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SOLITON STABILITY IN PLASMAS AND HYDRODYNAMICS

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Contents:

Abstract:

The stability of solitons is reviewed for nonlinear conservative media. The main attention is paid to the description of the methods: perturbation theory, inverse scattering transform, Lyapunov method. Its applications are demonstrated in detail for the nonlinear Schrödinger equation, the KdV equation, and their generalizations. Applications to problems in plasma physics and hydrodynamics are considered.

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Introduction

The ultimate aim of this paper is in an attempt to understand the role of localized nonlinear objects – solitons or solitary waves – in plasma physics. Being discovered in the last century on the surface of liquids (see ref. [1]) solitons for a long time remained of interest only for a small number of specialists in hydrodynamics and mathematics who tried to prove their existence. In the late 50s of our century the soliton concept penetrated into plasma physics. Here, due to work by Sagdeev [2], Gardner and Morikawa [3] and others, solitons were successfully used to construct the theory of a fine structure of shock waves under the conditions of rare collisions. Nowadays a great number of soliton types in plasmas are known. They are widely used for various theoretical speculations, and especially for the construction of different versions of strong turbulence theory. In order that these speculations should be real it is necessary for the considered solitons to be stable. Therefore the problem of soliton stability is of particular importance.

Let us consider the problem of soliton classification. For a long time only one type of soliton was considered – plane solitary waves whose profile with a one-dimensional localization is stationary in some system of reference. Such are the solitons on the liquid surface and also the first types of solitons discovered in plasmas – ion-acoustic and magnetosonic solitons. We will call them simple one-dimensional solitons. Similarly solitons with a stationary form in a definite reference system but localized in two or three dimensions will be called simple two- and three-dimensional solitons. Such stable solitons propagating along the magnetic field in a magnetized plasma with a low pressure were discovered in ref. [4].

Besides simple solitons, oscillating solitons can propagate in a plasma. Inside them there occur oscillations characterized by a definite frequency and wavelength. Such a soliton profile is, in the mean, stationary in some reference system.

Like the simple ones, oscillating solitons which can also be one-, two- and three-dimensional do not represent a specific phenomenon for plasma physics. They naturally occur in problems of quasimonochromatic wavepacket propagation in nonlinear media with dispersion including self-focusing problems. "An inner" wavelength of such solitons is much less than their size and so they are sometimes called "envelope solitons". Along with the envelope solitons in plasma physics one considers specific oscillating solitons whose inner scale is comparable with their size or is entirely absent (spatially homogeneous oscillations take place inside the solitons). Many authors tried to use these solitons for solving the very actual plasma physics problem of constructing a strong Langmuir turbulence theory. As a whole, the problem of the description of all plasma solitons, both simple and oscillating, is far from being solved.

Going over to the solition stability problem one should note that stability problems are naturally divided into two groups. First, there is the problem of soliton stability with respect to perturbations with the same dimension as the original soliton. Such are the problems of three-dimensional soliton stability (here perturbations have to be essentially three-dimensional ones) and also the problems of two-dimensional soliton stability relative to two-dimensional perturbations and one-dimensional soliton stability with respect to one-dimensional perturbations. These problems are usually not solved in explicit form and one has to limit oneself to variational estimates and qualitative methods.

It is evident that such an analysis is insufficient for one- and two-dimensional solitons. It is necessary to study the soliton stability against perturbations which break the symmetry relative to both neck and snake types along the direction of the original soliton homogeneity. Experience shows that this problem can be successfully solved in the limit when the perturbation wavelength exceeds significantly the soliton size.

It is well-known that a comparatively small number of mathematical models possessing a great degree of universality plays a very important role in soliton theory. Such are, for example, the Korteweg–de-Vries equation (KdV) with its multi-dimensional generalization, the Kadomtsev–Petviashvili equation (KP) describing simple solitons, and the nonlinear Schrödinger equation (NSE) which is the simplest model for defining oscillating solitons. Universal models also find a wide application in plasma physics and the presentation of the soliton stability theory should therefore begin with these very models. The first chapter of this review is devoted to a description of the most important universal models and those problems of plasma physics to which they may be applied. In chapter 2 the stability problem of solitons, both simple and oscillating, is considered in terms of universal models with respect to perturbations which do not change the soliton dimension. The principal results of this chapter are obtained by means of a variational method. Further, problems of soliton stability relative to perturbations breaking their symmetry are discussed in terms of universal models. Here in most cases it is possible to obtain explicit expressions for growth rates using a long-wavelength approximation. It should be noted that the results obtained in chapters 1 and 2 are of importance beyond the framework of plasma physics itself. They are of importance for hydrodynamics (this is reflected in the title of the present paper) and also for nonlinear optics, in particular, for theory of light self-focusing. Contrary to this, chapter 3 is devoted entirely to a detailed consideration of the important problem of Langmuir soliton stability in a plasma without an external magnetic field or in the presence of a weak field. In that chapter we show that Langmuir solitons are always unstable though a one-dimensional soliton is stable with respect to purely one-dimensional perturbations. The latter circumstance explains frequent observations of Langmuir solitons in one-dimensional numerical simulation of plasmas which once gave rise to the concept of soliton Langmuir turbulence. A one-dimensional Langmuir soliton is however unstable against transverse perturbations. This makes the concept of the soliton Langmuir turbulence unreal, though it does not exclude a possibility of solition turbulence realization for other cases when solitons are stable. To construct a realistic picture of turbulence it is necessary to study the nonlinear stage of Langmuir soliton instability development.

It is generally accepted nowadays that as a result of the nonlinear stage of the instability a Langmuir collapse takes place – a spontaneous concentration of the Langmuir wave energy in a small spatial region (of the order of a few Debye radii) with subsequent dissipation due to Landau damping. From the mathematical point of view collapse represents the formation of a singularity in the input equations which happens after a finite time. As it is getting clear, collapse is one of the standard ways of soliton instability development. The investigation of wave collapses is one of the most important problems of plasma physics and, in general, of wave physics in nonlinear media, but it is outside the framework of the present paper. In the last chapter we give a brief presentation of the principal concepts of the collapse theory.

1. Variational principle. Soliton existence

1.1. Basic equations

By solitons one usually means some solution of nonlinear equations which is spatially localized and keeps its form. The latter is of particular importance. It means specifically that solitons can exist only in conservative media.

Among conservative systems we are interested in systems possessing a Hamiltonian structure. It should be noted that for conservative systems Hamiltonian structure, as a rule, can be introduced in spite of the fact that there are no general methods of its introduction. (This situation is discussed in detail in ref. [5].) It should be emphasized that the Hamiltonian structure existence allows all the stability methods developed in classical mechanics to be extended to systems with a continuum number of degrees of freedom.

The universal nonlinear models [6–10] well-known nowadays are such as the KdV equation and the nonlinear Schrödinger equation (NSE). Their universality is explained by the fact that they describe a wide spectrum of phenomena in various nonlinear media; their fundamentality consists in a latent symmetry in the one-dimensional case which results in the integrability of the given equations by the inverse scattering transform. The methods of soliton stability studies are also universal within the framework of these models. The KdV equation arises when describing weakly nonlinear waves in media with a dispersion law $\omega(k)$ which is close to the linear one

$$\omega(k) = kc_s(1 + \alpha k^2)\,, \qquad \alpha k^2 \ll 1\,. \tag{1.1}$$

According to this law in a linear approximation the waves propagating all in the same direction are described with the help of the equation

$$\partial u/\partial t = -c_s \partial u/\partial x + c_s\, \alpha\, \partial^3 u/\partial x^3\,. \tag{1.2}$$

The first term on the right-hand side of this equation describes drift with a sound velocity, the second one is responsible for a slower process – dispersive diffusion of the wavepacket. For weakly nonlinear waves the local sound velocity will differ from the mean value. In the linear approximation in the amplitude one may assume $c_s(u) = c_s(1 + \beta u)$. Taking into account that βu is small as compared to unity, we get as a result the well-known KdV equation

$$u_t + c_s u_x - c_s(\alpha u_{xxx} - \beta u_x u) = 0\,. \tag{1.3}$$

This scheme of the KdV equation derivation is very convenient for concrete calculations since the constants α and β can be determined independently [6]. This equation can also be derived by a formal method (see, e.g. [11]) with the help of the introduction of a small parameter ε, slow coordinates and time. We shall illustrate the above by an example of ion-acoustic waves in a collisionless plasma when $T_e \gg T_i$. In this case for the ions one can neglect the thermal pressure and use for their description the hydrodynamic equations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} nv = 0\,, \tag{1.4}$$

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = -\,\frac{e}{M}\,\frac{\partial \varphi}{\partial x} \tag{1.5}$$

(where M is the ion mass). Under these conditions the electrons can be considered distributed according to the Boltzmann law as

$$n_e = n_0 \exp(e\varphi/T_e)\,. \tag{1.6}$$

The system of equations (1.4)–(1.6) is closed by the Poisson equation for the electric potential φ

$$\partial^2\varphi/\partial x^2 = -4\pi e(n - n_0 \exp(e\varphi/T_e)) . \tag{1.7}$$

We would remind ourselves that the equations (1.4)–(1.7) in a linear approximation describe waves for $k \to 0$ with the dispersion law (1.1):

$$\omega_k = \sqrt{\frac{T_e}{M}}\, k(1 - \tfrac{1}{2}k^2 r_d^2) , \qquad r_d^2 = \frac{T_e}{4\pi n e^2} , \qquad c_s^2 = \frac{T_e}{M} .$$

Hence we can conclude that in the long-wave limit for one-dimensional weakly nonlinear waves one can obtain the KdV equation. Formally it can be obtained if one seeks a solution of the equations (1.4)–(1.7) in the form of series in a small parameter ε

$$\begin{aligned} n &= n_0 + \sum_{k=1}^{\infty} \varepsilon^{2k} n_k(x', t') \\ v &= \sum_{k=1}^{\infty} \varepsilon^{2k} v_k(x', t') \\ \varphi &= \sum_{k=1}^{\infty} \varepsilon^{2k} \varphi_k(x', t') \end{aligned} \tag{1.8}$$

where $x' = \varepsilon(x - c_s t)$, $t' = \varepsilon^3 t$ are the slow coordinates and time.

Substituting (1.8) into eqs. (1.4)–(1.7) and equating to zero coefficients at every power in the equations we get an infinite set of equations for n_k, v_k and φ_k.

In the first order of ε algebraic relations arise:

$$\begin{aligned} &\frac{\partial}{\partial x'}(c_s n_1 - n_0 v_1) = 0 \\ &\frac{\partial}{\partial x'}\left(c_s v_1 - \frac{e\varphi_1}{M}\right) = 0 \\ &\frac{\partial}{\partial x'}\left(-n_1 + n_0 \frac{e\varphi_1}{T_e}\right) = 0 . \end{aligned} \tag{1.9}$$

The solvability condition of this system gives $c_s^2 = T_e/M$.

The next order of ε has the form

$$\begin{aligned} &\frac{\partial n_1}{\partial t'} + \frac{\partial}{\partial x'} n_1 v_1 = \frac{\partial}{\partial x'}(c_s n_2 - n_0 v_2) \\ &\frac{\partial v_1}{\partial t} + v_1 \frac{\partial}{\partial x'} v_1 = \frac{\partial}{\partial x'}\left(c_s v_2 - \frac{e\varphi_2}{M}\right) \\ &\frac{\partial}{\partial x'}\left(\frac{1}{4\pi e}\frac{\partial^2}{\partial x'^2}\varphi_1 - \frac{n_0}{2}\left(\frac{e\varphi_1}{T_e}\right)^2\right) = \frac{\partial}{\partial x'}\left(-n_2 + n_0\frac{e\varphi_2}{T_e}\right) . \end{aligned}$$

The right-hand side of the equations coincides with the linear system (1.9). Hence the KdV equation is obtained as the orthogonality condition of the solution conjugate to (1.9)

$$\frac{\partial n_1}{\partial t'} + \frac{c_s r_d^2}{2} \frac{\partial^3 n_1}{\partial x'^3} + \frac{c_s}{n_0} n_1 \frac{\partial}{\partial x'} n_1 = 0 . \tag{1.10}$$

After simple transformations the KdV equation in the form of (1.3) or (1.10) can be reduced to a standard form

$$u_t + u_{xxx} + 6uu_x = 0 . \tag{1.11}$$

A stationary solitary wave or soliton of the KdV equation (1.11)

$$u = 2\kappa^2/\cosh^2\kappa\,(x - 4\kappa^2 t - x_0) \tag{1.12}$$

is the simplest solution of the KdV equation (1.11). It plays a fundamental role in the evolution problem for an arbitrary initial distribution [6–9].

Together with the KdV equation we shall also consider its generalization

$$u_t + u_{xxx} + f'(u)\, u_x = 0 \tag{1.13}$$

where the prime denotes differentiation with respect to u. As for the function $f(u)$, we assume that it tends to zero as $u \to 0$ like αu^q ($\alpha, q > 0$) and increases faster than u as $u \to \infty$. This behaviour guarantees the existence of soliton solutions of the type $u = u(x - Vt)$ determined by the integration of the equation

$$u_{xx} = -f(u) + Vu . \tag{1.14}$$

In multi-dimensional cases the well-known Kadomtsev–Petviashvili (KP) equation [12] is a natural generalization of the KdV equation. The KP equation can be obtained if a characteristic transverse scale of sound disturbances is assumed to exceed significantly the longitudinal size (in this case along x):

$$\frac{\partial}{\partial x} (u_t + 6uu_x + u_{xxx}) = -3\beta^2 \Delta_\perp u , \qquad \Delta_\perp \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} . \tag{1.15}$$

The sign of β^2 on the right-hand side of the equation is opposite to the sign of the dispersion α in (1.1). All changes in comparison with the KdV equation in (1.15) are connected only with an additional term on the right-hand side, which describes the acoustic wave diffraction in the transverse direction. This equation is apparently valid for ion-acoustic waves, and for long-wavelength gravitational-capillary waves on the surface of a liquid of a finite depth. It is also valid for fast magnetosonic waves in the magnetized plasma with $\beta \equiv 8\pi nT/H^2 \ll 1$ propagating at angles distant from 0 and $\pi/2$ to the magnetic field. As to angles close to 0 or $\pi/2$, it is known that the dispersion of fast magnetosonic waves undergoes in these regions changes and eq. (1.14) becomes invalid.

All the above-mentioned equations are of a Hamiltonian type and can be presented in the form

$$u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u}$$

where the Hamiltonian H for the KdV equation is expressed as

$$H = \int_{-\infty}^{\infty} \left[\frac{u_x^2}{2} - u^3\right] \mathrm{d}x \; ; \tag{1.16}$$

for eq. (1.13) as

$$H = \int_{-\infty}^{\infty} \left[\frac{u_x^2}{2} - \phi(u)\right] \mathrm{d}x \, , \qquad \phi'(u) = f(u) \; ; \tag{1.17}$$

and for the KP equation as

$$H = \int_{-\infty}^{\infty} \left[\frac{u_x^2}{2} - \frac{3\beta^2}{2}(\nabla_\perp w)^2 - u^3\right] \mathrm{d}V \, , \qquad w_x = u \, . \tag{1.18}$$

The other simplest integrals for these equations are $M = \int_{-\infty}^{\infty} u \, \mathrm{d}x$ and $P = \frac{1}{2}\int u^2 \, \mathrm{d}x$ having the meaning of a total "mass" and the momentum along the x-axis.

The nonlinear Schrödinger equation is usually used for the description of the propagation of wavepackets with a small amplitude, i.e., when the field differs weakly from a harmonic one and nonlinear effects are small. This gives an opportunity to take into account dispersion and nonlinear effects separately for the derivation of the equation. Let a wavepacket propagate in an isotropic medium with a dispersion law $\omega = \omega(k)$ and the field in it change as $\psi(r, t)\exp(-\mathrm{i}\,\omega(k_0)\,t + \mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}))$. Here $\psi(r, t)$ is a wavepacket envelope, a slowly varying function of r and t. In its Fourier spectrum there are only frequencies and wavevectors, much smaller than $\omega(k_0)$ and k_0. It means that the frequency width of the wavepacket will be small and, therefore, in the dispersion law $\omega = \omega(k)$ the right-hand side may be expanded in a series [10]

$$\omega(k) = \omega(k_0) + u\kappa_z + \tfrac{1}{2}(\omega''\kappa_z^2 + (u/k_0)\kappa_\perp^2)$$

where $u = \partial\omega/\partial k|_{k=k_0}$ is the group velocity, $\omega'' = \partial^2\omega/\partial k^2|_{k=k_0}$; $\boldsymbol{\kappa} = \boldsymbol{k} - \boldsymbol{k}_0$ is the wavevector of the envelope; κ_z, $\kappa_\perp$ are the components of the wavevector along and across the direction of the packet propagation. Accomplishing the inverse Fourier transform with respect to $\Omega = \omega - \omega(k_0)$ and $\boldsymbol{\kappa}$ for the envelope ψ, we obtain

$$\mathrm{i}(\psi_t + u\psi_z) + \frac{\omega''}{2}\psi_{zz} + \frac{u}{2k_0}\Delta_\perp\psi = 0 \, .$$

Now we have to include the nonlinearity. It is clear that for quasiharmonic oscillations the total effect will be reduced to a nonlinear shift of the frequency $\omega(k_0)$, representing a certain functional depending only on $|\psi|^2$. Combining the linear and nonlinear terms we come to the generalized nonlinear Schrödinger equation

$$\mathrm{i}(\psi_t + u\psi_z) + \tfrac{1}{2}\omega''\psi_{zz} + \frac{1}{2k_0}\, u\,\Delta_\perp\psi + \Delta\omega_{\mathrm{nl}}\psi = 0\,,$$
$$\Delta\omega_{\mathrm{nl}} = \hat{f}(|\psi|^2)\,. \tag{1.19}$$

This equation describes, in particular, propagation of electromagnetic waves in nonlinear isotropic dielectrics, for example, in an isotropic plasma. In this case ψ has the meaning of a complex amplitude of the electric field and $f(|\psi|^2) \sim$ a nonlinear addition to the refraction index. For a Kerr nonlinearity $f(|\psi|^2) \propto |\psi|^2$.

In the simplest case when f depends on the local value of $|\psi|^2$, this equation in dimensionless variables in the system moving with the group velocity can be expressed as follows:

$$\mathrm{i}\psi_t + \Delta_\perp\psi + \alpha\psi_{zz} + f(|\psi|^2)\,\psi = 0\,, \qquad \alpha = k\omega''/u\,. \tag{1.20}$$

Equation (1.20) also belongs to the Hamiltonian type

$$\mathrm{i}\psi_t = \delta H/\delta\psi^*$$

with Hamiltonian

$$H = \int \{\alpha|\psi_z|^2 + |\nabla_\perp\psi|^2 - \phi(|\psi|^2)\}\,\mathrm{d}V\,, \qquad \phi(u) = \int_0^u f(u)\,\mathrm{d}u\,. \tag{1.21}$$

Other simple integrals of motion of (1.20) except H are the adiabatic invariant $N = \int |\psi|^2\,\mathrm{d}V$ which has the meaning of the total number of waves, the momentum $\boldsymbol{P} = -\tfrac{1}{2}\mathrm{i}\int(\psi^*\nabla\psi - \mathrm{c.c.})\,\mathrm{d}V$ and the angular momentum.

Equation (1.20) has in the one-dimensional case (ψ depends on z only) a solution in the form of a soliton dependent on four parameters:

$$\psi = g(z - Vt - z_0)\exp\mathrm{i}(\lambda^2 t + \tfrac{1}{4}V^2 t - \tfrac{1}{2}Vz - \phi_0) \tag{1.22}$$

where the function g satisfies the equation

$$-\lambda^2 g + \alpha g_{zz} + f(g^2)\,g = 0 \tag{1.23}$$

with the boundary condition $g \to 0$ as $|z| \to \infty$.

Such soliton solutions exist only when $\alpha f > 0$. When $f(|\psi|^2) = |\psi|^2$, $\alpha = 1$

$$g = \sqrt{2}\lambda/\cosh\lambda z\,. \tag{1.24}$$

Stationary waveguide configurations localized in the transverse direction and realizing energy propagation without diffraction divergence (which will later be called waveguides) are formally analogous to solitons. Such solutions exist only for $f > 0$, $f' > 0$. Solutions in the form of cylindrical waveguides are of great interest from the physical point of view.

The nonlinear Schrödinger equation describes some other physical phenomena. For example, for oscillations of a weakly non-ideal Bose gas the value ψ has the meaning of the condensate wavefunction, and eq. (1.20) is actually a Schrödinger equation. As a rule, the nonlinear Schrödinger equation describes, at least in the one-dimensional case, the evolution of long-wave oscillations with a quadratic dispersion law and a gap in the spectrum

$$\omega(k) = \omega_0 + \alpha k^2 .$$

Its applicability for the description of Langmuir oscillations will be discussed later in detail.

Equation (1.20) assumes the medium to be inertia-free, i.e. the nonlinearity follows the wavefield. But for many problems it is necessary to take into account a finite time of medium relaxation. Thus for propagation of electromagnetic radiation in an isotropic plasma the nonlinear frequency shift is caused by density modulation under the action of a powerful wave,

$$\mathrm{i}(\psi_t + u\psi_z) + \frac{u}{2k_0}\Delta_\perp \psi + \frac{\omega''}{2}\psi_{zz} = \frac{\partial \omega}{\partial n_0} n\psi .$$

A density variation n caused by ponderomotive forces is described by the equation (see, for example, [10])

$$n_{tt} - c_s^2 \nabla^2 n = \nabla^2 |\psi|^2$$

where

$$c_s^2 = T_e/M .$$

When the characteristic times of the nonlinear processes are much longer than the period of the ion-acoustic oscillations, $n \propto |\psi|^2$ and the given system is reduced to the nonlinear Schrödinger equation with a cubic nonlinearity. In dimensionless variables these equations assume the form

$$\mathrm{i}(\psi_t + u\psi_z) + \Delta_\perp \psi + \alpha\psi_{zz} = n\psi \tag{1.25}$$

$$n_t + \nabla^2 \phi = 0$$

$$\phi_t + n = -|\psi|^2 . \tag{1.26}$$

The system (1.25), (1.26) is also Hamiltonian

$$\mathrm{i}\psi_t = \delta H/\delta\psi^* , \qquad n_t = \delta H/\delta\phi , \qquad \phi_t = -\delta H/\delta n$$

where

$$H = \int \{\alpha|\psi_z|^2 + |\nabla_\perp \psi|^2 + \tfrac{1}{2}n^2 + \tfrac{1}{2}(\nabla\phi)^2 + n|\psi|^2\}\,\mathrm{d}V + P_z u$$

$$P_z = \tfrac{1}{2}\mathrm{i}\int (\psi\psi_z^* - \text{c.c.})\,\mathrm{d}V . \tag{1.27}$$

When electromagnetic waves propagate in a plasma, modulation of the density is often caused by plasma heating. In this case instead of the second equation (1.26) we have [13, 14]

$$\frac{\partial \theta}{\partial t} - \kappa \nabla^2 \theta = \nu_{ei} \frac{\omega_p^2}{\omega^2} \frac{E^2}{8\pi} - \delta^2 \theta \,. \tag{1.28}$$

Here θ is the temperature perturbation, κ is the thermal conductivity coefficient, ν_{ei} is the frequency of the electron-ion collisions. The first term on the right-hand side of (1.28) describes the collision damping of an electromagnetic wave while the second one describes the energy transfer to ions, radiation losses, etc. Due to a large group velocity the longitudinal scales of parameter variations exceed greatly the transversal ones. So in (1.28) only the derivatives transverse to the direction of radiation propagation remain. The ratio of θ and n is found from the pressure constancy condition

$$n/n_0 = -\theta/T_0 \,. \tag{1.29}$$

The written equations are valid when the mean free path is less than the soliton size (see, e.g. [14]). The obtained equations are not Hamiltonian but their investigation is carried out by similar methods.

1.2. Multi-dimensional solitons and their stability in models of the KdV type

In the previous section the simplest examples of solitons have been considered, whose forms are determined analytically. When the nonlinearity is of a more complicated character, the problem of soliton existence in one-dimensional cases is far from being trivial. Variational methods are most effective here. They give an opportunity to reach a number of conclusions about their stability.

Let us discuss these methods in detail by an example of the KdV equations (1.11) and (1.13). For simplicity the function $\phi(u)$ in (1.17) is considered as a power one, $\phi = au^n$. It follows from the formulation of the equations of motion in the Hamiltonian form that stationary solutions of the type $u = u_0(x - Vt)$ vanishing at infinity can be found from the following variational problem

$$\delta(H + PV) = 0 \,. \tag{1.30}$$

This expresses the fact that such solutions are stationary points of the Hamiltonian H for fixed P. In this case the velocity plays the role of a Lagrangian multiplier. Due to the boundedness of the solution as $|x| \to \infty$ the velocity V is positive.

For soliton solutions the relation between the Hamiltonian and momentum P can be found directly from the variational principle (1.29). First, let us consider a one-dimensional case ($a = 1$).

Multiplying the equation

$$\frac{\delta}{\delta u}(H + PV) = 0$$

by u and integrating it over x, we obtain

$$2VP + I_1 - nI_n = 0 \tag{1.31}$$

where $I_1 = \int u_x^2 \, dx$, $I_n = \int u^n \, dx$.

The other relation between the integrals can be found with the help of the virial theorem. Let us consider test functions of the form $u_0(\alpha x)$. Then, by virtue of (1.30) the following identity* must be valid

$$\frac{\partial}{\partial \alpha}(H+PV)|_{\alpha=1}=0\,.$$

As a result, we have

$$-VP+\tfrac{1}{2}I_1+I_n=0\,. \tag{1.32}$$

Combining (1.30) and (1.31) we obtain

$$I_1=2\,\frac{n-2}{n+2}\,VP\,,\qquad I_n=\frac{4}{n+2}\,VP\,,\qquad H=\frac{n-6}{n+2}\,VP\,. \tag{1.33}$$

Thus for small powers of nonlinearity, $n<6$, H is negative; for $n=6$ it turns to zero; for all other values of $n>6$, H is positive.

For the KP equation (1.14) the situation is more complicated. First, soliton solutions exist only for positive dispersion ($\beta^2=-1$). Their explicit form in the two-dimensional case ($d=2$) can be found by the inverse scattering transform method [17]:

$$u_0(x-Vt,\,y)=4v'\,\frac{1+v'^2y^2-v'^2(x-3v't)^2}{[1+v'^2y^2+v'(x-3v't)^2]^3}\,,\qquad V>0\,,\qquad v'=\frac{V}{3}\,. \tag{1.34}$$

As for three-dimensional solitons, they were found by Petviashvili by means of numerical simulation [18].

Second, to find the relation between the integrals H and P, the following two relations analogous to (1.31), (1.32)

$$\begin{aligned}&2PV+I_1+2I_2-3I_3=0\,,\\&-VP+\tfrac{1}{2}I_1+I_2+I_3=0\,,\end{aligned} \tag{1.35}$$

$$I_1=\int u_x^2\,\mathrm{d}V\,,\qquad I_3=\int u^3\,\mathrm{d}V\,,\qquad I_2=\int(\nabla_\perp w)^2\,\mathrm{d}V$$

are insufficient. To obtain the third one, it is necessary to analyse test functions of the form $u_0(x,\beta r_\perp)$. By analogy with (1.32) we have

$$(d-1)(VP+\tfrac{1}{2}I_1-I_3)+\tfrac{3}{2}(d-3)I_2=0\,. \tag{1.36}$$

From (1.36), (1.35) we get

$$H=PV(2d-5)/(7-2d)\,.$$

* The first time such an approach was used was apparently by Derrick [15] for the Klein–Gordon model. It should be noted that for a variation of α, $\delta u=(r\nabla)u_0\,\delta\alpha$. So, (1.32) can be obtained by multiplying the previous equation by $(r\nabla)u_0$ and integrating over x, ref. [16].

It is seen that for $d = 2$, $H = -PV/3$ and is negative,* in the three-dimensional case H is positive.

Particular attention should be paid to the nondiscrepancy of the ratios (1.32), (1.33), and (1.35), (1.36) which is to be considered as a necessary condition of the existence of soliton solutions. It is clear that this requirement does not replace a total proof of the solution existence.

Now let us turn to the stability of the soliton solutions. For a solution of this problem we use the Lyapunov theorem (see e.g. [110]) according to which in a dynamic system there exists at least one stable solution when some integral, for example, the Hamiltonian will be bounded from above or below. The sense of the theorem is very clear. The boundedness of the integral results in its absolute minimum or maximum existence. Let us consider the system in the state corresponding to the absolute minimum of this integral. Every variation of the solution must increase its value in contradiction to the integral conservation. Hence, the solution must be stable. Therefore for the soliton solution stability it is sufficient to prove the boundedness of H for fixed P (in this case from below, since the Hamiltonian is not bounded from above, H can be made arbitrarily large due to the presence of the integral I_1 for a given integral P).

First let us consider the scaling transforms

$$u(x, r_\perp) = \alpha^{-1/2}\beta^{(1-d)/2}u_0(x/\alpha, r_\perp/\beta)$$

which conserve P. For such transforms H becomes a function of the parameters α and β. In the one-dimensional case ($d = 1$) H depends only on α; in two- and three-dimensional geometrics for the KP equation it depends on two parameters:

$$H = \frac{1}{2\alpha^2} I_1 + \frac{3\alpha^2}{2\beta^2} I_2 - \alpha^{-1/2}\beta^{(1-d)/2} I_3 .$$

For $d = 1$ and a power-law nonlinearity $\phi = u^n$ in (1.13), H as a function of α has a minimum for $n < 6$ only. When $n = 6$ this function has no extremum. When $n > 6$, H is unbounded from below and a maximum appears instead of a minimum.

An analogous situation takes place for the KP equation. When $d = 2$ the Hamiltonian is bounded from below but in a three-dimensional case the opposite situation takes place. Instead of a minimum a saddle point is available, and the Hamiltonian as a function of two parameters is unbounded from below. In order to make sure of it, it is sufficient to consider the lines

$$\alpha^2 = c\beta .$$

It should be noted that the unboundedness of the function $H(\alpha, \beta)$, strictly speaking, does not mean the Hamiltonian boundedness. In principle, its boundedness is possible because of other integrals. However, in this case this remark is insignificant, since for $n > 6$ the equations (1.13) and the KP equation for $d = 3$ have no nontrivial integrals [111, 112].† It means that in these equations there are no absolutely stable solutions for the above values of the parameters. As for the other values of n and $d = 2$ in the KP equation, the scaling transformations indicate only the boundedness of H. Below we adduce a rigorous proof of this fact [19, 20]. For this purpose we will estimate the integral I_n through

* Certainly this result can also be obtained by a direct substitution of the solution (1.34) into the Hamiltonian and the momentum.

† The angular momentum in the KP equation cannot lead to the boundedness of H, since it turns to zero for cylindrical-symmetric distributions.

the integrals I_1, I_2 and P. For $d=1$ we have

$$\int u^n\,\mathrm{d}x \le (\max u^2)^{(n-2)/2} \int u^2\,\mathrm{d}x\,. \tag{1.37}$$

Using later the obvious inequality

$$\max u^2 \le 2 \int |u|\,|u_x|\,\mathrm{d}x \tag{1.38}$$

and Hölder's inequality, we obtain

$$\int u^n\,\mathrm{d}x \le C_n I_1^{(n-2)/2} \tilde{P}^{(n+2)/4} \tag{1.39}$$

where $C_n = 2^{(n-2)/2}$, $\tilde{P} = 2P = \int u^2\,\mathrm{d}x$. This estimation can be improved. For the multiplier C_n (1.39) one can find an exact lower boundary [21]

$$C_n = \operatorname{Inf} F[u] = \operatorname{Inf} I_1^{(n-2)/4} \tilde{P}^{(n+2)/4} / I_n\,. \tag{1.40}$$

To obtain Inf $F[u]$ it is necessary to choose among all stationary points $F[u]$ a minimizing one. It is easy to see that the Lagrange–Euler equation for F

$$u_{xx} = \frac{n+2}{n-2}\,\frac{I_1}{\tilde{P}}\,u - n\,\frac{I_1}{(n-2)\,I_n}\,u^{n-1} = 0$$

after a simple transformation $u = \tilde{u}(I_1/(n-2)I_n)^{1/(n-2)}$ coincides with eq. (1.14) or eq. (1.30) for the stationary solutions of the KdV equation with a power-law nonlinearity

$$\tilde{u}_{xx} = \frac{n+2}{n-2}\,\frac{I_2}{\tilde{P}}\,\tilde{u} - \tilde{u}^{n-1}\,. \tag{1.41}$$

It has a unique localized solution in the form of a soliton. Thus all integral relations (1.31), (1.32), (1.33) in which the value $(n+2)I_1/(n-2)\tilde{P}$ should be set instead of V, are valid for this solution. Hence, taking into account (1.39), (1.40) and (1.41) for H with $n<6$ we have the following estimation

$$H \ge \min[\tfrac{1}{2}I_1 - C_n I_1^{(n-2)/4} \tilde{P}^{(n+2)/4}] \ge \frac{n-6}{n+2}\,\frac{I_{1\mathrm{s}}}{2} = H_\mathrm{s}$$

where $I_{1\mathrm{s}}$ is the integral value I_1 for the soliton solution (1.41) or (1.14). Thus for $n<6$ the Hamiltonian H is bounded from below, while an absolute minimum of H (for fixed P) is attained for the soliton.

In the two-dimensional case the proof of the boundedness of H is based on the following inequalities. First, with the help of Hölder's inequality we have

$$\int u^3\,\mathrm{d}x\,\mathrm{d}y \le \left(\int u^2\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \left(\int u^4\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

Then we estimate $\int u^4\,\mathrm{d}x\,\mathrm{d}y$:

$$\int u^4\,\mathrm{d}x\,\mathrm{d}y \le \int_{-\infty}^{\infty} \max_x u^2\,\mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{y} u\,u_{y'}\,\mathrm{d}y'$$

(in the latter integral we change the integration over x and y' and integrate by parts):

$$2\int \mathrm{d}y \max_x u^2 \int \mathrm{d}y' \int u u_{y'}\,\mathrm{d}x = -2\int \mathrm{d}y \max_x u^2 \int \mathrm{d}y' \int u_x w_{y'}\,\mathrm{d}x\,.$$

Then, using the inequality (1.38) we obtain

$$I_3 \le 2\tilde{P}^{3/4} I_1^{1/2} I_2^{1/4}\,.$$

Substitution into the Hamiltonian gives the boundedness of H from below

$$H \ge \tfrac{1}{2}I_1 + \tfrac{3}{2}I_2 - 2\tilde{P}^{3/4} I_1^{1/2} I_2^{1/4} \ge -\tfrac{2}{3}\tilde{P}^3\,. \tag{1.42}$$

The inequality (1.42) and the relations (1.31), (1.36), (1.37) show that the nontrivial stationary soliton solution corresponds to the lower boundary of H. It is evident here that for fixed P, H has a unique minimum. It is this fact that proves the stability of solitons for the KdV equation (1.4) for $n<6$ [19] and for the two-dimensional KP equation [20].

1.3. Variational estimates for equations of the NSE type

Now let us consider the soliton stability in the nonlinear Schrödinger equation:

$$\mathrm{i}\psi_t + \nabla^2\psi + |\psi|^2\psi = 0\,. \tag{1.43}$$

Here the dispersion is regarded to be positive, $\alpha > 0$ in (1.20), while the nonlinearity is assumed to be cubic. Now let us seek a stationary solution of (1.43) in the form $\psi = \exp(\mathrm{i}\lambda^2 t)\,g(r)$. Such a type of solution with the multiplier oscillating in time is natural, because the interaction leads to a nonlinear frequency shift λ^2 general for all harmonics. The function g is determined from the equation

$$-\lambda^2 g + \nabla^2 g + g^3 = 0 \tag{1.44}$$

and represents the stationary point of H for a fixed number of waves N

$$\delta(H + \lambda^2 N) = 0 \tag{1.45}$$

where $H = \int \{|\nabla\psi|^2 - \tfrac{1}{2}|\psi|^4\}\,\mathrm{d}V$.

The ratio of H and N for the solution (1.44) can be found by analogy with (1.32):

$$H = \lambda^2 \frac{d-2}{4-d} N \tag{1.46}$$

from which it follows that only for $d = 1$ the Hamiltonian is negative. Scaling transformations show that only in a one-dimensional case the Hamiltonian boundedness is possible. The proof of this fact follows from the inequality (1.27), (1.39) generalized into complex-valued functions

$$\int |\psi|^4 \, dx \le C_4 I_1^{1/2} N^{3/2} \tag{1.47}$$

where $I_1 = \int |\psi_x|^2 \, dx$ and C_4 is defined from (1.40). Hence it follows that for $d = 1$ the Hamiltonian H for the NSE for fixed N is bounded from below, its lower boundary is attained for the soliton (1.24). Therefore in accordance with the Lyapunov theory the soliton in the one-dimensional NSE is absolutely stable.

For power-law nonlinearities $\phi = |\psi|^{2n}/n$ and an arbitrary dimensionality d the generalized inequalities (1.47) have the form [22, 23]:

$$\int |\psi|^{2n} \, d\mathbf{r} \le C_{n_\alpha} \left(\int |\nabla\psi|^2 \, d\mathbf{r} \right)^{d(n-1)/2} \left(\int |\psi|^2 \, d\mathbf{r} \right)^{(2n-nd+d)/2} \tag{1.48}$$

where a minimal value of the coefficient C_{n_α} is determined from solution of the variational problem coinciding with (1.45)

$$|\psi|^{2n-2}\psi + \nabla^2\psi - \frac{2n - nd + d}{d(n-1)} \frac{I_1}{N} \psi = 0 \, . \tag{1.49}$$

The localized solution (1.49) exists only in the case of positivity of $\lambda^2 = (2n - nd + d) I_1 / d(n-1)N$. In the three-dimensional case this requirement is violated when $n \ge 3$ (compare with [24]).

From (1.45) it is clear that a minimum for the corresponding functional H will be attained in the central-symmetrical distribution with a phase independent of r: $\psi = g\,e^{i\varphi}$, $\varphi = \varphi(t)$.

For these distributions the equation (1.49) can be rewritten in the form

$$\frac{d^2 g}{dr^2} + \frac{d-1}{r}\frac{dg}{dr} + g^{2n-1} - \lambda^2 g = 0 \, . \tag{1.50}$$

The soliton solution should satisfy the boundary condition

$$dg/dr|_{r=0} = g|_{r=\infty} = 0 \, .$$

Equation (1.50) is analogous to Newton's equation for a particle moving in the potential $U = g^{2n}/2n - \lambda^2 g^2/2$ (see fig. 1). Here g is the coordinate, r the time, the term $((d-1)/r)\, dg/dr$ plays the role of a friction force. The boundary conditions mean that the "particle" trajectory begins from the reflection point ($r = 0$) with a "velocity" equal to zero and ends at the origin of the coordinates. It is also obvious that in the one-dimensional case there exists only one soliton solution discussed above.

In two- and three-dimensional cases there exists a denumerable set of soliton solutions [25] where the m-soliton has m modes which corresponds to the number of particle oscillations in the potential. Its amplitude increases monotonically with the number of m. The soliton without nodes is called a ground-state soliton.

The minimum is reached for the ground-state soliton. Substitution of the inequality (1.48) in H

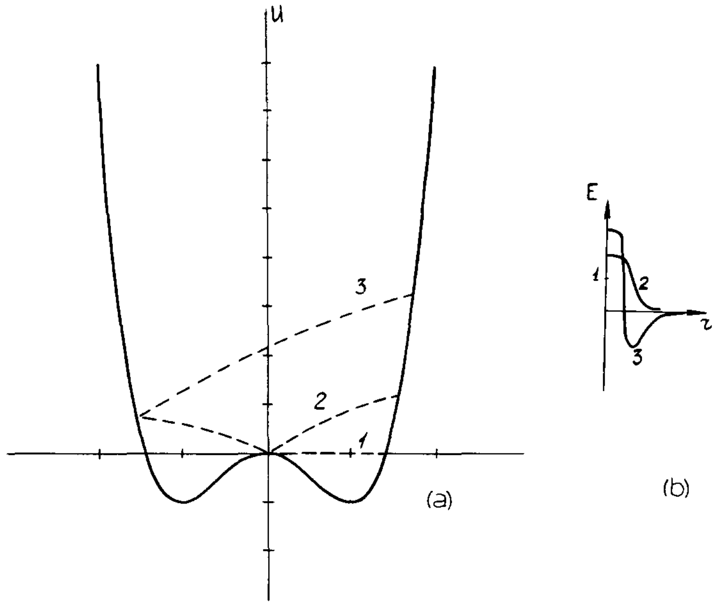

Fig. 1. (a) The "effective" potential well for eq. (1.50), ($n = 2$). The curves (1), (2), (3) correspond to one-dimensional, ground-state and first excited cylindrical solitons respectively. (b) Soliton solutions correspond to the curves (2), (3).

shows that boundedness of H takes place when

$$n < 2/d + 1 . \tag{1.51}$$

In this case the minimum of H is attained for the ground-state soliton which is stable for this reason. When $n > 2/d + 1$, H occurs to be unbounded from below.

When $n = 2/d + 1$ an additional symmetry arises in the nonlinear Schrödinger equation. In particular it is manifested in that for a scaling transformation conserving N, the dispersive and nonlinear terms in H are transformed in a similar way. For the soliton these terms are equal so that $H = 0$ (cf. (1.46)). A more general consequence of this symmetry is the existence at $n = 2/d + 1$ of a transformation translating the solution of ψ of NSE to another solution $\tilde{\psi}$:

$$\tilde{\psi}(r, t) = \left(\frac{\tau}{\tau - t}\right)^{d/2} \psi(r', t') \exp\left\{i \frac{r^2}{4(t - \tau)}\right\} ,$$
$$r' = r\tau/(\tau - t) , \qquad t' = t\tau/(\tau - t) . \tag{1.52}$$

The transform (1.52) includes the inversion transformation with respect to time and a scaling transformation with respect to the space coordinates. In the two-dimensional case when the NSE describes a stationary self-focusing in media with a Kerr nonlinearity, the transform (1.52) was found by Talanov [26]. It is easy to verify that a superposition of two transforms (1.52) with parameters $\mu_1 = \tau_1^{-1}$ and $\mu_2 = \tau_2^{-1}$ are transforms of the same type with $\mu = \mu_1 + \mu_2$, i.e. the relation (1.52) defines a one-parameter Abelian group relative to which NSE is invariant for $n = 2/d + 1$. Moreover, the given symmetry belongs to the Noether type; it leads to the invariance of the action,

$$I=\int \mathscr{L}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,, \qquad \mathscr{L}=\mathrm{i}\psi^*\psi_t-|\nabla\psi|^2+|\psi|^{2n}$$

and hence, according to the Noether theorem, generates an additional integral I. This integral can be found in the usual way (see [27]). It occurs also as a result of the transform of the Hamiltonian [28]:

$$\tilde{H}=\frac{1}{\tau^2}\,I(\psi,t-\tau)=\frac{1}{\tau^2}\left[H(t-\tau)^2+\tfrac{1}{4}\int r^2|\psi|^2\,\mathrm{d}V+\tfrac{1}{4}(t-\tau)\,\frac{\partial}{\partial t}\int r^2|\psi|^2\,\mathrm{d}V\right].$$

Differentiation of $\tilde{H}$ by t leads to the remarkable result obtained in [29]

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int r^2|\psi|^2\,\mathrm{d}V=8H\,. \tag{1.53}$$

This equality is sometimes called the virial theorem since it is a direct analogue of a well-known differential relation of classical mechanics after averaging of which the virial theorem is obtained [109].

The virial theorem (1.53) also takes place for arbitrary values of n and d (see e.g. [30])

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int r^2|\psi|^2\,\mathrm{d}V=8H-\frac{d}{2n}\left(n-\frac{2}{d}-1\right)\int|\psi|^{2n}\,\mathrm{d}\boldsymbol{r} \tag{1.54}$$

which can be established by a direct check.

The transformation (1.52) gives an opportunity to investigate the soliton solution $\psi_0(r,t)=g_0(r)\exp(\mathrm{i}\lambda^2 t)$ for stability (in problems of light self-focussing a stationary waveguide propagation corresponds to such a solution). If a solution is known, then its transformation with the help of the procedure (1.52) defines a one-parameter family of solutions. It is easy to see that the derivative of ψ_0 with respect to the parameter $\mu=1/\tau$ satisfies the equation linearized on the background of ψ_0. A perturbation

$$\delta\psi=\left.\frac{\partial\psi_0}{\partial\mu}\right|_{\mu=0}\delta\mu=\left(t\,\frac{d}{2}\,g-\frac{\mathrm{i}r^2}{4}\,g+\mathrm{i}\lambda^2t^2g+t(\boldsymbol{r}\nabla)g\right)\exp(\mathrm{i}\lambda^2t)$$

caused by the soliton solution $\psi_0(r,t)$ describes an instability of the power-law type [28]. The nonlinear stage of this instability development describes the formation of a singularity after finite time, collapse (again, one can see this with the help of formula (1.52)).

It should be added that the soliton solution instability for $d=2$ has been observed in numerical experiments [31, 32].

Therefore it follows from the above that multi-dimensional stable solitons can exist if the nonlinearity grows rather slowly with amplitude (see criterion (1.51)). In two- and three-dimensional cases such a situation arises for propagation in one direction with close group velocities of three resonance-interacting waves. In this case for a description of three wavepackets in equations of motion in comparison with the known Bloembergen system [33] it is necessary to take into account both dispersion and diffraction terms [34]:

$$\mathrm{i}(A_{1t}+v_1A_{1z})+\frac{v_1}{2k_1}\Delta_\perp A_1+\frac{\omega_1''}{2}A_{1zz}=-A_2A_3\,,$$
$$\mathrm{i}(A_{2t}+v_2A_{2z})+\frac{v_2}{2k_2}\Delta_\perp A_2+\frac{\omega_2''}{2}A_{2zz}=-A_1A_3^*\,, \qquad (1.55)$$
$$\mathrm{i}(A_{3t}+v_3A_{3z})+\frac{v_3}{2k_3}\Delta_\perp A_3+\frac{\omega_3''}{2}A_{3zz}=-A_1A_2^*\,.$$

This system is also a Hamiltonian one

$$\mathrm{i}\partial A_j/\partial t=\delta H/\delta A_j^*$$

with the Hamiltonian

$$H=\sum_j\int\left\{\mathrm{i}v_jA_j\,\frac{\partial A_j^*}{\partial z}+\frac{v_j}{2k_j}|\nabla_\perp A_j|^2+\frac{\omega_j''}{2}|A_{jz}|^2-(A_1^*A_2A_3+\mathrm{c.c.})\right\}\mathrm{d}\boldsymbol{r}\,.$$

Besides H (1.55) conserves the Manley–Row integrals

$$I_1=\int(|A_1|^2+|A_2|^2)\,\mathrm{d}\boldsymbol{r}\,,\qquad I_2=\int(|A_1|^2+|A_3|^2)\,\mathrm{d}\boldsymbol{r}\,.$$

It is not difficult to show that soliton solutions represent stationary points of H for fixed Manley–Row integrals. In the simplest case when all group velocities coincide and $\omega_j''>0$, estimates obtained with the help of the scaling transforms show that H is bounded from below in both two- and three-dimensional cases which is in agreement with the estimate (1.51).

Using inequalities similar to the above-mentioned ones, it is possible to state that the Hamiltonian is bounded from below [34]:

$$H\geq\tfrac{1}{24}m^3(I_1+I_2)^3$$

where

$$m=\min(v_j/2k_j,\,\omega_j'')\,.$$

Hence follows the existence of the stable localized stationary solutions (1.55). Here, according to [34], two-dimensional solitons prove to be unstable with respect to transverse perturbations while the three-dimensional solitons are absolutely stable. It should be also noted that apart from solitons realizing the minimum of H, there exist various stationary solutions corresponding to local extrema of H. Being unstable relative to finite perturbations, these solitons are of little physical interest.

A situation analogous to that in the one-dimensional Schrödinger equation takes place under the interaction of a quasimonochromatic HF wave with an acoustic one described by means of equations (1.25), (1.26). It is easy to see that the simplest stationary solution in the form of a soliton at rest

$$\psi=\exp(\mathrm{i}\lambda^2t)g(r)\,,\qquad n=-|\psi|^2\,,\qquad v=\nabla\phi=0 \qquad (1.56)$$

presents a stationary point H for fixed N

$$\delta(H + \lambda^2 N) = 0\,.$$

The boundedness of H (1.27), possible only in a one-dimensional case, follows from the estimation [35]

$$H_s = \int \left(|\nabla\psi|^2 + \frac{n^2}{2} + \frac{v^2}{2} + n|\psi|^2 \right) dV$$

$$V = \int \left\{ |\nabla\psi|^2 + \tfrac{1}{2}(n + |\psi|^2)^2 + \frac{v^2}{2} - \frac{|\psi|^4}{2} \right\} dV \ge \int \{ |\nabla\psi|^2 - \tfrac{1}{2}|\psi|^4 \} \, dV = H_{\mathrm{NSE}} \tag{1.57}$$

which becomes exact for $n = -|\psi|^2$ and $v = 0$ (cf. eq. (1.56)).

The inequality (1.57) shows that the Hamiltonian H (1.27) is always majorized by the Hamiltonian H_{NSE} for the NSE. As was shown above the boundedness of H_{NSE} of the form (1.57) is possible in a one-dimensional case only while the minimum H for fixed N is attained for the soliton solution. The latter proves the stability of the one-dimensional solution (1.56) with respect to one-dimensional perturbations [35, 36]. The stability of one-dimensional solitons moving with velocity v is set almost in the same way. For this purpose the Lyapunov functional is constructed in the following form (cf. [37])

$$\tilde{H} = H + \tfrac{1}{4}\beta^2 N - \beta P\,, \qquad P = \int [nv - \mathrm{i}\psi^* \psi_x] \, dx\,.$$

After the transformation $\psi(x, t) = \tilde{\psi}(x, t) \exp \mathrm{i}(\beta x/2 - \beta^2 t/4)$ at $\beta < 1$ for $\tilde{H}$ we have an estimation analogous to (1.57)

$$\tilde{H} = \int \left(|\tilde{\psi}_x|^2 + \frac{n^2}{2} + \frac{v^2}{2} + \beta nv + n|\psi|^2 \right) dx$$

$$= \int \left(|\tilde{\psi}_x|^2 + \tfrac{1}{2}(v + \beta n)^2 + \frac{1 - \beta^2}{2} \left(n + \frac{|\omega|^2}{1 - \beta^2} \right)^2 - \frac{|\psi|^4}{2(1 - \beta^2)} \right) dx \ge \int \left(|\tilde{\psi}_x|^2 - \frac{|\tilde{\psi}|^4}{2(1 - \beta^2)} \right) dx$$

from which the desired proof follows.

Now we will discuss the problem of stationary waveguide solutions arising due to thermal nonlinearities. In dimensionless variables the stationary equations (1.25), (1.28) and (1.29) (independent of t) may be presented in the form

$$\mathrm{i}\psi_z + \Delta_\perp \psi + \theta\psi = 0\,, \qquad -\Delta_\perp \theta + \eta^2 \theta = |\psi|^2\,. \tag{1.58}$$

A thermal self-focusing mechanism is quite clear.

For the usual striction self-focusing pushing the plasma out from the waveguide and hence increasing the refractive index is due to ponderomotive forces. In this case the plasma displacement is caused by a temperature increase and consequently by a pressure increase due to energy absorption in the region of radiation localization. As a rule, the term describing plasma cooling is small, $\eta < 1$, but, as is easily seen, after integration of the second equation in the system (1.58) over the volume, without this term it is impossible to obtain a stationary solution.

It may be concluded from the first equation (1.58) that for stationary waveguides the transverse size of the waveguide decreases with increasing amplitude. In its turn, it follows from the second equation that the effective nonlinearity in (1.58) grows more slowly than in a medium with a cubic nonlinearity. Hence according to the criterion (1.51) two-dimensional waveguides of the form $\psi = f(r_\perp)\exp(\mathrm{i}\lambda z)$ (having the meaning of two-dimensional solitons for the system (1.58)) should be stable with respect to stationary perturbations, i.e. in the frame of (1.58).

Let us show it in a more rigorous form. The system (1.58) can be written in Hamiltonian form

$$\mathrm{i}\psi_z = \delta H/\delta\psi^*$$

with an additional condition $\delta H/\delta\theta = 0$. The Hamiltonian H is of the form

$$H = \int \{|\nabla_\perp\psi|^2 + \tfrac{1}{2}(\nabla_\perp\theta)^2 + \tfrac{1}{2}\eta^2\theta^2 - \theta|\psi|^2\}\, \mathrm{d}\boldsymbol{r}_\perp .$$

Soliton solutions (1.58)

$$\psi = \exp(\mathrm{i}\lambda^2 z) f_0(r_\perp) , \qquad \theta = \theta_0(r_\perp) \tag{1.59}$$

as before, represent the stationary point of H for fixed $N = \int |\psi|^2\, \mathrm{d}\boldsymbol{r}$

$$\delta(H + \lambda^2 N) = 0 .$$

By analogy with (1.46) H appeared to be negative for the soliton solution:

$$H_s = -\tfrac{1}{2} \int (\nabla_\perp\theta)^2\, \mathrm{d}\boldsymbol{r}_\perp .$$

Then with the help of integral estimates one can show the boundedness of H from below [14]:

$$H \geq -N^3((1-\eta^2)/\eta^2)^3 .$$

This inequality indicates the existence of a stable in the framework of the system (1.58) stationary two-dimensional waveguide. It is rather obvious that the minimum of H will be attained for the ground-state soliton being cylindrically symmetric and having no nodes.

1.4. Stability of two-dimensional vortices

In concluding this chapter we indicate one important application of the methods considered above. We will study the problem of vortex stability in hydrodynamics. This question is of special importance since exact stationary solutions in the form of two-dimensional vortices have recently been found in geophysics and plasma physics. Up to the present time their stability is far from being completely studied. There is a wide bibliography devoted to this question; we indicate only some of these papers [38–43].

Descriptions of plane vortices in various physical situations have much in common. Below we restrict ourselves only to a consideration of the flow of an ideal incompressible fluid following mainly Arnold [44–46]. These results can also be extended to other problems.

Let us consider the two-dimensional Euler equation for an ideal flow in some region D^2:

$$\frac{\partial \Omega}{\partial t} = \frac{D(\Omega, \psi)}{D(x, y)} = \frac{\partial \Omega}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \Omega}{\partial y}\frac{\partial \psi}{\partial x} . \tag{1.60}$$

Here ψ is the stream function in terms of which the velocity and its curl can be expressed by means of the formulae

$$v_x = -\partial\psi/\partial y , \qquad v_y = \partial\psi/\partial x , \qquad \Omega = (\mathrm{curl}\, v)_z = \nabla^2\psi .$$

Similar to the universal nonlinear equations considered above, equation (1.60) is also a Hamiltonian one [47]. It can be written in the form (see [4, 48])

$$\partial\Omega/\partial t = \{\Omega, H\} ,$$

where the Hamiltonian H coincides with the fluid energy $H = \frac{1}{2}\int (\nabla\psi)^2\, \mathrm{d}x\, \mathrm{d}y$ and the Poisson brackets $\{F, G\}$ between two functionals represent brackets of the Kostant–Lie–Kirillov type

$$\{F, G\} = \int \Omega\, \frac{D(\delta F/\delta\Omega, \delta G/\delta\Omega)}{D(x, y)}\, \mathrm{d}x\, \mathrm{d}y .$$

Besides H, eq. (1.59) conserves integrals of the form $R_f = \int f(\Omega)\, \mathrm{d}x\, \mathrm{d}y$ with an arbitrary function f.

Consider now stationary solutions (1.60) of the form $\psi = \psi_0(x, y)$ defined from the condition

$$D(\psi_0, \nabla^2\psi_0)/D(x, y) = 0 . \tag{1.61}$$

The solution of (1.61) can obviously be written as

$$\psi_0 = F(\nabla^2\psi_0) \qquad \text{or} \qquad \nabla^2\psi = g(\psi_0) , \qquad g = F^{-1} \tag{1.62}$$

where F is an arbitrary function, and $g \equiv F^{-1}$ is the function which is the inverse of F.

For the Lyapunov functional L we take a combination of H and R_f of the form

$$L = \int [\tfrac{1}{2}(\nabla\psi)^2 + f(\Omega)]\, \mathrm{d}x\, \mathrm{d}y .$$

We choose the function f so that $f'(\Omega) = F(\Omega)$. Then one can directly verify that the stationary equation (1.62) is the Lagrange equation for L:

$$\delta L = \int [-\psi_0 + F(\Omega)]\, \delta\Omega\, \mathrm{d}x\, \mathrm{d}y .$$

It is easy to check that for any perturbation φ, $\psi = \psi_0 + \varphi$, the functional L may be presented in the form

$$L = L_0 + \int (\tfrac{1}{2}(\nabla\varphi)^2 + f(\Omega_0 + \omega) - f(\Omega_0) - f'(\Omega_0)\,\omega)\,dx\,dy\,.$$

For small perturbations $\omega = \Delta\varphi \ll \Omega_0$

$$G = f(\Omega_0 + \omega) - f(\Omega_0) - f'(\Omega_0)\,\omega \simeq \tfrac{1}{2}F'(\Omega_0)\,\omega^2\,;$$

from this there follows a sufficient criterion for the stability of two-dimensional solutions $\psi_0(x, y)$ with respect to small perturbations:

$$F'(\Omega_0) > 0\,.$$

Provided this derivative for all Ω is bounded from below by a positive constant a, then

$$G = f(\Omega + \omega) - f(\Omega) - f'(\Omega)\,\omega \geq \tfrac{1}{2}a\omega^2$$

is also positive and therefore the functional L has an absolute minimum for $\psi = \psi_0$ [45]. This result can also be shown by simple integration:

$$\int_0^{\omega} d\omega_1 \int_0^{\omega_1} d\omega_2\, f''(\Omega + \omega_2) = f(\Omega_0 + \omega) - f(\Omega_0) - f'(\Omega_0)\,\omega\,.$$

Thus L becomes positive when $f'' = F' > 0$.

The stability of two-dimensional vortex motions of a barotropic liquid has been studied similarly [49].* However, up to now the problem of the stability of two-dimensional vortex solutions with respect to three-dimensional perturbations remains an open one.

2. Stability of solitons with respect to small perturbations

2.1. General remarks

As we have seen in the previous chapter, in a number of cases it is possible to show soliton stability by means of a variational method constructing the corresponding Lyapunov functional. However, this approach is not suitable when the question is the stability of solitons realising local but not absolute extremums of functionals. In this case it is necessary to investigate soliton stability with respect to small perturbations. As a rule, the set of equations linearized on the background of a soliton solution represents a spectral problem for differential operators which are, generally speaking, non-self-conjugate ones. These problems are complicated in a technical sense; there is no general method for their solution. Nevertheless, there are several methods which give an opportunity to investigate soliton stability for a wide spectrum of problems. In this chapter we consider these techniques choosing very simple but at the same time sufficiently interesting examples. An effective investigation of soliton

* See also a recent review by the same authors [50].

stability is possible due to our knowledge of neutrally stable perturbations corresponding to small changes of stationary solution parameters. The use of this idea goes back to a paper by Barenblatt and Zel'dovich on the stability of combustion waves [51]. Pitaevsky used this approach to investigate the oscillation spectrum of vortex filaments in liquid helium [52]. Zastavenko [53] used it to study soliton stability for the nonlinear Klein–Gordon equation, and so on.

Note that the soliton dimensionality is often less than the dimensionality of the space in which it is considered. This is the case for the one- and two-dimensional solitons discussed in the previous chapter. All these solutions possess a continuous symmetry group (a translational group). Therefore the problem of soliton stability naturally is divided into two parts. First, it is necessary to study soliton stability with respect to perturbations which do not break the soliton symmetry, i.e. to perturbations with the same dimensionality. Next, if the soliton dimensionality is less than the space dimensionality, one should study soliton stability with respect to perturbations of a higher dimensionality, for example, with respect to "necks" and "snakes".

In the study of instabilities breaking the soliton symmetry a basic method is efficiently used which is based on expansion in a small parameter, the ratio of the transverse soliton size to the scale of perturbation assumed, therefore, to be a long-wavelength one. The proximity of these perturbations to neutrally stable modes allows us to construct a regular procedure for calculating their spectrum which has a great degree of universality. This method was first used successfully by Kadomtsev and Petviashvili for the investigation of one-dimensional soliton stability in weakly dispersive media [12].

2.2. *Perturbations without symmetry breaking*

In this section we will discuss the problem of the stability of stationary centrally-symmetric solutions of the NSE with respect to perturbations which do not break their symmetry, i.e. which have the same dimensionality as the solution itself.

From the methodical point of view we will start with the simplest problem – the problem of the stability of solitons described by the relativistically invariant nonlinear Klein–Gordon equation

$$u_{tt} - \nabla^2 u = f(u) = -m^2 u + \lambda^2 u^3 . \tag{2.1}$$

We consider two cases for positive and negative values of m^2 and λ^2. For the latter case eq. (2.1) is often called the Higgs equation. One-dimensional stationary solutions of (2.1), $u = u(x)$, obey the Newton equation

$$\mathrm{d}^2u/\mathrm{d}x^2 = -\partial\phi/\partial u \tag{2.2}$$

with the potential $\phi(u) = -\frac{1}{2}m^2u^2 + \frac{1}{4}\lambda^2u^4$ (fig. 1). Particle trajectories corresponding to solitons are represented by the line 1 in fig. 1. For $m^2 > 0$, $\lambda^2 > 0$ the solitons are analogous to the solitons (1.24) of the NSE

$$u_0(x) = \frac{\sqrt{2}m}{\lambda} \cosh^{-1} mx . \tag{2.3a}$$

When m^2 and λ^2 have an opposite sign solitons acquire the form of a shock wave or kink with width $\sim m^{-1}$:

$$u_0(x) = \frac{m}{\lambda} \tanh \frac{m}{\sqrt{2}} x . \tag{2.3b}$$

Now we consider the stability of stationary solutions with respect to small perturbations δu. Linearizing (2.1) on the background of the soliton solution (2.3) and assuming $\delta u = \psi(x) \exp(-i\Omega t + i\kappa r_\perp)$ we obtain a spectral problem for the Schrödinger operator

$$[-d^2/dx^2 - f'(u_0) - (\omega^2 - \kappa^2)]\, \psi = 0 . \tag{2.4}$$

Differentiating (2.2) with respect to x and comparing the result with (2.4) we see that $\partial u_0/\partial x$ is a neutrally stable perturbation with $\omega = \kappa = 0$. This fact is a consequence of the translational invariance and thus the perturbation $\psi = \partial u_0/\partial x$ corresponds to an infinitesimal soliton shift. For the solution (2.3a) an eigenfunction ψ_0 is zero in the soliton centre and hence it cannot be a ground state according to the oscillation theorem (see, e.g. ref. [54]). Therefore the ground state has a negative value $\omega^2 - \kappa^2$ and consequently corresponds to unstable perturbations. For $\kappa = 0$ the instability growth rate is of the order of the nonlinear frequency shift $\gamma_{max} \sim \Delta\omega_{nl}$; transverse perturbations with $\kappa^2 \le \gamma^2_{max}$ are also unstable. For the solution (2.3a) the Schrödinger operator spectrum is well-known [54]. For the instability growth rate we have

$$\gamma^2 = -\kappa^2 + 3m^2$$

and the corresponding eigenfunction

$$\psi = \cosh^{-2} mx .$$

For solutions in the form of a shock wave or kink $\psi_0 = \partial u_0/\partial x$ has no nodes and, therefore, represents the ground-state eigenfunction. Thus $\omega^2 - \kappa^2$ has only positive values and the solutions (2.3b) are also stable against transverse perturbations. In particular, the kink stability in the sine-Gordon model where $f(u) = \sin u$ follows from this.

Now we turn to the problem of the stability of cylindrically and spherically symmetric solutions of NSE

$$i\psi_t + \nabla^2\psi + |\psi|^2\, \psi = 0 \tag{2.5}$$

of the form $\psi = g(r) \exp(i\lambda^2 t)$ which obey the equation

$$-\lambda^2 g + \nabla^2 g + g^3 = 0 . \tag{2.6}$$

The quantity λ^2 has the meaning of a bound state energy. The S-solutions without nodes correspond to the ground state which must be the most stable one. Therefore, below, when this point is not discussed especially, we shall mean by the stationary solution the ground state (2.6). It is clear that the function g in this case can be considered to be real.

The solutions of eq. (2.5) will be sought in the form

$$\psi = (g + u + iv) \exp(i\lambda^2 t) = \tilde{\psi} \exp(i\lambda^2 t)$$

where u and v are real-valued functions. For such transformation

$$H \to H + \lambda^2 N .$$

Then assuming u and v to be small we linearize eq. (2.5). As a result for perturbations $u, v \propto \exp(\mathrm{i}\Omega t)$ we obtain the following spectral problem

$$\Omega^2 u = L_0 L_1 u$$

where the self-conjugated Schrödinger operators L_0 and L_1 have the form

$$L_0 = -\nabla^2 + \lambda^2 - g^2 , \qquad L_1 = -\nabla^2 + \lambda^2 - 3g^2 . \tag{2.7}$$

For instability it is sufficient to show that the minimum eigenvalue Ω^2 is negative.

Consider now some properties of these operators. From a comparison of (2.6) and (2.7) it is seen that the stationary solution $g(r)$ is an eigenfunction of L_0 with eigenvalue zero:

$$L_0 g = (-\nabla^2 + \lambda^2 - g^2)\, g = 0 . \tag{2.8}$$

Since g nowhere turns to zero, it is an eigenfunction corresponding to the ground state and the operator L_0 is nonnegative. This is clearly seen from the fact that L_0 may be represented in the form

$$L_0 = -\frac{1}{g} \nabla g^2 \nabla \frac{1}{g} . \tag{2.9}$$

A derivative of the stationary solution in some direction ξ is an eigenfunction of the operator L_1 with eigenvalue zero. This can be easily verified by differentiation of (2.6). Since $\partial g / \partial \xi$ vanishes on the line which passes through the soliton centre it cannot correspond to the ground state and hence L_1 has negative eigenvalues. The minimum value of Ω^2 can be found [55] as the functional minimum

$$\Omega^2 = \min \left[\langle u | L_1 | u \rangle / \langle u | L_0^{-1} | u \rangle \right] . \tag{2.10}$$

This minimum is taken on the class of functions orthogonal to zero-eigenvalue eigenfunction of the operator L_0: $L_0 u_0 = 0$, coinciding with g. The operator L_0 on this function class is positive definite, so to prove instability or stability it is necessary to determine the sign of the functional $\langle u | L_1 | u \rangle$. This problem is reduced to the solution of the spectral problem for L_1

$$L_1 u = \lambda u + \alpha g \tag{2.11}$$

with an undetermined Lagrange multiplier α and an additional orthogonality condition $\langle u \mid g \rangle = 0$. Following ref. [56] we expand u and g in terms of a complete orthonormalized system of eigenfunctions of the operator L_1, ψ_n ($L_1 \psi_n = \lambda_n \psi_n$). Substituting this expansion into (2.11) we obtain

$$u = \alpha \sum_n \frac{\langle g \mid \psi_n \rangle}{\lambda_n - \lambda} \psi_n .$$

The orthogonality condition gives

$$\alpha \sum_n \frac{\langle g \mid \psi_n \rangle \langle \psi_n g \rangle}{\lambda_n - \lambda} = \alpha\, f(\lambda) = 0 \,. \tag{2.12}$$

It should be noted that in this sum the term with $\lambda_1 = 0$ is absent due to $\langle \nabla g \mid g \rangle = 0$. Below the level $\lambda_1 = 0$ the operator L_1 has only a ground state which is an S-state with $\lambda_0 < 0$. Let us consider a λ-value between the first positive eigenvalue λ_2 and the negative eigenvalue λ_0. When λ increases from λ_0 to λ_2, $f(\lambda)$ changes monotonically from $-\infty$ to ∞ and, consequently, passes through zero. Therefore to define the sign of $\lambda_{\min}$ it is sufficient to determine $f(0)$. For $f(0) > 0$, $\lambda_{\min} < 0$ but for $f(0) < 0$, $\lambda_{\min} > 0$. From (2.12) it follows that

$$f(0) = \sum_n \frac{\langle g \mid \psi_n \rangle \, \langle \psi_n \mid g \rangle}{\lambda_n} = \langle g \mid L_1^{-1} \mid g \rangle \,.$$

Differentiating (2.6) with respect to λ^2 we obtain

$$L_1\, \partial g / \partial \lambda^2 + g = 0 \,,$$

whence

$$f(0) = -\langle g \mid \partial g / \partial \lambda^2 \rangle = -\tfrac{1}{2} \partial N / \partial \lambda^2 \,.$$

Thus, the solitons are unstable when [56]

$$\partial N / \partial \lambda^2 < 0 \,, \qquad N = \int |\psi|^2 \, \mathrm{d}V \tag{2.13}$$

and stable in the opposite case. It should be noted that the given criterion is valid for an arbitrary functional dependence $f(|\psi|^2)$. For media with a cubic nonlinearity $g(\lambda, r) = \lambda\, \varphi(\lambda, r)$ and $N \propto \lambda^{2-d}$, i.e. in the three-dimensional case $N \propto \lambda^{-1}$, in the one-dimensional case $N \propto \lambda$, while in the two-dimensional case N does not depend on λ. Thus, three-dimensional solitons in this situation are unstable while one-dimensional ones are stable in agreement with (1.51). In the two-dimensional case the criterion (2.13) does not answer the question about the stability; from (2.13) it follows that exponential instability is absent. This fact agrees with the result of section 1.3 according to which the two-dimensional instability relative to two-dimensional perturbations is of a non-exponential character. This instability is however weak. The instabilities considered below play a more essential role. They are connected with the transverse modulation of the soliton.

The above considerations are suitable for the investigation of soliton stability in a medium with a power-law nonlinearity

$$\mathrm{i}\psi_t + \nabla^2 \psi + |\psi|^{2s} \psi = 0 \,.$$

In this case $N(\lambda) \propto \lambda^{2/s - d}$. One can see that the cubic medium for $d = 2$ is on the stability boundary. The criterion obtained above gives us the possibility to determine only the very fact of instability. The characteristic growth rate of instability is of the order of the only characteristic frequency, the nonlinear frequency shift; $\gamma_{\max} \sim \Delta\omega_{\mathrm{nl}}$.

The criterion (2.13) may be obtained also from a consideration of the second variation of the Hamiltonian of the solution g:

$$\delta^2 H = \int [vL_0 v + uL_1 u]\, \mathrm{d}V \,.$$

In this case the functions u and v are the usual canonical variables

$$2u_t = \delta H'/\delta v \,, \qquad 2v_t = \delta H'/\delta u$$

and the Hamiltonian $H' = \delta^2 H$ is the sum of the terms $\langle v|L_0|v\rangle$ and $\langle u|L_1|u\rangle$ which can be considered as "kinetic" and "potential" energies. When each of these terms is positive (or negative) definite, the state being investigated is stable. If one of them is positive definite and the other is negative definite, instability takes place. Due to conservation of the number of waves $\int |\psi|^2\, \mathrm{d}V$ and of momentum $\mathrm{i}\int (\psi^* \nabla\psi - \mathrm{c.c.})\, \mathrm{d}V$ the function u must be orthogonal to g and v must be orthogonal to ∇g:

$$\int ug\, \mathrm{d}V = 0 \,, \qquad \int v\, \nabla g\, \mathrm{d}V = 0 \,. \tag{2.14}$$

For a linear problem the given conditions serve as solvability conditions. Taking (2.14) into account it is easy to see that the first term $\langle v|L_0|v\rangle$ is positive definite while the condition of the second term to be negative leads to the criterion (2.13). The soliton stability for a KdV equation of the type (1.4) can be considered in a similar way. In this case the instability criterion takes the form [19]

$$\partial P/\partial V < 0 \,.$$

Whence, in particular, there follows the soliton instability for $n > 6$ which is in agreement with the results presented in chapter 1. It should also be noted that the analogue of the criterion (2.13) in field theory was obtained in the work by Friedberg, Lee and Sirling [57].

2.3. Soliton stability with respect to transverse perturbations

In the present section we consider a number of problems which are difficult to investigate with the help of the variational principle. In particular, this is the problem of soliton stability with respect to perturbations breaking the translational symmetry. First the soliton stability described by NSE, will be discussed, following mainly ref. [58]. As we noted before two types of soliton solutions may be distinguished in this equation: solitons localized along the direction of wave propagation and waveguides localized in the transverse direction. For concreteness we will consider the problem of waveguide stability; the result of soliton stability studies can be obtained by a simple change of notation.

The waveguide instability

Let us start the discussion of the stability of flat waveguides which are the solutions of the following equation

$$\mathrm{i}\psi_t + \Delta_\perp \psi + \alpha\psi_{zz} + |\psi|^2 \psi = 0 , \qquad \alpha = k_0\omega''/u , \tag{2.15}$$

in the form (1.22), (1.24) $\psi_0 = g(t)\exp(\mathrm{i}\lambda^2 t)$. In contrast to the preceding section, here we also consider the case when $\omega''<0$. Linearizing eq. (2.15) on the background of a flat waveguide, for the perturbations $\psi = (g + u + \mathrm{i}v)\exp(\mathrm{i}\lambda^2 t)$; $u, v \propto \exp(-\mathrm{i}\Omega t + \mathrm{i}\kappa z)$ the following spectral problem is obtained:

$$\Omega^2 u = (L_0 + \alpha\kappa^2)(L_1 + \alpha\kappa^2)u \tag{2.16a}$$

or

$$\Omega^2 v = (L_1 + \alpha\kappa^2)(L_0 + \alpha\kappa^2)v , \tag{2.16b}$$

where L_0 and L_1 are the operators introduced in the previous section. The principal concept for further considerations is the following. First let us consider a perturbation with $\kappa = 0$. Then it is not difficult to define neutrally stable modes (2.13), corresponding to the eigenvalues $\Omega = 0$. They relate to infinitesimal variations of the soliton parameters. Further we consider long-wave modes (along z) locally slightly differing from the neutrally stable ones and define their spectrum using perturbation theory with neutrally stable modes as a first approximation.

Waveguides, as was mentioned above, are the four-parametric family of equations (1.22). The difference between two stationary solutions with close parameters is a neutrally stable mode. Differentiating the stationary solution with respect to parameters we introduce the following functions:

$$\begin{aligned} &u_0^- = g_x , \qquad v_0^- = -\tfrac{1}{2}xg , \\ &u_0^+ = -\partial g/\partial\lambda^2 , \qquad v_0^+ = g . \end{aligned} \tag{2.17}$$

Here the indices $\pm$ correspond to functions with different parity. Generally speaking, a waveguide with amplitude $\lambda_0 + \delta\lambda$ is not stationary because it has an additional nonlinear frequency shift $2\lambda\,\delta\lambda$. Therefore $\partial\psi_0/\partial\lambda^2$contains a term increasing linearly with time which does not make a contribution into (2.16).

As was mentioned above

$$L_0 v_0 = 0 . \tag{2.18}$$

Differentiating (2.18) with respect to x and λ^2 we get

$$L_1 u_0^- = 0 , \tag{2.19}$$

$$L_1 u_0^+ = v_0^+ . \tag{2.20}$$

It is easy to verify the relation

$$L_0 v_0^- = u_0^- . \tag{2.21}$$

We see that the functions $u_0^\pm$ are the solution of the equation

$$L_0 L_1 u_0 = 0 \tag{2.22a}$$

which is obtained from (2.16a) by putting $\kappa^2 = \Omega^2 = 0$, and the functions $v_0^{\pm}$ are the solutions of the conjugate equation:

$$L_1 L_0 v_0 = 0\,. \tag{2.22b}$$

Obviously

$$u_0 = c_1 u_0^+ + c_2 u_0^-\,, \qquad v_0 = c_1' v_0^+ + c_2' v_0^-$$

where c_1, c_2, c_1', c_2' are arbitrary constants.

The scalar product between functions of different parity is equal to zero. For functions of the same parity we have

$$\langle v_0^+ \,|\, v_0^+ \rangle = N\,, \qquad \langle v_0^+ \,|\, u_0^+ \rangle = -\tfrac{1}{2}\partial N/\partial\lambda^2\,, \qquad \langle v_0^- \,|\, u_0^- \rangle = \tfrac{1}{4}N\,. \tag{2.23}$$

The even and odd neutrally stable modes correspond for $\kappa^2 \neq 0$ to two branches of the spectrum of the operators $(L_0 + \alpha\kappa^2)(L_1 + \alpha\kappa^2)$, $\Omega_\pm^2(\kappa)$ and, respectively, the eigenfunctions $u^\pm(x)$. For $\alpha\kappa^2$ smaller than λ^2 we have

$$u^\pm(x) = u_0^\pm + u_1^\pm + \cdots\,, \qquad \Omega_\pm^2 = \Omega_{1\pm}^2 + \Omega_{2\pm}^2 + \cdots\,. \tag{2.24}$$

Substituting (2.24) into (2.16a) we obtain to first order in κ^2:

$$L_0 L_1 u_1^+ = (\Omega_{1+}^2 - \alpha\kappa^2(L_0 + L_1))\, u_0^+\,. \tag{2.25}$$

The solubility condition of eq. (2.22) is the orthogonality of its right-hand side to the solutions of the conjugated equation (2.22b). It is obviously sufficient to verify the orthogonality to the even solution of v_0^+. Multiplying (2.25) scalarly by v_0^+ and taking into account the relations (2.18), (2.20), (2.23) we obtain

$$\Omega_{1+}^2 = \alpha\kappa^2 \,\frac{\langle v_0^+ | L_0 + L_1 | u_0^+ \rangle}{\langle v_0^+ u_0^+ \rangle} = -2\alpha\kappa^2 \,\frac{N}{\partial N/\partial\lambda^2}\,. \tag{2.26}$$

Analogously, for the odd mode

$$\Omega_{1-}^2 = \alpha\kappa^2 \,\langle v_0^- | L_0 + L_1 | u_0^- \rangle / \langle v_0^- \,|\, u_0^- \rangle = 4\alpha\kappa^2 \,\langle u_0^- \,|\, u_0^- \rangle / N\,. \tag{2.27}$$

Using the specific form of the solution, we calculate $\langle u_0^- \,|\, u_0^- \rangle$ and obtain [58]

$$\Omega_{1+}^2 = -4\alpha\kappa^2\lambda^2\,; \qquad \Omega_{1-}^2 = \tfrac{4}{3}\alpha\kappa^2\lambda^2\,. \tag{2.28}$$

It is clear that for any sign of dispersion α the instability takes place. For $\alpha > 0$ the symmetric mode is unstable and for $\alpha < 0$ the antisymmetric mode is unstable. It follows directly from (2.16) that

$\Omega^2 \simeq (\alpha\kappa^2)^2$ for $\alpha\kappa^2 \gg \lambda^2$. The instability is limited at $\alpha\kappa^2 \sim \lambda^2$, and for the maximum growth rate of both modes we have the estimate $\gamma_{max} \sim \lambda^2 = \Delta\omega_{nl}$. The growth rate is of the order of the nonlinear frequency shift of the wave due to nonlinearity, and it is of the same order as the instability growth rate of the monochromatic wave. The symmetric neutrally stable mode has the meaning of an infinitesimal waveguide amplitude and phase modulation. The $\exp(i\kappa z)$ dependence leads to a successive decrease and increase of the amplitude with a period $2\pi/\kappa$. The increase of such perturbations leads to the bunching of the waveguide in a longitudinal direction (sausage-type instability), whereas the antisymmetric instability leads to waveguide bending (screw type instability). Qualitatively, the symmetric instability is analogous to the modulational instability of a monochromatic wave. For $\alpha < 0$ when the antisymmetric instability takes place, the interacting waves are similar to the particles being attracted to one another in a transverse direction and being repelled in a longitudinal one. The antisymmetric instability is therefore analogous to the instability of a rod compressed at both ends.

It can be easily verified that all the obtained results are valid for a description of media with an arbitrary nonlinearity $f(|\psi|^2)$. As $\partial N/\partial\lambda^2 \to 0$ the symmetric mode growth rate formally turns to infinity, and when $\partial N/\partial\lambda^2 < 0$ the instability disappears. In this case, however, the instability with respect to one-dimensional perturbations found in the previous chapter with the help of the variational method takes place. For $\partial N/\partial\lambda^2 < \alpha\kappa^2 N/\lambda^4$ the formula (2.26) is, of course, invalid. This case which is of particular importance for cylindrical geometry will be considered below.

To calculate the remaining terms of the expansions (2.24) it should be taken into account that the value $\Omega^2_{\pm n}$ appears as the solubility condition of the equation for $u_n^{\pm}$ and the knowledge of $u_{n-1}^{\pm}$ is necessary for its evaluation. To find $u_{n-1}^{\pm}$ it is, generally speaking, necessary to invert the operator $L_0 L_1$. Note, that the operator L_1 can take the form $g_x^{-1}(\mathrm{d}/\mathrm{d}x)g_x^2(\mathrm{d}/\mathrm{d}x)g_x^{-1}$ as follows from formula (2.19). This formula in combination with formula (2.9) demonstrates the possibility in principle to invert the operator $L_0 L_1$ and calculate any terms of the series (2.24) through quadratures. Let us present the results of the calculations for a one-dimensional waveguide with a cubic nonlinearity [58]

$$\Omega_+^2 = -4\alpha\kappa^2\lambda^2 + \tfrac{5}{3}(1+\pi^2/3)\alpha^2\kappa^4\,; \qquad \Omega_-^2 = \tfrac{4}{3}\alpha\kappa^2\lambda^2 + \tfrac{4}{9}(1+2\pi^2/9)\alpha^2\kappa^4\,.$$

It is clear that the growth rates reach their maximum and are bounded for $\alpha\kappa^2 \sim \lambda^2$. However, in this region all the terms in the series are of the same order and the given formulae can only be used for estimates. In a medium with a cubic nonlinearity one can also calculate the boundary value of κ_0 for which the instability disappears, and determine the growth rate structure near this point.

Let us first consider the case of positive dispersion, $\alpha > 0$. Equations (2.16) can be presented in the form of a set of equations, assuming $u, v \propto e^{\gamma t}$,

$$L\psi = M_1\psi + M_2\psi \tag{2.29}$$

where

$$L = \begin{pmatrix} L_1 + \alpha\kappa^2 & 0 \\ 0 & L_0 + \alpha\kappa^2 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}$$

$$M_2 = -\alpha\begin{pmatrix} \kappa^2 - \kappa_0^2 & 0 \\ 0 & \kappa^2 - \kappa_0^2 \end{pmatrix}, \qquad \psi = \begin{pmatrix} u \\ v \end{pmatrix}.$$

We seek the solution of (2.29) in the form of the series

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots .$$

The function ψ_0 is defined from (2.29) for $\gamma = \kappa = 0$

$$L\psi_0 = 0 . \tag{2.30}$$

Since the operator L_0 is nonnegative, $\alpha > 0$, the localized solution (2.30) is of the form $\psi_0 = (\begin{smallmatrix} \bar{u} \\ 0 \end{smallmatrix})$, where $\bar{u}$ is an eigenfunction of the bound state of the operator $L_1 = -\mathrm{d}^2/\mathrm{d}x^2 + \lambda^2 - 6\lambda^2/\cosh^2 \lambda x$. This operator has two bound states: a shift mode corresponding to the first excitation state and a ground state mode $\psi_0 = 1/\cosh^2 \lambda x$, $L_1\psi_0 = -3\lambda^2\psi_0$. Therefore, $\gamma(\kappa)$ for $\alpha > 0$ vanishes when $\alpha\kappa_0^2 = 3\lambda^2$, and $\kappa_0 = 0$. When $\alpha < 0$, $\gamma(\kappa)$ has only one zero for $\kappa_0 = 0$. Let us assume that in the neighbourhood of $\kappa = \kappa_0$, $\gamma \gg \alpha(\kappa^2 - \kappa_0^2)$. Then in the first approximation we have

$$L_1\psi_1 = M_1\psi_0 \qquad \text{or} \qquad (L_0 + \alpha\kappa_0^2)v_1 = \gamma u_0 .$$

The latter equation can be explicitly integrated by means of the representation (2.9) (where instead of g it is necessary to write $\psi_0 = \sqrt{2}\cos\sqrt{2}\lambda x + \tanh \lambda x \sin \sqrt{2}\lambda x)$. A dispersion equation is obtained from the solubility equation of the second approximation

$$L\psi_2 = M_1\psi_1 + M_2\psi_0$$

by taking the scalar product with the function ψ_0. As a result of rather cumbersome calculations one can get [60]:

$$\gamma^2 = 2\alpha(\kappa_0^2 - \kappa^2)(\tfrac{1}{6}\pi^2 - 1)^{-1} \simeq 3.1\alpha(\kappa_0^2 - \kappa^2) .$$

The plot of this instability growth rate obtained numerically [61] is shown in fig. 2.

In the case of a negative dispersion, as was shown above, in (2.30) there is no localized solution except $\kappa_0 = 0$. It is easy to verify that in this equation there exists the solution $u = \tanh \lambda x$ for $\alpha\kappa_0^2 = \lambda^2$ lying on the continuous spectrum boundary. It is impossible to construct a localized solution with finite γ for $\alpha\kappa^2 - \lambda^2 \ll \lambda^2$ which is likely an indication of the fact that the stability disappears stepwise for finite κ. This is in particular verified by the results of numerical calculations [62] given in fig. 3. The results of the numerical calculations of the work [63] are apparently incorrect. In fact, in accordance with these results the growth rate smoothly vanishes at the point $\alpha\kappa_0^2 = 1.09\lambda^2$. However, localized solutions must be absent for $\alpha\kappa^2 > \lambda^2$. It is also clear that perturbations of the continuous spectrum cannot be unstable. Such perturbations are obviously stable at infinity with respect to x. Thus the instability can be described only by localized solutions.

We have considered so far the instability of a flat waveguide against the onset of modulation along the z axis. It is easy to generalize these results by taking into account the possibility of modulation along the y axis. Assuming for the perturbations u, $v \propto \exp[\mathrm{i}\Omega t + \mathrm{i}\kappa(\cos\theta\, z + \sin\theta\, y)]$ we arrive at the same formulae as before in which, however, it is necessary to replace α by $\alpha_{\text{eff}} = \alpha\cos^2\theta + \sin^2\theta$. Here θ is the angle between the wavevectors of perturbation and the initial wave. We see therefore that in a medium with $\alpha > 0$ (medium with a positive dispersion) symmetric instability takes place for all

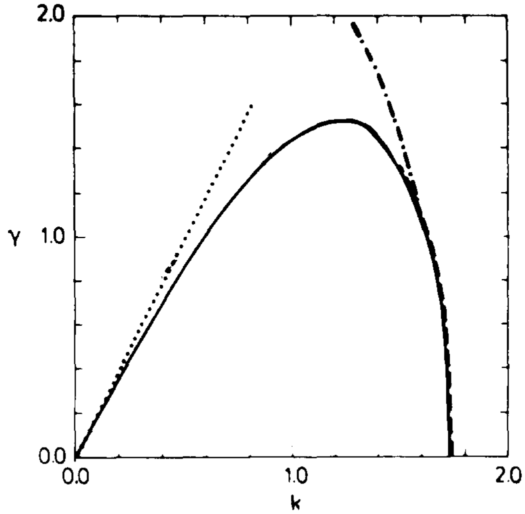

Fig. 2. Growth rate of the soliton instability versus the perpendicular wave number for the positive dispersion case [60]. The solid curve represents the numerical results of the paper by Anderson et al. [61]. The dotted line corresponds to the expansion near $k = 0$ (formula (2.28)), the dotted-dashed line the expansion near $k = k_0$.

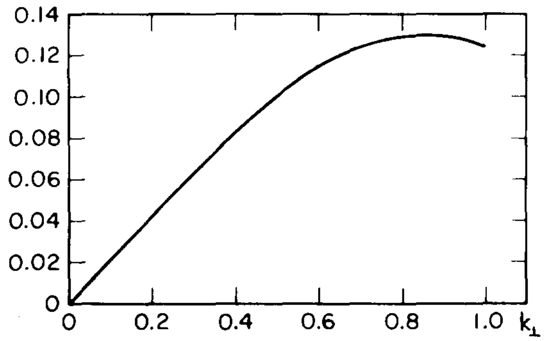

Fig. 3. Growth rate of the soliton instability versus the perpendicular wave number for the negative dispersion case [62].

propagation angles of the perturbation. In fact, it leads to a breakdown of the flat waveguide into three-dimensional bunches within which amplitude singularities in the form of wave collapses appear after a finite time [29, 30]. These collapsing bunches propagate with the group velocity. In a medium with negative dispersion $\omega'' < 0$ we have antisymmetric instability inside the cone $\tan^2\theta < |\alpha|$ and symmetric instability outside the cone. Their combination leads to fragmentation of the flat waveguide and to a subsequent energy scattering at large angles. The question of the existence of singularities for $\alpha < 0$ remains open.

The obtained results allow us to establish the fact of the "spatial" instability of a flat waveguide. Let us assume that $\psi \propto \exp(\mathrm{i}\lambda^2 t)$ in eq. (1.19) written down in the fixed coordinate system and neglect the term $\alpha\psi_{zz}$. In this case the well-known self-focusing equation arises:

$$\mathrm{i}\psi_z + \nabla^2\psi - \lambda^2\psi + f(|\psi|^2)\,\psi = 0\,.$$

A stationary flat waveguide represents the solution of this equation in the form $\psi = g(x)$. Let us consider a stationary solution close to it in the form $\psi = g(x) + (u + \mathrm{i}v)\exp(\mathrm{i}pz + \mathrm{i}\kappa y)$ and determine $p(\kappa)$. Obviously this problem is identical to that of the symmetric instability in a self-focusing medium for $\alpha = 1$; for the growth rate formula (2.28) is valid. The spatial instability leads to a fragmentation of the flat waveguide into cylindrical beams collapsing up to the formation of point focuses.

The problem of a stationary cylindrical waveguide instability against the onset of modulation along the z-axis leads to equations (2.16) where the operators L_0 and L_1 take the form

$$L_0 = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r} + \lambda^2 - g^2 + \frac{m^2}{r^2}\,, \qquad (f(\xi) = \xi)$$
$$L_1 = L_0 - 2g^2\,.$$

It is assumed here that the perturbation has an angular dependence $\exp(\mathrm{i}m\theta)$; r, θ are polar coordinates in the x, y plane. We consider first the case $m=1$. Introducing the notation $v_0^- = -\frac{1}{2}rg$ we verify that relations (2.19), (2.21) remain valid as before. This is natural since the mode $m=1$ corresponds to a displacement of the waveguide centre. Hence it follows that in the cylindrical case the instability according to the $m=1$ mode is perfectly analogous to the flat waveguide instability. For the growth rate we have a result analogous to (2.27)

$$\Omega^2 = 4\alpha\kappa^2\langle g_r^2\rangle / N .$$

From (2.17) we find the relation

$$\langle g_r^2\rangle = \lambda^2 N$$

and get

$$\Omega^2 = 2\alpha\kappa^2\lambda^2 .$$

The instability takes place in media with negative dispersion $\omega'' < 0$ and leads to a spontaneous bending of the waveguide.

Consider now the centrally symmetric mode $m=0$. The singularity of the growth rate (2.26) means the necessity to reconstruct a perturbation theory series.

To a first approximation we have

$$L_0 L_1 u_1^+ = \Omega^2 u_0^+ . \tag{2.31}$$

Since $\langle v_0^+ \mid u_0^+\rangle = \frac{1}{2}\partial N/\partial\lambda^2 = 0$, the first approximation solubility condition does not impose restrictions upon Ω^2. In the second approximation we have

$$L_0 L_1 u_2^+ = \Omega^2 u_1^+ - \alpha\kappa^2 (L_0 + L_1) u_0^+ . \tag{2.32}$$

To calculate u_1^+ we should note that u_0^+ can be presented in the form

$$u_0^+ = -\frac{\partial g}{\partial\lambda^2} = -\frac{1}{\lambda^2}\frac{\partial}{\partial r} rg .$$

Besides, L_0 can be written in the form ($m=0$)

$$L_0 = -\frac{1}{gr}\frac{\mathrm{d}}{\mathrm{d}r} rg^2 \frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{g}$$

and eq. (2.31) is integrated twice. We get

$$L_1 u_1^+ = \frac{\Omega^2}{8\lambda^2} r^2 g . \tag{2.33}$$

This relation proves to be sufficient for obtaining the dispersion relation. Multiplying (2.32) by v_0^+ we get

$$\Omega^2\langle v_0^+ \,|\, u_2^+\rangle = \alpha\kappa^2\langle v_0^+ L_1 u_0^+\rangle .$$

Further, taking into account that $v_0^+ = L_1 u_0^+$ and using the relation (2.33), we obtain

$$\Omega^4 = 16\alpha\kappa^2\lambda^4 N/\langle r^2 g^2\rangle . \tag{2.34}$$

It is obvious that an instability occurs with growth rate

$$\gamma^2 \sim \lambda^3\sqrt{\alpha\kappa^2} ,$$

which as before reaches its maximum at $\alpha\kappa^2 \sim \lambda^2$. This instability is also analogous to the modulational instability of a monochromatic wave.

When a medium with a nonlinearity close to cubic is considered, instead of (2.34) it is easy to obtain [58]

$$\Omega^2\left(\Omega^2 - \frac{8(\partial N/\partial\lambda^2)\lambda^4}{\langle r^2 g^2\rangle}\right) = \frac{16\alpha\kappa^2\lambda^4 N}{\langle r^2 g^2\rangle} . \tag{2.35}$$

For $\kappa = 0$ (2.35) describes two modes, a neutrally stable mode and the mode found in the previous section with the help of the variational principle.

2.4. Instability of a waveguide in a medium with an inertial nonlinearity

Equation (2.5) presupposes that the nonlinear medium is inertialess, i.e., the nonlinearity "follows" the wavefield instantaneously. In many physically important situations (see chapter 1) the nonlinearity has a finite relaxation time connected with the inertia of processes which occur in the medium under the influence of the wavefield. In this case eq. (2.5) must be replaced by a pair of equations which in the laboratory reference system take the form

$$\begin{aligned} &\mathrm{i}(\psi_t + \psi_z) + \Delta_\perp\psi + \alpha\psi_{zz} + P\psi = 0 , \\ &\hat{A}P = f(|\psi|^2) . \end{aligned} \tag{2.36}$$

Here $\hat{A}$ is a linear operator which is, generally speaking, nonlocal in the coordinates; it takes into account the delay of the nonlinearity. If $|\psi|^2$ does not depend on time, the operator $\hat{A} = 1$, and the system (2.36) has the same stationary solutions as (2.5).

Let us compare the effects of the inertia of the nonlinearity and the dispersion. Let the time of the inertial nonlinearity be τ. The inertialess instability of the waveguide has the largest growth rate $\gamma \sim \Delta\omega_{\rm nl}$ for $\alpha\kappa^2 \sim \Delta\omega_{\rm nl}$, i.e., when $\kappa \sim (\Delta\omega_{\rm nl}/\omega'')^{1/2}$. A perturbation of this scale drifts over a length of the order of its size within a reciprocal time

$$\kappa u \sim (\omega_k^2\,\Delta\omega_{\rm nl}/k^2\omega'')^{1/2} \gg \Delta\omega_{\rm nl} .$$

Obviously, the inertia of the nonlinearity can be neglected if $\tau^{-1} \gg \kappa u$, i.e. when $\Delta\omega_{\rm nl}/\omega \ll \omega'' k^2/\omega(\omega\tau)^2$; when $\omega'' \sim \omega/k^2$, $\Delta\omega_{\rm nl}/\omega \ll (\omega\tau)^{-2}$. This rather rigid condition is usually not satisfied in laser experiments. It is therefore natural to consider the opposite case $\Delta\omega_{\rm nl}/\omega \gg 1/(\omega\tau)^2$ with the inertia of nonlinearity as the decisive factor, and to neglect the dispersion term $\alpha\psi_{zz}$.

Taking this circumstance into account, linearization of the system (2.36) against the background of a stationary waveguide leads to the equations

$$L_0(L_1+\delta L)u=\Omega^2 u\,;\qquad L_0=-\Delta_\perp+\lambda^2-f(g^2)\,, \tag{2.37}$$

$$(L_1+\delta L)L_0 v=\Omega^2 v\,;\qquad L_1=L_0-2g^2 f'(g^2)\,. \tag{2.38}$$

Here $\delta L=2g[A^{-1}(\Omega,\kappa)-1]\,f'(g^2)\,g$; when $\Omega=0$, $A^{-1}=1$, $\delta L=0$.

As was shown in section 2.3, a flat waveguide as well as a cylindrical one in a medium close to cubic, experiences a "spatial" instability (for $\Omega=0$) which is conserved for media with any relaxation time. We therefore restrict ourselves to the case of a cylindrical waveguide in a medium with a strong saturation of the nonlinearity.

Multiplying (2.37) from the left by g we find that the symmetric instability mode is absent. Multiplying (2.38) from the left by gr and assuming $v=v_0^+=rg$, we obtain after simple transformations

$$(\Omega-\kappa u)^2=-\frac{4}{N}\,\langle gg_r|A^{-1}|f'(g^2)gg_r\rangle\,. \tag{2.39}$$

The applicability criterion of this formula is the condition

$$\Omega-\kappa u\ll\Delta\omega_{\mathrm{nl}}\,.$$

Let us consider the case of a medium with a nonlinearity relaxing in accordance with the law

$$\tau\,\partial P/\partial t=-P+f(|\psi|^2)\,.$$

In this case $A=1-\mathrm{i}\Omega\tau$ and eq. (2.39) in dimensional variables takes the form

$$(\Omega-\kappa u)^2=-\frac{4\mathrm{i}\Omega\tau}{1-\mathrm{i}\Omega\tau}\,(\Delta\omega_{\mathrm{nl}})^2\,c(\lambda)\,, \tag{2.40}$$

$c(\lambda)=\langle g^2g_r^2f'(g^2)\rangle/N\lambda^4>0$ is a dimensionless structure factor. For $\kappa=0$ eq. (2.40) has a neutrally stable solution $\Omega=0$ generating for $\kappa\neq0$ an unstable branch of the spectrum. For $\Omega\tau\ll1$ we have

$$\Omega=\kappa u+2\mathrm{i}\tau(\Delta\omega_{\mathrm{nl}})^2\,c(\lambda)\left[\sqrt{1+\frac{\mathrm{i}\kappa u}{\tau(\Delta\omega_{\mathrm{nl}})^2\,c(\lambda)}}-1\right]. \tag{2.41}$$

Expanding the radical for small κu, we obtain

$$\Omega=\frac{\mathrm{i}}{4}\,\frac{(\kappa u)^2}{(\Delta\omega_{\mathrm{nl}})^2\,\tau\,c(\lambda)}\,. \tag{2.42}$$

The positive sign of the imaginary part in (2.42) corresponds to instability.

The physical meaning of the instability is absolutely clear. During the waveguide bending the nonlinearity cannot, due to inertia, compensate the diffraction divergence and prevent further bending.

When the inertia is large, $\Delta\omega_{\mathrm{nl}}\tau\gg1$, formula (2.42) is valid up to $\kappa u\sim\Delta\omega_{\mathrm{nl}}$, where the instability growth rate attains the maximum value τ^{-1}. In this case only waveguides of a length $L < \lambda_0 \omega/\Delta\omega_{nl}$ are stable (λ_0 is the wavelength). In the opposite limiting case of small relaxation times $\Delta\omega_{nl} \ll 1$ one should neglect unity under the radical in formula (2.41). Then we have for the growth rate

$$\gamma \sim 2(\kappa u \tau\, c(\lambda))^{1/2}\, \Delta\omega_{nl}\,.$$

The maximum of it is reached for $\kappa u \sim \tau^{-1}$ and is of the order of $\gamma \sim \Delta\omega_{nl}$, the same as in the inertialess medium. The maximum length of the stable waveguide in this case is of the order of $L \sim \lambda_0 \omega\tau$.

Considerable physical interest is attached to self-focusing in a medium with striction nonlinearity, where the connection between P and ψ is given by the wave equation

$$-\frac{1}{s^2} P_{tt} + \Delta P = \Delta f(|\psi|^2)\,.$$

To investigate the waveguide instability in a medium with striction we use the previous results. Obviously, the operator A^{-1} takes the form

$$A = 1 + \frac{\Omega^2}{s^2}\Delta^{-1}\,.$$

For sufficiently small Ω^2, $\Omega^2 < k_0^2 s^2\, \Delta\omega_{nl}/\omega$, we can put

$$A^{-1} - 1 = -\frac{\Omega^2}{s^2}\Delta_\perp^{-1}\,. \qquad (2.43)$$

Substituting (2.43) into (2.39) we obtain in dimensional variables

$$\begin{aligned}(\Omega - \kappa u)^2 &= -\frac{\Omega^2 u^2}{s^2} c_1(\lambda)\,,\\ c_1(\lambda) &= \frac{1}{N}\langle g_r^2|-\Delta_\perp^{-1}|f'(g)\, g_r^2\rangle\,.\end{aligned} \qquad (2.44)$$

Here $c_1(\lambda)$ is a dimensionless structure factor, $c_1 \sim \delta\omega_{nl}/\omega$. The equation (2.44) has an unstable root

$$\Omega = \kappa u\left(1 + \mathrm{i}\,\frac{u}{s}\, c_1^{1/2}\right)\Big/\left(1 + \frac{u^2}{s^2}\, c_1\right). \qquad (2.45)$$

Just as before, the applicability condition (2.45) is

$$\Omega - \kappa u \ll \Delta\omega_{nl}\,.$$

Two limiting cases can be distinguished. For a low nonlinearity level $(\omega/k_0 s)^2\, \Delta\omega_{nl}/\omega \ll 1$

$$\Omega - \kappa u \simeq \mathrm{i}\kappa u\, \frac{u}{s}\, c_1^{1/2}\,.$$

The maximum instability growth rate is attained for $\kappa \sim k_0(s/u)(\Delta\omega_{nl}/\omega)^{1/2}$, $\gamma \sim \Delta\omega_{nl}$. As a result of instability the waveguide breaks up into elongated bunches $l_{\|} \sim (u/s)l_{\perp} > l_{\perp}$ which are particularly long in the case of striction self-focusing $l_{\|} \sim (c/s)l_{\perp}$. It is easy to see that the waveguide is stable with respect to shorter perturbations. In the case of a large nonlinearity $(u/s)^2\,\Delta\omega_{nl}/\omega \gg 1$ the instability becomes aperiodic:

$$\Omega = \mathrm{i}\kappa s(\Delta\omega_{nl}/\omega)^{-1/2}\,.$$

The maximum growth rate is reached for $\kappa \sim k_0\,\Delta\omega_{nl}/\omega$ and is equal to

$$\gamma \sim \omega_k(s/u)(\Delta\omega_{nl}/\omega)^{1/2} < \Delta\omega_{nl}\,.$$

For larger κ the growth rate remains approximately constant up to $\kappa \sim \lambda$, then the instability disappears.

In conclusion, we discuss two more examples in which the above-mentioned instability is manifested. Let us consider the waveguide instability in the case of thermal self-focusing of light. If the instability growth rate is sufficiently small, the temperature variation due to plasma heating by electromagnetic radiation is described by means of the heat-transfer equation [14]:

$$\alpha_1\,\partial\theta/\partial t + \nabla^2\theta = -\eta^2\theta^2 + |\psi|^2\,.$$

The evolution of the electric field is described by eq. (1.25). Here the same dimensionless variables as in the previous chapter are used, while the parameter $\alpha_{\perp} = n_0 u/2\kappa k_0 \sim v_{ei}c^2/\omega v_T^2$. The nonlinearity is of an inertial character. We have shown above that for thermal self-focusing, the nonlinearity is effectively a nonlinearity with saturation. Therefore as above, we shall only consider instability of the mode with $m = 1$.

The equation for small perturbations against the background of the solution (1.59) f_0 takes the form

$$(L_1 + \delta L)L_0 v = (\omega - \kappa u)^2\,v$$

where the operators L_0, L_1, δL are the following:

$$L_0 = \nabla^2 - \theta_0 - \lambda^2\,; \qquad L_1 = L_0 + 2f_0 A_0^{-1} f_0\,,$$
$$\delta L = 2f_0[A_\omega^{-1} - A_0^{-1}]f_0\,, \qquad A_\omega = -\Delta_\perp + \eta^2 - \mathrm{i}\alpha_1\omega\,, \qquad A_0 = -\Delta_\perp + \eta^2\,.$$

The spectrum $\Omega(\kappa)$ is obtained as a solubility condition to first order in κ^2 [14]

$$\Omega = \mathrm{i}\,\frac{2}{\alpha_1}\,\frac{\langle f_0\,|\,f_0\rangle}{\langle(\partial/\partial r)\,\Delta^{-1}f_0^2\,|\,(\partial/\partial r)\,\Delta_\perp^{-1}f_0^2\rangle}\,(\kappa u)^2\,. \tag{2.46}$$

We have stated above that thermal self-focusing is conditioned by pushing out plasma from the waveguide region due to its local heating. For greater κ the growth rates are so large that it is necessary to take into account the ion inertia. In so doing the evolution of the ion density is described by eq. (1.26) while the density variation due to the temperature modulation can be neglected. The instability growth rate is given by the formula (2.45).

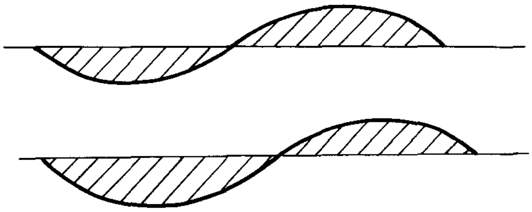

Fig. 4. Schematic representation of the waveguide before and after bending. The oscillations are carried out from the dashed regions due to the mismatch of the group velocities. As a result, the nonlinearity can no longer compensate for the diffraction expansion and the bending increases.

In the following example we consider the stability of waveguides conditioned by mutual focusing of three resonantly interacting waves [34]. A bending mode also proves to be unstable here. We will restrict ourselves to a brief description of its properties. For waveguide bending (see fig. 4) due to a great difference in group velocities $\Delta u > u(\Delta\omega_{\mathrm{nl}}/\omega)^{1/2}$ oscillations are taken away from the shaded regions. As a result, as in the inertial medium, the nonlinearity level decreases in these regions and cannot compensate the diffraction divergence.

2.5. Soliton instability in weakly dispersive media

To demonstrate the efficiency of the above-mentioned method let us consider the problem of soliton stability described by the KP equation with $\beta^2 < 0$. We study the stability of the two-dimensional soliton (1.34) with respect to a variation along the z-axis

$$\delta u \simeq \exp(-\mathrm{i}\omega t + \mathrm{i}kz)\psi(x - Vt, y) .$$

Linearization of the KP equation gives

$$\hat{A}\psi = \mathrm{i}\omega\, \partial\psi/\partial x - 3\kappa^2\psi ,$$

where the operator

$$\hat{A} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2} + 6u_0 - V\right) - 3\,\frac{\partial^2}{\partial y^2} .$$

As before we will investigate soliton stability in the long-wave limit considering neutrally stable modes

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots$$

as a first approximation. It is clear that the derivative $\partial u_0/\partial x$ is a neutrally stable mode like $\partial u_0/\partial y$. These perturbations are independent and can be discussed separately. As follows from the equation defining the soliton shape

$$\frac{\partial^2}{\partial x^2}\left(-V + 3u_0 + \frac{\partial^2}{\partial x^2}\right)u_0 = 3\,\frac{\partial^2 u_0}{\partial y^2} \tag{2.47}$$

a zeroth order eigenfunction of the operator conjugated with A takes the form

$$\psi_0 = w_0\,, \qquad w_{0x} = u_0\,.$$

As will be shown below, ω is a quantity of first order in κ, and therefore to the first approximation we have

$$\hat{A}\psi_1 = \mathrm{i}\omega\, \frac{\partial}{\partial x}\, u_{0x}\,. \tag{2.48}$$

Differentiating the stationary equation (2.47) with respect to V and comparing it with (2.48) we get

$$\psi_1 = \mathrm{i}\omega\, \partial u_0/\partial V\,.$$

In the second approximation we have

$$A\psi_2 + \omega^2\, \frac{\partial^2}{\partial x\,\partial V}\, u_0 = -3k^3\, \frac{\partial u_0}{\partial x}\,.$$

From this the spectrum $\omega(k)$ is obtained as the solubility condition. Multiplying this equation scalarly by the zeroth eigenfunction of the operator conjugated to A, we obtain

$$\omega^2 \int w_0\, \frac{\partial^2}{\partial x\,\partial V}\, u_0\, \mathrm{d}x\, \mathrm{d}y = -3k^2 \int w_0\, \frac{\partial u_0}{\partial x}\, \mathrm{d}x\, \mathrm{d}y\,.$$

Integrating by parts we get the expression

$$\frac{\omega^2}{2}\, \frac{\partial P}{\partial V} = -3k^2 P\,. \tag{2.49}$$

The dependence of P on V can easily be found. For this purpose it is sufficient to note that eq. (2.47) admits scaling transformations:

$$u_0 \to V u_0(x\sqrt{V},\, yV)\,.$$

From this we have $P \sim V^{1/2}$ or $\omega^2 = -12k^2V < 0$ [28]. Thus, the two-dimensional soliton is unstable. The one-dimensional soliton instability is established in a similar manner. In this case the expression for the growth rate follows directly from (2.49): $\omega^2 = -4k^2V$ (cf. ref. [12]).

The instability of both one- and two-dimensional solitons in positive dispersion media is explained by the decrease in the soliton velocity when its amplitude increases. It means that a local change in the soliton amplitude results in its bending and in wave self-focusing [64]. As a result, a self-focused type instability is developed, the nonlinear stage of which leads as for the NSE to collapse [65]. This instability is stabilized for $k \sim \sqrt{V}$. The exact value of k can be found for the one-dimensional soliton only. In this case we have a spectral problem for the fourth order operator:

$$\left[\frac{\mathrm{d}^4}{\mathrm{d}x^4} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\,(6u_0 - 4\kappa^2)\right] \psi = -3k_0^2\psi$$

where $u_0 = 2\kappa^2/\cosh^2\kappa(x - 4\kappa^2 t)$ is a one-soliton solution for $d = 1$.

If one seeks the solution of this equation in the form $\psi = \partial\varphi/\partial x$, the equation for ϕ can be written as

$$[(L_1 + \kappa^2)^2 - 4\kappa^4]\phi = -3k_0^2\phi \,. \tag{2.50}$$

Here $L_1 = -d^2/dx^2 - 6\kappa^2/\cosh^2\kappa x + \kappa^2$ is the well-known operator (2.7) with the known discrete spectrum $E_0 = 0$, $E_1 = -3\kappa^2$. From this it is evident that eq. (2.50) has two solutions with $k_0 = 0$ and $k_{0\,\max} = \kappa^2$ (cf. [66]). The function

$$\psi = \frac{\partial}{\partial x}\frac{1}{\cosh^2\kappa x}$$

corresponds to the latter value.

If one considers the one-dimensional soliton stability in a medium with negative dispersion (which corresponds to the selection of the opposite sign for the right-hand side of the KP equation), similar calculations provide neutral stability with respect to long-wave oscillations [12]:

$$\omega^2 = 4k^2V^2 > 0 \,. \tag{2.51}$$

Thus, the developed perturbation theory permits us to draw a conclusion about soliton instability. In the case of stable solitons the analysis of long-wave perturbations does not, of course, offer a comprehensive answer. The problem of the convergence of the perturbation theory series also remains uncertain. It may be solved either by means of a numerical solution of the spectral problems or with the help of exact methods, such as the inverse scattering transform method. Using this technique, below we will give an exact solution of the one-dimensional soliton stability problem for the two-dimensional KP equation following ref. [67],

$$\frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = 3\beta^2\frac{\partial^2 u}{\partial y^2} \,. \tag{2.52}$$

This problem was first solved in the paper [68].

Equation (2.52) represents the compatibility condition for a linear overdetermined system

$$\begin{gathered}(\beta\,\partial/\partial y - L(x))\psi = 0\,, \qquad (\partial/\partial t - A(x))\psi = 0\,, \\ L(x) = -\frac{\partial^2}{\partial x^2} - u\,; \qquad A = -u\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - 3u_x + 3\beta f\,; \qquad f_x = u_y\,.\end{gathered} \tag{2.53}$$

Along with (2.53) let us consider the set of two equations for some function $F(x, y, z, t)$:

$$\begin{gathered}(\beta\,\partial/\partial y - L(x) + L^+(z))\,F(x, y, z, t) = 0\,, \\ (\partial/\partial t - A(x) + A^+(z))\,F(x, y, z, t) = 0\,,\end{gathered} \tag{2.54}$$

where L^+, A^+ are the operators which are the conjugates of L and A. It is not difficult to see that by virtue of (2.53) this system is also compatible when $u(x, y, t)$ obeys the KP equation.

Let us show that the system (2.54) solves the stability problem with respect to small perturbations for any solution $u(x, y, t)$ of the KP equation (2.52) when the perturbation δu is defined as follows [67]:

$$\delta u(x, y, t) = (\partial/\partial x + \partial/\partial z)\, F(x, y, z, t)|_{x=z} .$$

This is verified by direct calculations. For this purpose it is necessary to apply the operator $\partial/\partial x + \partial/\partial z$ to the first equation (2.54), while the operator $\partial^2/\partial x^2 - \partial^2/\partial z^2$ should be applied to the second equation. The obtained result must be considered on the characteristic $x = z$.

An important feature of equations (2.54) is that they admit separation of variables. As a result, there appear spectral problems for operators of a lower order than those of the initial linearized equation. The advantage of the inverse scattering transform method for the solution of the stability problem lies just in the reduction of the order of the differential operators.

The stability problem is most simply solved for a one-dimensional soliton

$$u_0 = \frac{\kappa}{\cosh^2\kappa(x - 4\kappa^2 t)} .$$

At first it is necessary in equations (2.53) to turn to the system of a soliton at rest. As a result, the operator $\partial/\partial t$ will be replaced by $\partial/\partial t - 4\kappa^2(\partial/\partial x + \partial/\partial z)$. Then in equations (2.53) we make the separation of variables

$$F(x, y, z, t) = \exp(\Gamma t - \mathrm{i}ky)\, \psi(x)\, \chi(z)$$

where $\psi(x)$, $\chi(z)$ are the eigenfunctions of the known operator L_0 (2.7):

$$(\partial^2/\partial x^2 - 2\kappa^2/\cosh^2\kappa x + \mathrm{i}\beta k)\psi = -E\psi ,$$

$$(\partial^2/\partial z^2 - 2\kappa^2/\cosh^2\kappa z)\chi = -E\chi .$$

From this we have

$$\psi = \mathrm{e}^{-\nu x}[\nu/\kappa + \tanh \kappa x] , \qquad \chi = \mathrm{e}^{\eta z}[\eta/\kappa - \tanh \kappa z] , \tag{2.55}$$

where ν and η are connected by the relation

$$\nu^2 - \eta^2 = \mathrm{i}\beta k . \tag{2.56}$$

An expression for the growth rate Γ is obtained by the substitution of (2.55) into the second equation of the set (2.54). As a result we obtain

$$\Gamma = 4[\nu^3 - \eta^3 - \kappa^2(\nu - \eta)] . \tag{2.57}$$

To determine the spectrum of the linearized problem it is necessary to require the boundedness of the perturbation δu for all x:

$$\delta u(x, y, t) = \exp(\Gamma t - \mathrm{i}ky)\, \frac{\mathrm{d}}{\mathrm{d}x} \left\{\exp[(\nu - \eta)x]\left(\frac{\nu}{\kappa} + \tanh \kappa x\right)\left(\frac{\eta}{\kappa} - \tanh \kappa x\right)\right\} . \tag{2.58}$$

When $\nu \neq \pm\kappa$ and $\eta \neq \pm\kappa$ the solution will be bounded if $\mathrm{Im}\, p = 0$, $p = \mathrm{i}(\nu - \eta)$. It is not difficult to show that in this case independently of the sign of β^2, (2.57) proves Γ to be a purely imaginary quantity

$$\Gamma = \mathrm{i}p\left[\frac{3\beta^2 k^2}{p^2} - p^2 - 4\kappa^2\right] \tag{2.59}$$

which for $\kappa = 0$ goes over into the dispersion law for small oscillations. When $\eta = \kappa$ the solution (2.58) decreases exponentially when $\mathrm{Re}\,\nu < \kappa$, $|x| \to \infty$. This condition is in agreement with eq. (2.56) for positive dispersion $\beta^2 < 0$. In this case from (2.56), (2.57) we obtain the instability growth rate in the k-region [68, 67]: $\Gamma = 4|k|(\kappa^2 - |k|)^{1/2}$, which agrees well with all the limiting cases obtained earlier.

Unlike for the case of positive dispersion, for the case of negative dispersion there are no localized modes for finite values of k; there exist only solutions from the continuous spectrum with a frequency defined by (2.59). In fact for $\beta^2 > 0$ from (2.56) for $\eta = \kappa$ there follows an inequality

$$(\mathrm{Re}\,\nu)^2 = \kappa^2 + (\mathrm{Im}\,\nu)^2 > \kappa^2$$

which is incompatible with the boundedness condition for solutions $|\mathrm{Re}\,\nu| < \kappa$. As to the mode (2.51) obtained with the help of perturbation theory, its exact solution gives an exponential growth with index $\mathrm{Re}(\sqrt{\kappa^2 + \mathrm{i}\beta k} - \kappa)x$ which is small in the limit as $k \to 0$. A similar situation takes place for quantum-mechanical quasistationary states, when stationary solutions of the Schrödinger equation are considered formally [69]. The physical cause for this increase is connected with radiation. The given modes can be interpreted analogously; an exact analysis of the Cauchy problem of a linearized equation [70] proves this.

3. Stability of plasma solitons

In the present chapter we consider the problem of plasma soliton stability which is of great importance for plasma physics. Both for laboratory and space plasmas the situation is typical when the plasma turbulence level is so high that nonlinear effects are comparable with or exceed dispersion effects. In this case modulation instabilities usually develop in the plasma, giving rise to localized bunches of the electric field. Solitons are supposed to be formed from these bunches. Therefore turbulence can be represented as a soliton gas [71]. Such a turbulence picture can be realized when the main structural unit of it, a plasma soliton, is a stable formation. In the case of an alternative turbulence picture [30], the cavity formed due to the development of an instability does not reach a stationary state; it collapses in a finite time. In the final stage of the collapse the energy "trapped" in the cavity is transferred to the particles. Plasma turbulence is therefore an ensemble of cavities created by the pump and collapsing. It is important to emphasize that the difference between these turbulence pictures leads to absolutely different macroscopic manifestations. There is a great difference in absorption rates, absorption mechanisms, and, consequently, in the distribution functions of the heated particles. The problem of soliton stability is therefore of primary importance from the practical point of view.

By now the problem of plasma soliton stability seems to be rather clear – plasma solitons are almost always unstable. In a well-defined sense these instabilities are the continuation of the well-known first or second order decay instabilities (see the review [72]) or their modifications [73].*

* For stationary waves in the KP equation this correspondence is exactly determined [67] in terms of the inverse scattering transform method.

In the present chapter we discuss mainly the stability of high-frequency wave solitons in a plasma which are described with the help of equations generalizing the NSE. These equations are obtained after averaging the initial equations over a high frequency; they contain information about the vector structure of the plasma oscillations. Only one section is devoted to the stability of low-frequency solitons.

To study the solition stability we use all the earlier discussed methods with small modifications. When considering concrete plasma problems much attention is focused on the description methods as well as on the applicability boundaries of the obtained equations. Significant attention is also paid to the discussion of the physical meaning of the results.

3.1. *High-frequency solitons in an isotropic plasma and their stability*

We will start the description with high-frequency (HF) Langmuir waves in an isotropic plasma. The dispersion law for Langmuir oscillations is of the form

$$\omega_k = \omega_p(1 + \tfrac{3}{2}k^2 r_d^2) .$$

Here ω_p is the electron plasma frequency and r_d the Debye radius. Langmuir oscillations exist in the region $kr_d \ll 1$, where collisionless Landau damping is small. The criterion $kr_d \ll 1$ denote also that the electrons move as a whole and, therefore, can be described from a hydrodynamic point of view. The principal nonlinear mechanism in the region $E^2/8\pi = w \ll nT$ is the Langmuir wave scattering by low-frequency density fluctuations $\delta n < n_0$. This implies that the frequency of the nonlinear Langmuir waves is close to the plasma frequency $\omega_p(n_0)$. Due to this, by averaging the dynamic equations (hydrodynamic equations for the electrons plus the Maxwell equations) over the fast time ω_p^{-1}, it is possible to get shortened equations similar to (1.25), (1.26). For this purpose a complex envelope $\boldsymbol{E} = \frac{1}{2}(\tilde{\boldsymbol{E}} \exp(-\mathrm{i}\omega_p t) + \text{c.c.})$ of the electric field is introduced, for which from the Maxwell equations we get [74]:

$$\mathrm{i}\tilde{\boldsymbol{E}}_t + \tfrac{3}{2}\omega_p r_d^2 \nabla \operatorname{div} \tilde{\boldsymbol{E}} + \frac{c^2}{2\omega_p} \operatorname{curl}\operatorname{curl} \tilde{\boldsymbol{E}} = \omega_p \frac{\delta n}{n_0} \tilde{\boldsymbol{E}} . \tag{3.1}$$

This equation describes besides plasma waves long-wave electromagnetic oscillations with frequencies close to ω_p and their mutual transformation due to inhomogeneities. In this equation the ratio of the potential and the non-potential terms contains a large parameter c^2/v_T^2. This means that the electric field is approximately potential, $\tilde{\boldsymbol{E}} = -\nabla\varphi$, with the exception of small $kr_d \sim v_T/c$ [74]. To distinguish the potential term in (3.1) let us take the divergence from (3.1) and then assume $\tilde{\boldsymbol{E}} \simeq -\nabla\varphi$. As a result we get [30]:

$$\nabla^2(\mathrm{i}\varphi_t + \tfrac{3}{2}\omega_p r_d^2 \nabla^2\varphi) = \frac{\omega_p}{2} \operatorname{div} \frac{\delta u}{n_0} \nabla\varphi . \tag{3.2}$$

We should emphasize that it is impossible to substitute $\tilde{E} = -\nabla\varphi$ directly into (3.1) because due to the large coefficient c^2/v_T^2 the nonpotential part appears to be of the order of the potential term. In particular for this reason the equation

$$\mathrm{i}\tilde{E}_t + \tfrac{3}{2}\omega_p r_d^2 \nabla^2 \tilde{E} = \omega_p \frac{\delta u}{2n_0} \tilde{E}$$

may be considered only as a model one.

Under the action of ponderomotive forces the density variation δn is described by an equation similar to (1.26):

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2\right) \delta n = \frac{1}{16\pi M} \nabla^2 |\nabla\varphi|^2 . \tag{3.3}$$

Finally, introducing dimensionless variables

$$r = \tfrac{3}{2} r_d \sqrt{(M/m)}\, r' , \qquad t = \tfrac{3}{2}\omega_p^{-1}(M/m)t' ,$$
$$\delta n/n_0 = \tfrac{4}{3}(m/M)n , \qquad \varphi = (T/e)\sqrt{12}\,\varphi'$$

we obtain

$$\begin{aligned} &\nabla^2(\mathrm{i}\varphi_t + \nabla^2\varphi) - \mathrm{div}(n\nabla\varphi) = 0 , \\ &(\partial^2/\partial t^2 - \nabla^2)n = \nabla^2|\nabla\varphi|^2 . \end{aligned} \tag{3.4}$$

In the one-dimensional case equations (3.3) are identically the same as (1.25), (1.26) for $u = 0$ and have a four-parametric family of soliton solutions

$$\varphi_x = \sqrt{2(1-\beta^2)}\, \lambda \operatorname{sech} \lambda(x - \beta t - x_0) \exp \mathrm{i}[(\lambda^2 - \beta^2/4)t + \beta x/2 + \alpha_0] .$$

The character of these solutions depends significantly on the velocity β which in dimensional variables is equal to $3V_T k_0 r_d$ where k_0 is a wavenumber corresponding to the packet maximum. The electric field in the soliton for various values of $k_0 r_d$ is shown in fig. 5. For a soliton at rest $k_0 r_d \ll w/nT$ the electric field varies monotonically. When $(k_0 r_d)^2 \gg w/nT$ this soliton is an envelope soliton with a quasimonochromatic filling. In this case one can make the additional simplification (3.4) and, passing to the envelope, obtain the equations (1.19). The properties of solitons and their stability in this limit have

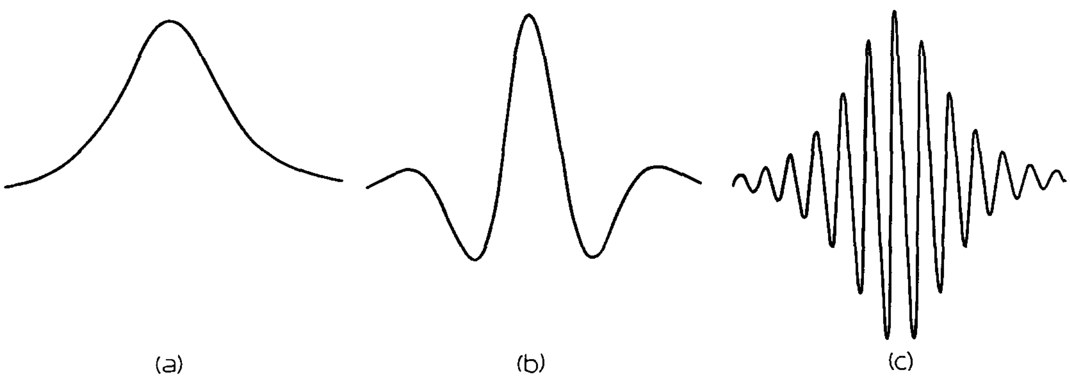

Fig. 5. Electric field in a Langmuir soliton. a, b, c correspond to the different values of the soliton velocities β/λ. a, 0; b, 4; c, 20. $\beta \ll 1$.

been studied above. In the present chapter we shall mainly restrict ourselves to consideration of solitons at rest.

In the static limit when $n_{tt} \ll \Delta n$, the system (3.4) is reduced to one equation

$$\nabla^2(\mathrm{i}\varphi_t + \nabla^2\varphi) + \operatorname{div}|\nabla\varphi|^2\,\nabla\varphi = 0\,. \tag{3.5}$$

This equation describes all stationary solutions except solitons which move with velocities of the order of the acoustic velocity or close to it.

Let us first consider multi-dimensional solitons. It is not difficult to find stationary central-symmetric solutions. In this case (3.5) is reduced to a second order equation similar to (1.50)

$$\begin{gathered} -\lambda^2 E + \frac{\mathrm{d}^2 E}{\mathrm{d}r^2} + \frac{d-1}{r}\frac{\mathrm{d}E}{\mathrm{d}r} - \frac{d-1}{r^2}E + E^3 = 0 \\ E = \mathrm{d}\varphi/\mathrm{d}r\,, \qquad \varphi = \varphi(r)\exp(\mathrm{i}\lambda^2 t)\,. \end{gathered} \tag{3.6}$$

The structure of solutions (3.6) is also investigated qualitatively as has been done for (1.50). However, in contrast to (1.50), eq. (3.6) contains an additional centrifugal term $((d-1)/r^2)E$ corresponding to the orbital quantum number $l = 1$. Therefore the "wavefunction" E as $r \to 0$ behaves as $r^l = r$. Thus, for centrally symmetric solutions the electric field in the centre is equal to zero. Thus, field distributions are energetically more preferable when the field is different from zero for $r = 0$ (see [75, 80]).

Consider now the soliton stability.

Equation (3.5) is like the NSE, a Hamiltonian one

$$\mathrm{i}\nabla^2\varphi_t = \delta H/\delta\varphi^*\,.$$

Besides the Hamiltonian

$$H = \int [|\nabla^2\varphi|^2 - \tfrac{1}{2}|\nabla\varphi|^4]\,\mathrm{d}V$$

eq. (3.5) conserves also the total number of plasma waves:

$$N = \int |\nabla\varphi|^2\,\mathrm{d}V\,.$$

As for the NSE, the soliton solution $\varphi = \varphi_0\exp(\mathrm{i}\lambda^2 t)$ represents a stationary point H for a fixed number of waves

$$\delta(H + \lambda^2 N) = 0$$

or

$$\nabla^2(-\lambda^2 + \nabla^2)\,\varphi_0 + \operatorname{div}|\nabla\varphi_0|^2\,\nabla\varphi_0 = 0\,.$$

By analogy with (1.46) it is easy to check that for the soliton solution

$$H = -\lambda^2 N(d-2)/(4-d)$$

i.e., the Hamiltonian is positive for $d = 3$ and negative for one-dimensional solitons. If one performs the N-conserving scaling transformations,

$$\varphi_0(r) = a^{(2-d)/2}\varphi_0(r/a)$$

then the Hamiltonian H as a function of the parameter a

$$H(a) = I_1/a^2 - I_2/a^d \qquad \left(I_1 = \int |\nabla^2\varphi_0|^2\, dV, I_2 = \tfrac{1}{2}\int |\nabla\varphi_0|^4\, dV\right) \tag{3.7}$$

has a maximum for $d = 3$, which corresponds to a three-dimensional soliton and it is not bounded from below as $a \to 0$. It is not difficult to see that the Hamiltonian H will be unbounded from above because H contains higher derivatives of φ_0 than N. This means that a three-dimensional soliton is unstable at least with respect to finite perturbations. As to a rigorous proof of the three-dimensional soliton instability against small perturbations, this question remains still open. The same may be said about two-dimensional solitons. Here a situation similar to the one for the NSE is likely to take place. Solitons must be unstable against perturbations of a non-exponential character. As far as one-dimensional solitions are concerned, they are obviously stable with respect to one-dimensional perturbations.

As has been shown above, stable solitons may exist in a medium with a slowly increasing nonlinearity. If one assumes the characteristic times of the nonlinear processes to exceed significantly the time of an ion passing through the cavity, then both electrons in slow movements and ions can be considered to have a Boltzmann distribution [30]

$$\tilde{n}_e = n_0 \exp\left\{\frac{1}{T}\left(e\tilde{\varphi} - \frac{|E^2|}{16\pi n_0}\right)\right\}, \qquad n_i = n_0 \exp\left(-\frac{e\tilde{\varphi}}{T}\right).$$

From this and using the condition of quasi-neutrality we obtain

$$\tilde{n}_e = n_0 \exp(-E^2/32\pi nT).$$

Substituting this expression into (3.4) we get the equation

$$\nabla^2(i\varphi_t + \nabla^2\varphi) + \operatorname{div}\nabla\varphi(\exp(-|\nabla\varphi|^2) - 1) = 0. \tag{3.8}$$

It is not difficult to verify that this equation has solutions which are stable due to nonlinearity saturation. In particular, such solutions have been discussed in the paper [76] (see also [77]). It should, however, be noted that nonlinearity saturation occurs when $w/nT \sim 1$; a typical soliton amplitude must be just of the same value. Consequently the soliton size must be comparable with the Debye radius. Under these conditions eq. (3.8) cannot be applied. As has been mentioned above, eq. (3.4) is valid to first order in $(kr_d)^2$ and w/nT. In the next approximation together with the non-linearity saturation it is necessary to take into account the variation of the dispersion law for Langmuir waves, electron nonlinearities [78], and Landau damping. In the region $k \sim \omega_p/c$ relativistic nonlinearities may be of importance. Thus, an analytical description of strong plasma turbulence is rather complicated. However, when the average level of turbulence is low, collapsing cavity evolution does not lead to the formation of multi-dimensional plasma solitons. Due to the ion inertia the cavity is compressed until its energy is absorbed owing to Landau damping. In ref. [79] these problems are discussed in more detail.

Recently a numerical simulation (3.16) has been carried out intensively in two- and three-dimensional geometries (see, e.g. the reviews [80, 81]) in connection with the Langmuir collapse problem. All these experiments confirm the instability of multi-dimensional plasma solitons. Plasma soliton instability has been also demonstrated in laboratory experiments (see, e.g. [82]). Localized bunches of the field may, however, be observed at high energy densities $w \sim nT$ in narrow cavities (of the order of the Debye radius) when a large fraction of the electrons is trapped in the cavity [79].

Now we turn to the one-dimensional soliton stability with respect to transverse perturbations.

The soliton solution for the potential φ_0 has the form

$$\varphi_0 = \sqrt{2}\exp(\mathrm{i}\lambda^2 t) \arctan \sinh \lambda x \, .$$

As before we seek a solution in the form

$$\varphi = \varphi_0 + \delta\varphi \exp(\mathrm{i}\lambda^2 t + \mathrm{i}k_\perp r_\perp) \, , \qquad \delta\varphi \ll \varphi_0 \, .$$

Separating the real and imaginary parts of $\delta\varphi_x$ and taking into account only the first terms essential with respect to $k_\perp$, we obtain

$$v_t = -(L_1 + L')u \, , \qquad u_t + (L_0 + L')v \, .$$

Here the operators L_0 and L_1 are of the form (2.7) where $|\partial\varphi_0/\partial x|^2$ is put instead of g^2, and the action of the operator L' on the function is defined as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x} L'u = k_\perp^2\left(-\lambda^2 + |\varphi_{0x}|^2 + 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\int_{-\infty}^{x} u \,\mathrm{d}x = k_\perp^2\left(-L_0 + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\int_{-\infty}^{x} u \,\mathrm{d}x \, .$$

Assuming $u, v \propto \exp(-\mathrm{i}\Omega t)$ we find

$$\Omega^2 u = (L_0 + L')(L_1 + L')u \simeq (L_0 L_1 + L_0 L' + L' L_1)u \, . \tag{3.9}$$

Assuming $k_\perp$ to be rather small and applying the results of section 2.2, let us consider the stability with respect to both symmetric and antisymmetric modes. It is not difficult to verify that an odd mode is stable. For symmetric perturbations we get the following dispersion equation

$$\Omega^2 = 2k_\perp^2 \frac{\langle \varphi_{0x}|L'|\varphi_{0x}\rangle}{\partial N/\partial\lambda^2} \, .$$

Performing a rather cumbersome integration we obtain [58]

$$\Omega^2 = -\lambda^2 k_\perp^2 (12 - 7\,\xi(3)) \simeq -3.6\lambda^2 k_\perp^2 \, .$$

Here

$$\xi(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}$$

is Riemann's zeta function.

As before the maximum growth rate is reached at the limit of the given perturbation theory. First, the smallness of the instability growth rate in comparison with the nonlinear frequency shift $\Omega < \lambda^2$ is necessary for its applicability. Second, it is necessary to fulfill the static approximation condition $\Omega < \lambda$. The former condition is more important for small amplitudes $\lambda < 1$ or $w/nT < m/M$, and, therefore, $\gamma_{max} \sim \lambda^2$, while the latter is more important for large ones $\lambda > 1$ or $w/nT > m/M$ and $\gamma_{max} \sim \lambda$. In dimensional variables

$$\gamma_{max} \sim \omega_p\, w/nT \qquad \text{for } w/nT < m/M\,, \tag{3.10}$$

$$\gamma_{max} \sim \omega_{pi}(w/nT)^{1/2} \qquad \text{for } w/nT > m/M\,. \tag{3.11}$$

For small amplitudes w/nT the growth rate maximum is reached for $(k_\perp r_d)^2 \sim w/nT$ and then for kr_d values of the same order the growth rate goes to zero. Thus, the given instability is similar to the soliton instability described by a NSE with a positive dispersion. For large w/nT the growth rate reaches the value (3.11) for $kr_d \sim \sqrt{m/M}$. It is clear that for large $k_\perp$ up to $k_\perp r_d \sim \sqrt{w/nT}$ the soliton is unstable as before. In fact, if the growth rate begins to decrease in comparison with (3.11), the static approximation is valid again and the instability takes place. On the other hand, if we neglect the term $c_s^2\nabla^2 n$ in eq. (3.3), it is not difficult to see that the instability growth rates do not exceed (3.11). This implies that in the interval $\sqrt{m/M} < k_\perp r_d < \sqrt{w/nT}$ the growth rate is approximately constant. As to order of magnitude the maximum growth rate is the same as that for an unstable monochromatic wave with $\boldsymbol{k} = 0$. This is natural, since the wavelength of the perturbations corresponding to the maximum growth rate is of the order of a soliton size. Note that the fact of soliton instability itself is nontrivial here.

We have considered above the instability of a soliton at rest. The somewhat more complicated problem of the stability of a soliton moving with an arbitrary velocity has been solved by Benilov [36] with the help of the same method. It is shown that the magnitude of the growth rate depends weakly on the motion of the soliton. We shall only note that for a moving soliton the unstable mode does not possess a definite parity.

As has been mentioned above the problem of plasma soliton stability was considered in many papers the results of which were often contradictory. A critical discussion of these works has been carried out in the review [83] and the paper [36]. The drawbacks are connected either with the consideration of a narrow class of perturbations or with inaccuracies of approximate methods used in them.

3.2. Effect of a weak magnetic field on the Langmuir soliton stability

Consider now the problem of the effect of a weak magnetic field on the Langmuir soliton stability. It is well-known that a plasma becomes anisotropic in the presence of a magnetic field. In such a situation the investigation of multi-dimensional solitons presents a complicated problem sensitive to the plasma geometry and parameters.

We shall limit ourselves to a discussion of the stability of one-dimensional solitons propagating along the magnetic field.

First we determine the effect of a weak magnetic field on the Langmuir soliton stability. A modification of the Langmuir oscillation dispersion law is observed in very moderate fields:

$$\omega_k = \omega_p\left(1 + \tfrac{3}{2}k^2 r_d^2 + \frac{1}{2}\frac{\omega_H^2}{\omega_p^2}\frac{k_\perp^2}{k^2}\right). \tag{3.12}$$

Here $\boldsymbol{k}_\perp$ is the wavevector component perpendicular to the magnetic field $\boldsymbol{H}_0$, $eH_0/mc = \omega_H$ is the electron cyclotron frequency. From the dispersion law (3.12) it is obvious that the transverse perturbation frequency is higher than that in an isotropic plasma and hence, the magnetic field must be a stabilizing factor for solitons. It should also be noted that the magnetic field is important for solitons with a small amplitude for which the nonlinear frequency shift $\omega_p\, w/nT$ does not exceed ω_H^2/ω_p. Here in the static limit all changes of the averaged equation will refer only to the linear terms

$$\nabla^2(\mathrm{i}\varphi_t + \tfrac{3}{2}\omega_p r_d^2 \Delta\varphi) - \frac{\omega_H^2}{2\omega_p^2}\nabla_\perp^2\varphi + \omega_p \operatorname{div}\left(\frac{|\nabla\varphi|^2}{32\pi nT}\nabla\varphi\right) = 0$$

or in dimensionless variables

$$\nabla^2(\mathrm{i}\varphi_t + \nabla^2\varphi) - \sigma\nabla_\perp^2\varphi + \operatorname{div}|\nabla\varphi|^2\nabla\varphi = 0\,, \qquad \sigma = \tfrac{3}{4}\omega_H^2/\omega_p^2\,. \tag{3.13}$$

Consider first a soliton moving with a sufficiently large group velocity $3v_T(k_0 r_d) > v_T(w/nT)^{1/2}$. As has been mentioned above, in this case one can turn to envelopes and obtain a NSE of the type (2.15)

$$\mathrm{i}\psi_t + \psi_{zz} + \left(1 + \frac{\omega_H^2}{3\omega_p^2 k_0^2 r_d^2}\right)\nabla_\perp^2\psi + |\psi|^2\psi = 0\,.$$

Hence it immediately follows that a one-dimensional envelope soliton is unstable with respect to transverse symmetric perturbations (see section 2.2). Therefore for fast solitons the magnetic field does not stabilize the instability. It is also obvious that the maximum value of the growth rate remains unchanged. The inclusion of the magnetic field leads only to an increase in the unstable perturbation wavelengths [84] by a factor $\omega_H/\omega_p k_0 r_d$.

Now we shall investigate the stability of a soliton at rest. Linearizing eq. (3.13) on the background of the solution (3.9) (the x-axis is taken along the magnetic field) one comes to the spectral problem (3.10) in which the operator L' acquires an extra term depending on the magnetic field

$$\frac{\mathrm{d}}{\mathrm{d}x}L'u = k_\perp^2\left(-L_0 + \frac{\partial^2}{\partial x^2} - \sigma\right)\int_{-\infty}^{x} u\,\mathrm{d}x'\,.$$

For an antisymmetric mode calculations for small $k_\perp$ give marginal stability:

$$\Omega_-^2 = 4k_\perp^2\left[\frac{\lambda^2}{3} + \frac{3}{4}\frac{\omega_H^2}{\omega_p^2}\right].$$

At $k_\perp \to 0$ for symmetric perturbations the square of the frequency is given by the expression

$$\Omega_+^2 = -k_\perp^2 \left[(12 - 7\,\xi(3))\lambda^2 - \frac{21}{8} \frac{\omega_H^2}{\omega_p^2}\,\xi(3) \right].$$

It is seen from this that in the long-wave limit a magnetic field stabilizes the instability when

$$\omega_H^2/\omega_p^2 > \lambda^2\, \tfrac{8}{21}\, (12 - 7\xi(3))/\xi(3) \simeq 0.86\lambda^2$$

or in dimensional variables

$$\omega_H^2/\omega_p^2 > 1.7 w/nT .$$

It should be emphasized that this result does not prove the soliton stability. Instability may appear when we take into account the next terms of the expansion in $k_\perp$. In a weak magnetic field the fact of the Langmuir soliton instability has been shown in calculations [85]; however, the growth rate structure has not been studied in detail.

3.3. HF solitons in a strong magnetic field

It is well-known that there exist two branches of potential electron HF oscillations in a plasma in a magnetic field: the upper and lower hybrid wave oscillations. Upper hybrid oscillations are as $H \to 0$ transformed into Langmuir waves, the dispersion law (3.12) just corresponds to the upper hybrid oscillations in a weak ($\omega_H < \omega_p$) magnetic field. The upper-hybrid wave dispersion law for arbitrary magnetic fields with neglect of thermal additions is of the form

$$\omega_k^+ = \tfrac{1}{2}\left(\sqrt{\omega_H^2 + \omega_p^2 + 2\omega_H\omega_p \cos\theta} + \sqrt{\omega_H^2 + \omega_p^2 - 2\omega_H\omega_p \cos\theta}\right) \tag{3.14}$$

where θ is the angle between the propagation direction and the magnetic field. The dispersion law for lower-hybrid waves ω_k^- differs from (3.14) only in the signs of the square ratio. Both in weak and strong magnetic fields ω_k^- has the same angular dependence

$$\omega_k^- = \omega_0 |\cos\theta| , \qquad \omega_0 = \begin{cases} \omega_p , & \omega_H \gg \omega_p \\ \omega_H , & \omega_p \gg \omega_H \end{cases} . \tag{3.15}$$

This expression is valid up to angles $\cos\theta \sim \sqrt{m/M}$. For quasitransverse propagation in (3.15) it is necessary to take into account the ion motion. As $\theta \to \pi/2$ the oscillation frequency tends to the low-hybrid frequency

$$\omega_{LH} = \omega_H \omega_{pi}/(\omega_H^2 + \omega_p^2)^{1/2} .$$

Near the lower hybrid frequency ($\omega - \omega_{LH} \ll \omega_{LH}$) in the dispersion law it is necessary to consider the following thermal corrections:

$$\omega_k^- = \omega_{\rm LH}(1 + \tfrac{1}{2}k^2R^2 + \tfrac{1}{2}\cos^2\theta\, M/m)$$

$$R^2 = \begin{cases} 3(T_i/T_e)r_d^2\,, & \omega_H \gg \omega_p \\ (\frac{3}{4} + 3T_i/T_e)r_H^2\,, & \omega_p > \omega_H \end{cases} \tag{3.16}$$

where $r_H = v_{Te}/\omega_H$ is the electron Larmor radius.

Note that (3.16) is similar to the dispersion law for Langmuir waves (3.12).

Consider now the upper-hybrid branch of the spectrum. The equations describing the evolution of the oscillations can be noticeably simplified as in an isotropic plasma by averaging them with respect to the high frequency. The structure of the low-frequency equations is analogous to (3.1) (see, e.g. [86–88]) with the difference that now it is necessary to take into account local frequency oscillation variations resulting from slow fluctuations of the magnetic field. The structure of the high-frequency equation is noticeably complicated due to the fact that even in the hydrodynamic limit of a magneto-active plasma there exist three branches of low-frequency oscillations. A high-frequency force which generates slow motions remains potential $F = \nabla\phi$; however, in ϕ, besides the usual term proportional to $|E|^2$ there becomes essential a term with a vector nonlinearity $\phi \sim [\nabla\psi, \nabla\psi^*]_z$ whose appearance results from the particle drift in the field of the electric oscillations. The general form of the closed system of equations describing the evolution of upper-hybrid oscillations can be found in the papers [10, 86–88]. The system of equations arising here has a wide range of soliton solutions (see, e.g. [86, 87]).

Now we shall discuss in detail solitons of upper-hybrid waves propagating across the magnetic field. First it should be noted that in this case soliton solutions exist under the condition $\omega_p^2 > 3\omega_H^2$. In the opposite case the thermal dispersion changes its sign [87] and localized solutions are absent. In this case equations in dimensionless variables which describe the upper-hybrid oscillation propagation are of the form [87, 89]:

$$-i\mu\, \partial\psi/\partial t + \partial^2\psi/\partial x^2 - n\psi = 0 \tag{3.17}$$

$$\partial^2 n/\partial t^2 - \partial^2 n/\partial x^2 + n = \partial^2|\psi|^2/\partial x^2 - \beta|\psi|^2 \tag{3.18}$$

where $\mu = \frac{2}{3}\omega_{pi}/\omega_H$, $\beta = 8\pi nT/H^2 \ll 1$. The properties of solitons of this system depend significantly on the characteristic scale of the electric field variations. For simplicity we shall restrict ourselves to a consideration of quasistationary motions when the term $\partial^2 n/\partial t^2$ in (3.18) may be neglected. Then for smooth distributions with a scale length $L > \beta^{-1/2}$ (in dimensional variables $l > c/\omega_p$) $n = -\beta|\psi|^2$ and the set (3.17), (3.18) is reduced to the NSE. Similarly for narrow distributions $L < 1$ (in dimensional variables $l < v_{Te}/\omega_H$) we obtain a NSE with a stronger nonlinearity $n = -|\psi|^2$. In the intermediate region $1 < L < \beta^{-1/2}$ the equation with a nonlocal nonlinearity

$$i\mu\psi_t + \psi_{xx} - \psi \frac{\partial^2}{\partial x^2}|\psi|^2 = 0 \tag{3.19}$$

takes place. For this equation the Hamiltonian H is equal to $H = \int \{|\psi_x|^2 - \frac{1}{2}(|\psi|_x^2)^2\}\,dx$. Equation (3.19) can be easily studied with the help of the above-described methods. Its soliton solutions represent stationary points of H for a fixed number of waves $N = \int |\psi|^2\,dx$:

$$\delta(H + \lambda^2 N) = 0\,.$$

The simplest scaling transformations show that the Hamiltonian is unbounded and therefore one should expect unstable soliton solutions. A rigorous proof of the soliton instability is not difficult to obtain with the help of the analogue of the Kolokolov–Vakhitov criterion (2.13). As was shown in ref. [87] the soliton collapse takes place due to instability even in the framework of the one-dimensional equations. As a result, the characteristic size of the soliton decreases rapidly down to the limits of applicability (3.19).

In a medium with dissipation and pumping an auto-oscillating soliton can be constructed [89]. As a rule, excitation of oscillations by an electromagnetic wave or a beam occurs in the long-wave part of the spectrum. Since the electric field here is described by a NSE, solitons are an essential structural unit of one-dimensional turbulence. When the pumping is rather small it leads to a slow growth only and, accordingly, to contraction of the soliton. When the soliton reaches $L \propto \beta^{-1/2}$, (3.19) becomes valid. The soliton collapses and its size decreases rapidly. In the short-wave part of the spectrum Landau damping becomes important. The field energy is absorbed in the soliton and it expands up to $L \gtrsim \beta^{-1/2}$, and so on. This auto-oscillating mode has been demonstrated in paper [89].

Consider now the problem of upper-hybrid soliton stability with respect to non- one-dimensional perturbations resulting from decay process into upper- and lower hybrid oscillations

$$\omega_k^+ = \omega_\kappa^+ + \omega_{k_0-\kappa}^- . \tag{3.20}$$

In a strong magnetic field $\omega_k^+ \simeq \omega_{\rm H}(1 + (\omega_{\rm p}^2/2\omega_{\rm H}^2)\sin^2\theta)$ and the minimum frequency ω_k^- is about $\omega_{\rm pi}$ for propagation perpendicular to the magnetic field. So the process (3.20) is allowed for $1 < \omega_{\rm H}/\omega_{\rm p} < \frac{1}{2}\sqrt{(M/m)}$. The growth rate of this process for a monochromatic wave can be simply estimated as to order of the magnitude (see e.g., ref. [10])

$$\gamma \sim (w/nn_0)^{1/2}\kappa_0 .$$

Also the process of induced scattering on ions is possible

$$\omega_k^+ = \omega_\kappa^+ + |\boldsymbol{k} - \boldsymbol{\kappa}|\, v_{T\rm i} \qquad (T_{\rm i} = T_{\rm e}) \tag{3.21}$$

with a growth rate (see e.g., ref. [10])

$$\gamma \sim \omega_{\rm p}^4 w/\omega_{\rm H}^3 nT .$$

The decay conditions (3.20), (3.21) do not impose restrictions on the value of the wavevector κ, the growth rate is approximately constant for $\kappa > \kappa_0$ and as a result a growth of the short-wave perturbations takes place. The soliton can be considered as a monochromatic wave with $k = 0$ for them and the above-mentioned instabilities can be developed. Let us emphasize that due to the finite size of the solitons, the perturbations carried out can stabilize the instability.

Therefore, the solitons propagating along the weak magnetic field are the main subject of interest. The modulational instability of such solitons was studied in the previous section.

Now let us turn to the lower hybrid branch of the spectrum. First we note that for waves with the dispersion law (3.15) for any ratio of $\omega_{\rm H}$ to $\omega_{\rm p}$ the decay processes

$$\omega_{k_0}^- = \omega_{k_0-\kappa}^- + \omega_\kappa^- \tag{3.22}$$

are permitted inside the branch. The growth rate of this process for a monochromatic wave as well as for the process (3.20) tends to a constant with increasing perturbation wavenumber. Due to the finite size of the soliton the instability can be suppressed by the perturbations carried out.

It should be noted that the decay instability may stabilize due to the finite plasma size. Just this situation probably took place in some well-known experiments [90]. Oscillations excited in them had a wavelength comparable with plasma radius a. The dispersion law for such waves may be written in the form

$$\omega_k = \omega_p k_z a/\sqrt{1 + k_z^2 a^2} ,$$

that is, they represent a resonator mode with a fixed value of $k_\perp \sim 1/a$. In this case for small $k_z a$, waves with a small amplitude are described by the KdV equation for which, as we have seen, the solitons are stable. It is probably just this fact that can explain the observation of solitons in these experiments.

In the case of a homogeneous plasma the process (3.22) will be forbidden only for quasitransverse propagation when the oscillation frequency is close to ω_{LH}. Let us consider this problem in some more detail.

The equations describing oscillations with frequencies close to the lower-hybrid one can be obtained in the usual way by averaging over the "fast" frequency ω_{LH}. The equation for the electric field potential is of the form [91] (see also ref. [10]):

$$\nabla_\perp^2\left(\mathrm{i}\frac{\partial}{\partial t} + \tfrac{1}{2}\omega_{LH}R^2\nabla_\perp^2\right)\psi - \tfrac{1}{2}\omega_{LH}\frac{M}{m}\psi_{zz} = \mathrm{i}\frac{M}{2m}\frac{\omega_{LH}^2}{n_0\omega_H}[\nabla\,\delta n, \nabla\psi]_z . \tag{3.23}$$

In the static limit

$$\gamma < (k_z v_{Te}, k v_{Ti}) \tag{3.24}$$

the density variation under the action of ponderomotive forces is given by the equation

$$\delta n = -\frac{\mathrm{i}e^2}{m\omega_H\omega_{LH}(T_i + T_e)}[\nabla\psi, \nabla\psi^*]_z .$$

As a result, turning to dimensionless variables, we obtain

$$\nabla_\perp^2\left(\mathrm{i}\frac{\partial}{\partial t} + \nabla_\perp^2\right)\psi - \psi_{zz} = \mathrm{div}([\nabla\psi\,\nabla\psi^*]_z, [\boldsymbol{h}\,\nabla\psi]) ; \qquad \boldsymbol{h} = \boldsymbol{H}_0/H_0 . \tag{3.25}$$

Note a number of characteristic features of this equation. The nonlinear term in it vanishes for one-dimensional and axially symmetric solutions. In this case it is necessary to take into account weaker nonlinear effects. A similar consideration has been given in ref. [92] where it was shown that the propagation of one-dimensional packets of lower-hybrid waves is described by a modified KdV equation. It is however clear that these solutions are unstable with respect to transverse modulations due to stronger nonlinear mechanisms.

Induced scattering by particles [93] can be given as an example of a concrete instability mechanism.

Equation (3.25) describes only three-dimensional problems. In fact, as has been mentioned above, for axially-symmetric solution the nonlinear term vanishes. Strictly speaking, a consideration of the planar solutions $\partial\psi/\partial z = 0$ contradicts the criterion (3.24). The analysis shows, however, that upon the development of the modulation instability the condition $(M/m)\psi_{zz} \simeq R^2\nabla_\perp^4\psi$ must be satisfied. Therefore, instead of (3.25), the equation

$$\nabla_\perp^2(\mathrm{i}\psi_t + \nabla_\perp^2\psi) - \mathrm{div}([\nabla\psi, \nabla\psi^*]_z, [\boldsymbol{h}, \nabla\psi]) = 0 \qquad (3.26)$$

can be considered as a model one. Equation (3.26) has soliton solutions of the form

$$\psi = \exp(\mathrm{i}\lambda^2 t)\exp(\mathrm{i}m\varphi)f(r)$$

where $f(r)$ is determined by the equation

$$\hat{k}\left(-\lambda^2 + \frac{1}{r}\hat{k}\right)f + 2m^2 f\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}f^2 = 0\,,$$

$$\hat{k} = \frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{m^2}{r}\,.$$

Both equations (3.25) and (3.26) are Hamiltonian with

$$H = \int \left(|\nabla^2\psi|^2 + \tfrac{1}{2}[\nabla\psi, \nabla\psi^*]_z^2\right)\mathrm{d}r_\perp\,.$$

Its group properties are close to the properties of the two-dimensional NSE. Therefore in this situation, as well as for other similar examples, one can expect instability of the solitons. In actual fact, the nonstationary nature of the evolution of an initial local perturbation and its collapse has been confirmed by numerical calculations [94].

3.4. Stability of low-frequency solitons

As we have seen above, one of the main reasons of HF soliton instability can be associated with processes of decay of high-frequency oscillations into high- and low-frequency ones. If low-frequency ion acoustic waves are considered in an isotropic plasma, decay processes are forbidden which is essentially the main reason of the stability of the one-dimensional small-amplitude solitons described by the Kadomtsev–Petviashvili equation. A concrete proof of this fact is given in chapter 2 of this review. The case is quite different for low-frequency waves in a magnetized plasma. It is well-known that in the region of frequencies less than the cyclotron ion frequency ω_{Hi}, there exist three branches of low-frequency waves of the acoustic type: Alfvén waves (A), fast (M) and slow (S) magneto-acoustic waves. Between these three wave types various decay processes are possible, the matrix elements of which can be calculated, e.g., with the help of a standard Hamiltonian approach [5, 95] or directly from the equations of motion [96]. All these methods provide an immediate opportunity to determine without concrete calculations which solitons can be unstable. For example, for a low-pressure plasma $\beta = 8\pi \bar{n}T/H^2 \ll 1$ one can state that the Alfén waves and fast magneto-acoustic solitons will be unstable due to decays into A- and M-waves and an S-wave. When $\beta \ll 1$ the S-waves play a role similar to that of ion-acoustic waves in the decay processes for Langmuir waves.

Therefore among the three types of waves stable solitons can be expected only for S-waves. Let us consider this situation in some more detail. If the plasma is collisionless, for frequencies $\omega \ll \omega_{\mathrm{Hi}}$ slow magneto-acoustic waves exist only in a non-isothermal plasma when $T_e \gg T_i$, that is, they represent magnetized ion-acoustic oscillations. In the long-wave range the dispersion law for these oscillations takes the form

$$\omega = k_z c_s (1 - \tfrac{1}{2} k^2 r_d^2 - \tfrac{1}{2} k_\perp^2 r_H^2) \tag{3.27}$$

where $r_d = V_{Te}/\omega_{pe}$ is the Debye radius, $r_H = c_s/\omega_{Hi}$ is the ion Larmor radius calculated with respect to the ion sound velocity $c_s = \sqrt{T_e/M}$. Here, the first dispersion term describes a deviation from quasineutrality while the second one describes the dispersion of the ion-cyclotron frequency. Since the oscillation frequency is less than the ion-cyclotron one, one can consider the ions in these oscillations to move along the magnetic field. Therefore the nonlinear equations can in this case be written as follows [4, 97]:

$$\begin{aligned} &\frac{\partial n}{\partial t} + \frac{\partial}{\partial z} (1 + r_H^2 \nabla_\perp^2) v_z + \frac{\partial}{\partial z} n v_z = 0 \,, \\ &\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} = -c_s \frac{\partial}{\partial z} [n - \tfrac{1}{2} n^2 + r_d^2 \nabla^2 n] \,. \end{aligned} \tag{3.28}$$

Here $n = \delta\rho/\rho_0$, $\delta\rho$ is the density fluctuation. According to (3.27) the group velocity in the long-wavelength region of magnetized ion-acoustic oscillations is directed along the magnetic field; in this case a weak interaction of waves propagating in opposite directions is observed. This gives an opportunity for using the procedure described in section 1.1 to reduce the system (3.28) to one equation [4]

$$\frac{\partial v_z}{\partial t} + c_s \frac{\partial}{\partial z} \left\{ 1 + \tfrac{1}{2}(r_H^2 + r_d^2)\nabla_\perp^2 + \tfrac{1}{2} r_d^2 \frac{\partial^2}{\partial z^2} + \frac{1}{2} \frac{v_z}{c_s} \right\} v_z = 0 \tag{3.29}$$

which describes ion-acoustic waves propagating in one direction along the magnetic field. Equation (3.29) is a generalized KdV equation. Passing to a coordinate system moving with the sound velocity along the magnetic field and introducing the variables

$$\begin{aligned} &r_d^{-1}(z - c_s t) \to z \qquad && (r_H^2 + r_d^2)^{-1/2} r_\perp \to r_\perp \\ &\tfrac{1}{2}\omega_{pi} t \to t && v_z/6c_s \to u \end{aligned}$$

we write eq. (3.29) in the dimensionless form:

$$u_t + \frac{\partial}{\partial z} (\nabla^2 + 3u) u = 0 \,. \tag{3.30}$$

Equation (3.30) can be written in Hamiltonian form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z} \frac{\delta H}{\delta u}$$

where the Hamiltonian

$$H = \int [\tfrac{1}{2}(\nabla u)^2 - u^3]\,\mathrm{d}V\,. \tag{3.31}$$

Besides H eq. (3.30) has some more very simple integrals:

$$M(r_\perp) = \int u\,\mathrm{d}z\,; \qquad P = \tfrac{1}{2}\int u^2\,\mathrm{d}V$$

$$I = \int \boldsymbol{r}u\,\mathrm{d}V - t\boldsymbol{n}\int u^2\,\mathrm{d}V\,.$$

The first integral has the meaning of the law of conservation of "mass" along the line $r_\perp = \text{const}$. The second integral is the law of conservation of momentum, and the third one is the law of conservation of centre-of-mass. From the latter it follows in particular that the centre-of-mass velocity is equal to $2P/\int M\,\mathrm{d}r_\perp$ and directed along the magnetic field.

Further let us consider stationary solutions of eq. (3.30) of the form $u = u(z - Vt, r_\perp)$ which obey the equation

$$\nabla^2 u = (v - 3u)u\,. \tag{3.32}$$

For $V > 0$ it has a solution decreasing exponentially as $r \to \infty$. In the one-dimensional case these are the well-known solutions of the KdV equation

$$u_0 = 2\kappa^2/\cosh^2\kappa(x - 4\kappa^2 t)\,.$$

In the three-dimensional case the simplest soliton is a spherical symmetric soliton without nodes. According to our classification it corresponds to the ground-state soliton. By analogy with (1.46) one can find the connection between H and P for the soliton. For this purpose we notice that eq. (3.32) can be represented in the form

$$\delta(H + PV) = 0$$

so that all its solutions are stationary points of the functional H for fixed P. From this it is easy to determine that for the soliton solution

$$H_s = VP_s(d - 4)/(6 - d)$$

is negative for any dimensions $d = 1, 2, 3$. Using integral estimates (1.48) it is not difficult to verify that central symmetric solutions of the soliton type without nodes for $d = 1, 2, 3$ are realizing the minimum of H for fixed P. Thus in the class of finite solutions solitons will be stable for each dimension of those mentioned above.

As to the three-dimensional case, it remains to consider the stability of one- and two-dimensional solitons with respect to three-dimensional perturbations. For the sake of simplicity let us consider only the stability of a one-dimensional soliton u_0 with respect to perturbations $\delta u = \psi(x - Vt)\exp(-i\omega t + ik_\perp r_\perp)$. For ψ we have the following spectral problem:

$$-\mathrm{i}\omega\psi - V\psi_x + \psi_{xxx} + 6\,\frac{\partial}{\partial x}\,(u\psi) - k_\perp^2\psi_x = 0\,.$$

Let the perturbations be long-wavelength ones $k_\perp \to 0$; we expand ψ in a series

$$\psi = \psi_0 + \psi_1 + \dots\,.$$

We choose the translational mode $\partial u_0/\partial x$ as ψ_0. To first order as for the KP equation (see (2.48)) we obtain

$$\psi_1 = \mathrm{i}\omega\,\partial u_0/\partial V\,.$$

To a second approximation we have

$$-\mathrm{i}\omega\psi_1 - k_\perp^2\psi_{0x} = A\psi_2 \tag{3.33}$$

where the operator

$$A = v\,\frac{\partial}{\partial x} - \frac{\partial^3}{\partial x^3} - 6\,\frac{\partial}{\partial x}\,u_0\,.$$

It is easy to see that the conjugated operator A has a zeroth eigenfunction u_0. Therefore the dependence of the frequency ω on k is defined as the solubility condition (3.33). Multiplying (3.33) scalarly by u_0 we get [113]

$$\omega^2 = -\tfrac{4}{15}k_\perp^2V^2 < 0\,.$$

Thus a one-dimensional soliton is unstable with respect to bending perturbations. The two-dimensional soliton instability is determined in a similar way.

The results of this section show that a three-dimensional ion-acoustic soliton with a ratio of longitudinal and transverse dimensions $(1 + r_\mathrm{H}^2/r_\mathrm{d}^2)^{-1/2}$ is the only stable soliton. Usually $r_\mathrm{d} \ll r_\mathrm{H}$, that is, the solitons are of a pancake form. All other solitons are unstable when $\beta \ll 1$.

4. On wave collapse

As we have seen, in many cases solitons are unstable, especially solitons in a plasma. In the present review we have often pointed out that this instability should most commonly lead to wave collapse. Mathematically it means an increasing of the wave field amplitude till infinity occurs in some space point after a finite time. From the physical point of view the collapse is a spontaneous concentration of wave energy in a small area of space with its consequent dissipation. We think that the wave collapse concept possesses a great degree of universality and collapses are as widely distributed in nature as solitons. This concept is rather advanced nowadays and is supported by many numerical experiments and for its detailed description one should need an article of the same volume as the present one.

Here we shall present the simplest facts.

The nonlinear Schrödinger equation (1.43) is the most important mathematical model of wave collapse in the case of space dimensions $d \geq 2$. As has been stated above for the two-dimensional case ($d = 2$) this equation describes a stationary self-focusing (if we take the longitudinal coordinate z as a variable t). In connection with its physical applications the two-dimensional equation (1.43) (usually an axial-symmetric one) has been studied numerically since the middle of the 1960's. In the pioneering experiments [99] it has been shown that for a sufficiently large initial laser beam intensity the field amplitude ψ increases without limits when approaching a certain time $t = t_0$. This phenomenon being interpreted as the formation of "point focuses" was used as the basis of the self-focusing theory by Lugovoi and Prokhorov [100] which helped to explain the majority of experimentally observed data. In fact, this was how the first example of wave collapse was discovered. Let us show the way to it from the analysis of eq. (1.43).

We begin with considerations insufficiently rigorous mathematically but possessing a considerable physical generality. We shall first notice that the Hamiltonian of (1.43)

$$H = \int (|\nabla\psi|^2 - \tfrac{1}{2}|\psi|^4)\, \mathrm{d}V$$

is not positive and can take negative values. Furthermore, as we have shown in section 1.3, for $d \geq 2$ its value for a fixed integral $N = \int |\psi|^2\, \mathrm{d}V$ can be infinitely large in absolute value. Formula (1.45) is the confirmation of this fact; it shows that for $d \geq 2$ the functional H takes nonnegative values at its stationary points (which exist only for $d < 4$). On the other hand, for waves of a small amplitude the Hamiltonian H is positive. Let us consider the evolution of a localized initial pulse for which $H < 0$. This pulse due to the conservation of the integrals of motion can neither be radiated to infinity in the form of waves of a small amplitude nor pass into one of the stationary states. The formation of a quasistationary oscillating state should be also excluded since it must be accompanied by energy radiation to infinity (with the exception of the extremely degenerate case of the "breather" type in the sine-Gordon equation).

The only possibility remaining is the formation of a singularity in the vicinity of which $H \to -\infty$.

These heuristic considerations for eq. (1.43) can be supported by a mathematical proof based on the virial theorem (1.54). In our case $n = 1$ it represents the relation

$$\frac{\partial^2}{\partial t^2} \int r^2 |\psi|^2\, \mathrm{d}V = 8H - (d-2) \int |\psi|^4\, \mathrm{d}V \tag{4.1}$$

from which for $d \geq 2$ there follows the inequality

$$\int r^2\, |\psi|^2\, \mathrm{d}V \leq 4Ht^2 + c_1 t + c_2 \tag{4.2}$$

(c_1, c_2 are some constants) which becomes an exact equality for $d = 2$. Let $H < 0$. Then for arbitrary values c_1, c_2 the equality (4.2) for sufficiently large t becomes contradictory; it signifies the existence of a singularity of the solution of eq. (1.43). That the field ψ increases to infinity for this singularity has been proved by Zhiber [101].

Unfortunately such an elegant proof of the existence of collapse is not known for all cases when collapse takes place. In a number of cases for equations integrated by the inverse scattering transform method the existence of collapse can be determined directly by explicit calculations of exact solutions which contain the collapse [6, 102].

In other cases the above considerations based on conservation laws should be supported by numerical experiments.

The question about the nature of the field near the collapse point is a matter of great importance. In the majority of the investigated situations this dependence is self-similar so that

$$|\psi|^2 \sim \frac{1}{f^{d-\varepsilon}(t_0 - t)} R\left(\frac{r}{f(t_0 - t)}\right). \tag{4.3}$$

Here $f(\xi)$ is some function with $f(0) = 0$. From the conservation of the integral $\int |\psi| \, dV$ it follows that the constant ε is nonnegative. Two radically different cases are possible.

When $\varepsilon = 0$ a finite fraction of the integral N (the number of quasiparticles) is incident upon the point of collapse. Such a collapse is called a strong one. If $\varepsilon > 0$ the number of quasiparticles trapped in the collapse process is formally equal to zero. In this case, at the moment of collapse an integrable singularity is formed at the point $r = 0$. Such a collapse is usually called a weak collapse.

Even for the simplest model of the nonlinear Schrödinger equation (1.43) the problem of the nature of the collapse has not been completely solved. The collapse is undoubtedly strong in the two-dimensional case. The form of the function $f(\xi)$ is not definitively determined but there exist rather convincing arguments in favour of the fact that as $\xi \to 0$ it possesses the asymptotic form $f(\xi) \propto \xi^{1/2}$. In the three-dimensional case the exact self-similar solution $\varepsilon = 1$, $f(\xi) \propto \xi^{1/2}$ corresponding to a weak collapse has been constructed. An integrable singularity $|\psi|^2 = c/r^2$ is formed in the case of such a collapse. However, recently another approximate solution of the quasiclassical type $f(\xi) \sim \xi^{2/5}$ [114] has been constructed for which the collapse is strong. The detailed picture of the collapse for the nonlinear Schrödinger equation can ultimately be solved by means of numerical experiments.

Physically the Langmuir wave collapse in a plasma is of the greatest interest; it arises, in particular, as a result of the development of the plasma soliton instability described above in great detail. The Langmuir collapse has recently been investigated extensively both analytically and numerically (see the review [80, 81] and papers [103–105]). The nature of the Langmuir collapse depends significantly on the oscillation energy level. For a small energy level $w/nT < m/M$ (the subsonic case) the nature of the collapse is roughly the same as for the three-dimensional nonlinear Schrödinger equation.

In the most interesting, so-called supersonic, case $w/nT > m/M$ the nature of the collapse has been determined with a high degree of certainty. Here the collapse is strong, with $f(\xi) \propto \xi^{2/3}$. This result is supported by a great number of numerical experiments (see, e.g., refs. [104, 105]).

According to our concepts the Langmuir collapse plays a great role in plasma turbulence physics. Multiple collapses are present in the majority of experimental and astrophysical situations in which Langmuir oscillations are excited by an electron beam, by variable electromagnetic field or by other techniques. They are difficult to observe because of the small size of the collapsing cavities and their short lifetime (though the fact of the existence of Langmuir collapse has recently been proved experimentally in the papers by Wong and his group [106]). However, they have a great effect on the whole picture of the turbulence. The main dissipation of the energy of Langmuir oscillations takes place in collapsing cavities with this energy being transferred to fast electrons which are frequently observed, e.g., in experiments on the laser heating of a plasma.

Collapsing cavities also generate intense acoustic oscillations (see ref. [107]).

Since the character of the instability of various plasma solitons, such as acoustic, lower- and upper-hybrid ones, is very similar to the Langmuir oscillation instability we can expect the existence of different kinds of collapse whose contribution to plasma turbulence physics must also be fairly important. The first results in this direction are by now available [108].

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