half of them, as in the case of KdV, are conservation laws. To make this clear, we rewrite equation (9.15) as

$$\chi_{xt} + \chi_{xxxx} + 3\chi_{yy} + 3(u\chi_x)_x + (q\chi)_x + 3(u\chi)_y = 0$$

Then equation (9.15) can be rewritten as follows:

$$\chi_x = \varepsilon(\chi_y + \chi_{xx} + \phi_x\chi) \tag{6.143}$$

$$\chi_1 = \phi \tag{6.144}$$

$$\chi_{2x} = \phi_y + \phi_{xx} + \frac{1}{2}\frac{\partial}{\partial x}\phi^2$$

Hence,

$$\int \chi_{2x} dxdy = 0 \tag{6.145}$$

$$\chi_2 = \partial^{-1}\phi_y + \phi_x + \frac{1}{2}\phi^2 \tag{6.146}$$

$$\chi_{3x} = \chi_{2y} + \chi_{2xx} + \phi_x\chi_2 \tag{6.147}$$

Then

$$\int \chi_{3x} dxdy = \int \phi_x\chi_2 dxdy = \int (\phi_x\partial^{-1}\phi_y + \phi_x^2 + \frac{1}{2}\phi_x\phi^2)dxdy$$
$$= \int (-\phi\phi_y + \phi_x^2 + \frac{1}{6}\frac{\partial}{\partial x}\phi^3)dx = \int \phi_x^2 dxdy$$

Finally, we obtain the expected result

$$\int u^2 dxdy = const \tag{6.148}$$

Further calculation of the motion integrals in terms of χ_i is too cumbersome. To simplify this procedure, we can replace equation (9.14) with the equivalent equation

$$\chi_t + 4\chi_{xxx} + 6u\chi_x + (3u_x + q)\chi + \frac{B}{\varepsilon}(\chi_y - \chi_{xx}) = 0 \tag{6.149}$$

and introduce

$$\chi = e^{\int_{-\infty}^{x} p\ dx}$$

Then equation (9.17) becomes

$$p = \varepsilon(p_x + p_t^2 u + \partial^{-1}p_y \tag{6.150}$$

$$p_t + \frac{B}{\varepsilon}p_y + \frac{\partial}{\partial x}[4p_{xx} + 12pp_x + 4p^3 + 3u_x + q - \frac{B}{\varepsilon}(p_x + p^2)] \tag{6.151}$$

¿From equation (3.25), one can see that

$$I = \int pdxdy \tag{6.152}$$

is the motion constant. Again, one can perform an expansion

$$p = \varepsilon p_1 + \varepsilon^2 p_2 + ...$$
$$p_1 = u$$

By plugging $p = \varepsilon p_1$ into (3.25) one can see that the zeroth-order terms are cancelled. Indeed,

$$3\partial^{-1}u_y + 3u_x + q - 3u_x = 0$$

because $q = 3\partial^{-1}u_y$.

Thus, to get conservation laws, we must forget about equation (9.27) and work only with equation (9.26). Then,

$$p_2 = u_x + \partial^{-1}u_y = u_x - \frac{1}{3}q$$

Notice that

$$\int p_2 dxdy = -\frac{1}{3}\int dxdy = \frac{1}{3}A\Big|_{y=-\infty}^{y=\infty} = 0$$

Again, all integrals of even p_n are zero. Now,

$$p_3 = p_{2x} + \partial^{-1}p_{2y} + u^2$$

Then,

$$\int p_3 dxdy = \int u^2 dxdy$$

Finally, only the odd integrals survive, and

$$I_k = \int p_{2k+1}dxdy$$

$$I_0 = \int u\, dx \tag{6.153}$$

$$I_1 = \int u^2 dx \tag{6.154}$$

(6.155)

$$I_3 = 2H \tag{6.156}$$

$$H = \int \left(-\frac{1}{2}u_x^2 + u^3 \pm \frac{1}{3}q^2\right) dxdy \tag{6.157}$$

For KP-1, the $\pm$ sign is positive, and for KP-2, it is negative.

6.7. Local and nonlocal $\bar{\partial}$ problem

Let λ be a coordinate on the complex plane and $\chi(\lambda, \bar{\lambda})$ be some function which is not necessarily analytic. Let us study the equation

$$\frac{\partial \chi}{\partial \bar{\lambda}} = f(\lambda, \bar{\lambda}) \tag{6.158}$$

To solve this equation, one must invert the operator $\frac{\partial}{\partial \bar{\lambda}}$. To do this, we should find the $\bar{\partial}$, derivative of rational functions.

What is $\frac{\partial}{\partial \bar{\lambda}}\frac{1}{\lambda}$? To answer this question, we present

$$\frac{1}{\lambda} = \lim_{\varepsilon \to 0} \frac{\bar{\lambda}}{\lambda\bar{\lambda} + \varepsilon^2}$$

$$\frac{\partial}{\partial \bar{\lambda}}\frac{1}{\lambda} = \frac{1}{\lambda\bar{\lambda} + \varepsilon^2} - \frac{\lambda\bar{\lambda}}{(\lambda\bar{\lambda} + \varepsilon^2)^2} = \frac{\varepsilon^2}{(\lambda\bar{\lambda} + \varepsilon^2)^2}$$

Now let $\lambda = re^{i\phi}$:

$$\frac{\partial}{\partial \bar{\lambda}}\frac{1}{\lambda} = \frac{\varepsilon^2}{(r^2 + \varepsilon^2)^2} \tag{6.159}$$

If $\varepsilon \to 0$, then $\frac{\partial}{\partial \bar{\lambda}} \frac{1}{\lambda}$ tends to zero everywhere except at a single point, $\lambda = 0$. Hence,

$$\frac{\partial}{\partial \bar{\lambda}} \frac{1}{\lambda} = c\delta(\lambda) \tag{6.160}$$

where $\delta(\lambda)$ is a two-dimensional delta function. To find c, we must integrate

$$\begin{aligned} c &= \int d\lambda d\bar{\lambda} \frac{\partial}{\partial \bar{\lambda}} \frac{1}{\lambda} = 2\pi \int_0^\infty \frac{r\varepsilon^2}{(r^2+\varepsilon^2)^2} dr \\ &= \pi \int_0^\infty \frac{\varepsilon^2}{(y+\varepsilon^2)^2} dy = -\pi \frac{\varepsilon^2}{y+\varepsilon^2} \bigg|_0^\infty = \pi \end{aligned}$$

Finally,

$$\frac{\partial}{\partial \bar{\lambda}} \frac{1}{\lambda} = \pi \delta(\lambda) \tag{6.161}$$

This is the Poincaré formula.

Now, we can present $f(\lambda, \bar{\lambda})$ as follows:

$$\begin{aligned} f(\lambda, \bar{\lambda}) &= \int \delta(\lambda - \xi) f(\xi, \bar{\xi}) d\xi d\bar{\xi} \\ &= \frac{1}{\pi} \frac{\partial}{\partial \bar{\lambda}} \int \frac{1}{\lambda - \xi} f(\xi, \bar{\xi}) d\xi d\bar{\xi} \end{aligned}$$

Then,

$$\chi = chi_0(\lambda) + \frac{1}{\pi} \int \frac{1}{\lambda - \xi} f(\xi, \bar{\xi}) d\xi d\bar{\xi} \tag{6.162}$$

In (10.5), we must assume that

$$\frac{1}{\lambda - \xi} = lim_{\varepsilon \to 0} \frac{\bar{\lambda} - \bar{\xi}}{(\lambda - \xi)^2 + \varepsilon^2}$$

In (10.5), $\chi_0(\lambda)$ is some analytic function of λ. If we do not want to acquire additional singularities, we can set $\chi_0 = const$.

We can say that $\chi(\lambda, \bar{\lambda})$ satisfies the linear $\bar{\partial}$ problem if it is a solution of the equation

$$\frac{\partial \chi}{\partial \bar{\lambda}} = f(\lambda, \bar{\lambda}) \chi(\lambda, \bar{\lambda}) \tag{6.163}$$

and

$$\int f(\lambda, \bar{\lambda}) d\lambda d\bar{\lambda} < const$$

This problem has a unique solution if we additionally demand that $\chi \to 1$ as $\lambda \to \infty$. Then,

$$\chi = \chi_0 = exp\left\{ \frac{1}{\pi} \int \frac{f(\xi, \bar{\xi}}{\lambda - \xi} d\xi d\bar{\xi} \right\} \tag{6.164}$$

If the function f has an asymptotic expansion at $\lambda \to \infty$,

$$\chi = P(\lambda) + \mathcal{O}(\lambda) \tag{6.165}$$

where $P(\lambda)$ is some polynomial. Then one has to set

$$\chi = P(\lambda) \chi_0(\lambda, \bar{\lambda}) \tag{6.166}$$

Hereafter, we denote $\lambda = \lambda_R + i\lambda_I$. Suppose that

$$f(\lambda, \bar{\lambda}) = f(\lambda_R) \delta(\lambda_I)$$

Then function $\chi(\lambda, \bar{\lambda})$ is analytic in the upper and lower half planes. It has limiting values

$$\chi \to \chi^+, \ \lambda_I \to 0^+$$
$$\chi \to \chi^-, \ \lambda_I \to 0^-$$

Now equation (10.6) reads

$$\chi^+ - \chi^- = \frac{1}{2} f(\lambda_R)(\chi^+ + \chi^-)$$

or

$$\chi^+ = g(\lambda_R)\chi^- \tag{6.167}$$

$$g(\lambda_R) = \frac{1 + \frac{1}{2} f(\lambda_R)}{1 - \frac{1}{2} f(\lambda_R)} \tag{6.168}$$

Condition (10.10) defined the local Reimann-Hilbert problem on the real axis.

As before, this problem can be solved exactly. It has a unique solution normalized by the condition $\chi^{\pm} \to 1$ at $|\lambda| \to \infty$, if we propose additionally that functions $\chi^{\pm}$ have no zeros in the half-planes of their analiticity. Then functions $ln\chi^+$ and $ln\chi^-$ are also analytic functions, and

$$ln\chi^+ - ln\chi^- = lng \tag{6.169}$$

Then,

$$ln\chi^+ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{lng(\xi)}{\xi - (\lambda_R + i\lambda_I)} d\xi \tag{6.170}$$

$$ln\chi^- = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{lng(\xi)}{\xi - (\lambda_R + i\lambda_I)} d\xi \tag{6.171}$$

If function χ^+ has zeros at points $\lambda = a_1, ..., a_n$, $Rea_k > 0$, one can arbitrarily choose the same amount of points $\lambda = \tilde{a_1}, ..., \tilde{a_n}$ in the lower half-plane and define

$$\chi^+ = \frac{(\lambda - a_1)...(\lambda - a_n)}{(\lambda - \tilde{a_1})...(\lambda - \tilde{a_n})} \tilde{\chi}^+$$

In the same way, if χ^- has zeros at points $\lambda = b_1, ..., b_m$, $Imb_k < 0$, one can arbitrarily choose the same amount of points $\lambda = \tilde{b_1}, ..., \tilde{b_m}$ in the upper half-plane and define

$$\chi^- = \frac{(\lambda - b_1)...(\lambda - b_m)}{(\lambda - \tilde{b_1})...(\lambda - \tilde{b_m})} \tilde{\chi}^-$$

Functions $\tilde{\chi}^{\pm}$ have no zeros. They satisfy the new Riemann-Hilbert problem

$$\tilde{\chi}^+ = \tilde{g} \tilde{\chi}^-$$

$$\tilde{g} = \frac{(\lambda - a_1)...(\lambda - a_n)(\lambda - b_1)...(\lambda - b_m)}{(\lambda - \tilde{a_1})...(\lambda - \tilde{a_n})(\lambda - \tilde{b_1})...(\lambda - \tilde{b_m})} g$$

This Riemann-Hilbert problem has a unique solution.

Now we define a nonlocal $\bar{\partial}$ problem assuming that function $\chi(\lambda, \bar{\lambda})$ satisfies the following integral equation:

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \int F(\lambda, \bar{\lambda}, \xi, \bar{\xi}) \chi(\xi, \bar{\xi}) d\xi d\bar{\xi} \tag{6.172}$$

The function must be properly normalized; for instance, one has to require that $\chi \to 1$ as $|\lambda| \to \infty$. In this case, χ is a solution of the following integral equation:

$$\chi = 1 + \int R(\lambda, \bar{\lambda}, \xi, \bar{\xi})\chi(\xi, \bar{\xi})d\xi d\bar{\xi} \tag{6.173}$$

Here,

$$R(\lambda, \bar{\lambda}, \xi, \bar{\xi}) = \frac{1}{\pi}\int \frac{F(\mu, \bar{\mu}, \xi, \bar{\xi})}{\lambda - \mu} d\mu d\bar{\mu}$$

In the general case, (10.14) is a regular Fredholm integral equation of the second kind. However, in the case of the local $\bar{\partial}$ problem, this equation becomes singular:

$$\chi = 1 + \frac{1}{\pi}\int \frac{f(\xi, \bar{\xi})\chi(\xi, \bar{\xi})}{\lambda - \xi} d\xi d\bar{\xi} \tag{6.174}$$

We have seen that this equation can be solved explicitly. The method of solution described above is known as the Wiener-Hopf method.

Another important example of a nonlocal $\bar{\partial}$ problem is the following: suppose that χ satisfies the equation

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \delta(\lambda, \bar{\lambda})\chi(-\lambda, -\bar{\lambda}) \tag{6.175}$$

This is the "reflected local $\bar{\partial}$ problem." Integral equation (10.14) now takes the form

$$\chi = 1 + \frac{1}{\pi}\int \frac{f(\xi, \bar{\xi})\chi(\xi, \bar{\xi})}{\lambda + \xi} d\xi d\bar{\xi} \tag{6.176}$$

This equation cannot be solved explicitly.

6.8. Dressing method for the KP equation

Let us define the nonlocal $\bar{\partial}$ problem

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \int \chi(\eta, \bar{\eta})T(\eta, \bar{\eta}, \lambda\bar{\lambda})d\eta d\bar{\eta} \tag{6.177}$$

Normalized to unity at $\lambda \to \infty$, such a χ satisfies the integral equation

$$\chi = 1 + \frac{1}{\pi}\int \frac{T(\xi, \bar{\xi}, \eta, \bar{\eta})}{\lambda - \xi}\chi(\eta, \bar{\eta})d\xi d\bar{\xi} d\eta d\bar{\eta} \tag{6.178}$$

We will assume that the homogeneous integral equation

$$\chi = \frac{1}{\pi}\int \frac{T(\xi, \bar{\xi}, \eta, \bar{\eta})}{\lambda - \xi}\chi(\eta, \bar{\eta})d\xi d\bar{\xi} d\eta d\bar{\eta} \tag{6.179}$$

only has a solution of zero and will also assume that the kernel T depends on coordinates x, y, and t as follows:

$$\begin{aligned} T(\xi, \bar{\xi}, \eta, \bar{\eta}) &= e^{\phi(\eta)}T_0(\xi, \bar{\xi}, \eta, \bar{\eta})e^{-\phi(\lambda)} \\ \phi(\lambda) &= \lambda x - \lambda^2 y - 4\lambda^3 t \end{aligned}$$

This means that T satisfies the system of linear equations

$$\frac{\partial T}{\partial x} + (\lambda - \eta)T = 0 \tag{6.180}$$

$$\frac{\partial T}{\partial y} - (\lambda^2 - \eta^2)T = 0 \tag{6.181}$$

$$\frac{\partial T}{\partial t} - 4(\lambda^3 - \eta^3)T = 0 \tag{6.182}$$

Equation (11.1) can be rewritten symbolically as

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \chi * T \tag{6.183}$$

Let us introduce the differential operators

$$D_1\chi = \frac{\partial \chi}{\partial x} + \lambda\chi \tag{6.184}$$

$$D_2\chi = \frac{\partial \chi}{\partial y} - \lambda^2\chi \tag{6.185}$$

$$D_3\chi = \frac{\partial \chi}{\partial t} - 4\lambda^3\chi \tag{6.186}$$

These commute with operator $\frac{\partial}{\partial \bar{\lambda}}$. Applying D_i to (11.5), we can see that

$$\frac{\partial}{\partial \bar{\lambda}} D_i\chi = D_i\chi * T \tag{6.187}$$

Let $D\chi = D(D_1, D_2, D_3)\chi$ be any polynomial on D_1, D_2, D_3. Its coefficients are functions of x, y, and t. By induction, we can prove that

$$\frac{\partial}{\partial \bar{\lambda}} D\chi = D\chi * T \tag{6.188}$$

Now we construct the differential operator

$$L_1\chi = (D_2 + D_1^2 + u)\chi \tag{6.189}$$

$$= \frac{\partial \chi}{\partial y} - \lambda^2\chi + (\frac{\partial}{\partial x} + \lambda)^2\chi + u\chi \tag{6.190}$$

$$= \frac{\partial \chi}{\partial y} + \frac{\partial^2 \chi}{\partial x^2} + 2\lambda\frac{\partial \chi}{\partial x} + u\chi \tag{6.191}$$

in the neighborhood of infinity,

$$\chi = 1 + \frac{\chi_1}{\lambda} + \frac{\chi_2}{\lambda^2} + ... \tag{6.192}$$

and

$$L\chi \to 2\frac{\partial \chi_1}{\partial x} + u$$

Hence, if

$$u = -2\frac{\partial \chi_1}{\partial x} \tag{6.193}$$

$$L_1\chi \to 0 \ \ at \ \ \lambda \to \infty \tag{6.194}$$

However, $L_1\chi$ is a solution of the non-local $\bar{\partial}$ problem with zero asymptotics at infinity. Therefore, either $L_1\chi = 0$, which can be rewritten

$$(D_2 + D_1^2 + u)\chi = 0 \tag{6.195}$$

or

$$\frac{\partial \chi}{\partial y} + \frac{\partial^2 \chi}{\partial x^2} + 2\lambda \frac{\partial \chi}{\partial x} + u\chi = 0$$

In the same way, we can construct the operator

$$L_2\chi = (D_3 + D_1^3 + 6uD_1 + 3u_x + q)\chi \tag{6.196}$$

or

$$L_2\chi = (\frac{\partial}{\partial t} + 4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x + q + 12\lambda^2\frac{\partial}{\partial x} + 12\lambda\frac{\partial^2}{\partial x^2} + 6u\lambda)\chi \tag{6.197}$$

Substituting (11.10) into (11.11) and setting $\lambda \to 0$, we obtain

$$L_2\chi \to \lambda\left(12\frac{\partial \chi_1}{\partial x} + 6u\right) + 12\frac{\partial \chi_2}{\partial x} + 12\frac{\partial^2 \chi_1}{\partial x} + 6u\chi_1 + 3u_x + q \tag{6.198}$$

The term in (11.14) that is linear in λ is cancelled by virtue of (11.11). Then, substituting (11.10) into (11.12), we get

$$2\frac{\partial \chi_2}{\partial x} = -\frac{\partial \chi_1}{\partial y} - \frac{\partial^2 \chi_1}{\partial x^2} - u\chi_1 \tag{6.199}$$

Substituting (11.15) into (11.14), we see that

$$L_2\chi \to -6\frac{\partial \chi_1}{\partial y} + q$$

If we assume that $q \to -6\partial^{-1}u_y$, then $L_2\chi \to 0$ at $\lambda \to 0$; hence

$$L_2\chi = 0 \tag{6.200}$$

Equations (11.12) and (11.16) are identical to equations (9.12) and (9.13). Thus we see that by using the nonlocal $\bar{\partial}$ problem one can construct exact solutions of the KP equation.

6.9. Solitonic solutions of the KdV equation

Let us assume that

$$T(\eta, \bar{\eta}, \lambda, \bar{\lambda}) = T(\lambda, \bar{\lambda})\delta(\eta + \lambda)\delta(\bar{\eta} + \bar{\lambda}) \tag{6.201}$$

Now the $\bar{\partial}$ problem (11.1) is reduced to

$$\frac{\partial \chi}{\partial \bar{\lambda}} = T(\lambda, \bar{\lambda})e^{-2\lambda x + 8\lambda^3 t}\chi(-\lambda, -\bar{\lambda}) \tag{6.202}$$

In this case, the dependence on χ is cancelled and the KP equation reduces to the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0 \tag{6.203}$$

The integral equation (11.2) now reads

$$\chi = 1 + \frac{1}{\pi}\int \frac{T_0(\eta, \bar{\eta})e^{-2\eta x + 8\eta^3 t}\chi(-\eta, -\bar{\eta})}{\lambda - \eta} d\eta d\bar{\eta} \tag{6.204}$$

or

$$\chi = 1 + \frac{1}{\pi}\int \frac{T_0(-\eta, -\bar{\eta})e^{2\eta x - 8\eta^3 t}\chi(\eta, \bar{\eta})}{\lambda + \eta} d\eta d\bar{\eta} \tag{6.205}$$

Again, let

$$D_1\chi = \frac{\partial \chi}{\partial x} + \lambda\chi$$
$$D_2\chi = \frac{\partial \chi}{\partial t} - 4\lambda^3 t$$
$$D\chi = P(D_1, D_2, \lambda^2)$$

where $P(D_1, D_2, \lambda^2)$ is any polynomial of the three variables. The coefficients are functions of x and t. Them

(6.206) $$\frac{\partial}{\partial \bar{\lambda}} D\chi = T(\lambda, \bar{\lambda}) e^{-2\phi} P\chi(-\lambda, -\bar{\lambda})$$

Here, $\phi = \lambda x - 4\lambda^3 t$. If $P\chi \to 0$ at $\lambda \to \infty$, then

(6.207) $$P\chi = 0$$

In particular, one can choose

$$P_1\chi = L_1\chi = (D_1^2 - \lambda^2 - u)\chi$$
$$P_2\chi = L_2\chi = (D_2 + D_1^3 + 6uD_1 + 3u_x)\chi$$

We have obtained the result of chapter 8: the KdV equation (12.3) is a compatibility condition for the linear equations

(6.208) $$\frac{\partial^2 \chi}{\partial x^2} + 2\lambda \frac{\partial \chi}{\partial x} + u\chi = 0$$

(6.209) $$\frac{\partial \chi}{\partial t} + 4\frac{\partial^3 \chi}{\partial x^3} + 12\lambda \frac{\partial^2 \chi}{\partial x^2} + 12\lambda^2 \frac{\partial \chi}{\partial x} + 6u\left(\frac{\partial \chi}{\partial x} + \lambda\chi\right) + 3u_x\chi$$

Now suppose that

(6.210) $$T(\lambda, \bar{\lambda}) = \pi \sum_{k=1}^{N} M_k^2 \delta(\lambda - \lambda_k)\delta(\bar{\lambda} - \bar{\lambda}_n)$$

(6.211) $$\chi = 1 + \sum_{n=1}^{N} \frac{f_n(x,t)}{\lambda - \lambda_n}$$

$$\frac{\partial \chi}{\partial \bar{\lambda}} = \pi \sum_{n=1}^{N} f_n(x,t)\delta(\lambda - \lambda_k)\delta(\bar{\lambda} - \bar{\lambda}_n)$$
$$\chi(-\lambda, -\bar{\lambda}) = 1 - \sum_{n=1}^{N} \frac{f_n(x,t)}{\lambda + \lambda_n}$$

Let us denote

(6.212) $$g_n = f_n e^{\phi_n}$$

Functions $g_n(x,t)$ satifies the linear system

(6.213) $$g_n = M_n^2 \sum_{m=1}^{N} \frac{g_m e^{-(\phi_n + \phi_m)}}{\lambda_n + \lambda_m} = M_n^2 e^{-\phi_n}$$

The determinant of this system

$$\Delta = det\|\delta_{nm} + M_n^2 \frac{e^{-(\phi_n+\phi_m)}}{\lambda_n + \lambda_m}\| = det\,\Lambda_{nm} \tag{6.214}$$

Then

$$\chi_1 = \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{N} g_n e^{-\phi_n} \tag{6.215}$$

We know the fundamental fact that

$$\chi_1 = -\frac{\Delta_x}{\Delta} \tag{6.216}$$

Thus,

$$u = 2\frac{\partial^2}{\partial x^2} ln\,\Delta \tag{6.217}$$

To prove this fact, we must remember that

$$\Delta_x = \sum_{n=1}^{N} \Delta_n$$

where Δ_n are the determinants obtained from Δ by differentiating along the n^{th} column. As a result, the n^{th} column of Δ_n is

$$\begin{bmatrix} M_1^2 e^{-\phi_1-\phi_n} \\ M_2^2 e^{-\phi_2-\phi_n} \\ . \\ . \\ . \\ M_n^2 e^{-2\phi_n} \end{bmatrix} = e^{-\phi_n} \begin{bmatrix} M_1^2 e^{-\phi_1} \\ M_2^2 e^{-\phi_2} \\ . \\ . \\ . \\ M_n^2 e^{-\phi_n} \end{bmatrix} \tag{6.218}$$

Now, using Kramer's rule, we can conclude that

$$-g_n e^{-\phi_n} + \frac{\Delta_n}{\Delta}$$

Hence,

$$\sum g_n e^{-\phi_n} = -\frac{1}{\Delta}\sum \Delta_n = -\frac{\Delta_x}{\Delta}$$

It is interesting that the whole solution of the linear system (12.8), (12.9) can be presented in the determinant form

$$\chi = \frac{\tilde{\Delta}}{\Delta} \tag{6.219}$$

where

$$\tilde{\Delta} = \begin{vmatrix} 1 & \frac{e^{-\phi_1}}{\lambda-\lambda_1} & \cdots & \frac{e^{-\phi_n}}{\lambda-\lambda_n} \\ M_1^2 e^{-\phi_1} & & & \\ . & & & \\ . & & \Delta_{nm} & \\ . & & & \\ M_n^2 e^{-\phi_n} & & & \end{vmatrix} \tag{6.220}$$

In the case when there is a single pole λ_0,

$$\Delta = 1 + e^{-2S} \tag{6.221}$$

$$S = \lambda_0(x - x_0) - 4\lambda_0^3 t \tag{6.222}$$

$$e^{2\lambda_0 x_0} = \frac{M^2}{2\lambda_0} \tag{6.223}$$

$$x_0 = \frac{1}{2x_0} ln \frac{M^2}{2\lambda_0} \tag{6.224}$$

Now,

$$\frac{\Delta_x}{\Delta} = -\frac{2\lambda}{e^{2S} + 1} \tag{6.225}$$

$$u = \frac{2\lambda^2}{cosh^2[(x - x_0) - 4\lambda^2 t]} \tag{6.226}$$

This is a soliton.

The determinant Δ is called the “tan function” and is denoted τ. Let us look for solutions of the KdV equation with the form

$$u = 2\frac{\partial^2}{\partial x^2} ln\, \tau \tag{6.227}$$

If $u = v_x$, then

$$v = 2\frac{\partial}{\partial x} ln\, \tau = 2\frac{\tau_x}{\tau}$$

v satisfies the equation

$$v_t + 3v_x^2 + v_{xxx} = 0 \tag{6.228}$$

Then,

$$v_t = \frac{2}{\tau^2}(\tau_{xt}\tau - \tau_x\tau_t) \tag{6.229}$$

$$v_x = \frac{2}{\tau^2}(\tau_{xx}\tau - \tau_x^2) \tag{6.230}$$

$$v_{xx} = \frac{2}{\tau^2}(\tau_{xxx}\tau - \tau_x\tau_{xx}) - \frac{4\tau_x}{\tau^3}(\tau_{xx}\tau - \tau_x^2) \tag{6.231}$$

$$= \frac{2}{\tau^2}(\tau_{xxx}\tau - 3\tau_x\tau_{xx}) + \frac{4\tau_x^3}{\tau^3} \tag{6.232}$$

$$v_{xxx} = \frac{2}{\tau^2}[\tau_{xxxx}\tau - 2\tau_x\tau_{xxx} - 3\tau_{xx}^2] + \frac{12\tau_x^2\tau_{xx}}{\tau^3} - \frac{12\tau_x^4}{\tau^4} \tag{6.233}$$

It is remarkable that, after substitution of (12.25) into (12.24), only quadratic terms survive and equation (12.24) takes the form

$$\tau_{xt}\tau - \tau_x\tau_t + \tau_{xxxx}\tau - 4\tau_{xxx}\tau_x + 3\tau_{xx}^2 = 0 \tag{6.234}$$

This is the famous Hirota equation.

One can check that the Hirota equation has the solution

$$\tau = 1 + ce^{-2\lambda x + 8\lambda^3 t}$$

corresponding to a simple soliton.

If we make the transformation

$$\tau \to Ae^{Bx}\tau \tag{6.235}$$

where A and B are arbitrary constants, then $ln\,\tau \to ln\,A + x\,ln\,B + ln\,\tau$, and

$$\frac{d^2}{dx^2}T \to \frac{d^2}{dx^2}ln\,\tau$$

A solution of the KdV equation is invariant with respect to transformation (12.27).

6.10. Scattering of solitons

Let us study a two-soliton solution characterized by the position of poles λ_1 and λ_2 and parameters M_1^2, M_2^2. Now,

$$\Delta = \begin{vmatrix} 1 + M_1^2 \frac{e^{-2\phi_1}}{2\lambda_1} & M_1^2 \frac{e^{-(\phi_1+\phi_2)}}{\lambda_1+\lambda_2} \\ M_2^2 \frac{e^{-(\phi_1+\phi_2)}}{\lambda_1+\lambda_2} & M_2^2 \frac{e^{-2\phi_2}}{2\lambda_2} \end{vmatrix} \tag{6.236}$$

$$= 1 + e^{\xi_1} + e^{\xi_2} + \frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)^2} e^{(\xi_1+\xi_2)} \tag{6.237}$$

Here,

$$\xi_1 = -2\lambda_1(x - 4\lambda_1^2 t - x_{10}) \tag{6.238}$$

$$\xi_2 = -2\lambda_2(x - 4\lambda_2^2 t - x_{20}) \tag{6.239}$$

$$x_{10} = \frac{1}{2\lambda_1} ln\,\frac{M_1^2}{2\lambda_1} \tag{6.240}$$

$$x_{20} = \frac{1}{2\lambda_2} ln\,\frac{M_2^2}{2\lambda_2} \tag{6.241}$$

We assume that λ_1 and λ_2 are real numbers, and that $\lambda_1 \neq \pm\lambda_2$.

Let both λ_1 and λ_2 start off positive and that initially $\lambda_1 > \lambda_2 > 0$. Let $t \to \infty$. In this area, $x \approx 4\lambda_1^2 t$ and $\xi_1 \approx 1$. In the same area,

$$\xi_2 \sim -8\lambda_2(\lambda_1^2 - \lambda_2^2)t$$

So, $\xi_2 \to \infty$ and $e_2^{\xi} \to \infty$. Hence, in this region, $\xi_1 \approx 1$, and the two-soliton solution is reduced to one soliton:

$$\Delta \to 1 + e^{\xi_1} \tag{6.242}$$

This is the "fast" soliton. Its center is positioned at $x = x_{10} + 4\lambda_1^2 t$.

The "slow" soliton at $t \to \infty$ is positioned in a region where $\xi_2 \approx 1$. In this region,

$$\xi_1 \sim -8\lambda_1(\lambda_2^2 - \lambda_1^2)t = 8\lambda_1(\lambda_1^2 - \lambda_2^2)t \to \infty$$

if $t \to \infty$. In this region, we can replace the determinant (13.1) with an equivalent determinant

$$\Delta \to \tilde{\Delta} = 1 + \frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)^2} e^{\xi_2} + e^{-\xi_1}(1 + e^{\xi_2}) \tag{6.243}$$

$$\simeq 1 + \frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)^2} e^{\xi_2} = 1 + e^{\tilde{\xi}_2} \tag{6.244}$$

$$\tilde{\xi}_2 = \xi_2 + ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)^2} \tag{6.245}$$

Equations (13.6) and (13.7) mean that the slow soliton is positioned at the point

$$\tilde{x}_{20} = x_{20} - \frac{1}{2\lambda_2} ln \frac{(\lambda_1 + \lambda_2)^2}{(\lambda_1 - \lambda_2)^2} \tag{6.246}$$

Now let $t \to -\infty$. The center of the "slow" soliton is located at the point x_{20}, while the center of the "fast" soliton is at the point

$$\tilde{x}_{10} = x_{10} - \frac{1}{2\lambda_1} ln \frac{(\lambda_1 + \lambda_2)^2}{(\lambda_1 - \lambda_2)^2} \tag{6.247}$$

At $t \to \pm\infty$, the fast and slow solitons are far separated. In some intermediate time, they undergo a collision. As a result of this collision, the fast soliton acquires a positive shift $\frac{1}{2\lambda_1} ln \frac{(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2}$, while the slow soliton acquires a negative shift, $-\frac{1}{2\lambda_1} ln \frac{(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2)^2}$. The center of mass of the solitons,

$$< x >= \frac{\lambda_1 x_{10} + \lambda_2 x_{20}}{\lambda_1 + \lambda_2} \tag{6.248}$$

remains unchanged.

To continue further, we must first learn how to calculate the determinant

$$\Delta(\lambda_1, ..., \lambda_m) = \begin{vmatrix} \frac{1}{2\lambda_1} & \frac{1}{\lambda_1+\lambda_2} & \cdots & \frac{1}{\lambda_1+\lambda_m} \\ \frac{1}{\lambda_1+\lambda_2} & \frac{1}{2\lambda_2} & \cdots & \frac{1}{\lambda_2+\lambda_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\lambda_1+\lambda_m} & \cdots & \cdots & \frac{1}{2\lambda_m} \end{vmatrix} \tag{6.249}$$

Actually, $\Delta(\lambda_1, ..., \lambda_m) = 0$ if $\lambda_i = \lambda_j$, and $\Delta(\lambda_1, ..., \lambda_,) = \infty$ if either $\lambda_i = 0$ or $\lambda_i + \lambda_j = 0$. We can thus conclude that

$$Delta(\lambda_1, ..., \lambda_m) = C \frac{\prod(\lambda_i - \lambda_j)^2}{\prod \left[(\lambda_i + \lambda_j)^2\right] 2^n \lambda_1 ... \lambda_n} \tag{6.250}$$

where C is some constant. To determine this constant, we send $\lambda_m \to 0$ and find that, in this limit,

$$\Delta(\lambda_1, ..., \lambda_m) \to \frac{1}{2\lambda_m} \Delta(\lambda_1, ..., \lambda_{m-1})$$

Repeating this procedure, we find that $C = 1$. Now we can present a general function τ as the sum

$$\tau = \Delta = 1 + \tau_1 + \tau_2 + ... + \tau_n \tag{6.251}$$

$$\tau_1 = \sum_{k=1}^{N} e^{\xi_k}$$

$$\xi_k = -2\lambda_k (x - 4\lambda_k^2 t - x_{0k}) \tag{6.252}$$

$$x_{0k} = \frac{1}{2\lambda_k} ln \frac{M_k^2}{2\lambda_k}$$

$$\tau_2 = \sum_{i_2 > i_1} \frac{(\lambda_{i_1} - \lambda_{i_2})^2}{(\lambda_{i_1} + \lambda_{i_2})^2} e^{\xi_{i_1} + \xi_{i_2}}$$

where, in this sum, $\lambda_i \neq \lambda_j$. In the same way,

$$\tau_k = \sum_{i_1, ..., i_k} \frac{\prod(\lambda_{i_p} - \lambda_{i_q})^2}{\prod(\lambda_{i_p} - \lambda_{i_q})^2} e^{\xi_{i_1} + ... + \xi_{i_n}} \tag{6.253}$$

inside the products, i_p and i_q run through all possible pairs of entries $i_1, ..., i_k$.

Representation (13.12) makes it possible to study analytically the asymptotic behavior of any n-solitonic solution. Suppose that the position of the poles are ordered

$$\lambda_1 > \lambda_2 > ... > \lambda_n \tag{6.254}$$

Let $t \to \infty$. In the area $\xi_1 \approx 1$, $\xi_i \to -\infty$ for $i > 1$. Then the solitonic solution can be simplified to the form

$$\Delta = 1 + e^{\xi_1}$$

and the fastest soliton is placed at $x = x_{01}$.

To look for the position of one of the slower solitons with number k, $n > k > 1$, we should divide the τ function by the factor $e^{\xi_1 + \dots + \xi_{k-1}}$ so that the term proportional to e^{ξ_k} becomes the major. Then we send $t \to \infty$ to make sure that all exponents $e^{\xi_l - \xi_k}$, $l > k$, are exctinct. We end up with the one-soliton solution

$$\Delta \to const \left(1 + \prod_{l>k} \frac{(\lambda_l - \lambda_k)^2}{(\lambda_l + \lambda_k)^2} e^{\xi_k} \right)$$

We can conclude that the soliton of rank k at time $t \to \infty$ is shifted backward up to distance

$$\delta x_k^+ = \frac{1}{2\lambda_k} \sum_{l>k} ln \frac{(\lambda_l + \lambda_k)^2}{(\lambda_l - \lambda_k)^2}$$

Repeating this consideration in the limit $t \to -\infty$, we find that, in this limit, the slowest soliton takes a leading position, while the faster solitons are shifted backward by a distance

$$\delta x_k^- = \frac{1}{2\lambda_k} \sum_{l<k} ln \frac{(\lambda_l + \lambda_k)^2}{(\lambda_l - \lambda_k)^2}$$

As a result, we find the remarkable conclusion that the shift of a soliton in an n-soliton solution between $t \to -\infty$ and $t \to \infty$ is an algebraic sum of shifts of this soliton in pair collisions with other solitons:

$$\delta x_k^+ - \delta x_k^- = \frac{1}{2\lambda_k} \left(\sum_{l<k} ln \frac{(\lambda_l + \lambda_k)^2}{(\lambda_l - \lambda_k)^2} - \sum_{l>k} ln \frac{(\lambda_l + \lambda_k)^2}{(\lambda_l - \lambda_k)^2} \right)$$

6.11. More on solitonic solutions: Transplantation

Suppose that the pole in a single-soliton solution is located on the negative half axis $\lambda_1 = -\eta_1$, $\eta_1 > 0$. To avoid singularities, we must demand that M_1 is imaginary and replace $M_1^2 \to -M_1^2$. Then,

$$\Delta = 1 + e^{2\eta_1(x + x_{10} - 4\eta_1^2 t)} \tag{6.255}$$

where, as before,

$$x_{10} = \frac{1}{2\eta_1} ln \frac{M_1^2}{2\eta_1} \tag{6.256}$$

Let us present Δ in the form

$$\Delta = e^{2\eta(x + x_{10} - 4\eta_1^2 t)} \left(1 + e^{-2\eta_1(x + x_{10} - 4\eta_1^2 t)} \right) \sim \left(1 + e^{-2\eta_1(x + x_{10} - 4\eta_1^2 t)} \right) \tag{6.257}$$

This is a soliton of amplitude η_1 located at $x = -x_{10}$.

Now we consider a two-soliton solution and assume that one pole is located on the negative half-axis, $\lambda_1 = -\eta_1$, and the second pole is located on the positive half-axis. As before, we replace $M_1^2 \to -M_1^2$. Now,

$$\Delta = 1 + e^{\xi_1} + e^{\xi_2} + \left(\frac{\eta_1 + \lambda_2}{\eta_1 - \lambda_2}\right)^2 e^{\xi_1+\xi_2} \tag{6.258}$$

$$\xi_1 = 2\eta_1(x + x_{10} - 4\eta_1^2 t)$$
$$\xi_2 = -2\eta_2(x - x_{20} - 4\eta_2^2 t)$$

Now we multiply (14.3) by e^{ξ_1} and obtain the equivalent determinant

$$\tilde{\Delta} = 1 + e^{-\xi_1} + \left(\frac{\eta_1 + \lambda_2}{\eta_1 - \lambda_2}\right)^2 e^{\xi_2} + e^{-\xi_1+\xi_2} \tag{6.259}$$

Now we replace

$$\xi_2 \to \tilde{\xi}_2 = \xi_2 + ln\left(\frac{\eta_1 - \lambda_2}{\eta_1 + \lambda_2}\right)^2$$

Then we return to the "standard" form of the two-soliton solution (13.6), but the position of the solitons are shifted.

The soliton with eigenvalue η_1 is located at

$$\tilde{x_{10}} = -\frac{1}{2\eta_1} ln\,\frac{M_1^2}{2\eta_1}$$

The second soliton is located at the point

$$x_{20} = \frac{1}{2\eta_1} ln\,\frac{M_2^2}{2\eta_2} + \frac{1}{2\eta_2} ln\,\frac{(\lambda_1 + \eta_2)^2}{(\eta_1 - \lambda_2)^2}$$

This process can be called "transplantation of solitons." If k solitons with eigenvalues $\lambda_i = -\eta_i$, $i = 1, ..., k$ are located on the negative half-axis ($\eta_i > 0$), the transplantation can be made by multiplying Δ by the factor

$$e^{-(\xi_1 + ... + \xi_k)} \frac{\prod(\eta_i - \eta_j)^2}{\prod(\eta_i + \eta_j)^2} \tag{6.260}$$

where $1 \le i \le k$ and $1 \le j \le k$. After this multiplication, we obtain the same solution of the KdV equation. It must be the solitonic solution with n "positive" poles (this is a theorem!). But determination of the soliton's position $\tilde{x}_{10}$ is an as-yet unsolved problem.

Now we consider the two-soliton solution and assume that the poles are complex: $\lambda_1 = p + iq$ and $\lambda_2 = p - iq$, $p > 0$. Then,

$$\frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2} = -\frac{q^2}{p^2}$$

and

$$\Delta = 1 + e^{\xi} + e^{\bar{\xi}} - \frac{q^2}{p^2} e^{(\xi + \bar{\xi})}$$

If $x \to \infty$, then

$$\Delta \to -\frac{q^2}{p^2} e^{2p(x - x_0 - 4(\lambda^2 + \bar{\lambda}^2)t)} \to -\infty$$

If $x \to -\infty$, then $\Delta \to 1$. Then, inevitably, Δ has a pole at some point $x = x_0$. Near this pole,

$$u \simeq \frac{2}{(x-x_0)^2}$$

$$\chi(-\lambda) = 1 - \sum \frac{\tilde{\chi}_m(\lambda,t)}{\lambda+\lambda_m} - \frac{1}{2\pi}\int \frac{\rho(s,x,t)}{\lambda+is}dk \tag{6.261}$$

On the imaginary axis, $\chi(-\lambda) \to \chi^+(ik)$ if $\lambda > 0$, or $\chi(-\lambda) \to \chi^-(ik)$ if $i\lambda < 0$.

There is some ambiguity in choosing a version of χ on the imaginary axis. We will assume that

$$\frac{\partial \chi}{\partial \bar{\lambda}} = T(\lambda)e^{2\lambda(x-4\lambda^2 t)}\chi^+(-\lambda) \tag{6.262}$$

where χ^+ is a function that is analytical in the right half-plane with exceptions at points $\lambda = \lambda_l$, where this function has simple poles.

Now we will decipher equation (14.12). Putting together terms proportional to $\delta(k)$, we get

$$\chi_l + \sum_{m=1}^{n} \frac{\tilde{\chi}_m}{\lambda_l+\lambda_m} M_l^2 e^{-2\psi_l} = M_l^2 e^{-2\psi_l}\left(1 - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\rho(s,x,t)}{\lambda_l+is}ds\right) \tag{6.263}$$

Terms proportional to $\delta(\eta)$ give

$$\rho(k,x,t) = f(k)e^{-2ik(x+4k^2t)}\left(1 - \sum_{m=1}^{n}\frac{\tilde{\chi}_m(x,t)}{ik+\lambda_m} - \frac{1}{2\pi}\int \frac{\rho(s,x,t)}{\varepsilon + i(k+s)}ds\right) \tag{6.264}$$

Now we define

$$\phi(k,x,t) = \rho(k,x,t)e^{ikx} \tag{6.265}$$

$$\phi_m(x,t) = \tilde{\chi}_m e^{\lambda_m x} \tag{6.266}$$

and introduce the functions

$$K(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(k,x,t)e^{-ikx}dk + \sum_{m=1}^{\infty}\phi_m(x,t)e^{-\lambda_m x} \tag{6.267}$$

$$F(x+y,t) = \frac{1}{2\pi}\int f(x,t)e^{-ik(x+y)+8ik^3t}dk + \sum_{m=1}^{n} M_m^2 e^{-\lambda_m(x+y)-8\lambda_m^3 t} \tag{6.268}$$

There exists a fundamental theorem that says functions K and F satisfy the equation

$$K(x,y,t) + \int_x K(x,z,t)F(x+z,t)dz = F(x+y,t) \tag{6.269}$$

To prove this theorem, we write down a system of algebraic equations for ϕ_l:
(6.270)

$$\phi_l + M_l^2 e^{-\lambda_l x + 8\lambda_l^3 t}\sum_{m=1}^{n}\frac{\phi_m e^{-\lambda_n x}}{\lambda_n+\lambda_m} = M_l^2 e^{-\lambda_l x+8\lambda_l^3 t}\left(1 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\phi(s,x,t)e^{-isx}}{\lambda+is}ds\right)$$

and a single integral equation for $\phi(k,x,t)$,
(6.271)

$$\phi(k,x,t)+f(k)e^{-ikx-8ik^3t}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\phi(s,x,t)e^{-isx}}{\varepsilon+i(k+s)}ds + \sum_{m=1}^{n}\frac{\phi_m e^{-\lambda_m x}}{ik+\lambda_m}\right\} = f(k)e^{-ikx-8ik^3t}$$

Hereafter, we will use the following obvious equalities

$$\frac{e^{-(\lambda_l+\lambda_m)x}}{\lambda_l+\lambda_m} = \int_x^\infty e^{-(\lambda_l+\lambda_m)z}dz \tag{6.272}$$

$$\frac{e^{-(\lambda_l+is)x}}{\lambda_l+is} = \int_x^\infty e^{-(\lambda_l+is)z}dz \tag{6.273}$$

$$\frac{e^{-i(k+s)x-\eta x}}{i(k+s)+\eta} = \int_x^\infty e^{-i(k+s)z-\eta z}dz \tag{6.274}$$

Then we multiply each term of system (14.19) by $e^{-\lambda_l x}$, multiply equation (14.20) by $\frac{1}{2\pi}e^{-ikx}$, sum over l, integrate over k from $-\infty$ to ∞, and add the final results. We end up with the Gelfand-Marchenko equation, using formulae (14.21):

$$K(x,y,t) + \int_x^\infty K(x,z,t)F(z+y,t) = F(x+y,t)$$

This equation is correct only when $y \geq x$. Notice that, according to (14.10),

$$\begin{aligned} K(x,x,t) &= \frac{1}{2\pi}\int_{-\infty}^\infty \phi(k,x,t)e^{-ikx}dx + \sum_{m=1}^n \phi_m(x,t)e^{-\lambda_m x} \\ &= \frac{1}{2\pi}\int \rho(k,x,t)dk + \sum_{m=1}^n \tilde{\chi}_m(x_1,t) = \chi_1 \end{aligned}$$

Here, χ_1 is the first term in the expansion

$$\chi = 1 + \frac{\chi_1}{\lambda} + ...$$

Then,

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x,t)$$

6.12. Dressing in the Marchenko equation

Let us begin with the equation

$$K(x,y) + \int_x^\infty K(x,z)F(z,y)dz = F(x,z) \tag{6.275}$$

where $y > x$ and $F(x,y)$ satisfies the equation

$$L_0 F = \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} + (u_0(x)+u_0(y))F = 0 \tag{6.276}$$

By applying this operator to (15.1) we obtain

$$L_0 K + \int_x^\infty \left(\frac{\partial^2 K(x,z)}{\partial x^2} + u_0(x)K(x,z)\right)F(z,y)dz - \int_x^\infty K(x,z)\left(\frac{\partial^2 F(z,y)}{\partial y^2} + u_0(y)F(z,y)\right)dz \tag{6.277}$$

$$-\frac{d}{dx}K(x,x)F(x,y) - \frac{\partial K(x,x)}{\partial x}F(x,y) = 0 \tag{6.278}$$

Then we replace

$$-\int_x^\infty K(x,z)\left(\frac{\partial^2 F(z,y)}{\partial y^2} + u_0(y)F(z,y)\right)dy = -\int_x^\infty K(x,z)\left(\frac{\partial^2 F(z,y)}{\partial z^2} + u_0(z)F(z,y)\right)dy \tag{6.279}$$

We integrate this equation by parts to find that
(6.280)
$$-\int_x^\infty K(x,z)\frac{\partial^2 F(z,y)}{\partial z^2}dz = -\int_x^\infty \frac{\partial^2 K(x,z)}{\partial z^2}F(z,y)dz + K(x,x)\frac{\partial F(x,z)}{\partial x} - \frac{\partial K(x,z)}{\partial z}\Big|_{z=x}F(x,z)$$

Putting it all together, we end up with the relation

$$L_0 K(x,z) - 2\frac{d}{dx}K(x,x)F(x,z) + \int L_0 K(x,z)F(z,y)dz = 0 \tag{6.281}$$

or, expressing F using (15.1),

$$LK(x,z) + \int LK(x,z)F(z,y)dz = 0 \tag{6.282}$$

where

$$LK = L_0 K - 2\frac{d}{dx}K(x,x)K$$

which can be rewritten as

$$LK = \frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial y^2} + (u(x) - u_0(y))K(x,y) \tag{6.283}$$

where

$$u = u_0 - 2\frac{d}{dx}K(x,x)$$

Hereafter, we will assume that the homogeneous equation (15.7) has only zeros as solutions. This implies $LK = 0$, and the kernel K then satisfies the equation

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial y^2} + (u(x) - u_0(y))K(x,y) = 0$$

Equation (15.1) realizes "dressing" on the nonzero background $u_0(x)$. In the case where $u_0 = 0$, one can choose $F(x,y) = F(x+y)$, and we return to the equation described in the previous chapter. To make our consideration as close as possible to the material of chapter 14, we will introduce the "light cone variables"

$$\begin{aligned} \xi &= \frac{1}{2}(x+y) \\ \eta &= \frac{1}{2}(x-y) \\ x &= \xi + \eta \\ y &= \xi - \eta \end{aligned}$$

In terms of the light cone variables, equation (15.2) takes the form

$$\frac{\partial^2 F}{\partial \xi \partial \eta} + [u(\xi+\eta) - u(\xi-\eta)]F = 0 \tag{6.284}$$

Now we demand that $\frac{\partial F}{\partial u} \to 0$ as $\xi \to 0$. Suppose that $|u(x)| < c$ is a bounded function. We could then write a solution of equation (15.2) in the form of a degenerative kernel,

$$F(x,y) = \sum_n M_n^2 \phi_n(x)\phi_n(y) \tag{6.285}$$

Here, ϕ_n is a solution of the equation

$$\frac{\partial^2}{\partial x^2}\phi_n + u\phi_n = \lambda_n^2 \phi_n \tag{6.286}$$

with the boundary condition $\phi_n \to e^{-\lambda_n x}$ as $x \to -\infty$. Here, the λ_n are, at this point, arbitrary constants.

We can look for solutions of equation (15.1) of the form

$$K(x,y) = \sum_{l=1}^{n} \Psi_l(x)\phi_l(y) \tag{6.287}$$

$$\Psi_l(x) + M_l^2 \sum_{m=1}^{n} \int_x^\infty \phi_l(s)\phi_m(s)ds = M_l^2 \phi_l(x) \tag{6.288}$$

We can see that, in the case $u_0(x) = 0$, $\phi_n(x) = e^{-\lambda_n x}$.

Now,

$$K(x,x) = \sum_{l=1}^{n} \Psi_l(x)\phi_l(x) \tag{6.289}$$

Using the same consideration as in the previous chapter, we can see that

$$K(x,x) = -\frac{\partial}{\partial x} ln\, \Delta \tag{6.290}$$

where

$$\Delta = det\, \Delta_{nm} \tag{6.291}$$

$$\Delta_{nm} = \delta_{lm} + M_l^2 \int_x^\infty \phi_l(s)\phi_m(s)ds \tag{6.292}$$

Finally,

$$u(x) = u_0(x) - 2\frac{\partial^2}{\partial x^2} ln\, \Delta \tag{6.293}$$

Notice that $\Psi_l(x)$ satisfies the equation

$$\frac{\partial^2 \Psi_l}{\partial x^2} + u(x)\Psi_l = \lambda_l^2 \Psi_l \tag{6.294}$$

Moreover, $\Psi_l \to 0$ as $|x| \to \pm\infty$. This means that Ψ_l is an eigenfunction of the L operators with eigenvalues λ_l^2. There are two possible cases now. It may be that λ_l^2 is already and eigenvalue of the "seed" L operator. In this case, the dressing procedure does not change the eigenvalue; it will only deform the potential and the eigenfunction. If λ_l is not an eigenvalue of the initial "seeding" potential, then the dressing procedure adds a new eigenvalue together with its corresponding eigenfunction. In this case, the dressing procedure can be treated as the addition of a finite number of solitons to the initial solution of the KdV equation.

Suppose that all the λ_l are eigenvalues of the "seeding potential." In other words, all $\phi_l(x) \to 0$ at $x \to -\infty$. Thus, Ψ_l are eigenfunctions. in this case, we can define $M_n^2 = \varepsilon \tilde{M}_n^2$ and send $\varepsilon \to 0$. Now, $\Delta \to 1$ and $\Psi_l \to \varepsilon \tilde{M}_n^2 \phi_l$. Thus, $u_0 \to u_0 + \delta u$, and

$$\delta u = -2\varepsilon \frac{\partial}{\partial x} \sum_{l=1}^{n} M_l^2 \phi_l^2(x) \tag{6.295}$$

This is a particular case of the "infinitesimal dressing." Another example of this type of dressing is presented by the following construction. Let us define a solution of equation

$$\frac{\partial^2\Psi}{\partial x^2} + u(x)\Psi + k^2\Psi = 0 \tag{6.296}$$

as the "Yost function" if $\Psi(k,x) \to e^{ikx}$ as $x \to \infty$. Then we can present a general "infinitesimal dressing" of the form

$$\delta K(x,x) = \varepsilon \sum_{l=1}^{n} M_l^2 \phi_l^2(x) + \int_{-\infty}^{\infty} g_k \Psi^2(k,x) dk \tag{6.297}$$

Hereafter, we will show that (15.19) is just the expansion of $\delta K(x,x)$ in the generalized Fourier transform in a new set of functions

$$F_l = \phi_l^2(x) \tag{6.298}$$

$$F_k = \Psi^2(k,x) \tag{6.299}$$

We will prove that, for an orthogonal system, these functions are eigenfunctions of a very interesting "recursion operator."

6.13. Triangularization

The Marchenko equation (15.1) can be treated as a result of Gaussian triangularization of ???

In equation (15.1), we must replace $K \to -K^+$. We obtain

$$K^+(x,y) + F(x,y) + \int_x^\infty K^+(x,z)F(z,y)dz = 0 \tag{6.300}$$

This equation can be interpreted in the following way. Each nongenerative operator $1+F$ can be presented as a product of triangular operators

$$1 + F = (1+K^+)^{-1}(1+K^-) \tag{6.301}$$

where K^+ and K^- are triangular operators:

$$K^+ f = \int_x^\infty K^+(x,s)f(s)ds, \quad K^+(x,s) = 0 \;\; s > x \tag{6.302}$$

and

$$K^- f = \int_{-\infty}^x K^-(x,s)f(s)ds, \quad K^-(x,s) = 0 \;\; s < x \tag{6.303}$$

Now, multiplying by $(1+K^+)$, we obtain the equation

$$K^+(x,y) + F(x,y) + \int_x^\infty K^+(x,z)F(z,y)dz = K^-(x,y) \tag{6.304}$$

On the half plane $y > x$, equality (16.4) gives (16.1). For $y < x$, we obtain the expression for K^-,

$$K^-(x,y) = F(x,y) + \int_x^\infty K^+(s,z)F(s,y)ds \tag{6.305}$$

In equality (16.5), $y < x$.

Let the function $\phi_0(x,\lambda)$ be a solution of the equation

$$\frac{\partial^2\phi_0}{\partial x^2} + u_0(x)\phi_0 = \lambda\phi_0 \tag{6.306}$$

Then the function

$$\phi(x) = \phi_0(x) - \int_x^\infty K(x,s)\phi_0(s)ds \tag{6.307}$$

is a solution of the equation

$$\frac{\partial^2\phi}{\partial x^2} + u(x)\phi = \lambda\phi \tag{6.308}$$

We can check this equality by direct implementation of the operator $L - \lambda = \frac{\partial^2}{\partial x^2} + u(x) - \lambda$ to equation (16.7) and use of equality (15.8). Doing so, we find

$$(L-\lambda)\phi = (u-u_0)\phi_0 - \frac{d}{dx}K(x,x)\phi_0(x) + \frac{\partial K(x,s)}{\partial x}\phi_0(s) - \int_x^\infty \left(\frac{\partial^2 K}{\partial x^2} + u(x)K - \lambda K\right)\phi_0(s)ds \tag{6.309}$$

$$= -2\frac{d}{dx}K(x,x)\phi_0 + \frac{d}{dx}K(x,x)\phi_0 + \frac{\partial K(x,s)}{\partial x}\phi_0(s)\big|_{s=x} \tag{6.310}$$

$$- \int \left(\frac{\partial^2 K}{\partial s^2} + u_0(s)K - \lambda K\right)\phi_0(s)ds \tag{6.311}$$

Integrating (15.5) by parts twice gives us

$$(L_0 - \lambda)\phi_0 = 0 \tag{6.312}$$

Therefore, dressing transforms the solution of equation (16.10) into solutions of the equation $(L-\lambda)\phi = 0$. Equation (16.7) can be rewritten

$$\phi = (1-K)\phi_0 = (1+K^+)\phi_0$$

Sometimes one must use the dual triangularization,

$$1 + F = (1+M^+)(1+M^-)^{-1} \tag{6.313}$$

Now, at $x > y$,

$$F + M^- + FM^- = 0$$

or

$$F(x,y) + M^-(x,y) + \int_x^\infty F(x,z)M^-(z,y)dz = 0 \tag{6.314}$$

In this equation, M^- in an unknown function of the first variable, while y is a parameter.

Suppose that F is a symmetric operator, $F(x,y) = F(y,x)$. This can be achieved if

$$F(x,y) = \sum_{l=1}^n M_l^2\phi_l(x)\phi_l(y) + \int \alpha(k)\psi_k(x)\psi_k(y)dk \tag{6.315}$$

In this case, $M^-(x,y) = K^+(y,x)$. Of course, one could perform another factorization,

$$1 + F = (1+K^-)^{-1}(1+K^+)$$

This factorization leads to the alternative Marchenko equation,

$$F + K^- + K^-F = 0$$

or

$$F(x,y) + K^-(x,y) + \int_{-\infty}^x K^-(x,s)F(s,y)ds = 0$$

Now the right and left half-planes interchange their roles.

Suppose we perform a dressing on the zero background. We can choose $\phi_0 = e^{ikx}$ and $\lambda = -k^2$. Now,

$$\psi = e^{ikx} - \int_x^\infty K(x,s)e^{iks}ds \tag{6.316}$$

is the function analytic in the upper-half plane. It has asymptote $\phi \to e^{ikx}$ as $x \to \infty$. We will call this function the "Yost function."

6.14. Second Lax pair for the KdV equation

Let $\psi(k,x)$ be a solution of the equation

$$\psi'' + u\psi = \lambda\psi$$

Let us calculate

$$\frac{d^3}{dx^3}\psi^2 = 2\psi\psi''' + 6\psi'\psi'' = 2\psi[(\lambda-u)\psi]' + 6\psi'(\lambda-u)\psi$$

Then we define $w = \frac{\partial}{\partial x}\psi^2$. This function satisfies the equation

$$\mathcal{L}w = 4\lambda w \tag{6.317}$$

$$\mathcal{L}w = w'' + 4uw + 2u'\partial^{-1}w \tag{6.318}$$

We will call $\mathcal{L}$ the recursive operator. Let us consider the equation

$$\frac{\partial u}{\partial t} + \mathcal{L}\frac{\partial u}{\partial x} = 0 \tag{6.319}$$

One can see that this is nothing but the KdV equation,

$$\frac{\partial u}{\partial t} + 6uu_x + u_{xxx} = 0 \tag{6.320}$$

Moreover, equation

$$\frac{\partial u}{\partial t} + \mathcal{L}^2\frac{\partial u}{\partial x} = 0$$

is nothing but the next KdV equation,

$$\frac{\partial u}{\partial t} + u^{(IV)} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x = 0 \tag{6.321}$$

Notice that the linearized KdV equation appearing from (16.18) by substituting $u \to u + \varepsilon w$ and linearization,

$$\frac{\partial w}{\partial t} + w_{xxx} + 6uw_x + 6u_xw = 0 \tag{6.322}$$

is compatible with the spectral problem (16.15). These equations compose the second Lax pair for the KdV equation. Compatibility of (16.15) and (16.19) can be proven by direct calculation; however, we can prove this fact in a more clever way. Equation (15.2) can be written as follows:

$$(L^{(x)} - L^{(y)})F = 0 \tag{6.323}$$

where $L^{(x)}$ and $L^{(y)}$ are Lax operators for the KdV equation, acting along x and y, respectively.

This equation is compatible with the following evolution equation:

$$\frac{\partial F}{\partial t} + (A^x + A^y)F = 0 \tag{6.324}$$

where A^x and A^y are A operators acting along the x and y axes. Apparently, $\delta F = \varepsilon\psi(x,\lambda)\psi(y,\lambda)e^{8\lambda^3 t}$ is a compatible solution of both equations (16.20) and (16.21). On the other hand,

$$w = -2\varepsilon\frac{\partial}{\partial x}\psi^2(x,\lambda)e^{8\lambda^3 t}$$

is an infinitesimally small dressing which must satisfy both (16.16) and (16.19).

6.15. Direct and inverse scattering from the Scrödinger equation

Let us denote $\lambda = -k^2$ and $v(x) = -u(x)$. Now, $v(x,t)$ satisfies the following KdV equation:

$$v_t - 6vv_x + v_{xxx} = 0 \tag{6.325}$$

Suppose that $\psi(x,t)$ satisfies the overdetermined system of linear equations

$$\psi_{xx} + (k^2 - v(x))\psi = 0 \tag{6.326}$$

$$\psi_t = (2v + 4k^2)\psi_x - u_x\psi \tag{6.327}$$

Equation (17.2) is the "classical" Schrödinger equation with potential $v(x)$. Solutions with real values of k correspond to functions of continuous spectrum while special values of $k = i\kappa_n$ such that there is a solution $\psi = 0\psi_n \to 0$ at $|x| \to \infty$ compose the discrete spectrum of the Schrödinger operator. We will assume from now on that

$$\int_{-\infty}^{\infty} |x||v(x)|dx < \infty \tag{6.328}$$

In other words, the number of discrete eigenvalues is finite.

Suppose that the compatible common solution of equations (17.2) and (17.3) has asymptotes

$$\psi \to \alpha(k,t)e^{-ikx}, \ \ x \to -\infty \tag{6.329}$$

$$\psi \to \beta(k,t)e^{-ikx} + \gamma(k,t)e^{ikx}, \ \ x \to \infty \tag{6.330}$$

At $|x| \to \infty$, equation (17.3) degenerates to the form

$$\psi_t = 4k^2\psi_x$$

Hence,

$$\alpha(k,t) = \alpha_0(k)e^{-4ik^3 t} \tag{6.331}$$

$$\beta(k,t) = \beta_0(k)e^{-4ik^3 t} \tag{6.332}$$

$$\gamma(k,t) = \gamma_0(k)e^{4ik^3 t} \tag{6.333}$$

Then we define

$$a(k) = \frac{\beta(k,t)}{\alpha(k,t)} = a_0(k) \tag{6.334}$$

$$b(k) = \frac{\gamma(k,t)}{\alpha(k,t)} = b_0(k)e^{8ik^3 t} \tag{6.335}$$

Let us find a solution of equation (17.2) defined by the asymptote

$$\phi_1(x,k) \to e^{-ikx}, \ \ x \to -\infty \tag{6.336}$$

Then, as $x \to \infty$,

$$\phi_1 \to a(k)e^{-ikx} + b(k,t)e^{ikx} \tag{6.337}$$

$\phi_1(k)$ can be called "the left Yost function." The second left Yost function is defined by asymptote

$$\phi_2(k) \to e^{ikx}, \quad x \to -\infty$$

Actually,

$$\phi_2(k) = \phi_1(-k) = \bar{\phi}_1(k) \tag{6.338}$$

In the same way, one can define the right Yost functions:

$$\psi_1 \to e^{-ikx} \quad x \to \infty \tag{6.339}$$

$$\psi_2 \to e^{ikx} \quad x \to \infty \tag{6.340}$$

(6.341)

The Yost function defined in the previous chapter is exactly $psi_2(x,k)$. Again,

$$\psi_1(k,x) = \psi_2(-k,x) = \bar{\psi}_2(k,x) \tag{6.342}$$

¿From (17.9) we get

$$\phi_1 = a(k)\psi_1(k) + b(k)\psi_2(k) \tag{6.343}$$

As long as

$$\bar{\phi}_1(-k) = \phi_1(k) \tag{6.344}$$

$$\bar{\psi}_i(-k) = \psi_i(k) \tag{6.345}$$

$$\bar{a}(-k) = a(k) \tag{6.346}$$

$$\bar{b}(-k) = b(k) \tag{6.347}$$

both the left and right Yost functions constitute a basis, and

$$\phi_i = T_{ik}\psi_k$$

Here, T_{ik} is the monodromy matrix

$$T_{ik} = \begin{bmatrix} a(k) & b(k) \\ b(\bar{k}) & a(\bar{k}) \end{bmatrix}$$

Let f_1, f_2 denote the Wronskian of two functions f_1 and f_2:

$$f_1, f_2 = f_1 f_{2x} - f_{1x} f_2$$

Recall that if f_1 and f_2 are solutions of equation (17.2)

$$\frac{d}{dt} f_1, f_2 = 0$$

Hence,

$$\phi_1, \phi_2 = \psi_1, \psi_2 = 2ik$$

and

$$|a(k)|^2 + |b(k)|^2 = 1 \tag{6.348}$$

We have seen already that the Yost function ψ_2 is analytic in the upper half-plane. This fact can be proven directly. We can transform the equation for ψ_2 to an integral equation using the method of constant variation. We present it in the form

$$\psi_2 = c_1(x)e^{ikx} + c_2(x)e^{-ikx}$$
$$c_1' e^{ikx} + c_2' e^{-ikx} = 0$$

and end up with the equation

$$f(k,x) = 1 - \frac{1}{2ik}\int_x^\infty (e^{2ik(s-x)} - 1)v(x)f(k,s)ds \tag{6.349}$$

where $f(k,x) = \psi(k,x)e^{-ikx}$. Suppose $k = k_0 + i\eta$, $\eta > 0$. Then,

$$|e^{2ik(s-x)}| < e^{-2\eta(s-x)}, \;\; s > x$$

Equation (17.16) is of the Volterra type and solvable for all η. The same is true for the derivative f_k. Heance, $f(k)$ is analytic.

In the limit $|k| \to \infty$,

$$\psi_2 = e^{ikx}\left(1 + \frac{1}{2ik}\int_x^\infty v(y)dy + ...\right) \tag{6.350}$$

In the same way, one can prove that ψ_1 and ϕ_2 are analytic in the lower half-plane, while ϕ_1 is analytic in the upper half-plane. Now,

$$\psi_1, \psi_2 = \phi_1, \phi_2 = 2ik$$

Then,

$$a(k) = \frac{\phi_1(k,x), \psi_2(k,x)}{2ik} \tag{6.351}$$

As long as both function ϕ_1 and ψ_2 are analytic in the upper half-plane, $a(k)$ is analytic there also, and $a(k) \to 1 + \mathcal{O}(\frac{1}{k})$ at $k \to \infty$, $\mathcal{I}\Updownarrow k > 0$. On the imaginary axis, $k = i\kappa$, and

$$\psi_2(\kappa,\lambda) \to e^{-\kappa x}, \;\; x \to \infty$$
$$\phi_1(\kappa,\lambda) \to e^{\kappa x}, \;\; x \to -\infty$$

In the point of the discrete spectrum, ψ_2 and ϕ_1 are proportional to each other, therefore $a(i\kappa_n) = 0$. Hereafter, we denote

$$\psi_2(i\kappa_n) = \psi_n \to e^{-\kappa_n x}, \;\; x \to \infty \tag{6.352}$$

$$\psi_n \to c_n e^{\kappa_n x}, \;\; x \to -\infty \tag{6.353}$$

where the c_n are real constants. If the eigenvalues are ordered

$$\kappa_1 > \kappa_2 > ... > \kappa_n$$

then

$$c_1 > 0, \; c_2 < 0, \; c_3 > 0 \; etc.$$

The analyticity of ψ_2 in the upper half-plane implies that ψ_2 has a triangular representation

$$\psi_2 = e^{ikx} + \int_x^\infty K(x,y)e^{ikx}dy \tag{6.354}$$

After sending $k \to \infty$, one can calculate the integral in (17.18) approximately:

$$\psi_2 = e^{ikx}\left(1 - \frac{1}{ik}K(x,x)\right) \tag{6.355}$$

By comparison with (17.16), we get

$$V = d\frac{d}{dx}K(x,x)$$

or

$$U = -2\frac{d}{dx}K(x,x)$$

Now we introduce the transmission and reflection coefficients $d(k) = \frac{1}{a(k)}$ and $c(k) = \frac{b(k)}{a(k)}$ which satisfy the condition of unitarity,

$$|d(k)|^2 + |c(k)|^2 = 1$$

Equation (17.13) can be rewritten as follows:

$$\frac{\phi_1(x,k)}{a(k)} - e^{-ikx} = \psi_1 - e^{-ikx} + c(k)\psi_2(k,x)$$

$$= \int_x^\infty K(x,z)e^{-ikz}dz + c(k)\left(\int_x^\infty K(x,z)e^{ikz}dz + e^{ikx}\right)$$

If we multiply this equation by $\frac{1}{2\pi}e^{iky}$, $y > x$, and integrate by k from $-\infty$ to ∞, we get

$$F_0(x+y) + K(x+y) + \int_x^\infty K(x,z)F_0(z+y)dz = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}\left(\frac{\phi_1(x,k)}{a(k)} - e^{ikx}\right)dk = G$$

$$F_0(\xi) = \frac{1}{2\pi}\int_{-\infty}^\infty c(x)e^{ik\xi}dk$$

$\frac{\phi_1(x,k)}{a(k)}$ is analytic in the upper half-plane and behaves like e^{-ikx} at $k \to \infty$. Function

$$e^{ikx}\left(\frac{\phi_1}{a} - e^{-ikx}\right)dx \simeq e^{-ik(x-y)}\mathcal{O}(\frac{1}{k})$$

is also analytics and for $y > x$ it decays along the imaginary axis. Hence,

$$G = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}\left(\frac{\phi_1(x,k}{a(k)} - e^{-ikx}\right)dx = i\sum e^{-\kappa_n x}\frac{\phi_1(x,i\kappa_n)}{a'(i\kappa_n)}$$

The integral is calculated using residues.

However, $\phi_1(x,i\kappa_n) = c_n\psi_2(i\kappa_n)$. Therefore,

$$G = i\sum\frac{e^{-\kappa_n x}c_n\psi_2(\kappa_n)}{a'(i\kappa_n)}$$

Hereafter, we prove that $\frac{ic_n}{a'(i\kappa_n)}$ are negative real numbers. Then we denote

$$F_1(x+y) = \sum_{n=1}^N M_n^2 e^{-\kappa_n(x+y)}$$

$$\psi_2(i\kappa_n) = e^{-\kappa_n x} + \int_x^\infty K(x,z)e^{-\kappa_n z}dz$$

We end up with the equation

$$F(x+y) + K(x,y) + \int_x^\infty K(x,z)F(z+y)dz = 0$$

$$F = F_0 + F_1$$

6.16. On the transmission coefficient

We know that the transmission coefficient $a(k)$ is an analytic function in the upper half-plane. It has zeros at $k_n = i\kappa_n$. Thus, we can present it in the form

$$a(k) = \sum_{n=1}^{N} \frac{k = i\kappa_n}{k + i\kappa_n} \tilde{a}(k) \tag{6.356}$$

$\tilde{a}(k)$ is analytic in the upper half-plane, having no zeros there, so $ln\, \tilde{a}(k)$ is also an analytic function. Moreover, as long as $\tilde{a}(k) \to 1$ at $k \to \infty$, $\ln\, \tilde{a}(k) \to 0$ as $k \to \infty$ and one can present

$$ln\, \tilde{a}(k) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{ln|a(q)|}{q-k} dq$$
$$\mathcal{I}\Updownarrow k > 0$$

On the imaginary axis, $k = i\kappa$, so

$$ln\, \tilde{a}(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ln|a(q)|}{\kappa + iq} dq$$

as long as $a(-k) = \bar{a}(k)$,

$$|a(-k)| = |a(k)|$$
$$ln\, |a(q)| = ln\, |a(-q)|$$

then

$$ln\, \tilde{a}(\kappa) = \frac{1}{\pi} \int_{0}^{\int} ln\, |a(q)| \left(\frac{1}{\kappa + iq} + \frac{1}{\kappa - iq} \right) dq = \frac{2\kappa}{\pi} \int_{0}^{\infty} \frac{ln\, |a(q)|}{\kappa^2 + q^2} dq \tag{6.357}$$

We see that $ln\, |\tilde{a}(q)|$ is a real and positive function, therefore $\tilde{a}(k)$ is real and positive. Furthermore, $\tilde{a}(\kappa) \geq 1$.

Let us calculate $a'(i\kappa_n)$. Apparently,

$$a'\big|_{i\kappa_n} = \frac{1}{2i\kappa_n} \prod_{m \neq n} \frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m} \tilde{a}(i\kappa_n)$$

Hence,

$$M_n^2 = \frac{2\kappa_n c_n}{\tilde{a}(i\kappa_n)} \prod_{m \neq n} \frac{\kappa_n + \kappa_m}{\kappa_n - \kappa_m} \tag{6.358}$$

Notice that

$$\phi_1(i\kappa_n) \to e^{\kappa_n x} \quad as \quad x \to -\infty$$
$$\phi_1(i\kappa_n) \to c_n e^{-\kappa_n x} \quad as \quad x \to \infty$$

Let us recall that $\kappa_1 > \kappa_2 > ... > \kappa_n$. $\phi_1(i\kappa_1, x)$ is the wave function of the ground state; hence, $c_1 > 0$. Then, $\kappa_1 > \kappa_m$, $m \geq 2$, and $M_1^2 > 0$. Now one can see that c_k and $\prod_{m \neq k} \frac{\kappa_k + \kappa_m}{\kappa_k - \kappa_m}$ change sign simultaneously. Finally, we obtain $M_n^2 > 0$ for all n.

Let us recall the differential equation for ϕ_1,

$$\phi_1'' + k^2 \phi_1 = V \phi_1 \tag{6.359}$$

and differentiate it by k:

$$\phi_{1k}'' + k^2 \phi_{1k} = V \phi_{1k} \tag{6.360}$$

Then we multiply (17.6) by ψ_{1k}, multiply (17.7) by ϕ_1, and subtract the results, and we get the relation

$$\frac{d}{dx}\phi_{1k}, \phi_1 = -2k\phi_1^2 \tag{6.361}$$

Here, $\phi_{1k}, \phi_1 = \phi'_{1k}\phi_1 - \phi'_1\phi_{1k}$ is the Wronskian of ϕ_{1k} and ϕ_1. At $x \to -\infty$, $\phi_1 \to e^{-ikx}$, and $\phi_{1k} \to -ike^{-ikx}$. Hence, $\phi_{1k}, \phi_1 \to 0$ as $x \to -\infty$. Now we perform the following trick: we cut off the potential U and set $U \equiv 0$, $|x| > L \to \infty$. At any finite value of L, $\phi_{1,2}$ and $\psi_{1,2}$ are entire functions of k. b_k is an entire function as well. This means that the relation

$$\phi_1(k, x) = a(k)\psi_1(k, x) + b(k)\psi_2(k, x)$$

can be analitically continued up to $k = i\kappa_n$. Asymptotically at $x \to \infty$,

$$\phi_1(k, x) \to a(k)e^{-ikx} + b(k)e^{ikx} \tag{6.362}$$

We can set $b(i\kappa_n) = c_n$:

$$\phi_1(i\kappa_n, x) \to c_n e^{-\kappa_n x}$$
$$\phi'_1(i\kappa_n, x) \to -\kappa_n c_n e^{-\kappa_n x}$$

Then,

$$\phi_{1k}(k, x) = a'(k)e^{-ikx} - (be^{ikx})_k - ia(k)e^{-ikx}$$

As long as $a(i\kappa_n) = 0$, we can write

$$\phi_{1k}\big|_{k=i\kappa_n} \to a'(i\kappa_n)e^{\kappa_n x}$$
$$\phi'_{1k} \to \kappa_n a'(i\kappa_n)e^{\kappa_n x}$$

Integration of equation (17.8) leads to the relation

$$\phi_{1k}, \phi_1 = -2i\kappa_n \int_{-\infty}^{\infty} \phi_1^2 dx$$

or

$$2\kappa_n c_n a'(i\kappa_n) = -2i\kappa_n \int_{-\infty}^{\infty} \phi_1^2 dx$$

Now remember that

$$c_n = ia'(i\kappa_n)M_n^2 \tag{6.363}$$

$$M_n^2(a'(i\kappa_n))^2 = -\int_{-\infty}^{\infty} \phi_1^2 dx \tag{6.364}$$

As long as $a'(i\kappa_n)$ is pure imaginary, $M_n^2 > 0$. As long as $c_n = b(i\kappa_n)$,

$$c_n(t) = c_n(0)e^{8\kappa_n^3 t}$$

and

$$M_n^2 = M_n^2(0)e^{8\kappa_n^3 t} \tag{6.365}$$

This means that the norm of ϕ_1 grows exponentially in time.

Let us continue with the trick where we cut off the potential and expand equation (17.9) in the upper half-plane. Then we send $x \to \infty$. Asymptotically,

$$\phi_1 \to a(k)e^{-ikx} \;\; as \;\; x \to \infty, \;\; \mathcal{I}\Updownarrow k > 0$$
$$ln\, \phi_1 \to ln\, a(k) - ikx \;\; as \;\; x \to \infty$$

Now we remember that ϕ_1 can be presented in the form

$$\phi_1 = e^{\int_{-\infty}^{x} q(x,k)dx - ikx}$$

$$ln\, \phi_1 = \int_{-\infty}^{x} q(x,k)dx - ikx$$

Then,

$$\int_{-\infty}^{\infty} p(x,k)dx = ln\, a(k) = I(k)$$

$$p(x,k) = \sum_{n=1}^{\infty} \frac{p_n}{(-2ik)^{\Gamma}}$$

where the p_n are the motion integrals connected to integrals q_n, and introduced in chapter 8. By relation, coming from the replacement $k \to -k$, $p_n = (-1)^n q_n$. Then, $ln\, a(k)$ is

$$ln\, a(k) = \sum_{n=1}^{N} ln\, \frac{k - i\kappa_l}{k + i\kappa_l} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ln\, |a(k')|}{k' - k - i\sigma} dk'$$

If $k \to \infty$, then $ln\, a(k)$ can be represented by an asymptotic series, including only the odd powers:

$$ln\, a(k) = 2i \sum_{j=0}^{\infty} \frac{1}{k^{2j+1}} \left\{ \frac{(-1)^{j+1}}{2j+1} \sum_{l} \kappa_l^{2j+1} + \frac{2}{\pi} \int_{0}^{\infty} k^{2j} ln\, |a(k)| dk \right\}$$

Thus,

$$p_{2j+1} = I_j = \frac{2^{2(j+1)}}{2j+1} \sum_{i=1}^{N} \kappa_l^{2j+1} + 2^{2(j+1)}(-1)^{j+1} \frac{1}{\pi} \int_{0}^{\infty} k^{2j} ln\, |a(k)| dk$$

In particular,

$$I_0 = \int_{-\infty}^{\infty} u dx = -\int_{-\infty}^{\infty} V dx = 4 \sum_{l=1}^{N} \kappa_l - \frac{4}{\pi} \int_{0}^{\infty} ln\, |a(k)| dx$$

$$I_1 = \int_{-\infty}^{\infty} u^2 dx = \int_{-\infty}^{\infty} V^2 dx = \frac{16}{3} \sum_{l=1}^{\infty} \kappa_l^3 + \frac{16}{\pi} \int_{0}^{\infty} ln\, |a(k)| dk$$

As we know, $|a(k)| > 1$ and $ln\, |a(k)| > 0$. Also,

$$I_2 = \int (-u_x^2 + 2u^3) dx = -\int_{-\infty}^{\infty} (V_x^2 + 2V^3) dx$$

$$= \frac{64}{5} \sum \kappa_l^5 - \frac{64}{\pi} \int_{0}^{\infty} k^4 ln\, |a(k)| dk$$

For the Hamiltonian $H = -\frac{1}{2} I_2$, we obtain

$$H = -\frac{32}{5} \sum_{l=1}^{N} \kappa_l^5 + \frac{32}{\pi} \int_{0}^{\infty} k^4 ln\, |a(k)| dk$$

Notice that we have proven the theorem at $p_{2l} = 0$.

6.17. Poisson structure of scatering data

In this chapter we will find the variational derivative of scattering data by potential and we determine the Poisson brackets between scattering data. Let us introduce the variational derivatives of the Yost functions,

$$G^{(1)}(x,y,k) = \frac{\delta\phi_1(k,y)}{\delta u(x)} \tag{6.366}$$

$$G^{(2)}(x,y,k) = \frac{\delta\psi_2(k,y)}{\delta u(x)} \tag{6.367}$$

These satsify two inhomogeneous differential equations,

$$\left(\frac{\partial^2}{\partial y^2} + k^2 - u(y)\right) G^{(1)}(x,y,k) = \phi_1(k,x)\delta(x-y) \tag{6.368}$$

$$\left(\frac{\partial^2}{\partial y^2} + k^2 - u(y)\right) G^{(2)}(x,y,k) = \phi_2(k,x)\delta(x-y) \tag{6.369}$$

Both functions are continuous and cancel on the diagonal $x = y$, thus

$$G^{(1)}(x,x,k) = G^{(2)}(x,x,k) = 0$$

Moreover, $G^{(1)}(x,x,k) \equiv 0$ if $y < x$, and $G^{(2)}(x,x,k) \equiv 0$ if $y > x$. The derivatives $\frac{\partial G^{(1)}}{\partial y}$, $\frac{\partial G^{(2)}}{\partial y}$ have jumps on the diagonal:

$$\frac{\partial G^{(1)+}}{\partial y} - \frac{\partial G^{(1)-}}{\partial y} = \phi_1(k,x) \tag{6.370}$$

$$\frac{\partial G^{(2)+}}{\partial y} - \frac{\partial G^{(2)-}}{\partial y} = \psi_2(k,x) \tag{6.371}$$

Here, the sign $+$ means that $y = x + \varepsilon$, while the sign is $-$ if $y = x - \varepsilon$, $\varepsilon \to 0$. Apparently, $\frac{\partial G^{(1)-}}{\partial y} = 0$ and $\frac{\partial G^{(2)+}}{\partial y=0}$, hence

$$\frac{\partial G^{(1)+}}{\partial y} = \phi_1(k,x) \tag{6.372}$$

$$\frac{\partial G^{(2)-}}{\partial y} = -\psi_2(k,x) \tag{6.373}$$

Equation (17.17) reads

$$a(k) = \frac{1}{2ik}\psi_1(k,y)\psi_2'(k,y) - \psi_2(k,y)\psi_1'(k,y)$$

Notice that $a(k)$ does not depend on y. Now we calculate

$$\frac{\delta a(k)}{\delta u(x)} = \frac{1}{2ik}\frac{\delta}{\delta u}\psi_1(k,y)\psi_2'(k,y) - \psi_2(k,y)\psi_1'(k,y) \tag{6.374}$$

and set $y = x - \varepsilon$. The variation of $\psi_1(k,y)$ in this area gives zero, so $G^{(2)}(x, x - \varepsilon, k) \to 0$ at $\varepsilon \to 0$. The only surviving term in (19.7) is

$$\frac{\delta a(k)}{\delta u(x)} = \frac{1}{2ik}\left\{\psi_1(k,y), \frac{\partial G^{(2)}(x,y,k)}{\partial u}\right\}_{y=x-\varepsilon}$$

Finally,

$$\frac{\delta a(k)}{\delta u(x)} = -\frac{1}{2ik}\psi_1(k,x)\psi_2(k,x) \tag{6.375}$$

In the same way, one can prove that

$$\frac{\delta b(k)}{\delta u(x)} = \frac{1}{2ik}\psi_1(k,x)\psi_1(k,x) \tag{6.376}$$

If ϕ_1 and ψ_2 are analytic in the upper half-plane then in (19.8) one can set $\mathcal{I}\Updownarrow k > 0$. In (19.9) one has to assume that k is strictly on the real axis.

Further consideration is based on the following basic relation. Suppose f_1 and g_1 are two linearly independent solutions of the equation

$$\frac{\partial^2\phi}{\partial x^2} + k^2\phi = u\phi \tag{6.377}$$

while f_2 and g_2 are solutions of the equation

$$\frac{\partial^2\phi}{\partial x^2} + k_1^2\phi = u\phi \tag{6.378}$$

Then, $\{f_1, f_2\} = f_1 f_{2x} - f_2 f_{1x}$ and $\{g_1, g_2\} = g_1 g_{2x} - g_2 g_{1x}$. One can prove by direct calculation that

$$\{f_1 g_1, f_2 g_2\} = f_1 g_1 \frac{d}{dx} f_2 g_2 - f_2 g_2 \frac{d}{dx} f_1 g_1 = \frac{1}{k^2 - k_1^2}\frac{d}{dx}\{f_1, f_2\}\{g_1, g_2\} \tag{6.379}$$

The singularity in the right hand of (19.12) is fictive. If $k_1^2 = k^2$, equations (19.10) and (19.11) coincide and the Wronskians $\{f_1, f_2\}$ and $\{g_1, g_2\}$ turn into constants, and the derivative on the right hand side of (19.12) annihilates.

Now recall that the Poisson brackets between functionals F and G (depending on $u(x)$) is defined

$$\{F, G\} = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\frac{\delta F}{\delta u(x)}\frac{\partial}{\partial x}\frac{\delta G}{\delta u(x)} - \frac{\delta G}{\delta u(x)}\frac{\partial}{\partial x}\frac{\delta F}{\delta u(x)}\right\}dx$$

Hence,

$$\{a(k), a(k_1)\} = -\frac{1}{8kk_1(k^2 - k_1^2)}\phi_1(k,x), \phi_1(k_1,x)\psi_2(k,x), \psi_2(k_1,x)_{-\infty}^{\infty} = \lim_{L\to\infty}[A(L) - A(-L)] \tag{6.380}$$

where

$$A(L) = -\frac{1}{8kk_1(k^2 - k_1^2)}\phi_1(k,x), \phi_1(k_1,x)\psi_2(k,x), \psi_2(k_1,x)_{x=L} \tag{6.381}$$

and

$$A(-L) = -\frac{1}{8kk_1(k^2 - k_1^2)}\phi_1(k,x), \phi_1(k_1,x)\psi_2(k,x), \psi_2(k_1,x)_{x=-L} \tag{6.382}$$

Hereafter, we will denote

$$\{\phi_1(k,x), \phi_1(k_1,x)\} = W^{(1)}(k, k_1, x) \tag{6.383}$$

and

$$\{\psi_2(k,x), \psi_2(k_1,x)\} = W^{(4)}(k, k_1, x) \tag{6.384}$$

At $x \to \infty$,

$$\psi_2(k) \to e^{ikx} \tag{6.385}$$

$$\psi_2(k_1) \to e^{ik_1x} \tag{6.386}$$

$$W^{(4)} \to -i(k-k_1)e^{i(k+k_1)x} \tag{6.387}$$

Then,

$$\phi_1(k,x) \to a(k)e^{-ikx} + b(k)e^{ikx} \tag{6.388}$$

$$\phi_1(k_1,x) \to a(k_1)e^{-ik_1x} + b(k_1)e^{ik_1x} \tag{6.389}$$

(6.390)
$$W^{(1)} \to i(k-k_1)[a_k a_{k_1} e^{-i(k+k_1)x} - b_k b_{k_1} e^{i(k+k_1)x}] + i(k+k_1)[a_k b_{k_1} e^{-i(k-k_1)x} - a_{k_1} b_k e^{i(k-k_1)x}]$$

Now we can calculate $A(L)$:
(6.391)
$$A(L) = -\frac{k-k_1}{8kk_1(k+k_1)}(a_k a_{k_1} - b_k b_{k_1} e^{2i(k+k_1)L}) - \frac{1}{8kk_1}(a_k b_{k_1} e^{2ik_1L} - a_{k_1} b_k e^{2ikL})$$

To calculate $A(L)$, we first notice that $\phi_1(k) \to e^{-ikx}$ and $\phi_1(k_1) \to e^{-ik_1x}$, so:

$$W^{(1)}(k,k_1,x) \to i(k-k_1)e^{-i(k+k_1)x} \tag{6.392}$$

$$W^{(1)}(k,k_1,L) \to i(k-k_1)e^{-i(k+k_1)L} \tag{6.393}$$

Then, at $x \to -\infty$,

$$\psi_2 \to -\bar{b}(k)e^{-ikx} + a(k)e^{ikx}$$

$$\psi_2'(k_x) \to ik(\bar{b}(k)e^{-ikx} + a(k)e^{ikx})$$

$$W^{(4)}(k,k_1,x) \to -i(k-k_1)[a_k a_{k_1} e^{i(k+k_1)x} - b_k b_{k_1} e^{-i(k+k_1)x}] + i(k+k_1)[a_k \bar{b}(k_1) e^{i(k-k_1)x} - a_{k_1} \bar{b}(k) e^{-i(k-k_1)x}]$$

Then,
(6.394)
$$A(L) = -\frac{k-k_1}{8kk_1(k+k_1)}(a_k a_{k_1} - b_k b_{k_1} e^{2i(k+k_1)L}) - \frac{1}{8kk_1}(a_k \bar{b}(k_1) e^{2ik_1L} - a_{k_1} \bar{b}(k) e^{2ikL})$$

The we remember that $\mathcal{I}\Updownarrow k > 0$, $\mathcal{I}\Updownarrow k_1 > 0$, and all exponents in (14.20) and (14.22) become extinct. Finally, we obtain

$$A(L) = A(-L) = -\frac{k-k_1}{8kk_1(k+k_1)}a_k a_{k_1}$$

Hence,

$$\{a_k, a_{k_1}\} = 0$$

Now we calculate b_k, b_{k_1}:

$$\{b_k, b_{k_1}\} = -\frac{1}{8kk_1(k^2-k_1^2)} W^{(1)}_{k,k_1,x} W^{(2)}_{k,k_1,x}\Big|_{-\infty}^{\infty}$$

$$W^{(2)}_{k,k_1,x} = \psi_1(k,x), \psi_1(k_1,x) = \psi_1(k,x)\psi_1'(k_1,x) - \psi_1(k_1,x)\psi_1'(k,x)$$

at $x \to \infty$,

$$\psi_1(k,x) \to e^{-ikx}$$

$$\psi_1(k_1,x) \to e^{-ik_1x}$$

$$W^{(2)}_{k,k_1,x} = i(k-k_1)e^{-i(k+k_1)x}$$

Now,

$$A(L) = \frac{k - k_1}{8kk_1(k + k_1)}[a_k a_{k_1} e^{-2i(k+k_1)L} - b_k b_{k_1}] + \frac{1}{8kk_1}(a_k b_{k_1} e^{-2ikL} - a_{k_1} b_k e^{-2ikL})$$

At $x \to -\infty$,

$$\psi_1 \to \bar{a}(k)e^{-ikx} - b(k)e^{ikx}$$
$$\psi_1' \to -ik(\bar{a}(k)e^{-ikx} + b(k)e^{ikx})$$

Then,

$$W^{(2)}(k, k_1, x) = i(k-k_1)[\bar{a}_k \bar{a}_{k_1} e^{-i(k+k_1)x} - b_k b_{k_1} e^{i(k+k_1)x}] - i(k+k_1)[\bar{a}_k b_{k_1} e^{-i(k-k_1)x} - \bar{a}_{k_1} b_k e^{i(k-k_1)x}]$$

Using (14.21), we obtain

$$A(-L) = \frac{k - k_1}{8kk_1(k + k_1)}[\bar{a}_k \bar{a}_{k_1} e^{2i(k+k_1)L} - b_k b_{k_1}] + \frac{1}{8kk_1}(\bar{a}_k b_{k_1} e^{2ikL} - \bar{a}_{k_1} b_k e^{2ikL})$$

Then,

$$\{b_k, b_{k_1}\} = A(L) - A(-L) = \frac{k - k_1}{8kk_1(k + k_1)}[-\bar{a}_k \bar{a}_{k_1} e^{2i(k+k_1)L} + a_k a_{k_1} e^{-2i(k+k_1)L}] + \frac{1}{8kk_1}[-\bar{a}_k b_k e^{2ikL} + a_k b_{k_1} e^{-2ikL}] +$$

6.18. Complete integrability of the KdV equation

In equation (14.25) we must send $L \to \infty$. Hereafter, we will use a well-known relation from the theory of generalized functions,

$$\lim_{L \to \infty} \frac{sin\, kL}{k} = \pi \delta(k) \tag{6.395}$$

The first term in (14.25) is not zero if $k_1 = -k$. Recall that $a_{-k} = \bar{a}_k$; hence, $a(0) = \bar{a}(0)$, which is a real number. For now, we will assume that $a(0) < \infty$, i.e., it is a finite real number. Then we can set

$$\{b_k, b_{k_1}\} = -\frac{\pi i}{2k}|a_k|^2 \delta(k + k_1) + \frac{\pi i a(0)}{4k^2}[b_k \delta(k_1) - b_{k_1} \delta(k)] \tag{6.396}$$

Hereafter, we will assume that $k \neq 0$ and $k_1 \neq 0$. In this case, we can simplify equation (20.2) to the form

$$\{b_k, b_{k_1}\} = -\frac{\pi i}{2k}|a_k|^2 \delta(k + k_1) \tag{6.397}$$

Dividing by $|b_k|^2$ gives

$$\{ln\, b_k, ln\, b_{k_1}\} = -\frac{\pi i}{2k} \frac{|a_k|^2}{frac b_k|^2} \delta(k + k_1) \tag{6.398}$$

but

$$ln\, b_k = ln\, |b_k| + i\, arg\, b_k$$

and

$$|b_k|^2 = |a_k|^2 - 1$$

Thus,

$$\{ln\, |b_k|, ln\, |b_{k_1}|\} \simeq \{|a_k|^2, |a_{k_1}|^2\} = 0$$

by virtue of the equality

$$\{a_k, a_{k_1}\} = 0$$

¿From (20.4), we get

$$\{ln\,|b_k|^2, arg\,b_{k_1}\} + \{arg\,b_k, ln\,|b_{k_1}|\} = -\frac{\pi}{k}\frac{|a_k|^2}{|b_k|^2}$$

Now remember that $b(-k) = \bar{b}_k$. Thus,

$$arg\,b(-k) = -arg\,b_k$$

or

$$\{ln\,|b_k|^2, arg\,b_{k_1}\} = \frac{\pi}{2k}\frac{|a_k|^2}{|b_k|^2}\delta_{k-k_1}$$

Then,

$$|b_k|^2\frac{\delta}{\delta u}ln\,|b_k|^2 = |a_k|^2\frac{\delta}{\delta u}ln\,|a|^2$$

Finally,

$$\{ln\,|a_k|^2, arg\,b_{k_1}\} = \frac{\pi}{2k}\delta_{k-k_1}$$

Now we denote

$$\phi_k = arg\,b_k \tag{6.399}$$

$$n_k = \frac{2k}{\pi}ln\,|a_k|^2 \tag{6.400}$$

$$\{\phi_k, \phi_{k_1}\} = 0 \tag{6.401}$$

$$\{n_k, n_{k_1}\} = 0 \tag{6.402}$$

$$\{n_k, \phi_{k_1}\} = \delta_{k-k_1} \tag{6.403}$$

In the absence of a discrete spectrum, the Hamiltonian for the KdV equation is

$$H = 8\int_0^\infty k^3 n(k)dk \tag{6.404}$$

Remember that, according to (17.7),

$$arg\,b(k,t) = arg\,b_0(k) + 8k^3 t$$

$$\frac{\partial q_k}{\partial t} = 8k^3 = \frac{\delta H}{\delta n_k} \tag{6.405}$$

On the other hand,

$$\frac{\partial p_k}{\partial t} = 0 = \frac{\delta H}{\delta \phi_k} \tag{6.406}$$

Thus, p_k and q_k are "action-angle variables" for the KdV equation.

So far, we have not taken the discrete spectrum into consideration. In order to do this, we first need to calculate the following Poisson bracket

$$\{a(k), b(k_1)\} = \frac{1}{8kk_1}\frac{1}{k^2 - k_1^2}W^{(1)}_{k,k_1,x}W^{(3)}_{k,k_1,x}\Big|_{-\infty}^{\infty} \tag{6.407}$$

where

$$W^{(3)}_{k,k_1,x} = \psi_2(k,x), \psi_1(k_1,x) \ \ at \ \ x \to \infty$$
$$W^{(3)}_{k,k_1,x} = -i(k+k_1)e^{i(k-k_1)x} \ \ at \ \ x \to -\infty$$
$$\psi_2(k) \to -\bar{b}(k)e^{-ikx} + a(k)e^{ikx}$$
$$\psi_1(k_1) \to \bar{a}(k_1)e^{-ik_1x} - b(k_1)e^{ik_1x}$$

Hereafter, we assume that $\mathcal{I}\Updownarrow k > 0$; hence, $e^{ikx} \to 0$ as $x \to -\infty$, and

(6.408)
$$W^{(3)} \to i(k-k_1)a(k)b(k_1)e^{i(k+k_1)x} - i(k+k_1)a(k)\bar{a}(k_1)e^{i(k-k_1)x} \ \ as \ \ x \to -\infty$$

Thus, if $a(k), b(k_1) = A(L) - A(-L)$,

$$(6.409) \quad A(-L) = \frac{1}{8kk_1(k^2-k_1^2)} - (k-k_1)^2 a(k)b(k_1) + (k^2-k_1^2)a(k)\bar{a}(k_1)e^{2ik_1L}$$

To calculate $A(L)$ we notice that now terms proportional to e^{ikL} are extinct, so

$$W^{(1)}(k,k_1,x) \to i(k-k_1)a_k a_{k_1} e^{-i(k+k_1)L} + i(k+k_1)a_k b_{k_1} e^{-i(k-k_1)L}$$

At the same time,

$$W^{(3)} \to -i(k-k_1)e^{i(k-k_1)L}$$

So,

$$A(L) = \frac{1}{8kk_1(k^2-k_1^2)}(k+k_1)^2 a_k b_{k_1} + (k^2-k_1^2)e^{-2ikL}$$

Finally,

$$(6.410) \qquad \{a(k), b(k_1)\} = \frac{1}{4kk_1}\frac{k^2+k_1^2}{k^2-k_1^2}a_k b_{k_1} - \frac{\pi i}{4}a(0)a_k\delta(k_1)$$

Notice that this formula differs from the last formula on page 41 of the book "Theory of Solitons."

Let us turn our attention to the discrete spectrum. In a neighborhood of $k = i\kappa_n$, $a(k) \simeq a'(i\kappa_n)(k-k_n)$, and

$$\frac{\delta a_k}{\delta u} = -a'(\kappa_n)\frac{\delta \kappa_n}{\delta u}$$

Hence,

$$(6.411) \qquad \frac{\delta\kappa_n}{\delta u(x)} = \frac{1}{2ia'_{i\kappa_n}}\phi_1(k,x)\psi_2(k,x) = \frac{1}{2ia'_{i\kappa_n}c_n}\phi_1^2(i\kappa_n,x)$$

because

$$\psi_2(i\kappa_n) = \frac{1}{c_n}\phi_1(i\kappa_n)$$

According to the results of chapter 18,

$$\frac{1}{ua'_{i\kappa_n}c_n}\phi_1^2(i\kappa_n,x) = \chi_n^2(x)$$

where χ_n is the normalized eigenfunction

$$\int_{-\infty}^{\infty}\chi_n^2 dx = 1$$

Thus,

$$\frac{\delta}{\delta u(x)}\kappa_n^2 = \chi_n^2(x) \tag{6.412}$$

This is the classical formula of perturbation theory.

As long as $\chi_n^2(x) \to 0$ as $x \to \pm\infty$, the following equalities are obvious:

$$\{\kappa_n^2, \kappa_m^2\} = 0$$
$$\{\kappa_n^2, p(k)\} = 0$$
$$\{\kappa_n^2, q(k)\} = 0$$

To construct angle variables for the discrete spectrum, one must calculate the variational derivatives $\frac{\delta c_n}{\delta u(x)}$. To do this, we use the standard trick: introduce a cutoff potential $u(x)$ outside of an interval $|x| < L$ and set $c_n = b(i\kappa_n)$. Then we can use the previous formulae with a small modification. In (19.25), one must replace $\bar{a}(k) \to a(-k)$ and $\bar{b}(k) \to b(-k)$. Now suppose that $k = i\kappa_n$, $\mathcal{I} \Updownarrow k_1 = 0$. As long as $a(k) = a(i\kappa_n) = 0$,

$$\{b_{i\kappa_n}, b_{k_1}\} = 0$$

Hence,

$$\{c_n, b_{k_1}\} = 0 \tag{6.413}$$

If $k = i\kappa_n$ and $k_1 = i\kappa_m$, then $a(k) = 0$ and $a(k_1) = 0$. All remaining terms are cancelled if $L \to \infty$. Hence,

$$\{c_n, c_m\} = 0$$

Equation (20.16) can be rewritten

$$\{c_n, ln\, b_{k_1}\} = 0$$

The real and imaginary parts of this equation give

$$\{c_n, n(k)\} = 0$$
$$\{c_n, \phi(k)\} = 0$$

To calculate $\{\kappa_n^2, c_m\}$, we will use equation (20.19). Suppose $k = i\kappa$ and $k_1 = i\kappa_m$. If $a(i\kappa_n) = 0$ and $\delta a_k = a'(k) i\delta\kappa_n$, we immediately obtain

$$\{\kappa_n^2, c_m\} = 0 \;\; if \;\; m \neq n$$

The case $n = m$ needs special consideration. We will use equation (20.13) and set $k_1 = i\kappa_n$, $k = i\kappa_n + \varepsilon$ then send $\varepsilon \to 0$. On the left hand, we get

$$a = a'(\kappa_n)(k - i\kappa_n)$$
$$-ia'(i\kappa_n)\kappa_n, c_n = \frac{\kappa_n^2}{4\kappa_n^2} + \frac{\kappa_n^2 \varepsilon a'(i\kappa_n}{2i\kappa_n\varepsilon} c_n$$
$$\{\kappa_n, c_n\} = \frac{c_n}{4\kappa_n}$$
$$\{\kappa_n^2, p_n\} = 1$$
$$p_n = 2ln\,|c_n|$$

Now the higher motion constants are presented as follows:

$$H_j = \frac{1}{2}p_{2j+1} = \frac{2^{2j}}{2j+1}\sum p_n^{\frac{2j+1}{2}} + 2^{2j}(-1)^{j+1}\int_0^\infty k^{2j-1} p_k dk$$

Apparently,

$$\{H_i, H_j\} = 0$$

where the H_i are the Hamiltonians for higher members of the KdV hierarchy.

6.19. The recursive operator

The equations of the KP hierarchy for v can be presented as follows:

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\frac{\delta H_i}{\delta u} = 0 \tag{6.414}$$

Here, $H_i = \frac{1}{2}I_i$, $i = 0, 1, \ldots.$ The I_i are presented by formulae (8.17) and (8.18) with replacement $u \to -v$:

$$I_1 = \frac{1}{2}\int v^2 dx \tag{6.415}$$

$$I_2 = -\frac{1}{2}\int (v_x^2 + v^3) dx \tag{6.416}$$

According to the first equation in series (21.1),

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0 \tag{6.417}$$

The second equation is

$$\frac{\partial v}{\partial t} - 6v\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0 \tag{6.418}$$

In terms of canonical variables,

$$H_i = \frac{2^{2j+1}(-1)^{j+1}}{2j+1}\sum_{i=1}^{N} p_n^{\frac{2j+1}{2}} + 2^{2j-1}\int_0^\infty k^{2j-1} n(k) dk$$

Thus,

$$H_1 = \frac{8}{3}\sum_{i=1}^{\infty} p_n^{\frac{3}{2}} + 2\int_0^\infty k n(k) dk$$

$$H_2 = -\frac{32}{5}\sum_{i=1}^{\infty} p_n^{\frac{5}{2}} + 8\int_0^\infty k^3 n(k) dk$$

So, equation (21.4) is equivalent to the system

$$\frac{\partial q_n}{\partial t} = \frac{\delta H_1}{\delta p_n}$$
$$\frac{\partial q_k}{\partial t} = \frac{\delta H_1}{\delta q_k}$$
$$\frac{\partial q_n}{\partial t} = 4p_n^{\frac{1}{2}}$$
$$\frac{\partial q_k}{\partial t} = 2kn_k$$

Equation (21.5) is equivalent to the system

$$\frac{\partial q_n}{\partial t} = \frac{\delta H_2}{\delta p_n}$$
$$\frac{\partial q_k}{\partial t} = \frac{\delta H}{\delta n_k}$$

or

$$\frac{\partial q_n}{\partial t} = -16p_n^{\frac{3}{2}} \tag{6.419}$$

$$\frac{\partial q_k}{\partial t} = 8k^3 n_k \tag{6.420}$$

Then equations $\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\frac{\delta H^{i+1}}{\delta v} = 0$ can be written as follows:

$$\frac{\partial q_n}{\partial t} = 4p_n^{\frac{1}{2}}(-4p_n)^i \tag{6.421}$$

$$\frac{\partial q_k}{\partial t} = 2kk^{2i}n_k \tag{6.422}$$

In other words, for the i^{th} equation,

$$\frac{\partial q^{(i)}}{\partial t} = L^i \frac{\partial q^{(1)}}{\partial t}$$

Operator L acts in the following way: it multiplies $\frac{\partial q_n}{\partial t}$ by $(-4p_n) = -4\kappa_n^2$, and it multiplies $\frac{\partial q_k}{\partial t}$ by $4k^2$. The positions of discrete eigenvalues remain unchanged.

To figure out how this equation looks in the coordinate representation, we must present the KdV equation (21.5) in the form

$$\frac{\partial v}{\partial t} + L\frac{\partial v}{\partial x} = 0 \tag{6.423}$$

$$L = \frac{\partial^2}{\partial x^2} - 4v - 2\frac{\partial v}{\partial x}\partial^{-1} \tag{6.424}$$

This operator was introduced in chapter 16; it is just the "second Lax operator," (16.15).

Now we have proven that any member of the KdV hierarchy can be presented in the form

$$\frac{\partial v}{\partial t} + L^n\frac{\partial v}{\partial x} = 0, \quad n = 0, 1, ... \tag{6.425}$$

We mentioned this fact in chapter 16, but as a conjecture, without proof. Let us study the operator

$$K^{(1)} = L\frac{\partial}{\partial x}f = \frac{\partial^3}{\partial x^3}f - 4v\frac{\partial f}{\partial x} - 2v_x f \tag{6.426}$$

Suppose F and G are two functionals on $u(x)$. We can define the Poisson bracket as follows:

$$\{F, G\} = \int_{-\infty}^{\infty} \frac{\delta F}{\delta u}K^{(1)}\frac{\delta G}{\delta u}dx \tag{6.427}$$

Operator $K^{(1)}$ is skew-symmetric. The skew-symmetry of its linear part is obvious. As for the nonlinear part, we can see that

$$\int_{-\infty}^{\infty} g(2v\frac{\partial f}{\partial x} + v_x f)dx = -\int f(2v\frac{\partial g}{\partial x} + v_x g)dx$$

in that

$$\int_{-\infty}^{\infty} 2(gv\frac{\partial f}{\partial x} + fv\frac{\partial g}{\partial x} + v_x fg)dx = 2\int_{-\infty}^{\infty} \frac{\partial}{\partial x} fgv\, dx = 0$$

It is just a little bit more difficult to prove by direct calculation that the Jacobi equality holds,

$$F, G, H + H, F, G + G, H, F = 0$$

Thus, (21.13) is a real Poisson bracket.

In the same way, operators $K^{(n)} = L^{(n)}\frac{\partial}{\partial x}$ form Poisson brackets, and the n^{th} KdV equation can be written as

$$\frac{\partial v}{\partial t} + u, {H_1}^{(n)} = 0 \tag{6.428}$$

Here, the symbol $^{(n)}$ means that we use the n^{th} Poisson bracket. Hence each member of the KdV hierarchy has a plethora of Poisson brackets. Notice that any Hamiltonian system with one degree of freedom also has a plethora of Poisson brackets.

A system with one degree of freedom is always integrable, and action-angle variables I, ϕ can always be introduced such that

$$H = H(I) \tag{6.429}$$

$$\frac{\partial \phi}{\partial t} = \frac{\delta H}{\delta I} = w(I) \tag{6.430}$$

$$\frac{\partial I}{\partial t} = -\frac{\delta H}{\delta I} = 0 \tag{6.431}$$

Now the Poisson brackets of functions $F = F(I,\phi)$ and $G = G(I,\phi)$ are standard:

$$F, G = \frac{\partial F}{\partial I}\frac{\partial G}{\partial \phi} - \frac{\partial F}{\partial \phi}\frac{\partial G}{\partial I}$$
$$F, G = -G, F$$

We can introduce the Poisson bracket as follows:

$$F, G^* = A(I)\left(\frac{\partial F}{\partial I}\frac{\partial G}{\partial \phi} - \frac{\partial F}{\partial \phi}\frac{\partial G}{\partial I}\right)$$

where $A(I)$ is an arbitrary function of one variable. Apparently this Poisson bracket is also skew-symmetric. Moreover, one can easily check that it satifies the Jacobi identity. Equations (21.5) are written

$$\frac{\partial \phi}{\partial t} = \phi, H$$
$$\frac{\partial I}{\partial t} = I, H$$

They can also be written

$$\frac{\partial \phi}{\partial t} = \phi, H^*$$
$$\frac{\partial I}{\partial t} = \phi, H^*$$

where $H^* = \int_0^I \frac{1}{A(I)} \frac{\partial H}{\partial I} dI$. Hence, a system with one degree of freedom admits an infinite number of Hamiltonian structures.

Suppose we consider an integrable system of n degrees of freedom. This system admits the introduction of action-angle variables $I_1, ..., I_n$ and $\phi_1, ..., \phi_n$. In a general case, the Hamiltonian is a function on all actions

$$h = H(I_1, ..., I_n)$$

We call the system a separable integrable system if

$$H = H(I_1, ..., I_n) = \sum H_k(I_k) \tag{6.432}$$

Apparently, such a system admits an infinite number of Hamiltonian structures. We have seen that the KdV equation is a separable Hamiltonian system if $|u| \to 0$ at $|x| \to \infty$.

What about integrability of KdV if $u(x)$ is just a bound function $|u(x)| < c$, $-\infty < x < \infty$? Little is known!

6.20. Spectral singularity

Let us consider the Schrödinger equation at $k = 0$:

$$\psi'' = u(x)\psi, \ \ |u(x)| \to 0 \ \ as \ \ |x| \to \infty$$

Now, $\phi_1 \to 1$ as $x \to -\infty$. At a general position,

$$\psi \to c_1 + c_2 x, \ \ x \to \infty \tag{6.433}$$

If $c_2 \neq 0$, both coefficients $a(k)$ and $b(k)$ have a singularity at $k = 0$. Indeed,

$$\phi \to a(k)e^{-ikx} + b(k)e^{ikx}$$
$$|a(k)|^2 - |b(k)|^2 = 1$$

The asymptotic behavior (21.17) presumes that, at $k \to 0$,

$$a(k) \to \frac{ic_2}{2k} + \frac{1}{2}\left(c_1 + \frac{1}{c_1}\right)$$
$$b(k) \to -\frac{ic_2}{2k} + \frac{1}{2}\left(c_1 - \frac{1}{c_1}\right)$$

$a(k)$ and $b(k)$ have simple poles at $k = 0$. This means that the degree of freedom corresponding to $k = 0$ must be excluded from consideration. At $k \to 0$, $n(k) \simeq -ln\,|k|$. This is an integrable singularity.

Now we can address the following question: under what conditions are all integrals I_n finite? For sure, the necessary condition is

$$\int_{-\infty}^{\infty} [u^{(n)}]^2 dx < \infty \tag{6.434}$$

Thus, the potential $u(x)$ must be infinitely smooth. We can formulate the following conjecture: condition (21.18) is also sufficient for the existence of all integrals.

Notice that condition (21.18) does not guarantee that the condition of finiteness of the discrete spectrum,

$$\int_{-\infty}^{\infty} |xu(x)|dx < \infty \tag{6.435}$$

is satisfied. For instance, the potential could have "oscillatory tails"

$$u(x) \simeq \frac{1}{x^{\frac{1}{2}+\varepsilon}}|cos\, k_0 x, \ \ \varepsilon > 0 \tag{6.436}$$

In the presence of such tails, $a(k)$ must have an essential singularity at $k = 0$. Moreover, $a(k)$ must have poles on the real axis.

Suppose the reflection coefficient $c(k)$ is given by

$$c^2(k) = \frac{e^{-\alpha k^2}}{(k^2 - a^2)^2 + e^{-\alpha k^2}} \tag{6.437}$$

Now,

$$|a(k)|^2 = 1 + \frac{e^{-\alpha k^2}}{(k^2 - a^2)^2} > 0$$

Now if

$$arg\, a(k) = \frac{1}{2\pi} \int \frac{1}{k' - k} ln \left(1 + \frac{e^{-\alpha k'^2}}{(k'^2 - a^2)^2} \right) dk' \tag{6.438}$$

is analytic in the upper half-plane and has no zeros there, the singularity of $ln|a(k)|^2$ at $k = a$ is integrable. Hence there is no discrete spectrum and all I_n are finite. In particular,

$$\int_{-\infty}^{\infty} u^2 dx < \infty$$

However, $c^2(\pm a) = 1$. This means that the potential $u(x)$ realizes complete reflection of probe particles with $k = \pm a$. It would be extremely interesting to explore the asymptotic behavior of this potential at $|x| \to \pm\infty$.

6.21. Symmetric reflectionless potentials

¿From the point of view of experts in quantum mechanics, solitonic solutions of the KdV equation are given by "reflectionless potentials" characterized by the cancelling of reflection coefficient $c(k)$. In this chapter, we describe a particularly important class of reflectionless potentials given by even functions of coordinates. We start with a one-soliton solution. Let us set $x_0 = 0$; then,

$$\Delta(x) = 1 + e^{-2\eta x} = 2e^{-\eta x} cosh\, \eta x$$

We can therefore replace $\Delta(x) \to cosh\, \eta x$. Now,

$$u = -2\frac{d^2}{dx^2} ln\, cosh\, \eta x = -\frac{2\eta^2}{cosh\, \eta x} \tag{6.439}$$

This is an even function. Any given soliton is even in a certain moment of time.

Now we study a general two-soliton solution,

(6.440)
$$\Delta(x) = 1 + q_1 e^{-2\eta_1 x} + q_2 e^{-2\eta_2 x} + q_1 q_2 \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 e^{-2(\eta_1+\eta_2)x}, \ \ q_1 > 0, \ \ q_2 > 0$$

Now we set

$$q_1 = q_2 = \left|\frac{\eta_1+\eta_2}{\eta_1-\eta_2}\right| = q^{-1}$$
$$q = \left|\frac{\eta_1-\eta_2}{\eta_1+\eta_2}\right|$$

This determinant $\Delta(x)$ is equivalent to

$$\Delta = cosh\ (\eta_1+\eta_2)x + q^{-1} cosh\ |\eta_1-\eta_2|x$$

Without loss of generality, we can replace $x \to \frac{x}{\eta_1+\eta_2}$. We end up with the following simple expression:

(6.441)
$$\Delta(x) = cosh\ x = \frac{1}{q} cosh\ qx$$

The symmetry of this function is obvious. Let us calculate the derivatives of $\Delta(x)$ at $x=0$:

$$\Delta(0) = 1 + \frac{1}{q}$$
$$\Delta'(0) = 0$$
$$\Delta''(0) = 1 + q$$
$$\Delta'''(0) = 0$$
$$\Delta^{(IV)}(0) = 1 + q^3$$

Now we recall that

(6.442)
$$u(x) = -2\frac{\Delta''\Delta - \Delta'^2}{\Delta^2}$$

(6.443)
$$u(0) = -2\frac{\Delta''}{\Delta} = 2q$$

Then,

$$u'(x) = -2\frac{\Delta'''\Delta - \Delta''\Delta'}{\Delta^2} + \frac{4}{\Delta^3}(\Delta''\Delta - \Delta'^2)\Delta'$$
$$u'(0) = 0$$

To calculate $u''(0)$, we should remember that $\Delta'''(0) = 0$ and $\Delta'(0) = 0$. So,

$$u''(0) = 2\left(-\frac{\Delta^{(IV)}}{\Delta} + \frac{3\Delta''^2}{\Delta^2}\right) = -2q(q^2 - 4q + 1)$$

The quadratic $q^2 - 4q + 1$ has root $q_0 = 2 - \sqrt{3} \simeq 0.268$. Notice that if $q = q_c = \frac{1}{3}$,

$$\Delta \simeq (cos\ \frac{1}{3}x)^3$$

and the soliton is given by

$$u = -\frac{6\eta^2}{cosh\ \eta x}$$

Now, $\eta_2 = 2\eta_1$.

We see that the configuration of the symmetric reflectionless potential essentially depends on the parameter

$$q = \frac{\eta_1-\eta_2}{\eta_1+\eta_2},\quad \eta_1 > \eta_2$$

!!INSERT ONE-SOLITON FIGURE HERE!!

!!INSERT TWO-SOLITON FIGURE HERE!!

varying inside the interval $0 < q < 1$. Let us plot qualitative pictures of symmetric two-soliton solutions in different subintervals of q:

Case 1: If $q_0 < q < 1$, where $q_0 = 2 - \sqrt{3}$, the solution has only one maximum, In the special case $q = \frac{1}{3}$, the two-soliton solution has the same shape as a single soliton multiplied by a factor of 3.

Case 2: If $0 < q < q_0$, the solution has two maxima and, in the limiting case $q \to 0$, $\eta_2 \to \eta_1$, it splits into two soliton of amplitude close to η_1.

A general n-soliton solution depends on $2n$ parameters - amplitudes $\eta_1, ..., \eta_n$ and phase factors $q_i = e^{-2\eta_i x_{0i}}$. In the symmetric solution, the phases are completely defined by the amplitudes; namely,

$$q_i = \prod_{k \neq i} \frac{\eta_k + \eta_i}{|\eta_k - \eta_i|}$$

Without loss of generality, one can set

$$\sum_{i=1}^{n} \eta_i = 1, \quad \eta_i > \eta_j \; if \; i < j$$

A general n-soliton solution consists of 2^n terms; each of these terms is an exponent. The symmetric solution consists of 2^{n-1} cosine functions with possible arguments $\eta_1 \pm \eta_2 \pm ... \pm \eta_n$. For instance, the three-soliton solution looks like

$$\Delta(x) = cos(\eta_1 + \eta_2 + \eta_3)x + q_1 cos(\eta_1 - \eta_2 - \eta_3)x + q_2 cos(\eta_2 - \eta_1 - \eta_3)x + q_3 cos(\eta_3 - \eta_1 - \eta_2)x$$

$$q_1 = \frac{\eta_2 + \eta_1}{|\eta_2 - \eta_1|} \frac{\eta_3 + \eta_1}{|\eta_3 - \eta_1|}$$

$$q_2 = \frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} \frac{\eta_3 + \eta_2}{|\eta_3 - \eta_2|}$$

$$q_3 = \frac{\eta_1 + \eta_3}{\eta_1 - \eta_3} \frac{\eta_2 + \eta_3}{\eta_2 - \eta_3}$$

In the particular case $\eta_1 = 3\eta$, $\eta_2 = 2\eta$, $\eta_3 = \eta$,

$$q_1 = 6$$
$$q_2 = 15$$
$$q_3 = 10$$

and

$$\Delta(x) = (cos\, \eta x)^6$$
$$u(x) = \frac{12\eta^2}{cos^2\, \eta x}$$

It is very interesting to study a symmetric potential with a large number of solitons, $n \to \infty$, all packed inside the interval $\eta_{min} < \eta < \eta_{max}$. Now the n-soliton solution can be characterized by a distribution function,

$$dF = f(\eta) d\eta$$

dF is the number of solitons with amplitudes within the interval $[eta, \eta + d\eta]$. The symmetric reflectionless potential describes the maximally-compressed state of the solitonic gas with a given distribution function. A certain (still unknown) distribution function will be present as a periodic choidal wave. The study of symmetric reflectionless potentials must be done via a massive amount of analytical calculation on a computer using the "Mathematica" software.

6.22. Symmetric reflectionless potentials - continued

Let us again consider the symmetric two-soliton solution,

$$\Delta(x) = cosh\, x + \frac{1}{q} cosh\, qx, \quad 0 < x < \infty \tag{6.444}$$

and study the case $q \to 0$. Now, $u(0) = q \to 0$, and the solution is concentrated at large q. Therefore we can replace (23.1) by ???.

For $u'(x)$, we get

$$u'(x) = -\frac{2}{\Delta^3} \Delta''' \Delta^2 + 3\Delta'' \Delta' \Delta - 4\Delta'^3 \tag{6.445}$$

For small-enough q, the solution is a superposition of two soliton-like humps, located at $x = \pm x_m$. Strictly speaking, to find them we must find the zeros of $u'(x)$. However, in the limit $q \to 0$, the problem can be simplified. Indeed, at $x = 0$, $u = q \to 0$, and the hump is concentrated at $x \simeq ln\, \frac{2}{q}$. In this area, expression (23.1) can be simplified to

$$\Delta(x) \simeq e^x + \frac{2}{q}$$

or, in terms of the original variables,

$$\begin{aligned} \Delta(x) &\simeq 1 + e^{-2\eta(x - x_0)} \\ \eta &= \frac{\eta_1 + \eta_2}{2} \\ x_0 &= \frac{1}{2\eta} ln\, \frac{2}{q} \end{aligned}$$

$$x_0 = \frac{1}{\eta_1 + \eta_2} ln\, 2 \frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} \tag{6.446}$$

$2x_0$ is the minimum distance between two solitons with close values of η_1 and η_2. This formula is correct up to terms of order $\left(q\, ln\, \frac{1}{q}\right)^2$.

Let us now study the three-soliton solution and consider only the simplest case, $\eta_1 = 1+q$, $\eta_2 = 1$, $\eta_3 = 1-q$. In this case,

$$q_1 = \frac{(\eta_1+\eta_2)(\eta_1+\eta_3)}{(\eta_1-\eta_2)(\eta_1-\eta_3)} = \frac{2+q}{q^2}$$

$$q_2 = \frac{(\eta_2+\eta_1)(\eta_2+\eta_3)}{(\eta_1-\eta_2)(\eta_2-\eta_3)} = \frac{4-q^2}{q^2}$$

$$q_3 = \frac{(\eta_1+\eta_3)(\eta_2+\eta_3)}{(\eta_1-\eta_3)(\eta_2-\eta_3)} = \frac{2-q}{q^2}$$

$$\Delta(x) = cosh\, 3x + q_1 cosh\, (1-2q)x + q_2 cosh\, x + q_3 cosh\, (1+2q)x$$

$$\Delta(0) = 1 + q_1 + q_2 + q_3 = 1 + \frac{8}{q^2}$$

$$-\Delta''(0) = 9 + q_1(1-2q)^2 + q_2 + q_3(1+2q)^2 = \frac{8}{q^2}(1+q^2)$$

Hence,

$$-2\frac{\Delta''}{\Delta} = \frac{16(1+q^2)}{8+q^2} \simeq 2$$

at $q \to 0$. Also,

$$u_{max} = -2\frac{\Delta''}{\Delta} \to 2\left(1 + \frac{7}{8}q^2\right)$$

Remember that, for $0 < q < 1$, the soliton in the center always exists.

To find the position of the second soliton, we assume that $q \to 0$. In the area of this soliton,

$$\Delta(x) \simeq \frac{e^x}{2}\left(e^{2x} + \frac{16}{q^2}\right)$$

Now we see that the second soliton has amplitude $\eta = 1$, while the "central" soliton has amplitude

$$\eta_{max} \simeq 1 + \frac{7}{4}q^2 > 1 \tag{6.447}$$

Only in the limit $q \to 0$ do both amplitudes become equal.

6.23. Knoidal wave

Let us study stationary waves in the framework of the KdV equation,

$$v_t - 6vv_x + v_{xxx} = 0$$

After assuming that $v_t = -cv_x$, we end up with the equation

$$v_{xx} - 3v^2 - cv = 0$$

Integrating this gives

$$\frac{1}{2}v_x^2 - v^3 - \frac{c}{2}v^2 = E \tag{6.448}$$

This equation has solution

$$v = -\frac{c}{6} + 2P(x + i\omega_2) \tag{6.449}$$

Here, $P(x)$ is the elliptic Weierstrass function satisfying the equation

$$P_x^2 = 4P^3 - g_2 P - g_3 \tag{6.450}$$

!!INSERT PLOT OF POTENTIAL WELL HERE!!

$$g_2 = \frac{c^2}{12}$$
$$g_3 = -\frac{c^3}{6^3} - \frac{E}{2}$$

Equation (23.6) can be rewritten

$$P_x^2 = 4(P - l_1)(P - l_2)(P - l_3) \tag{6.451}$$

Equation (23.4) can be treated as an energy balance equation for a particle with mass unity moving in the potential well $u = u(v)$:

$$\frac{1}{2}v_x^2 + u(v) = E \tag{6.452}$$

$$u(v) = -v^3 - \frac{c}{2}v^2 \tag{6.453}$$

This potential well is plotted in Figure (23.1): The solution is periodic if $E < 0$; in this case, all roots l_1, l_2, l_3 are real. We will assume $l_1 < l_2 < l_3$.

The Weierstrass elliptic function $P = P(x, \omega_1, \omega_2)$ is double-periodic. One period, $2\omega_1$, is real, and the second one is imaginary:

$$P = P(x + 2\omega_1) = P(x + 2i\omega_2)$$

Then,

$$\omega_1 = \int_{l_1}^{l_2} \frac{dy}{\sqrt{4y^2 - g_2 y - g_3}} \tag{6.454}$$

$$i\omega_2 = \int_{l_2}^{l_3} \frac{dy}{\sqrt{4y^2 - g_2 y - g_3}} \tag{6.455}$$

Function $P(x)$ has double poles on the real axis. Function $P(i\omega_2 + x)$ is real and regular at $-\infty < x < \infty$.

Let us consider the function

$$f(x) = -\frac{\pi^2}{cosh^2\, \pi x} = -\frac{\pi^2}{cos^2\, \pi y} = \frac{\pi^2}{sin^2\, \pi(y - \frac{1}{2})}, \quad y = ix$$

This function can be presented as a sum of partial fractions,

$$f(x) = -\sum_{n=-\infty}^{\infty} \frac{1}{\left(y - n - \frac{1}{2}\right)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{\left(x - i(n - \frac{1}{2})\right)^2}$$

Now we introduce $\eta = \frac{2\omega_2}{\pi}$ and study the series

$$w(x) = -2\eta^2 \sum_{n=-\infty}^{\infty} \frac{1}{cosh^2\, \eta(x - 2\omega_1 n}$$

This function has double poles at points $x_{nm} = i\left(n + \frac{1}{2}\right)\omega_2 + \omega_1 m$ with asymptotes $w \to \frac{2}{(x - x_{nm})^2}$ at $x \to x_{nm}$. We can state that

$$w(x) = 2P(x + i\omega_2) + C$$

where C is some constant. Later on, we will prove that $C = -\frac{c}{6}$, thus $w(x) = v(x)$, which is a knoidal wave.

Now we consider the following infinite product:

$$\tilde{i}(x) = (1 + z^2) \prod (1 + h^{2n} z^2)(1 + h^{2n} z^{-2})$$
$$z = e^{-\eta x}$$
$$h = e^{-\eta x_0}$$
$$x_0 > 0$$

After simple calculation, we see that this is exactly the τ function for the knoidal wave.

Appendix

A.1. Tests of Canonicity of transformations

The Hamiltonian equations for discrete systems

$$\begin{aligned} \frac{\partial q_j}{\partial t} &= \frac{\partial H}{\partial p_j}, \\ \frac{\partial p_j}{\partial t} &= -\frac{\partial H}{\partial q_j} \end{aligned} \tag{A.456}$$

as well as the Hamiltonian equations

$$\begin{aligned} \frac{\partial q(\mathbf{r},t)}{\partial t} &= \frac{\delta H}{\delta p(\mathbf{r},t)}, \\ \frac{\partial p(\mathbf{r},t)}{\partial t} &= -\frac{\delta H}{\delta q(\mathbf{r},t)} \end{aligned} \tag{A.457}$$

for continuous media can be transformed to the same type of equations

$$\begin{aligned} \frac{\partial Q_j}{\partial t} &= \frac{\partial H}{\partial P_j}, \\ \frac{\partial P_j}{\partial t} &= -\frac{\partial H}{\partial Q_j} \end{aligned} \tag{A.458}$$

and

$$\begin{aligned} \frac{\partial Q(\mathbf{r},t)}{\partial t} &= \frac{\delta H}{\delta P(\mathbf{r},t)}, \\ \frac{\partial P(\mathbf{r},t)}{\partial t} &= -\frac{\delta H}{\delta Q(\mathbf{r},t)} \end{aligned} \tag{A.459}$$

in new variables (Q_j, P_j) and $(Q(\mathbf{r},t), P(\mathbf{r},t))$, respectively provided the transformations ???

A.2. Variational Derivatives

Let $\phi(x)$, $-\infty < x < \infty$ be a smooth function of one variable, and let $L[\phi] : \phi \to \mathbb{R}^1$ be a functional. If $L[\phi]$ is a *linear* functional such that

$$L[\alpha_1\phi_1 + \alpha_2\phi_2] = \alpha_1 L[\phi_1] + \alpha_2 L[\phi_2]$$

then we can write it in the integral form as

$$L[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)dx. \tag{A.460}$$

Here, $f(x)$ is the generalized function (distribution). For instance, if $L[\phi] = \phi(a)$, then $f(x) = \delta(x-a)$, where $\delta(x)$ is the Dirac delta function, see Appendix A.3.

We define that $f(x)$ is the *variational derivative* (also called by the functional derivative) of the functional L over $\phi(x)$,

$$f(x) = \frac{\delta L}{\delta \phi} \tag{A.461}$$

Now, let $L[\phi]$ be a *nonlinear* functional. We can add to $\phi(x)$ a small variation

$$\phi(x) \to \phi(x) + \delta\phi(x)$$

and consider the functional

$$\Delta L[\phi, \delta\phi] = L[\phi + \delta\phi] - L[\phi] \tag{A.462}$$

If $\|\delta\phi\| \to 0$, ΔL becomes a linear functional on $\delta\phi$. Hence,

$$\lim_{\|\delta\phi\|\to 0} \Delta L = \int_{-\infty}^{\infty} f[\phi, x]\delta\phi(x)dx \tag{A.463}$$

Now,

$$f[\phi, x] = \frac{\delta L}{\delta \phi} \tag{A.464}$$

$f[\phi, x]$ is a variational derivative of L by ϕ. Notice that $f[\phi, x]$ is simultaneously a generalized function on x and a functional on ϕ.

A chain rule for the variational derivative of functionals, i.e. of the functional F of the functional G is defined by

$$\frac{\delta F[G[\phi]]}{\delta \phi(x)} = \int dx' \frac{\delta F[G[\phi]]}{\delta G[\phi(x')]} \frac{\delta G[\phi(x')]}{\delta \phi(x)} \tag{A.465}$$

which is the generalization of the chain rule of the differentiation of the composition of the functions f and g as follows

$$\frac{\partial f(g(x_1), \ldots, g(x_n))}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial f(g(x_1), \ldots, g(x_n))}{\partial g(x_k)} \frac{\partial f(g(x_k))}{\partial x_i}. \tag{A.466}$$

Examples

1. Suppose that $L[\phi]$ is a quadratic functional

$$L[\phi] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x,y)\phi(x)\phi(y)dxdy, \tag{A.467}$$

where $K(x,y)$ is the kernel function (also sometime called by the integral kernel) and we assume that $K(x,y) = K(y,x)$.

$$\frac{\delta L}{\delta \phi} = 2\int_{-\infty}^{\infty} K(x,y)\phi(y)dy \tag{A.468}$$

2. Let $l[\phi]$ be a *monomial* functional

(A.469) $$L = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} K(x_1, ..., x_n)\phi(x_1)...\phi(x_n)dx_1...dx_n$$

where the kernel function $K(x_1, ..., x_n)$ assumed to be invariant with respect to all permutations $P(x_1, ..., x_n)$, i.e. $K(x_1, ..., x_n) = K(x_{j_1}, ..., x_{j_n})$ for any in

Then,

(A.470) $$\frac{\delta L}{\delta \phi} = n \int K(x, x_1, ..., x_{n-1})\phi(x_1)...\phi(x_{n-1})dx_1...dx_{n-1}$$

3. Let

(A.471) $$L[\phi] = \int_{-\infty}^{\infty} F(\phi(x))dx$$

Then,

(A.472) $$\frac{\delta L}{\delta \phi} = F'(\phi(x))$$

Equation (2.10) is an example of local functionals. In a general case, the local functional which depends on the highest nth derivative can be presented as follows:

(A.473) $$L = \int_{-\infty}^{\infty} F(\phi, \phi', ..., \phi^{(n)})dx$$

Then,

(A.474) $$\Delta L = \int_{-\infty}^{\infty} \left[\frac{\partial F}{\partial \phi}\delta\phi + \frac{\partial F}{\partial \phi'}\delta\phi' + ... + \frac{\partial F}{\partial \phi^{(n)}}\delta\phi^{(n)} \right] dx$$

Integrating (2.13) by parts, we obtain

$$\Delta L = \int_{-\infty}^{\infty} \frac{\delta L[\phi, x]}{\delta \phi}\delta\phi(x)dx$$

where

(A.475) $$\frac{\delta L}{\delta \phi} = \frac{\partial F}{\partial \phi} - \frac{\partial}{\partial x}\frac{\partial F}{\partial \phi'} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial \phi''} + ... + (-1)^n \frac{\partial^n}{\partial x^n}\frac{\partial F}{\partial \phi^{(n)}}$$

In particular, if

$$L = \frac{1}{2}\int \phi'^2(x)dx$$

then

$$\frac{\delta L}{\delta \phi} = -\phi''(x)$$

This construction is immediately generalized to functionals on functions of several variables. For instance, if $\phi = \phi(x_1, ..., x_n)$ and

(A.476) $$L = \frac{1}{2}\int (\nabla\phi)^2 d\vec{r}, \quad d\vec{r} = dx_1...dx_n$$

then

$$\frac{\delta L}{\delta \phi} = -\Delta\phi \tag{A.477}$$

More generally, if

$$L = \frac{1}{2}\int A(\nabla\phi)^2 d\vec{r}$$

then

$$\frac{\delta L}{\delta \phi} = -\nabla\cdot(A\nabla\phi) = -\mathrm{div}(A\nabla\phi). \tag{A.478}$$

A.3. Differential Manifolds

A.4. Dirac delta function and Fourier transform

Fourier transform (FT) $f_{\mathbf{k}}$ of a complex-value function $f(\mathbf{r}),\ \mathbf{r}\in\mathbb{R}^D$ is defined by the integral

$$\mathcal{F}^{-1}(f)(\mathbf{k}) := f_{\mathbf{k}} = \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} f(\mathbf{r})e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}, \tag{A.479}$$

where $\mathbf{k}\in\mathbb{R}^D$. The integral in equation (A.479) can be understood as the usual Riemann integral for Riemann-integrable $f(\mathbf{r})$. More generally, FT is valid for Lebesgue absolutely integrable functions $f(\mathbf{r})$, i.e. $f(\mathbf{r})\in L^1(\mathbb{R}^D)$ because $f(\mathbf{r})\in L^1(\mathbb{R}^D)$ together with equation (A.479) implies that $|f_{\mathbf{k}}|\le \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D}|f(\mathbf{r})|d\mathbf{r}<\infty$. Here $L^p(\mathbb{R}^D),\ p\ge 1$ is the space of functions $f(\mathbf{r})$ such that $|f(\mathbf{r})|^p$ is Lebesgue integrable, i.e. the norm

$$\|f(\mathbf{r})\|_{L^p} := \left(\int\limits_{\mathbb{R}^D}|f(\mathbf{r})|^p d\mathbf{r}\right)^{1/p},\ p\ge 1 \tag{A.480}$$

is finite, $\|f(\mathbf{r})\|_{L^p}<\infty$.

Wave systems often implies that $f(\mathbf{r})$ is not only in $L^1(\mathbb{R}^D)$ but infinitely differentiable with the Lebesgue integral in FT (A.479) is then reduced to the Riemann integral. FT is especially convenient to work with the space $S(\mathbb{R}^D)$ of infinitely differentiable functions which decay faster than any inverse power of $|\mathbf{r}|$ for $|\mathbf{r}|\to\infty$). FT (A.479) of $f(\mathbf{r})\in S(\mathbb{R}^D)$ results in $f_{\mathbf{k}}$ being in $S(\mathbb{R}^D)$ as the function of $\mathbf{k}\in\mathbb{R}^D$, i.e. FT maps $S(\mathbb{R}^D)$ into $S(\mathbb{R}^D)$ (and even more, FT is one-to-one mapping of $S(\mathbb{R}^D)$ onto $S(\mathbb{R}^D)$) [**Rud91**].

Also $f(\mathbf{r})\in S(\mathbb{R}^D)$ ensures the existence of inverse FT

$$\mathcal{F}^{-1}(f_{\mathbf{k}})(\mathbf{r}) := \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} f_{\mathbf{k}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}, \tag{A.481}$$

which recovers $f(\mathbf{r})$ from $f_{\mathbf{k}}$ pointwise for any $\mathbf{r}\in\mathbb{R}^D$, i.e. $\mathcal{F}^{-1}(\mathcal{F}(f))\equiv f$, or

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} f_{\mathbf{k}}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}, \tag{A.482}$$

which is referred to as Fourier inversion theorem.

Example. FT of the Gaussian function $f_G(\mathbf{r}) = e^{-\epsilon|\mathbf{r}|^2},\ \epsilon > 0$ is given by

$$\mathcal{F}f_G = \frac{1}{(2\pi)^{D/2}} \int\limits_{\mathbb{R}^D} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}-\epsilon|\mathbf{r}|^2} d\mathbf{r} = \frac{1}{(2\epsilon)^{D/2}} e^{-\frac{|\mathbf{k}|^2}{4\epsilon}}, \tag{A.483}$$

where we used the integral (A.502) with $a = \epsilon$ and $b = -\mathrm{i}k_i, i = 1, \ldots, D$ for the integration along each coordinate $r_i,\ i = 1, \ldots, D$.

For more general case $f(\mathbf{r}) \in L^1(\mathbb{R}^D)$, Fourier inversion theorem (A.482) is still valid almost everywhere (except the set of measure zero in $\mathbf{r} \in \mathbb{R}^D$) provided $f_\mathbf{k} \in L^1(\mathbb{R}^D)$ [**Rud91**]. For example, if $D = 1$, $f(\mathbf{r}) \in L^1(\mathbb{R}), \mathbf{r} := r$ and $f(r)$ is a piecewise smooth with a finite number of discontinuities then equation (A.482) is valid for any r for which $f(r)$ is continuous while $\mathcal{F}^{-1}(f_\mathbf{k})(r) = [f(r^+)+f(r^-)]/2$ at discontinuities. Here $f(r^+)$ and $f(r^-)$ are one-sided limits at the discontinuities and the integral in equation (A.481) is understood as the improper Riemann integral (the Lebesgue integral may not be defined in that case).

Plancherel's theorem states that $\int\limits_{\mathbb{R}^D} f(\mathbf{r})\bar{g}(\mathbf{r})d\mathbf{r} = \int\limits_{\mathbb{R}^D} f_\mathbf{k}\bar{g}_\mathbf{k}d\mathbf{k}$ for complex-valued functions $f(\mathbf{r})$ and $g(\mathbf{r})$ with $\mathbf{r} \in \mathbb{R}^D$ (bar here means complex conjugation) provided $f(\mathbf{r}),\ g(\mathbf{r}) \in L^1(\mathbb{R}^D) \cap L^2(\mathbb{R}^D)$. A particular case of Plancherel's theorem for $g(\mathbf{r}) = f(\mathbf{r})$ resulting in $\int\limits_{\mathbb{R}^D} |f(\mathbf{r})|^2 d\mathbf{r} = \int\limits_{\mathbb{R}^D} |f_\mathbf{k}|^2 d\mathbf{k}$ is called Parseval's identity.

FT is also extended into the space of generalized functions, or distributions. Distributions $f(\mathbf{r})$ generalize the notion of classical functions to continuous linear functionals which we designate by $\langle f, \varphi\rangle$. These functionals map a space of test functions $\varphi \in \mathcal{D}(\mathbb{R}^D)$ into the set of complex numbers, where $\mathcal{D}(\mathbb{R}^D)$ is the set of infinitely differentiable functions with compact support. The set of all such functionals define the space of distributions $\mathcal{D}'(\mathbb{R}^D)$. If $f(\mathbf{r})$ is the locally integrable (Lebesgue integrable over every compact subset of $\mathbb{R}^D$) function then $\langle f, \varphi\rangle = \int\limits_{\mathbb{R}^D} f(\mathbf{r})\phi(\mathbf{r})d\mathbf{r}$. A simplest nontrivial example of the distribution is the Dirac delta function (distribution) $\delta(\mathbf{r})$, defined by $\langle \delta, \varphi\rangle = \varphi(\mathbf{0})$. In this book we abuse notation and formally replace $\langle f, \varphi\rangle$ by $\int\limits_{\mathbb{R}^D} f(\mathbf{r})\phi(\mathbf{r})d\mathbf{r}$ for all distributions (not only for locally integrable functions) meaning that e.g. *Dirac delta function* (or simply δ function) is defined by

$$\int\limits_{\mathbb{R}^D} \delta(\mathbf{r} - \mathbf{r}_0)\varphi(\mathbf{r})d\mathbf{r} = \varphi(\mathbf{r}_0),\ \mathbf{r}_0 \in \mathbb{R}^D. \tag{A.484}$$

Using the space $\mathcal{D}(\mathbb{R}^D)$ for defining FT of distributions is not convenient because FT of $\varphi \in \mathcal{D}(\mathbb{R}^D)$ generally does not belong to $\mathcal{D}(\mathbb{R}^D)$ (FT of compactly supported function is not compactly supported function of $\mathbf{k}$ variable). Instead of the space $\mathcal{D}(\mathbb{R}^D)$ we choose a larger Schwartz space $S(\mathbb{R}^D)$ as the space of test functions φ. Obviously $\mathcal{D}(\mathbb{R}^D) \subset S(\mathbb{R}^D)$ as well as $\mathcal{D}(\mathbb{R}^D)$ is dense in $S(\mathbb{R}^D)$. The space of test functions $\varphi \in S(\mathbb{R}^D)$ defines the space of *tempered distributions* $S'(\mathbb{R}^D)$ as the set of continuous linear functionals $\langle f, \varphi\rangle$. All distributions from $S'(\mathbb{R}^D)$ belong to $\mathcal{D}'(\mathbb{R}^D)$, i.e. $S'(\mathbb{R}^D) \subset \mathcal{D}'(\mathbb{R}^D)$ [**Rud91**]. Similar to the more general space of distribution $\mathcal{D}(\mathbb{R}^D)$, we abuse the notation for linear functionals through the integral symbol and define FT of $f(\mathbf{r}) \in S'(\mathbb{R}^D)$ by equation (A.479) and inverse FT by equation (A.482). FT is one-to-one continuous mapping of $S'(\mathbb{R}^D)$ onto $S'(\mathbb{R}^D)$ with $\mathcal{F}^{-1}(\mathcal{F}(f)) \equiv f$.

Example. Consider FT of Dirac delta function $f(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_0)$: $f_{\mathbf{k}} = \frac{1}{(2\pi)^{D/2}} \int\limits_{\mathbb{R}^D} \delta(\mathbf{r}-\mathbf{r}_0) e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} = \frac{e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_0}}{(2\pi)^{D/2}}$. Taking inverse FT of that expression by using equation (A.482) we obtain that $\mathcal{F}^{-1}\frac{e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_0}}{(2\pi)^{D/2}} = \frac{e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_0}}{(2\pi)^{D}} \int\limits_{\mathbb{R}^D} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} = \delta(\mathbf{r}-\mathbf{r}_0)$, i.e. we obtain the formal identity

$$\int\limits_{\mathbb{R}^D} e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)} d\mathbf{k} = (2\pi)^D \delta(\mathbf{r}-\mathbf{r}_0), \quad \mathbf{r},\ \mathbf{r}_0 \in \mathbb{R}^D, \tag{A.485}$$

and similar

$$\int\limits_{\mathbb{R}^D} e^{\mathrm{i}(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}} d\mathbf{r} = (2\pi)^D \delta(\mathbf{k}-\mathbf{k}_0), \quad \mathbf{k},\ \mathbf{k}_0 \in \mathbb{R}^D. \tag{A.486}$$

Equations (A.485) and (A.486) are very useful for formal manipulations with FT throughout this book. For the additional insight one can also obtain equations (A.485) and (A.486) as $\epsilon \to 0$ limit of the following Gaussian integral

$$\delta_\epsilon(\mathbf{k}) := \frac{1}{(2\pi)^D} \int\limits_{\mathbb{R}^D} e^{\mathrm{i}(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}-\epsilon|\mathbf{r}|^2} d\mathbf{r} = \frac{1}{(4\pi\epsilon)^{D/2}} e^{-\frac{(\mathbf{k}-\mathbf{k}_0)^2}{4\epsilon}}, \tag{A.487}$$

where equation (A.501) from Appendix A.5 was used. It follows from equation (A.486) that $\lim\limits_{\epsilon\to 0} \delta_\epsilon(\mathbf{k}) = \delta(\mathbf{k})$ because for any $\varphi \in S(\mathbb{R}^D)$ or $\varphi \in \mathcal{D}(\mathbb{R}^D)$ one obtains that

$$\lim_{\epsilon\to 0} \int\limits_{\mathbb{R}^D} \varphi(\mathbf{k})\delta_\epsilon(\mathbf{k}) d\mathbf{k} = \varphi(\mathbf{k}_0) \lim_{\epsilon\to 0} \int\limits_{\mathbb{R}^D} \delta_\epsilon(\mathbf{k}) d\mathbf{k} = \varphi(\mathbf{k}_0).$$

Any distribution $f \in \mathcal{D}'(\mathbb{R}^D)$ is infinitely differentiable with the derivatives defined as

$$\langle \frac{\partial^{\alpha_1}}{\partial r_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial r_2^{\alpha_1}} \cdots \frac{\partial^{\alpha_D}}{\partial r_1^{\alpha_D}} f, \varphi\rangle = (-1)^{\alpha_1+\alpha_2+\ldots+\alpha_D} \langle f, \frac{\partial^{\alpha_1}}{\partial r_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial r_2^{\alpha_1}} \cdots \frac{\partial^{\alpha_D}}{\partial r_1^{\alpha_D}} \varphi\rangle \tag{A.488}$$

In particular derivatives of Dirac delta function for $D=1$ are defined as

$$\langle \delta', \varphi\rangle = -\varphi'(0) \tag{A.489}$$

for the first derivative and $\langle \delta^{(n)}, \varphi\rangle = (-1)^n \varphi^{(n)}(0)$ for the nth derivative. Also $\Theta'(x) = \delta(x)$, $x \in \mathbb{R}$, where $\Theta(x)$ is the *Heaviside step function* defined by $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$. The Heaviside step function is also called by the unit step function.

The definition (A.479) ensures that FT transforms a convolution $f * g$

$$(f * g)(\mathbf{r}) := \int\limits_{\mathbb{R}^D} f(\mathbf{r}-\mathbf{r}')g(\mathbf{r}')d\mathbf{r}' \tag{A.490}$$

of the function $f(\mathbf{r})$ and $g(\mathbf{r})$ into the (pointwise) product of FTs as follows

$$\mathcal{F}(f * g)(\mathbf{k}) = (2\pi)^{D/2} f_{\mathbf{k}} g_{\mathbf{k}}, \tag{A.491}$$

which is sometimes called by the convolution theorem.

The convolution theorem (A.491) is valid for $f, g \in L^1(\mathbb{R}^D)$ which implies the absolute convergence $(f * g) \in L^1(\mathbb{R}^D)$ by Fubini's theorem [**Rud91**] as follows

$$\|f * g\|_{L^1(\mathbb{R}^D)} = \int\limits_{\mathbb{R}^D}\int\limits_{\mathbb{R}^D} |f(\mathbf{r}-\mathbf{r}')g(\mathbf{r}')|d\mathbf{r}d\mathbf{r}' = \int\limits_{\mathbb{R}^D}\int\limits_{\mathbb{R}^D} |g(\mathbf{r}')||f(\mathbf{r}-\mathbf{r}')|d\mathbf{r}d\mathbf{r}'$$
$$= \int\limits_{\mathbb{R}^D} |g(\mathbf{r}')|\|f\|_{L^1(\mathbb{R}^D)}d\mathbf{r}' = \|f\|_{L^1(\mathbb{R}^D)}\|g\|_{L^1(\mathbb{R}^D)}$$

and interchanging the order of integration we obtain (A.491) as follows

$$\mathcal{F}(f * g)(\mathbf{k}) = \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\left[\int\limits_{\mathbb{R}^D} f(\mathbf{r}-\mathbf{r}')g(\mathbf{r}')d\mathbf{r}'\right]$$
$$= \frac{1}{(2\pi)^{D/2}}\int\limits_{\mathbb{R}^D} f(\mathbf{r}-\mathbf{r}')e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}d\mathbf{r}\int\limits_{\mathbb{R}^D} g(\mathbf{r}')e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}d\mathbf{r}' = (2\pi)^{D/2}f_{\mathbf{k}}g_{\mathbf{k}}.$$

Also $S(\mathbb{R}^D)$ is closed under convolution.

Different definitions of FT. We note that generally instead of equations (A.479) and (A.482), FT and inverse FT pair can be defined using two arbitrary real constants a and b as follows

$$\begin{aligned} f_{\mathbf{k}} &= \frac{|b|^{D/2}}{(2\pi)^{D/2-a}}\int\limits_{\mathbb{R}^D} f(\mathbf{r})e^{-\mathrm{i}b\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}. \\ f(\mathbf{r}) &= \frac{|b|^{D/2}}{(2\pi)^{D/2+a}}\int\limits_{\mathbb{R}^D} f_{\mathbf{k}}e^{\mathrm{i}b\mathbf{k}\cdot\mathbf{r}}d\mathbf{r} \end{aligned} \tag{A.492}$$

which is immediately verified using equation (A.486). E.g. $a = D/2$, $b = 1$ is commonly used in quantum mechanics [**LL76**] an generally in theoretical physics for FT over the spatial variable $\mathbf{r}$. Throughout this book we use definitions equations (A.479) and (A.482) for FT over the spatial variable $\mathbf{r}$.

We define Fourier transform (FT) over time as follows

$$f_\omega := \int\limits_{-\infty}^{\infty} f(t)e^{i\omega t}dt. \tag{A.493}$$

Respectively, inverse FT is given by

$$f(t) := \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f_\omega e^{-i\omega t}d\omega \tag{A.494}$$

Note that we chose for FT (A.493) to have zeroth power of 2π (contrary to the definition of FT over $\mathbf{r}$ in equation (A.479)) to be consistent with the standard definition of the dielectric permittivity in theoretical physics as e.g. in Ref. [**LL84**]. In terms of the general definition (A.492) it corresponds to $D = 1$, $a = 1/2$ and $b = -1$. Also the sign in exponent is opposite for FT over time (A.493) compare with FT over $\mathbf{r}$ (A.479). These changes of the normalization constant and the sign of exponent are convenient for this book and do not change FT properties (except trivial changes in normalization constants) so FT (A.493) has the same properties as (A.479) for $D = 1$.

The definition (A.493) ensures that FT transforms a convolution $f * g$

$$(f * g)(t) := \int_{-\infty}^{\infty} f(t-t')g(t')dt' \tag{A.495}$$

of the function $f(t)$ and $g(t)$ into the product of FTs as follows

$$\mathcal{F}(f * g) = f_\omega g_\omega. \tag{A.496}$$

This expression is similar to equation (A.491) except the absence of the normalization constant in power of 2π.

FT of the derivative of function and FT of the envelope. Assume that $f(t)$ is the envelope of $g(t)$ as follows $g(t) = f(t)e^{-i\omega_0 t}$. Then FT of g results in FT of f with the shifted argument as follows

$$\mathcal{F}(f(t)e^{-i\omega_0 t}) = \int_{\mathbb{R}} f(t)e^{-i\omega_0 t}e^{i\omega t}dt = \int_{\mathbb{R}} f(t)e^{i(\omega-\omega_0)t}dt = f_{\omega-\omega_0}. \tag{A.497}$$

Looking for FT of the derivative of f together with the integration by parts and the condition $f(t) \to 0$ for$|t| \to \infty$ we obtain that

$$\mathcal{F}\left(e^{-i\omega_0 t}\frac{\partial}{\partial t}f(t)\right) = \int_{\mathbb{R}} e^{-i\omega_0 t}\frac{\partial}{\partial t}f(t)e^{i\omega t}dt = -i(\omega-\omega_0)\int_{\mathbb{R}} f(t)e^{i(\omega-\omega_0)t}dt \tag{A.498}$$

$$= -i(\omega-\omega_0)f_{\omega-\omega_0}. \tag{A.499}$$

In a similar way,

$$\mathcal{F}\left(e^{-i\omega_0 t}\frac{\partial^n}{\partial t^n}f(t)\right) = [-i(\omega-\omega_0)]^n f_{\omega-\omega_0},\ n = 0, 1, \ldots \tag{A.500}$$

which provides the explicit expression how to invert FT of the expression $[-i(\omega - \omega_0)]^n f_{\omega-\omega_0}$.

A.5. Gaussian integrals

Consider the integral $I_b = \int_{\mathbb{R}} e^{-ax^2+bx}dx,\ a > 0$. One can change the integration from the variable x to the the variable $z = a^{1/2}x - ba^{-1/2}2^{-1}$ aiming to bring the expression in the exponent to the full square as follows: $-ax^2 + bx = -(a^{1/2}x - ba^{-1/2}2^{-1})^2 + \frac{b^2}{4a} = -z^2 + \frac{b^2}{4a}$. If $b \in \mathbb{R}$ then

$$I_b = \int_{\mathbb{R}} e^{-ax^2+bx}dx = \frac{1}{a^{1/2}}\int_{\mathbb{R}} e^{-z^2+\frac{b^2}{4a}}dz = \left(\frac{\pi}{a}\right)^{1/2} e^{\frac{b^2}{4a}}, \tag{A.501}$$

where we used the standard integral $\int_{\mathbb{R}} e^{-z^2}dz = \sqrt{\pi}$. The expression in r.h.s of equation (A.501) is also valid for the arbitrary complex constant $b = b_r + ib_i \in \mathbb{C}$ except that z is now complex-valued and the integration is now performed over the horizontal line $-\infty < Re(z) < \infty$, $Im(z) = -iba^{-1/2}2^{-1}$ in the complex plane. The function $f_0(z) = e^{-z^2}$ is entire holomorphic function of $z \in \mathbb{C}$ which decays to zero,

$f_0(z) \to 0$ for $|z| \to \infty, |Arg(z)| < \pi/4$ and $|Arg(z) - \pi| < \pi/4$ which ensures that the integration contour can be moved from the real line to $Im(z) = -\mathrm{i}ba^{-1/2}2^{-1}$ without changing the value of integral. We conclude that

$$\int_{\mathbb{R}} e^{-ax^2+bx}dx = \left(\frac{\pi}{a}\right)^{1/2} e^{\frac{b^2}{4a}}, \ a > 0, \ b \in \mathbb{C}. \tag{A.502}$$

Multidimensional Gaussian integrals in $\mathbb{R}^n$ are given by

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum\limits_{i,j=1}^{n} A_{ij}x_ix_j + \mathbf{b}\cdot\mathbf{x}} d\mathbf{x} = \frac{(2\pi)^{n/2}}{(\det A)^{1/2}} e^{\frac{1}{2}\sum\limits_{i,j=1}^{n} A^{-1}_{ij}b_ib_j}, \quad \mathbf{b} \in \mathbb{R}^n, \tag{A.503}$$

where $A \in \mathbb{R}^{n\times n}$ is the positive-definite symmetric matrix, i.e. $\sum\limits_{i,j=1}^{n} A_{ij}x_ix_j > 0$ for any $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} \in \mathbb{R}^n$, $A^T = A$ and $\det A$ is the determinant of A. To derive equation (A.503) one can use the orthogonal transformation $\mathbf{y} = U^T\mathbf{x}$ to the new integration variable $\mathbf{y}$, where U is the orthogonal matrix $U^T = U^{-1}$ which diagonalizes the matrix A as follows $D = U^TAU$. Here $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix $D_{ij} = \lambda_j\delta_{ij}, \lambda_j > 0, \ i,j = 1,\dots,n$. Existence of U is ensured because A is the symmetric matrix. Then the quadratic form $\sum\limits_{i,j=1}^{n} A_{ij}x_ix_j = \sum\limits_{i=1}^{n} \lambda_i y_i^2$ and the integration over the new variable $\mathbf{y}$ is performed using equation (A.502) for each component $y_j, \ j = 1,\dots,n$.

Problem 1: generalize the integral (A.503) to the complex fields: $A \in \mathbb{C}^{n\times n}$ is the positive-definite Hermitian matrix, i.e. $A = \bar{A}^T$ and $\sum\limits_{i,j=1}^{n} A_{ij}\bar{x}_ix_j > 0$ for any $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} \in \mathbb{C}^n$. In other words, evaluate the integral

$$\int_{\mathbb{C}^n} e^{-\frac{1}{2}\sum\limits_{i,j=1}^{n} A_{ij}\bar{x}_ix_j + \mathbf{b}\cdot\mathbf{x}} d\mathbf{x}, \quad d\mathbf{x} = \prod_{k=1}^{n} dRe(x_k)dIm(x_k), \quad \mathbf{b} \in \mathbb{C}^n.$$

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