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Integrability of Nonlinear Systems and Perturbation Theory

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1. Introduction

The theory of so-called integrable Hamiltonian wave systems arose as a result of the inverse scattering method discovery by *Gardner, Green, Kruskal* and *Miura* [1] for the Korteveg–de Vries equation. This discovery was initiated by the pioneering numerical experiment by *Kruskal* and *Zabusky* [2]. After a pragmatic phase, which was devoted to finding new soliton equations, the theory became rather complicated. One of its branches may be called the "qualitative theory of infinite-dimensional Hamiltonian systems", to which the results reviewed in this paper belong. We consider only Hamiltonian systems possessing Hamiltonians with a quadratic part which may be transformed in normal variables to the form

$$H_0 = \sum_{\alpha=1}^{N} \int \omega_{\boldsymbol{k}}^{(\alpha)} a_{\boldsymbol{k}}^{(\alpha)} a_{\boldsymbol{k}}^{*(\alpha)} d\boldsymbol{k} \,. \tag{1.1.1}$$

Here, $a_{\boldsymbol{k}}^{(\alpha)}$ are normal coordinates of the α-th linear mode (usually simply expressed through Fourier components of physical fields): $\boldsymbol{k} = (k_1, \ldots, k_d)$ is the wave vector; d is the dimension of space; and $\omega_{\boldsymbol{k}}^{(\alpha)}$ is the dispersion law of the α-th mode. Corresponding Hamiltonian systems, i.e., those having Hamiltonians of the form

$$H = H_0 + H_{\text{int}} \,, \tag{1.1.2}$$

are called "Hamiltonian wave systems". The majority of nonlinear wave theory problems may be mapped into this class. The crucial property of systems (1.1.2) is that they make a weak nonlinear approximation possible. Our approach is based on treating H_{int} as a perturbation; besides, we assume that H_{int} is an analytic functional of the fields $a_{\boldsymbol{k}}^{(\alpha)}$. This is not very limiting since it is usually true at the weak nonlinear approximation.

The qualitative theory of infinite-dimensional Hamiltonian systems now being developed stems from the qualitative theory of ordinary differential equations; all existing methods can somehow be linked with this theory. The approach used in the papers by *Newell, Tabor* and by *Siggia*, and *Ercolani*, presented in this volume, actually originates from the analytic theory of ordinary differential equations, while Makhailov, Sokolov and Shabat's method can be traced to the Sophus Lie symmetry theory. Our own work stems from Poincaré's proof of the nonexistence of the invariants of motion, analytic in a small parameter, and from Birkhoff's results on the canonical transformations of Hamiltonians to the normal form near the equilibrium. The main theorem in Sect. 2.2 is an infinite-dimensional generalization of the well-known *Poincaré* theorem [3] which determines the sufficient conditions for the nonexistence of an additional motion invariant; the theorem in Sect. 2.6 should be considered as a theorem which in analogy with Birkhoff's result determines the conditions for a Hamiltonian wave system to be reducible to the form of the Birkhoff's infinite-dimensional integrable chain. In the infinite-dimensional case a new notion arises, which is absent in the finite-dimensional case: the degenerative dispersion laws.

The Painlevé test method is based on the study of solution singularities and works effectively both in one-dimensional and in multidimensional cases. It may be used to determine whether a given equation is solvable exactly. If the equation satisfies the test, a Lax representation may be found for it. The "Lie-Bäcklund symmetry approach" is used for one-dimensional systems with functional freedom: it permits conclusions about the existence or absence of additional *local* motion invariants and symmetries, thus making possible a choice of "good" equations among those of a given functional form. This method is, however, inappropriate for finding L-A pairs.

Our approach does not permit functional arbitrariness in an equation but effectively proves the nonexistence of additional motion invariants analytic in $a_k^{(\alpha)}$ independent of its locality or nonlocality and the dimensionality. For reasons which will be explained below, this method is often simpler in multidimensional spaces.

An approach based on perturbation theory has another important advantage. It concerns the definition of the content of the concept "integrable equations". It leads to a natural subdivision into two classes of all systems of the form (1.1.2) with additional integrals: i) exactly solvable but not integrable in Liouville's sense and ii) exactly solvable and completely integrable. For example, the Kadomtsev-Petviashvili (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x = 3\alpha^2 u_{yy} \tag{1.1.3}$$

with $\alpha^2 = 1$, belongs to the first class, while this equation with $\alpha^2 = -1$ and the well-known Davey-Stewartson equation (DS),

$$\begin{aligned} &\mathrm{i}\Psi_t + \Psi_{yy} - \Psi_{xx} + \Phi\Psi = 0 \\ &\Phi_{xx} + \Phi_{yy} = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)|\Psi|^2 , \end{aligned} \tag{1.1.4}$$

belong to the second class [4–7].

This method of classification, properties of the equations from the first and the second classes, interrelations between solvability (existence of commutation representation and infinite number of conservation laws) and complete integrability (introduction of virtual action-angle variables which do not disappear at periodic boundary conditions) are considered in Sect. 2.6 for the general case of periodic boundary conditions.

Besides the above-mentioned direct methods, other approaches may be effectively applied in some cases: the Walquist-Estabrooq method of finding L-A pairs, the method of searching for alternative commutation representation (when a linear operator defining time dynamics arises as the Gateux derivative of the original equation; see *Chen, Lie, Lin* [8]), etc.

Our paper is organized as follows: Chap. 2 is self-contained; it is devoted to the description of the general theory in the case of zero boundary conditions at infinity with the exception of Sect. 2.6, in which periodic boundary conditions are explored. Chapter 3 contains some information about the physics giving rise to various universal, exactly solvable equations (Sect. 3.1) and their properties from the viewpoint of the general theory (Sects. 3.2, 3); it also offers examples of verification of the integrability of some particular systems (Sects. 3.4, 5). The appendices contain proofs of the most important theorems.

2. General Theory

2.1 The Formal Classical Scattering Matrix in the Solitonless Sector of Rapidly Decreasing Initial Conditions [6]

Consider a homogeneous medium of d dimensions, where the waves of N types can propagate, and their dispersion laws are $\omega_k^{(\alpha)}$, $\alpha = 1, \ldots, N$. The Hamiltonian of such a medium can be represented in the form (1.2) (see Sect. 3.1), with H_0 of the form (1.1.1) and H_{int} practically always being the functional series in the complex normal coordinates $a_k^{(\alpha)}$, $a_k^{*(\alpha)}$, $\alpha = 1, \ldots, N$. The $a_k^{(\alpha)}$ indicate the wave amplitudes for corresponding linear modes with wave vector $\boldsymbol{k}$. Amplitudes $a_k^{(\alpha)}$ obey the equations

$$\mathrm{i}\dot{a}_k^{(\alpha)} = \omega_k^{(\alpha)} a_k^{(\alpha)} + \frac{\delta H_{\text{int}}}{\delta a_k^{*(\alpha)}} . \tag{2.1.1}$$

In analogy with the quantum scattering theory, let us consider the system with interaction, adiabatically decreasing as $t \to \pm\infty$:

$$H = H_0 + H_{\text{int}} \mathrm{e}^{-\epsilon|t|} . \tag{2.1.2}$$

For the system (2.1.1), the global solvability theorem may not be fulfilled, and asymptotic states may not exist as $t \to \pm\infty$. However, for the system with the Hamiltonian (2.1.2) at finite and sufficiently small a_k, they do exist, i.e., the solution of (2.1.1) turns asymptotically into the solution of the linear equation:

$$a_k^{(\alpha)}(t) \longrightarrow \left[a_k^{(\alpha)}(t)\right]^{\pm} = \left[C_k^{(\alpha)}\right]^{\pm} \mathrm{e}^{-\mathrm{i}\omega_k^{(\alpha)} t} . \tag{2.1.3}$$

Furthermore, asymptotic states may contain solitons, which certainly cannot exist at finite ε. So our consideration should be restricted to the class of initial states without solitons and with smooth $C_{\boldsymbol{k}}^{-}$. We shall call this class the solitonless sector. Although our consideration is restricted to a special class of initial states, the result will be very useful because the structure obtained for the formal series for the S-matrix provides us with the structure of motion invariants (Sect. 2.5).

Functions $C_{\boldsymbol{k}}^{(\alpha)\pm}$ are not independent; there is a nonlinear operator $S_\varepsilon^{(\alpha)}[C^-]$, transforming them into each other. To study this operator we go as usual to the interaction representation:

$$a_{\boldsymbol{k}}^{(\alpha),s}(t) = b_{\boldsymbol{k}}^{(\alpha),s}(t)\mathrm{e}^{-\mathrm{i}s\omega_{\boldsymbol{k}}^{(\alpha)}t} \,. \tag{2.1.4}$$

Here, $s = \pm 1$, $a_{\boldsymbol{k}}^1(t) = a_{\boldsymbol{k}}(t)$, $a_{\boldsymbol{k}}^{-1}(t) = a_{\boldsymbol{k}}^*(t)$. The motion equations now take the form

$$s\mathrm{i}\dot{b}_{\boldsymbol{k}}^{(\alpha),s} = \frac{\delta H_{\mathrm{int}}}{\delta b_{\boldsymbol{k}}^{(\alpha),-s}} \,. \tag{2.1.5}$$

In (2.1.5), H_{int} is the interaction Hamiltonian expressed in the variables $b_{\boldsymbol{k}}^s$. Equation (2.1.5) is equivalent to an integral equation:

$$b_{\boldsymbol{k}}^{(\alpha),s} = \left[C_{\boldsymbol{k}}^{(\alpha),s}\right]^- - \frac{\mathrm{i}s}{2}\int_{-\infty}^{t} dt_1 \frac{\delta H_{\mathrm{int}}}{\delta b_{\boldsymbol{k}}^{(\alpha),-s}(t_1)}\mathrm{e}^{-\varepsilon|t_1|} \,. \tag{2.1.6}$$

Equation (2.1.6) gives a map $C_{\boldsymbol{k}}^{(\alpha),-} \to b_{\boldsymbol{k}}^{(\alpha),s}(t)$ which may be written in the form

$$b_{\boldsymbol{k}}^{(\alpha),s} = S_\varepsilon^{(\alpha),s}(-\infty, t)\left[\{C_{\boldsymbol{k}}^-\}\right] \,. \tag{2.1.7}$$

As $t \to +\infty$ in (2.1.7), one finds

$$C_{\boldsymbol{k}}^{(\alpha),+} = S_\varepsilon\left[\{C_{\boldsymbol{k}}^-\}\right] \,, \tag{2.1.8}$$

where $S_\varepsilon^{(\alpha)} = S_\varepsilon^{(\alpha)}(-\infty, \infty)$.

At finite ε and sufficiently small $a_{\boldsymbol{k}}^{(\alpha)}$, operators $S_\varepsilon^{(\alpha)}(-\infty, \infty)$ and $S_\varepsilon^{(\alpha)}$ may be obtained in the form of a convergent series by iterations of (2.1.6). Let $\varepsilon \to 0$ now in each term of the series. As we shall see, the expression obtained is finite in the sense of generalized functions. The series obtained for the operator $S_\varepsilon^{(\alpha)}(-\infty, t)$ as $\varepsilon \to 0$ will be called the classical transition matrix. We shall refer to the corresponding series for S as the formal classical scattering matrix. Let us designate

$$S^{(\alpha)}(-\infty, t) = \lim_{\varepsilon\to 0} S_\varepsilon^{(\alpha)}(-\infty, t) \tag{2.1.9}$$

$$S^{(\alpha)} = \lim_{\varepsilon\to 0} S_\varepsilon^{(\alpha)}(-\infty, \infty) \,,$$

where the limits are to be understood in the above-mentioned sense.

Before proceeding, let us introduce a more convenient notation. For the function $\Pi_{\boldsymbol{k}_1\ldots\boldsymbol{k}_n}^{\pm s_1\ldots\pm s_n}$ we will write simply $\Pi_{\pm 1,\ldots,\pm n}$. Moreover, we will deisgnate

$$\Pi_{\pm 1,\ldots,\pm n}\delta(\pm s_1\boldsymbol{k}_1 \pm \ldots \pm s_n\boldsymbol{K}_n) = \hat{\Pi}_{\pm 1,\ldots,\pm n}$$

and

$$\hat{\Pi}_{\pm 1,\ldots,\pm n}\delta\left(\pm s_1\omega_{\boldsymbol{k}_1} \pm \ldots \pm s_n\omega_{\boldsymbol{k}_n}\right) = \hat{\hat{\Pi}}_{\pm 1,\ldots,\pm n} \,.$$

This notation reduces the length of the formulae and makes their structure visible. In addition we will use the special notation

$$\begin{aligned} E_{\pm 1,\ldots,\pm n} &= \pm s_1\omega_{\boldsymbol{k}_1} \pm \ldots \pm s_n\omega_{\boldsymbol{k}_n} \\ \boldsymbol{P}_{\pm 1,\ldots,\pm n} &= \pm s_1\boldsymbol{k}_1 \pm \ldots \pm s_n\boldsymbol{k}_m \,. \end{aligned} \tag{2.1.10}$$

As $\varepsilon \to 0$, the series for $S_\varepsilon(-\infty, t)$ and for S_ε are generally speaking divergent and formal. Consider the structure of the classical scattering matrix in the simplest case of a cubic interaction Hamiltonian H_{int} and only one mode:

$$\begin{aligned} H_{\mathrm{int}} = &\frac{1}{3!}\sum_{ss_1s_2}\int V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{ss_1s_2}a_{\boldsymbol{k}}^s a_{\boldsymbol{k}_1}^{s_1}a_{\boldsymbol{k}_2}^{s_2} \\ &\times \delta(s\boldsymbol{k} + s_1\boldsymbol{k}_1 + s_2\boldsymbol{k}_2)d\boldsymbol{k}\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2 \,. \end{aligned} \tag{2.1.11}$$

From the fact that the Hamiltonian is real, it follows that

$$V_{*\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{-s-s_1-s_2} = V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{ss_1s_2} \,. \tag{2.1.12}$$

Besides, coefficient functions V possess an evident symmetry,

$$V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{ss_1s_2} = V_{\boldsymbol{k}\boldsymbol{k}_2\boldsymbol{k}_1}^{ss_2s_1} = {}_{\boldsymbol{k}_1\boldsymbol{k}\boldsymbol{k}_2}^{s_1ss_2} \,. \tag{2.1.13}$$

In the interaction representation, we have the integral equation

$$\begin{aligned} \mathrm{i}s\left(b_{\boldsymbol{k}}^s(t) - c_{\boldsymbol{k}}^s\right) = &\frac{1}{2}\sum_{s_1s_2}\int_{-\infty}^{t} dt_1 \int d\boldsymbol{k}_1 d\boldsymbol{k}_2 V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{-ss_1s_2}(t_1) \\ &\times b_{\boldsymbol{k}_1}^{\delta_1}(t_1)b_{\boldsymbol{k}_2}^{\delta_2}(t_2)\delta\left(-s\boldsymbol{k} - s_1\boldsymbol{k}_1 + s_2\boldsymbol{k}_2\right) \end{aligned} \tag{2.1.14}$$

$$V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{ss_1s_2}(t) = V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{ss_1s_2}\exp\left(\mathrm{i}E_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{ss_1s_2}t - \varepsilon|t|\right) \tag{2.1.15}$$

$$E_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{ss_1s_2} = s\omega_{\boldsymbol{k}} + s_1\omega_{\boldsymbol{k}_1} + s_2\omega_{\boldsymbol{k}_2} \,. \tag{2.1.16}$$

Equation (2.1.14) may be symbolically represented in graphical form:

$$s \;{=\!=\!=\!=\!=}\; s \;-\cdot-\cdot-\; -\frac{\mathrm{i}}{2}\;\text{———}0{\genfrac{}{}{0pt}{}{=\,=\,=\,=\,=}{=\,=\,=\,=\,=}} \,, \tag{2.1.17}$$

where $=\!=\!=\!=\!=$ indicates the two-component over the index s unknown value $b_{\boldsymbol{k}}^s$, $s = \pm 1$; $-\cdot-\cdot-$ designates $c_{\boldsymbol{k}}^{-s}$; ——— corresponds to the factor $\exp\{-\mathrm{i}E_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{-ss_1s_2}\}$; 0 indicates $V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{-ss_1s_2}\delta(-s\boldsymbol{k} + s_1\boldsymbol{k}_1 + s_2\boldsymbol{k}_2)$, and summation is assumed over s_1 and s_2. Using (2.1.17), certain graphical expressions (diagrams) may be attributed to each term of the series arising when iterating (2.1.14). These graphical expressions are connected graphs, having no loops; they are, in other words, "trees".

Each graph consists of two types of elements: lines and vertices; the former are subdivided into inner and external lines. One of the external lines is different from the others (we shall call it a "root"); the other ones may be called "leaves". Each tree, corresponding to the n-th iteration, contains exactly n vertices and $n+2$ leaves. Inner lines are usually called "branches". They correspond to both the external and internal lines, a certain value of wave vector $\boldsymbol{k}_i$ and the index s_i. The "external" value of $\boldsymbol{k}$ and s corresponds to the root. Integration goes over all $\boldsymbol{k}_i$ except $\boldsymbol{k}_i = \boldsymbol{k}$; the summation goes over all s_i except $s_i = s$. To each leaf with the wave vector $\boldsymbol{k}_q$ and index s_q corresponds a factor $c_{\boldsymbol{k}_q}^{-s_q}$.

The graph corresponding to the N-th iteration contains N integrations over time variables $t_1, \ldots, t_N$. Each time variable t_i in the diagram for the transition matrix corresponds to its own branch. The external time t corresponds to the root. The presence of the root leads to partial ordering of the graph elements. From each vertex in which three lines meet there is a unique path to the root. The line leading to the root we shall designate as the exiting line. Let the corresponding wave vector and index be $\boldsymbol{k}_\alpha$ and s_α. The other two lines are entering. Let them correspond to the wave vector $\boldsymbol{k}_\beta, \boldsymbol{k}_\gamma$ and indices s_β, s_γ. It is important that both entering lines correspond with one and the same time variable t_q. Corresponding to this vertex factor is

$$V^{-s_\alpha s_\beta s_\gamma}_{\boldsymbol{k}_\alpha \boldsymbol{k}_\beta \boldsymbol{k}_\gamma} \exp\left[\mathrm{i}E^{-s_\alpha s_\beta s_\gamma}_{\boldsymbol{k}_\alpha \boldsymbol{k}_\beta \boldsymbol{k}_\gamma} - \varepsilon|t_q|\right] \delta\left(-s_\alpha \boldsymbol{k}_\alpha + s_\beta \boldsymbol{k}_\beta + s_\gamma \boldsymbol{k}_\gamma\right) . \tag{2.1.18}$$

Let us cut the graph across the line exiting from the vertex. Now that part of the graph which is cut off from the root is to be integrated over the variable t_q in the limits $-\infty < t_q \le t_p$. In fact this method of ordering is equivalent to the chronological ordering used in quantum field theory.

To conclude our description of the diagram technique let us note that the set of digarams which correspond to the n-th iteration consists of all possible trees containing n-vertices and fixed roots. In front of each diagram there is a numerical factor i/p. The number p is equal to the number of the symmetry group elements for the diagram under consideration, i.e. the number of rotations at different vertices which leaves the diagram unchanged, identity transformation included.

At finite $\varepsilon > 0$, the actual calculation of diagrams is a rather difficult task. However, it becomes much simpler as $\varepsilon \to 0$. We shall refer to integration over the time variable t_1 closest to the root as outer integration; all the other integrations will be called inner integrations. It is important that when integrating over any inner variable t_q, one may make the replacement

$$\mathrm{e}^{-\varepsilon|t_q|} \to \mathrm{e}^{\varepsilon t_q} . \tag{2.1.19}$$

We shall not prove this statement here. The analogous statement has been proved in the quantum field theory (see [10], for example). It is important to notice that using (2.1.19), all the integrations over inner times may be carried out explicitly, greatly simplifying the diagram technique.

Consider an inner branch with the wave vector $\boldsymbol{k}_p$ and the index s_p such that when cutting it, we may separate a tree having m leaves ($m \ge 2$) from the root. Let these leaves have wave vectors $\boldsymbol{k}_i$ and indices s_i, $i = 1, \ldots, m$. Let the vertex, from which this tree grows, be entered from the other sides by lines (branches or leaves) with the wave vectors and indices $\boldsymbol{k}_q$, $\boldsymbol{k}_r$ and s_q, s_r. Then the expression corresponding to this vertex is as follows (the line with $\boldsymbol{k}_q$, s_q is the exiting line):

$$V^{-s_q s_r s_p}_{\boldsymbol{k}_q \boldsymbol{k}_r \boldsymbol{k}_p} \delta\left(-s_q \boldsymbol{k}_q + s_r \boldsymbol{k}_r + s_p \boldsymbol{k}_p\right) , \tag{2.1.20}$$

while the expression corresponding to the branch with the wave vector $\boldsymbol{k}_p$ and the index s_p is

$$G_m = \lim_{\varepsilon \to 0} \frac{\exp(\mathrm{i}E_m t + m\varepsilon t)}{\mathrm{i}(E_m - \mathrm{i}m\varepsilon)} = \frac{\exp(\mathrm{i}E_m t)}{\mathrm{i}(E_m - \mathrm{i}0)} . \tag{2.1.21}$$

Consider now the last (outer) integration over t_1. We have

$$S_{N\varepsilon}(-\infty, t) = W_N \int_{-\infty}^{t} \exp\left[-\varepsilon|t_1| + \mathrm{i}E_N t_1\right] dt_1 . \tag{2.1.22}$$

Here,

$$W_N = W^{-s, s_1 \ldots s_N}_{\boldsymbol{k}, \boldsymbol{k}_1 \ldots \boldsymbol{k}_N} \delta(-s\boldsymbol{k} + s_1 \boldsymbol{k}_1 + \ldots + s_N \boldsymbol{k}_N) \tag{2.1.23}$$

is some expression which tends to the constant in the limit $\varepsilon \to 0$. At finite t we have, from (2.1.22),

$$S_N(-\infty, t) = \lim_{\varepsilon \to 0} S_{N\varepsilon}(-\infty, t) = \frac{W_N \mathrm{e}^{\mathrm{i}E_N t}}{\mathrm{i}(E_N - \mathrm{i}0)} ; \tag{2.1.24}$$

as $t \to +\infty$, we have

$$S_N = \lim_{\varepsilon \to 0} S_{N\varepsilon}(-\infty, \infty) = 2\pi\delta(E_N) W_N . \tag{2.1.25}$$

So the expressions for the $S_N(-\infty, t)$ and S_N have the singularity on a manifold defined by the equations

$$\begin{aligned} \boldsymbol{P}_N &= -s\boldsymbol{k} + s_1 \boldsymbol{k}_1 + \ldots + s_N \boldsymbol{k}_N = 0 \\ E_N &= -s\omega_{\boldsymbol{k}} + s_1 \omega_{\boldsymbol{k}_1} + \ldots + s_k \omega_{\boldsymbol{k}_N} = 0 . \end{aligned} \tag{2.1.26}$$

Equation (2.1.26), depending on the choice of the $s, s_1, \ldots, s_N$, splits into a set of relations:

$$\begin{aligned} &\boldsymbol{k} + \boldsymbol{k}_1 + \ldots + \boldsymbol{k}_n = \boldsymbol{k}_{n+1} + \ldots + \boldsymbol{k}_{n+m} \\ &\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}_1} + \ldots + \omega_{\boldsymbol{k}_n} = \omega_{\boldsymbol{k}_{n+1}} + \ldots + \omega_{\boldsymbol{k}_{n+m}} . \end{aligned} \tag{2.1.27}$$

Equation (2.1.27) determines a manifold which we shall call the resonant manifold $\Gamma^{n+1,m}$. We designate the corresponding entity W_N via

$$W^{n+1,m}_{k,k_1,\ldots,k_{n+1},\ldots,k_{n+m}} = W^{n+1,m} .$$

It is important to notice that W_N is regular on the manifold (2.1.27) in the points of a general position. However it has singularities on the submanifolds of lower dimension on which at least one of the entities E_m becomes zero, which corresponds to one of the inner lines of any diagram constituting the $W^{n+1,m}$. As can be seen from (2.1.21), these singularities may be of two types, in agreement with the two terms in (2.1.21). The first item in (2.1.21) is distributed over all of $\Gamma^{n+1,m}$, while the second one is localized on a manifold (to be more precise, on a set of manifolds):

$$-s_p\omega_{k_p} + s_1\omega_{k_1} + \ldots + s_m\omega_{k_m} = 0 \qquad (2.1.28)$$

$$-s_p k_p + s_1 k_1 + \ldots + s_m k_m = 0 .$$

Manifolds (2.1.28) may be considered the youngest resonant manifolds in comparison with (2.1.27). Equations (2.1.28) together with (2.1.27) determine a set of submanifolds of $\Gamma^{n+1,m}$ having the codimension unity. The division of two items in (2.1.21) has a certain physical meaning. One may say that the first item describes processes which go via virtual waves while the second item describes processes going via real intermediate particles. The elements of a classical S-matrix with interactions going via real waves may be called singular. They decompose on the singularity powers, depending on the number of inner lines in which the Green function G_m denominator becomes zero and on the correspondent codimension of the younger resonant manifold. For any concrete dispersion law there is an element of the scattering matrix possessing maximal singularity.

Let us now set some additional symmetry property of the amplitudes of the classical scattering matrix, i.e., let us consider the equation

$$\mathrm{i}s\dot{a}^s_k = \omega_k a^s_k + \frac{\delta H_{\mathrm{int}^*}}{\delta a^{-s}_k} , \qquad (2.1.29)$$

where H_{int^*} may be obtained from H_{int} in (1.1.2) by the substitution of complex conjugated Hamiltonian coefficients, for example, into (2.1.6): $V^{ss_1s_2}_{kk_1k_2} \to V^{-s-s_1-s_2}_{kk_1k_2}$. As before, we shall assume the interaction to be the adiabatically switched on and off. Then as $t \to \pm\infty$, the solutions of (2.1.29) and of (1.1.2) as well will degenerate into those of the linear equation.

Let us consider the solution of (2.1.29), which becomes $C^{*+}_k \exp(-\mathrm{i}\omega_k t)$ as $t \to -\infty$:

$$a_k \to C^-_{*k}\mathrm{e}^{-\mathrm{i}\omega_k t} = C^{*+}_k \mathrm{e}^{-\mathrm{i}\omega_k t} .$$

As in (1.1.2), (2.1.29) possesses a classical scattering matrix, $C^+_{*k} = S_*[C^-_{*k}]$. One should note here that (2.1.29) is derived from (1.1.2) by complex conjugation

and change of the time sign. So, on account of the unique solution of the Cauchy problem for (1.1.2) and also for (2.1.29), $S_*[C^{*+}_k] = C^{*-}_k$.

Substituting the definition of the classical scattering matrix (2.1.8), we get

$$S_*\left[S^*\left[C^-_k\right]\right] = C^{*-}_k . \qquad (2.1.30)$$

Identity (2.1.30) is analogous to the unitarity condition for the scattering matrix in quantum mechanics.

Nonlinear operator S_* can be easily calculated. It coincides with the operator S, where the Hamiltonian coefficient function V is substituted for the complex conjugated in each vertex of a diagram. It is convenient for us to introduce operator R by the following formula:

$$S = 1 + R . \qquad (2.1.31)$$

Then from (2.1.30) we obtain the following condition for R :

$$R_*\left[C^{*-}_k\right] + R^*\left[C^-_k\right] + R_*\left[R^*\left[C^-_k\right]\right] = 0 . \qquad (2.1.32)$$

One may also simply verify that

$$W^*_{m,n+1} = -\frac{m}{n+1}W_{n+1,m} . \qquad (2.1.33)$$

It follows from (2.1.33) in particular that the amplitude $W_{m,n}$ is asymmetric relative to the permutation of the m-indices, so that the diagram root does not really occur as a marked line. From physical considerations it is clear that the classical scattering matrix we have constructed coincides with the quantum one, were radiation corrections are not taken into account, and only diagrams of the "tree type" are retained.

Formulae for the case of many modes can be obtained from those above by ascribing mode numbers α, $\alpha = 1, \ldots, N$, to the field variables, coefficient functions V of the Hamiltonian and other objects. We will do so in what follows without further explanation.

2.2 Infinite-Dimensional Generalization of Poincaré's Theorem. Definition of Degenerative Dispersion Laws [4,5,6]

The classical scattering matrix introduced in Sect. 2.1 may be used to understand what restrictions should be imposed on the Hamiltonian system in order for additional motion invariants to exist. Indeed, let the system (2.1.1) have a Hamiltonian

$$H_i = \frac{1}{3!}\sum_{s_1s_2s_3}\int \hat{V}^{(\alpha_1),s_1(\alpha_2),s_2(\alpha_3),s_3}_{k_1k_2k_3} a_1a_2a_3 dk_1dk_2dk_3 . \qquad (2.2.1)$$

The cubic term in

$$I[a] = I_0 + \ldots = \sum_{\alpha_j} f_k^{\alpha_j} |a_k^{\alpha_j}|^2 dk + \ldots \quad (2.2.2)$$

is

$$I_1 = \frac{1}{3!} \int \hat{I}_{123} a_1 a_2 a_3 dk_1 dk_2 dk_3 \,.$$

Using the condition $dI/dt = 0$ and motion equations (2.1.1), we find, after collecting terms cubic in a_k:

$$E_{123} I_{123} = \hat{V}_{123} F_{123} \,, \quad (2.2.3)$$

where

$$F_{123} = s_1 f_{k_1}^{\alpha_1} + s_2 f_{k_2}^{\alpha_2} + s_3 f_{k_3}^{\alpha_3} \,. \quad (2.2.4)$$

The existence of the integral $I[a]$ depends on the presence of the limit of the right-hand side of (2.2.3) as $E_{123} = s_1 \omega_{k_1}^{\alpha_1} + \ldots + s_3 \omega_{k_3}^{\alpha_3} \to 0$. We remember that $\hat{V} = V \delta(P_{123})$.

Now two cases are possible. Consider a system of equations,

$$P_{123} = s_1 k_1 + s_2 k_2 + s_3 k_3 = 0 \quad (2.2.5)$$

$$E_{123} = s_1 \omega_{k_1}^{\alpha_1} + s_2 \omega_{k_2}^{\alpha_2} + s_3 \omega_{k_3}^{\alpha_3} = 0 \,. \quad (2.2.6)$$

If this system has no solution, the formula (2.2.4) gives the nonsingular expression for I_{123} and there is no nontrivial information available in this order. If the system (2.2.5, 6) has nontrivial solutions, it determines the simplest possible resonant surface on which the coefficient functions of a new motion invariant may have singularities. One of the following alternatives should take place in the absence of this singularity on the resonant surface (2.2.5, 6): either

$$V_{123} = V_{k_1 k_2 k_3}^{(\alpha_1), s_1(\alpha_2), s_2(\alpha_3), s_3} = 0 \,, \quad (2.2.7)$$

or

$$F_{123} \equiv s_1 f_{k_1}^{\alpha_1} + s_2 f_{k_2}^{\alpha_2} + s_3 f_{k_3}^{\alpha_3} = 0 \,. \quad (2.2.8)$$

In the latter case, if a nontrivial solution of (2.2.8) exists, we call the set of dispersion laws $\{\omega_k^{\alpha_1}, \omega_k^{\alpha_2}, \omega_k^{\alpha_3}\}$ degenerate with respect to the process (2.2.5, 6).

If there is only one type of waves in the system with the dispersion law ω_k satisfying $\omega_k > 0$ (the absence of waves with negative energy), the system (2.2.5, 6) is reduced to the equation

$$\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2) \,. \quad (2.2.9)$$

If this equation is solvable, the dispersion law is called decaying.

The alternative (2.2.7, 8) allows a generalization to higher orders of perturbation theory. To do this it is necessary to use the classical scattering matrix

introduced in Sect. 2.1. The result stated below is really the infinite-dimensional generalization of the well-known Poincaré theorem [3].

Theorem 2.2.1. For the existence of an additional motion invariant of (2.1.1) $I[a]$ of the form [6]

$$I[a] = I_0[a] + I_1[a] + \ldots \,, \quad I_0 = \sum_{\alpha_j} f_k^{\alpha_j} |a_k^{\alpha_j}|^2 dk$$

it is necessary that on each resonant surface,

$$E_{1\ldots q} = 0 \,, \quad P_{1\ldots q} = 0 \,, \quad (2.2.10)$$

in the points of general position, the following alternative occurs: either the amplitude W of the classical scattering matrix, corresponding to (2.2.10), equals zero,

$$W_{1\ldots q} = 0 \,, \quad (2.2.11a)$$

or the following condition holds:

$$F_{1\ldots q} \equiv \sum_{1}^{q} s_j f_{k_j}^{\alpha_i} = 0 \,. \quad (2.2.11b)$$

Proof. The conservation of the integral $I[a]$ results in the equality of its limit values as $t \to \pm\infty$:

$$\lim_{t \to -\infty} I\left[b_k e^{-i\omega_k t}\right] = \lim_{t \to +\infty} I\left[b_k e^{-i\omega_k t}\right] \,, \quad (2.2.12)$$

where the limits in (2.2.12) should be understood in terms of distributions.

By definition of the classical scattering and transition matrices (2.1.7, 8) we have: $b_k(t) = S_k(-\infty, t)[C^-]$ $C_k^+ = S_k[C^-]$. Now let us insert this formula into (2.2.12), taking into account (2.1.32) and the explicit form of the integral quadratic part I_0. By doing so we reduce two limit points, $t = \pm\infty$, to only one point, $t = -\infty$, and obtain

$$\lim_{t \to -\infty} \sum^{n} f_k^{(\alpha)} \left[C_k^{-(\alpha)^*} R_k^{(\alpha)}[C^-](t) + C_k^{*-(\alpha)} R_k^{(\alpha)}[C^-](t) \right] dk = \mathcal{D}_k \,. \quad (2.2.12a)$$

Here we have already used the fact that $\lim_{t \to -\infty} b_k = C_k^-$. In (2.2.12a) we keep an explicit dependence of $R_k(t) = S(-\infty, t) - 1$ on t, because this dependence leads to the important fact that each term in (2.2.12a) is localized on the corresponding resonant surface.

The $\mathcal{D}_k$ contains the term $\sum \int f_k^{(\alpha)} R_{\alpha^k}^{(\alpha)^*} R_k^{(\alpha)} dk$ resulting from the I_0 and all terms resulting from the higher orders in $I[a]$. As we have already seen in

Sect. 2.1, R_k is a series, and each of its terms, $S^{n,m}_{k_1 \dots k_{n+m}}$ corresponding to some nonlinear processes "$n \to m$", has the structure (2.1.24).

Recalling now (2.1.3) and using the well-known identity from distribution theory,

$$\lim \frac{e^{iEt}}{E - i0} = \pi\delta(E) ,$$

we see that the integrand in $S^{n,m}_k$ takes the form

$$\hat{W}^{n,m}_{k_1 \dots k_{n+m}} C_1 \dots C_{n+m} \delta(s_1\omega_1 + \dots + s_{n+m}\omega_{n+m}) ,$$

and each term resulting from the left-hand side of (2.2.12a) is localized on the resonant surface. As to $\mathcal{D}_k$, each contributing it term contains at least one additional δ-function of frequencies and is therefore localized on the submanifold of codimensionality 1 or more.

To see this, consider an arbitrary term in $\mathcal{D}_k$, for example one resulting from the cubic part of $I[a]$:

$$\lim_{\to -\infty} \frac{1}{3!} \sum_{s_i, \alpha_i} \int f_k \hat{V}^{(\alpha), s(\alpha_1), s_1(\alpha_2), s_2}_{k k_1 k_2} R^{(s,\alpha)}_k C^{-(s_1,\alpha_1)}_{k_1} C^{-(s_2,\alpha_2)}_{k_2}$$
$$\times \left[\exp\left[i\left(E^{s s_1 s_2}_{k k_1 k_2}\right) t\right] \left(E^{s s_1 s_2}_{k k_1 k_2}\right)^{-1}\right] dk\, dk_1\, dk_2 .$$

This term has two δ-functions of frequencies: one resulting from $R^{(s,\alpha)}_k$ and the other from the expression in squared brackets. Certainly, the integrand is localized on a submanifold of a codimensionality 1 of the whole resonant manifold, and in points of a general position this term should not be taken into account. Analogously, each term constituting $\mathcal{D}_k$ possesses the property.

Now consider points of a general position ($\mathcal{D}_k = 0$) of a resonant surface for terms (on the left-hand side) which contain a combination of fields (C for C^-):

$$C^{(\alpha_1)}_{k_1} \dots C^{(\alpha_n)}_{k_n} C^{*(\tilde{\alpha}_1)}_{\tilde{k}_1} \dots C^{*(\tilde{\alpha}_m)}_{\tilde{k}_m} .$$

By symmetrizing these terms we obtain

$$\int \left[f^{(\alpha_1)}_{k_1} + \dots + f^{(\alpha_n)}_{k_n} - f^{(\tilde{\alpha}_1)}_{\tilde{k}_1} - \dots - f^{(\tilde{\alpha}_m)}_{\tilde{k}_m}\right] W^{\alpha_1 \dots \alpha_n \tilde{\alpha}_1 \dots \tilde{\alpha}_m}_{k_1 \dots k_n \tilde{k}_1 \dots \tilde{k}_m}$$
$$\times c^{(\alpha_1)}_{k_1} \dots c^{*(\tilde{\alpha}_m)}_{\tilde{k}_m} \delta\left(k_1 + \dots + k_n - \tilde{k}_1 - \dots - \tilde{k}_m\right)$$
$$\times \delta\left(\omega^{(\alpha_1)}_{k_1} + \dots + \omega^{(\alpha_n)}_{k_n} - \omega^{(\tilde{\alpha}_1)}_{\tilde{k}_1} - \dots - \omega^{(\tilde{\alpha}_m)}_{\tilde{k}_m}\right)$$
$$\times dk_1 \dots dk_n d\tilde{k}_1 \dots d\tilde{k}_m = 0 .$$

Hence, due to the arbitrariness of C^-_k, we obtain that in points of a general position of the resonant surface

$$k_1 + \dots - \tilde{k}_m = 0 , \quad \omega^{(\alpha_1)}_{k_1} + \dots - \omega^{(\tilde{\alpha}_m)}_{\tilde{k}_m} = 0 , \qquad (2.2.13)$$

the following equality should hold true:

$$\left[f^{(\alpha_1)}_{k_1} + \dots + f^{(\alpha_n)}_{k_n} - \dots - f^{(\tilde{\alpha}_1)}_{\tilde{k}_1}\right] W^{\alpha_1 \dots \alpha_n \tilde{\alpha}_1 \dots \tilde{\alpha}_m}_{k_1 \dots k_n \tilde{k}_1 \dots \tilde{k}_m} = 0 ,$$

from which the alternative (2.11a–b) follows. (End of proof.)

We now present a more general definition of degenerative dispersion laws.

Definition. The set of dispersion laws

$$\{\omega^{\alpha_1}_k , , \omega^{\alpha_q}_k\} , \quad \alpha_j = 1, \dots , N \qquad (2.2.14)$$

is called degenerative with respect to the process (2.2.13) in the point Q of a manifold (2.2.12), if (2.2.11b) in the neighbourhood of the point Q on (2.2.13) has a nontrivial solution, i.e., $f^\alpha_k \neq (v, k) + A\omega^\alpha_k$+const. The set (2.2.14) is called degenerative in the domain Ω in (2.2.13) if it is degenerative in each point of Ω. And the set (2.2.14) is called completely degenerative (or simply "degenerative") on (2.2.13) if it is degenerative in each point of the manifold. If the domain Ω does not exist, the set (2.2.14) is called nondegenerative with respect to (2.2.13).

If Ω exists but does not coincide with (2.2.13), the set (2.2.14) is called particularly degenerative and if an additional integral exists, the scattering amplitude outside Ω should become zero according to (2.2.11a). If all functions $\omega^{\alpha_j}_k$ from the degenerative set of dispersion laws coincide, the correspondent dispersion law is called degenerative.

Degenerative and particularly degenerative dispersion laws and degenerative sets represent in themselves exclusive phenomena. The properties of such exclusive ω^α_k will be described in the next paragraph.

2.3 Properties of Degenerative Dispersion Laws [6]

Properties of degenerative dispersion laws differ strongly in spaces of dimensionality $d = 1$, $d = 2$ and $d \geq 3$. For this reason we shall describe them separately.

2.3.1 Dimension $d = 1$ In this case any three functions $w^{\alpha_i}_k$, $i = 1, 2, 3$, $\alpha = 1, \dots , N$ form a degenerative set with respect to the process

$$k = k_1 + k_2$$
$$\omega^{\alpha_1}_{k_1} = \omega^{\alpha_2}_{k_2} + \omega^{\alpha_3}_{k_3} , \qquad (2.3.1)$$

if such a process is possible.

Actually, (2.3.1) defines the one-dimensional manifold in a three-dimensional space (k_1, k_2, k_3) so that locally $k_i = k_i(\xi)$, $i = 1, 2, 3$. Consider any two functions $f^{(2)}_k$ and $f^{(3)}_k$. On the surface (2.3.1), we define $f^{(1)}_k(\xi)$ by the equality $f^{(1)}_k(\xi) = f^{(2)}_{k_2}(\xi) + f^{(3)}_{k_3}(\xi)$. Then we have to invert the equality $k_1 = k_1(\xi)$ to obtain the function $f^{(1)}_{k_1} = f^{(1)}(\xi(k_1))$ which, together with $f^{(2)}_k$ and $f^{(3)}_k$, forms a nontrivial solution of (2.2.8).

Any dispersion law ω_k in a one-dimensional case is degenerative with respect to the scattering process "2 waves into 2 waves" ("2 → 2"):

$$k_1 + k_2 = k_3 + k_4 \qquad \omega_1 + \omega_2 = \omega_3 + \omega_4 \,. \tag{2.3.2}$$

In fact, (2.3.2) defines the two-dimensional manifold in the four-dimensional space ($\boldsymbol{k}_i$, $i = 1, \dots, 4$). On the other hand it is obvious that (2.3.2) is satisfied by the substitution

$$\left.\begin{matrix} k_1 = k_3 \\ k_2 = k_4 \end{matrix}\right\} \text{ or } \left.\begin{matrix} k_1 = k_4 \\ k_2 = k_3 \end{matrix}\right\} , \tag{2.3.3}$$

corresponding to the trivial scattering. The manifolds of trivial scattering prove to be very important when constructing action-angle variables.

Manifolds (2.3.2) and (2.3.3) obviously coincide. But on (2.3.3), any function f_k obeys the corresponding equation (2.2.11b), namely,

$$f_1 + f_2 = f_3 + f_4 \,; \tag{2.3.4}$$

this is proof of nondegeneracy. For the process "2 → 2" with several modes, this is in general not so. For example, a set $\omega_k^{(1)} = k^2$, $\omega_k^{(2)} = c|k|$ is only degenerative to the process

$$k_1 + q_1 = k_2 + q_2 \qquad \omega_{k_1}^{(1)} + \omega_{q_1}^{(2)} = \omega_{k_2}^{(1)} + \omega_{q_2}^{(2)} \,. \tag{2.3.5}$$

The manifold (2.3.5) is split onto two parts, $\Gamma_1^{\pm}$ and $\Gamma_2^{\pm}$. The first corresponds to the forward scattering of a sound wave and the second, to backward scattering. Corresponding parametrization has the form [11]:

for $\Gamma_1^{\pm}$,

$$k_1 = \tfrac{1}{2}(\pm c + \xi) \qquad q_1 = \tfrac{1}{2}(\eta - \xi)$$
$$k_2 = \tfrac{1}{2}(\pm c - \xi) \qquad q_2 = \tfrac{1}{2}(\eta + \xi) \,, \tag{2.3.6a}$$

and for $\Gamma_2^{\pm}$,

$$k_1 = \tfrac{1}{2}(\eta \pm 2c\xi) \qquad q_1 = \xi(\eta \mp c)$$
$$k_2 = \tfrac{1}{2}(\eta \mp 2c\xi) \qquad q_2 = \xi(\eta \pm c) \,. \tag{2.3.6b}$$

It happens that the set $\omega^{(1)}$, $\omega^{(2)}$ is degenerative on (2.3.6a) and nondegenerative on (2.3.6b). The solution of a corresponding equation (2.2.11b) on $\Gamma_1^{\pm}$,

$$f_{k_1}^{(1)} + f_{q_1}^{(2)} = f_{k_2}^{(1)} + f_{q_2}^{(2)} \,, \tag{2.3.7}$$

has the form

$$f^{(1)}(\xi) = \mu\left(\xi - \frac{c}{2}\right) + A\xi^2 + (B - Ac)\xi$$

$$f^{(2)}(\xi) = B\xi \,, \quad \forall \mu(\xi) = \mu(-\xi) \,.$$

Consider two dispersion laws: $\omega_k^{(1)} = c_1 k^2$, $\omega_k^{(2)} = c_2 k^2$. When $c_1 \neq \pm c_2$, the manifold (2.3.5) is nondegenerative. Indeed, let $\varrho = c_2/c_1 \neq \pm 1$. Manifold (2.3.5) then allows the following rational parametrization [12]:

$$k_1 = \frac{\varrho - 1}{2} q_1 + \frac{\varrho + 1}{2} q_2$$
$$k_2 = \frac{\varrho + 1}{2} q_1 + \frac{\varrho - 1}{2} q_2 \,. \tag{2.3.8}$$

Substituting (2.3.8) into (2.3.7), differentiating two times in q_1 and one in q_2 and setting $q_1 = q_2 = \xi/\varrho$, we obtain

$$(\varrho^2 - 1)(\varrho - 1) f^{(1)'''}(\xi) = (\varrho^2 - 1)(\varrho + 1) f^{(1)'''}(\xi) \,.$$

Hence, at $\varrho \neq \pm 1$, $f^{(1)'''} = 0$:

$$f^{(1)} = A^{(1)}\xi^2 + B^{(1)}\xi + C^{(1)} \,; \tag{2.3.9}$$

i.e., the set $\{c_1 k^2, c_2 k^2, c_1/c_2 \neq \pm 1\}$ is nondegenerative to (2.3.5). At $\varrho = \pm 1$, (2.3.5) is degenerative.

Processes with more than four waves have not been very well studied, in spite of some special results. It is certainly clear that degeneracy in such processes is an exclusive phenomenon. For example, the same set $(c_1 k^2, c_2 k^2)$ is nondegenerative with respect to a "3 → 3" process:

$$k_1 + k_2 + k_3 = k_4 + k_5 + k_6 \qquad \omega_1 + \omega_2 + \omega_3 = \omega_4 + \omega_5 + \omega_6 \tag{2.3.10}$$

at any ϱ. The proof can be performed by using a rational parametrization of (2.3.10) of the form [12]:

$$\begin{aligned}
k_1 &= \frac{3P\varrho}{1+2\varrho} + R\left[u + \frac{1}{u} - \frac{1}{v} + (1+2\varrho)v\right] \\
k_2 &= \frac{3P\varrho}{1+2\varrho} + R\left[u + \frac{1}{u} + \frac{1}{v} - (1+2\varrho)v\right] \\
k_3 &= \frac{3P}{1+2\varrho} - \frac{2R}{u} - 2Ru \\
k_4 &= \frac{3P}{1+2\varrho} + \frac{2R}{u} - 2Ru \\
k_5 &= \frac{3P\varrho}{1+2\varrho} + R\left[u - \frac{1}{u} + \frac{1}{v} + (1+2\varrho)v\right] \\
k_6 &= \frac{3P\varrho}{1+2\varrho} + R\left[u - \frac{1}{u} - \frac{1}{v} - (1+2\varrho)v\right] \,.
\end{aligned} \tag{2.3.11}$$

Parametrization (2.3.11) should be substituted into the condition corresponding to (2.2.11b):

$$f^{(1)}_{k_1} + f^{(1)}_{k_2} + f^{(2)}_{k_3} = f^{(2)}_{k_4} + f^{(1)}_{k_5} + f^{(1)}_{k_6} \,. \tag{2.3.12}$$

After this, the proof of nondegeneracy can be obtained by three differentiations and by subsequently taking a corresponding limit to obtain a differential equation from the functional one.

2.3.2 Dimensionality $d = 2$. Consider the simplest nonlinear process: decay of the wave into two waves of the same type. If the correspondent dispersion law is decaying, corresponding manifold (2.2.9) defines a three-dimensional manifold $\Gamma^{1,2}$ in a four-dimensional space (k_1, k_2). As an example of a decaying dispersion law, one can consider an isotropic function,

$$\omega_k = \omega(|k|) \,, \quad \omega(0) = 0 \,, \quad \omega' > 0 \,. \tag{2.3.13}$$

The equation (2.2.8) then takes a simple form,

$$f_{k_1+k_2} = f_{k_1} + f_{k_2} \,. \tag{2.3.14}$$

Let us now show that the degenerative decaying dispersion laws exist at $d = 2$. We designate components of a vector k via (p, q) and let $\omega(p, q)$ be defined parametrically by formulae

$$p = \xi - \xi \,; \quad q = a(\xi) - a(\xi) \,; \quad \omega_k = b(\xi) - b(\xi) \,, \tag{2.3.15}$$

where $a(\xi)$ and $b(\xi)$ are arbitrary functions of one variable. (The natural appearance of a parametrization of this type in exactly solvable systems from an underlying linear problem was shown by *Manakov* in [38].) We consider the three-dimensional manifold $\tilde{\Gamma}^{1,2}$ defined by

$$\begin{array}{ll} p_1 = \xi_1 - \xi_3 & p_2 = \xi_3 - \xi_2 \\ q_1 = a(\xi_1) - a(\xi_3) & q_2 = a(\xi_3) - a(\xi_2) \,. \end{array} \tag{2.3.16}$$

Now

$$p = p_1 + p_2 = \xi_1 - \xi_2$$

$$q = q_1 + q_2 = a(\xi_1) - a(\xi_2) \,,$$

and in accordance with (2.3.15),

$$\begin{aligned} \omega_{k_1+k_2} &= b(\xi_1) - b(\xi_2) = b(\xi_1) - b(\xi_3) \\ &\quad + b(\xi_3) - b(\xi_2) = \omega_{k_1} - \omega_{k_2} \,. \end{aligned} \tag{2.3.17}$$

Thus, the manifold $\tilde{\Gamma}^{1,2}$ is a domain in $\tilde{\Gamma}^{1,2}$.

Consider now a function $f(p, q)$ parametrized by

$$p = \xi_1 - \xi_2 \,, \quad q = a(\xi_1) - a(\xi_2) \,, \quad f = c(\xi_1) - c(\xi_2) \,, \tag{2.3.18}$$

where $c(\xi)$ is an arbitrary function. Obviously $f(p, q)$ obey (2.3.14) on $\tilde{\Gamma}^{1,2}$, and the law (2.3.15) is at least particulary degenerative. Its complege degeneracy should be considered separately.

Let $a(\xi) = \xi^2$, $b(\xi) = 4\xi^3$ in (2.3.15). Then

$$\omega(p, q) = p^3 + \frac{3q^2}{p} \,. \tag{2.3.19}$$

This is a dispersion law of the Kadomtsev-Petviashvily equation (1.1.3) (referred to in the following as KP-1) with $\alpha^2 = 1$.
Equation (2.3.13) now takes the form:

$$(p_1 + p_2)^2 = \left(\frac{q_1}{p_1} - \frac{q_2}{p_2}\right)^2 \,, \tag{2.3.20}$$

and it is clear that it consists of two parts. Simple analysis shows that $\tilde{\Gamma}^{1,2}$ coincides with the $\Gamma^{1,2}_{+}$ part given by the formulae

$$p_1 + p_2 = \frac{q_1}{p_1} - \frac{q_2}{p_2} \,. \tag{2.3.21}$$

Dispersion law (2.3.19) can also be obtained from a parametrization $a(\xi) = -\xi^2$, $b(\xi) = 4\xi^3$. Now $\tilde{\Gamma}^{1,1}$ coincides with $\Gamma^{1,2}_{-}$ when

$$p_1 + p_2 = -\frac{q_1}{p_1} + \frac{q_2}{p_2} \,. \tag{2.3.22}$$

Thus the dispersion law (2.3.19) is proved to be completely degenerative.

Now let $\xi_1 - \xi_2 = \delta \ll 1$ in (2.3.15–18). Then in the first order in δ relations,

$$\frac{q}{p} = a'(\xi_2) \,, \quad \frac{\omega}{p} = b'(\xi_2) \tag{2.3.23}$$

also define the degenerative dispersion law, and it is the homogeneous function of degree one,

$$\omega = p\Phi\left(\frac{q}{p}\right) \,. \tag{2.3.24}$$

We should note that (2.3.24) together with the function (2.3.15) are not analytic at $p \to 0$. Thus, the homogeneous function of degree one dispersion law is degenerative. The manifold $\Gamma^{1,2}$ for dispersion law (2.3.24) is

$$\frac{q_1}{p_1} = \frac{q_2}{p_2} = \frac{q}{p} \,,$$

which means that k_1 and k_2 are parallel and unidirected.

When many modes exist, there are three sets of dispersion laws $\{\omega^{(\alpha_1)}, \omega^{(\alpha_2)}, \omega^{(\alpha_3)}\}$ degenerative with respect to decay processes, too:

$$\begin{aligned} &\omega^{(\alpha_1)}_{k_1} = \omega^{(\alpha_2)}_{k_2} + \omega^{(\alpha_3)}_{k_3} \\ &k_1 = k_2 + k_3 \,. \end{aligned} \tag{2.3.25}$$

They are defined parametrically by the formulae

$$p_1 = \xi_1 - \xi_2\,, \quad p_2 = \xi_1 - \xi_3\,, \quad p_3 = \xi_3 - \xi_2$$
$$q_1 = a_1(\xi_1) - a_2(\xi_2)\,, \quad q_2 = a_1(\xi_1) - a_3(\xi_3)\,, \quad q_3 = a_3(\xi_3) - a_2(\xi_2)$$
$$\omega^{(\alpha_1)} = b_1(\xi_1) - b_2(\xi_2)\,, \quad \omega^{(\alpha_2)} = b_1(\xi_1) - b_3(\xi_3)\,, \quad \omega^{(\alpha_3)} = b_3(\xi_3) - b_2(\xi_2)\,. \tag{2.3.26}$$

Now the solutions corresponding to (2.2.11b),

$$f^{(\alpha_1)}_{\boldsymbol{k}_1} = f^{(\alpha_2)}_{\boldsymbol{k}_2} + f^{(\alpha_3)}_{\boldsymbol{k}_3}\,, \tag{2.3.27}$$

have a parametric form in (2.3.26):

$$\begin{aligned} f^{(\alpha_1)}_{\boldsymbol{k}_1} &= c_1(\xi_1) - c_2(\xi_2) \\ f^{(\alpha_2)}_{\boldsymbol{k}_2} &= c_1(\xi_1) - c_3(\xi_3) \\ f^{(\alpha_3)}_{\boldsymbol{k}_3} &= c_3(\xi_3) - c_2(\xi_2)\,. \end{aligned} \tag{2.3.28}$$

We should recall the fact mentioned above with respect to the specific case of the KP-1 equation. Namely, if in (2.3.15) we replace $p \to p$, $a(\xi) \to -a(-\xi)$; $b(\xi) \to -b(-\xi)$, such a dispersion law will be also degenerative with respect to the process (2.3.13). In the case of KP-1 these two parametrizations together cover the entire manifold $\Gamma^{1,2}$. It is still an open question as to whether these two parametrizations cover the whole degenerative piece of resonant manifold in all cases.

In addition, all homogeneous dispersion laws of the weight 1 (2.3.24) are degenerative to any decay processes $1 \to n$,

$$\begin{aligned} \omega &= \omega_1 + \ldots + \omega_n \\ \boldsymbol{k} &= \boldsymbol{k}_1 + \ldots + \boldsymbol{k}_n\,. \end{aligned} \tag{2.3.29}$$

The question naturally arises as to whether degenerative dispersion laws exist which differ from (2.3.15). The following theorems are true.

Theorem 2.3.1 (Local Uniqueness Theorem) [6]. All dispersion laws of the form

$$p = \xi_1 - \xi_2\,, \quad q = a(\xi_1) - a(\xi_2) \tag{2.3.30a}$$

$$\omega = b(\xi_1) - b(\xi_2) + \sum_1^\infty \varepsilon^n \omega_n(\xi_1, \xi_2)\,,$$

satisfying the degeneracy condition (2.3.14) with

$$f = c(\xi_1) - c(\xi_2) + \sum_1^\infty \varepsilon^n f_n(\xi_1, \xi_2)\,, \tag{2.3.30b}$$

$\varepsilon \ll 1$, will belong to the class (2.3.15).

This statement means that there are variables $\eta_1(\xi_1, \xi_2)$ and $\eta_2(\xi_1, \xi_2)$ in which terms the degenerative dispersion law (2.3.30a), with (2.3.30b) holds true, taking the form (2.3.15).

Theorem 2.3.2. If $\omega(p, q)$, degenerative with respect to the dispersion law (2.2.9), is analytic in the neighborhood of $p = q = 0$, then the corresponding function $f(p, q)$ cannot be analytic in the same domain.

Theorem 2.3.3. Let the dispersion law $\omega(\boldsymbol{k})$ near the point $\boldsymbol{k}_0$ admit the expansion

$$\omega(\boldsymbol{k}_0 + \boldsymbol{\kappa}) = \omega(\boldsymbol{k}_0) + (\boldsymbol{\nu}, \boldsymbol{\kappa}) + \sum A_{ij}\kappa_i\kappa_j\,. \tag{2.3.31}$$

Then in some domain near $\boldsymbol{k}_1 = \boldsymbol{k}_2 = \boldsymbol{k}_3 = \boldsymbol{k}_4 = \boldsymbol{k}_0$, the dispersion law (2.3.31) is nondegenerative to the process

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4 \tag{2.3.32}$$
$$\omega_{\boldsymbol{k}_1} + \omega_{\boldsymbol{k}_2} = \omega_{\boldsymbol{k}_3} + \omega_{\boldsymbol{k}_4}\,;$$

i.e., the equation

$$f_{\boldsymbol{k}_1} + f_{\boldsymbol{k}_2} = f_{\boldsymbol{k}_3} + f_{\boldsymbol{k}_4} \tag{2.3.33}$$

does not have nontrivial solutions.

Theorem 2.3.4 (Global Theorem). If $\omega^{(\alpha i)}(p, q)$ is a system of dispersion laws, degenerative with respect to the process (2.3.25), and if the equation (2.3.27) has at least *three* independent nontrivial solutions, the system $\omega^{(\alpha i)}(p, q)$ either belongs to the class (2.3.26) or could be obtained from it by some limiting process.

Now let $\omega(0) = 0$. From Theorem 2.3.3 it follows that the dispersion laws admitting expansions (2.3.31) are nondegenerative to the process

$$\sum^n \boldsymbol{k}_j = \sum^m \boldsymbol{k}_i\,, \quad \sum^n \omega_i = \sum^m \omega_l\,, \tag{2.3.34}$$
$$n \geq 2\,, \quad m \geq 2\,.$$

To see this, one can consider the neighbourhood of the manifold (2.3.32), putting the "extra" wave vectors equal to zero. Thus, only homogeneous functions of degree one dispersion laws can be degenerative and only to the processes (2.3.29).

From the above it follows that there is no unique dispersion law completely degenerative with (2.3.32). It is very doubtful that $\omega_{\boldsymbol{k}}$ exist which are degenerative to (2.3.32) in particular.

Let a dispersion law $\omega_{\boldsymbol{k}}$ be decaying. Then the manifold $\Gamma^{2,2}$ contains a submanifold $\Gamma^{2,2}_M$ of codimension one given by the equations

$$\begin{aligned} \boldsymbol{k}_1 + \boldsymbol{k}_2 &= \boldsymbol{k}_3 + \boldsymbol{k}_4 = \boldsymbol{q} \\ \omega_{\boldsymbol{k}_1} + \omega_{\boldsymbol{k}_2} &= \omega_{\boldsymbol{k}_3} + \omega_{\boldsymbol{k}_4} = \omega_{\boldsymbol{q}}\,. \end{aligned} \tag{2.3.35}$$

If the dispersion law is degenerative to a "$1 \to 2$" process, then the correspondent degenerative function $f(\boldsymbol{k})$ obeys on $\Gamma_M^{2,2}$ the equation

$$f_{k_1} + f_{k_2} = f_{k_3} + f_{k_4} = f_q \,. \tag{2.3.36}$$

This certainly does not mean that dispersion law ω_k is even particulary degenerative. For degeneracy to take place it is necessary that (2.3.33) be true on (2.3.32) in the neighbourhood of at least one point of the manifold (2.3.35). Degeneration, as we know, is possible only at $d = 2$, and the corresponding dispersion laws are given by (2.3.15).

Consider now the neighbourhood of any point on $\Gamma_M^{2,2}$. It can be defined as

$$p_1 = \xi_1 - \xi_2 \,, \quad p_2 = \xi_2 - \xi_3 \,, \quad p_3 = \xi_1 - \xi_4 \,, \quad p_4 = \xi_4 - \xi_3 \,,$$

$$q_1 = a(\xi_1) - a(\xi_2) \,, \quad q_2 = a(\xi_2 + \nu_1) - a(\xi_3 + \nu_1)$$

$$q_3 = a(\xi_1 + \nu_2) - a(\xi_4 + \nu_2) \,, \quad q_4 = a(\xi_4 + \nu_3) - a(\xi_3 + \nu_3)$$

$$\omega_1 = b(\xi_1) - b(\xi_2) \,, \quad \omega_2 = b(\xi_2 + \nu_1) - b(\xi_3 + \nu_1)$$

$$\omega_3 = b(\xi_1 + \nu_2) - b(\xi_4 + \nu_2) \,, \quad \omega_4 = b(\xi_4 + \nu_3) - b(\xi_3 + \nu_3) \,.$$

The resonant conditions are

$$\left[a'(\xi_2) - a'(\xi_3)\right] \nu_1 = \left[a'(\xi_1) - a'(\xi_3)\right] \nu_2 + \left[a'(\xi_4) - a'(\xi_3)\right] \nu_3$$

$$\left[b'(\xi_2) - b'(\xi_3)\right] \nu_1 = \left[b'(\xi_1) - b'(\xi_3)\right] \nu_2 + \left[b'(\xi_4) - b'(\xi_3)\right] \nu_3 \,.$$

The condition of degenerativeness gives another relation:

$$\left[c'(\xi_2) - c'(\xi_3)\right] \nu_1 = \left[c'(\xi_1) - c'(\xi_3)\right] \nu_2 + \left[c'(\xi_4) - c'(\xi_3)\right] \nu_3 \,.$$

If functions a, b, c are linear independent, this equation has only zero solutions. It follows from this that the manifold $\Gamma_M^{2,2}$ cannot be locally extended with preservation of degeneracy.

Consider now any process "n into m" given by the resonant conditions (2.3.34), and let ω_k be decaying and degenerative to "1 into 2" [see (2.3.15)]. In the corresponding manifold $\Gamma^{n,m}$ one can point out a set of manifolds $\Gamma_M^{n,m}$ which we can call minimal. To describe these manifolds, we recall that the scattering amplitude $W^{n,m}$ is given by a diagram of the tree type with a finite number of vertices and inner lines.

Let us designate via $\boldsymbol{p}_i$, $\boldsymbol{s}_i$ the outer wave vectors and their directions. Let some vertex contain vectors $\boldsymbol{p}_i$, $\boldsymbol{s}_i$, $\boldsymbol{p}_j$, $\boldsymbol{s}_j$, $\boldsymbol{p}_l$, $\boldsymbol{s}_l$. Then we have

$$\boldsymbol{s}_i \cdot \boldsymbol{p}_i + \boldsymbol{s}_j \cdot \boldsymbol{p}_j + \boldsymbol{s}_l \cdot \boldsymbol{p}_l = 0 \,. \tag{2.3.37}$$

We require another condition to be fulfilled:

$$\boldsymbol{s}_i \omega_{p_i} + \boldsymbol{s}_j \omega_{p_j} + \boldsymbol{s}_l \omega_{p_l} = 0 \,. \tag{2.3.38}$$

From (2.3.37, 38), it follows naturally that (2.3.34) is true but (2.3.37, 38) determine the manifold of smaller dimensionality – one of the minimal ones, $\Gamma_M^{n,m}$.

If ω_k is degenerative, in each vertex the equation

$$\boldsymbol{s}_i f_{p_i} + \boldsymbol{s}_j f_{p_j} + \boldsymbol{s}_l f_{p_l} = 0$$

will be true and, as a consequence, so will (2.2.11b), but it is impossible to enlarge the dimensionality of $\Gamma_M^{n,m}$ under condition (2.2.11b).

2.3.3 Dimensionality $d \geq 3$.

In higher dimensions the possibility of degeneracy is strongly limited in comparison with $d = 1, 2$. Only the homogeneous functional of degree one dispersion law

$$\omega(c\boldsymbol{k}) = c\omega(\boldsymbol{k}) \tag{2.3.39}$$

is degenerative to (2.3.29) only. Its degeneracy does not depend on the space dimensionality: (2.3.29) is solvable only if all $\boldsymbol{k}_i$, $i = 1, 2, \ldots$ are collinear to $\boldsymbol{k}$. So the corresponding manifold has smaller dimensionality than for decaying dispersion laws of general form; e.g., at $d = 3$, the dimensionality of (2.2.9) equals five while that of (2.3.29) equals four. On the basis of the following local theorem, it can be stipulated that no other degenerative laws exist.

Theorem 2.3.5. Let the dispersion law ω_k, $\boldsymbol{k} = (p, q, r)$ be parametrized in the neighbourhood of $r = 0$ by

$$p = \xi_1 - \xi_2 \,, \quad q = a(\xi_1) - a(\xi_2) \tag{2.3.40}$$

$$\omega(p, q, r) = b(\xi_1) - b(\xi_2) + r \sum_0^\infty r^n \omega_n(\xi_1, \xi_2)$$

and the dimensionality of $\Gamma^{1,2}$ be equal to five. Then $\omega_0 = \text{const}$, $\omega_n = 0$, $n > 0$. For the proof, see Appendix I.

2.4 Properties of Singular Elements of a Classical Scattering Matrix. Properties of Asymptotic States [6]

Let us examine the "$n \to m$" process. We shall choose another notation for wave vectors, and designate the nonlinear part of the amplitude via $S^{n,m}_{k_1 \ldots k_n; \tilde{k}_1 \ldots \tilde{k}_m}$.

We consider a diagram describing the process (2.3.34) in which the inner line (Green function) with the wave vector $\boldsymbol{q}$ is replaced by a δ-function. Let vector $\boldsymbol{q}$ be directed from the "root" of the diagram and "to the right of it"; i.e., further away from the root, there are external lines with vectors $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n_1}$, $\tilde{\boldsymbol{k}}_1, \ldots, \tilde{\boldsymbol{k}}_{m_1}$, $n_1 < n$, $m_1 < m$. Now the following equations are added to (2.3.34):

$$\boldsymbol{k}_1 + \ldots + \boldsymbol{k}_{n_1} = \tilde{\boldsymbol{k}}_1 + \ldots + \tilde{\boldsymbol{k}}_{m_1} + \boldsymbol{q} \tag{2.4.1}$$

$$\omega_{\boldsymbol{k}_1} + \ldots + \omega_{\boldsymbol{k}_{n_1}} = \omega_{\tilde{\boldsymbol{k}}_1} + \ldots + \omega_{\tilde{\boldsymbol{k}}_{m_1}} - \omega_q \, .$$

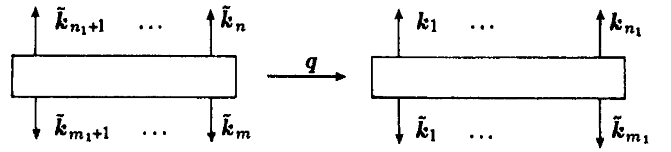
Moreover, it holds that

$$\boldsymbol{k}_{n_1+1} + \ldots + \boldsymbol{k}_n = \tilde{\boldsymbol{k}}_{m_1+1} + \ldots + \tilde{\boldsymbol{k}}_m - q \tag{2.4.2}$$

$$\omega_{\boldsymbol{k}_{n_1+1}} + \ldots + \omega_{\boldsymbol{k}_n} = \omega_{\tilde{\boldsymbol{k}}_{m_1+1}} + \ldots + \omega_{\tilde{\boldsymbol{k}}_m} - \omega_q \, .$$

Let us designate via $\tilde{S}^{n,m}_{\boldsymbol{k}_1 \ldots \boldsymbol{k}_n ; \tilde{\boldsymbol{k}}_1 \ldots \tilde{\boldsymbol{k}}_m}$ the singular part of the amplitude of the "n into m" process corresponding to equations (2.4.1), (2.4.2).

We obtain the expression for $\tilde{S}^{n,m}$ as a result of summing of all diagrams of the form



Then we have the relation:

$$\tilde{S}^{n,m}_{\boldsymbol{k}_1 \ldots \boldsymbol{k}_n , \tilde{\boldsymbol{k}}_1 \ldots \tilde{\boldsymbol{k}}_m} = \pi \mathrm{i} \int S^{n_1, m_1+1}_{\boldsymbol{k}_1 \ldots \boldsymbol{k}_{n_1} , \tilde{\boldsymbol{k}}_1 \ldots \tilde{\boldsymbol{k}}_{m_1} q} S^{n-n_1+1, m-m_1}_{\boldsymbol{k}_{n_1+1} \ldots \boldsymbol{k}_n , q ; \tilde{\boldsymbol{k}}_{m_1+1} \ldots \boldsymbol{k}_m} dq \, .$$

The formula shows that the singular amplitude $\tilde{S}^{n,m}$ is factorized through the composition of the two nonsingular amplitudes of lower order. It is clear that the analogous statement holds for the amplitude of any degree of singularity when there are several additional equations of the structure (2.4.1). All of them are factorized in the form of the composition of the finite number of the nonsingular amplitudes of lower orders. In particular, the maximum singular elements of the scattering matrix defined by the diagrams where all "Green functions" of the internal lines are substituted for δ-functions, are factorized in the form of the composition of the simplest scattering amplitudes "one into two".

These facts have a simple physical meaning. The substitution of one of the internal "Green functions" of a δ-function means that the corresponding wave is the eigenoscillation of the system (a "real particle"), and the process with such a wave occurs stage by stage, combined out of the process of the lowest order.

Now let the dynamical system under consideration possess the additional motion integral, and let the dispersion law be nondegenerative relative to all nonlinear processes. Then all nonsingular elements of the scattering matrix on the resonance surfaces are vanishing. As mentioned above, all singular amplitudes are vanishing too. Thus, in this case, the classical scattering matrix is trivial and the asymptotic states coincide, i.e.,

$$c^+_{\boldsymbol{k}} = c^-_{\boldsymbol{k}} \, . \tag{2.4.3}$$

In particular, this holds for the Kadomtsev-Petviashvili equation with $\alpha^2 = -1$, or KP-2, as was first noted in [9].

We have seen in Sect. 2.3 that in the two-dimensional case the situation in which the dispersion law is degenerative relative to the lowest-order process "one into two" and nondegenerative relative to all higher-order processes is typical. All degenerative dispersion laws constructed in Sect. 2.3 possess this property. In such a situation the classical scattering matrix S is nontrivial, but only its most singular part is nonvanishing, factorizing into the composition of the three-wave processes. This applies to the KP-1 equation, too.

It is very important in this case to find the scattering matrix in the explicit form in some sense. Let us note that for the most singular part of the S-matrix, one can cancel all inner Green functions and make a replacement in every vorticity:

$$\begin{aligned} &V^{-s_p s_q s_r}_{\boldsymbol{k}_p \boldsymbol{k}_q \boldsymbol{k}_r} \delta\left(-s_p \boldsymbol{k}_p + s_q \boldsymbol{k}_q + s_r \boldsymbol{k}_r\right) \\ &\quad \longrightarrow \pi \mathrm{i} V^{-s_p s_q s_r}_{\boldsymbol{k}_p \boldsymbol{k}_q \boldsymbol{k}_r} \delta\left(-s_p \boldsymbol{k}_p + s_q \boldsymbol{k}_q + s_r \boldsymbol{k}_r\right) \delta\left(-s_p \omega_{\boldsymbol{k}_p} + s_q \omega_{\boldsymbol{k}_q} + s_r \omega_{\boldsymbol{k}_r}\right) \, . \end{aligned} \tag{2.4.4}$$

This modified vorticity will be denoted symbolically as $\hat{\hat{V}}$.

Now we must rememeber that the entire set of diagrams has the factor 2πi. So we can write symbolically

$$c^+ = c^- + 2\left\{\hat{\hat{V}}\left[c^-, c^-\right] + \ldots\right\} \, . \tag{2.4.5}$$

The expression in curly brackets is the whole set of diagrams. Formulae (2.4.5) can be rewritten in the form:

$$\frac{c^+ + c^-}{2} = c^- \hat{\hat{V}}\left[c^-, c^-\right] + \ldots \, , \tag{2.4.6}$$

or

$$c^- = \frac{c^+ + c^-}{2} - \hat{\hat{V}}\left[\frac{c^+ + c^-}{2}, \frac{c^+ + c^-}{2}\right] \, . \tag{2.4.7}$$

Finally, we have

$$\frac{c^+ - c^-}{2} = \hat{\hat{V}}\left[\frac{c^+ + c^-}{2}, \frac{c^+ + c^-}{2}\right] \, ;$$

or, more detailed:

$$\begin{aligned} c^{+^s}_{\boldsymbol{k}} - c^{-^s}_{\boldsymbol{k}} = &\frac{\pi \mathrm{i}}{2} \int \sum_{s_1 s_2} V^{-s s_1 s_2}_{\boldsymbol{k} \boldsymbol{k}_1 \boldsymbol{k}_2} \delta(s\boldsymbol{k} - s_1 \boldsymbol{k}_1 - s_2 \boldsymbol{k}_2) \\ &\times \delta\left(s\omega_{\boldsymbol{k}} - s_1 \omega_{\boldsymbol{k}_1} - s_2 \omega_{\boldsymbol{k}_2}\right) \left(c^{+^{s_1}}_{\boldsymbol{k}_1} + c^{-^{s_1}}_{\boldsymbol{k}_1}\right) \\ &\times \left(c^{+^{s_2}}_{\boldsymbol{k}_2} + c^{-^{s_2}}_{\boldsymbol{k}_2}\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 \, . \end{aligned} \tag{2.4.8}$$

Formula (2.4.8) gives a direct connection between asymptotic states in the case of the degenerative dispersion law.

The equation similar to (2.4.8) applies in the one-dimensional case if one type of wave is involved. In the one-dimensional case, any dispersion law is degenerative to the process of two-particle scattering. For simplicity we consider the Hamiltonian (2.1.11); we have

$$\frac{c_k^+ - c_k^-}{2} = \pi \int T_{kk_1k_2k_3}\delta(k + k_1 - k_2 - k_3) \times \delta\left(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}\right)\left(\frac{c_{k_1}^{*+} + c_{k_1}^{*-}}{2}\right)\left(\frac{c_{k_2}^+ + c_{k_2}^-}{2}\right) \times \left(\frac{c_{k_3}^+ + c_{k_3}^-}{2}\right) dk_1 dk_2 dk_3 \,. \tag{2.4.9}$$

It follows from (2.4.9) that the squared module of the classical S-matrix is equal to unity:

$$|c_k^+|^2 = |c_k^-|^2 \;;$$

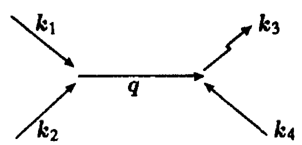
but in general, $\arg c_k^+ \neq \arg c_k^-$. Actually, it is well known that in such one-dimensional systems, the interaction is reduced to a phase shift only.

Now let us return to the two-dimensional case with the decaying degenerative dispersion law, and consider the amplitude of the "two into two" process with the resonant conditions

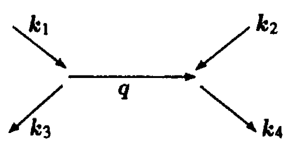
$$k_1 + k_2 = k_3 + k_4 \tag{2.4.10}$$

$$\omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4} \,.$$

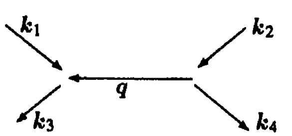
This amplitude is described by three diagrams:

1. 

$q = k_1 + k_2$

2. 

$q = k_3 - k_1$

3. 

$q = k_4 - k_2$

As we have stated above, the nonsingular part of the amplitude localized on the whole manifold (2.4.7) must be identically zero. On the other hand, this amplitude becomes infinity near resonant manifolds corresponding to an interaction via real waves. (The singular part of the amplitude is localized on these very manifolds). These manifolds are different for the three diagrams above. They are defined by the formulae:

$$\omega_{k_1+k_2} = \omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4} \tag{2.4.11a}$$

for diagram 1;

$$\omega_{k_1-k_3} = \omega_{k_1} - \omega_{k_3} = \omega_{k_4} - \omega_{k_2} \tag{2.4.11b}$$

for diagram 2; and

$$\omega_{k_3-k_1} = \omega_{k_3} - \omega_{k_1} = \omega_{k_2} - \omega_{k_4} \tag{2.4.11c}$$

for the diagram 3. Since the amplitude of the process (2.4.7) becomes zero, the singularities localized near manifolds (2.4.11) must cancel each other. This cancellation can only occur if the manifolds coincide, at least partially.

The resonant surface "one into two" for KP-1 consists of two connected parts [see (2.3.21–22)]. A simple analysis shows that each of the two parts described by one of the equations (2.4.11a–c) coincides with some parts described by another of these three equations. This results in the number of connected manifolds, defined by (2.4.11a–c), being equal to three, but not six. The statement about pair compatibility of (2.4.11a–c) is a general one for the degenerative dispersion laws and could be used for enumeration of such laws. It is worth noticing that the coincidence of manifolds (2.4.11a–c) (in the above-mentioned sense) is only a necessary but not sufficient condition for the singularities in (2.4.7) to cancel each other. Rather rigid conditions imposed on the coefficient functions $V_{kk_1k_2}^{ss_1s_2}$ of the three-wave Hamiltonian (2.1.11) should be satisfied. We have checked these conditions for KP-1 equations. We should also note that checking for the cancellation of singularities is a useful and simple way to analyse the existence of the additional motion invariants for the particular systems.

2.5 The Integrals of Motion [5]

One of the important statements in the present paper is that the existence of an infinite set of additional integrals follows from the existence of one such integral of system (2.1.1). Let us prove this fact and find the integral of motion in the form of a formal integropower series:

$$G = \int g_k |a_k|^2 dk + \sum_q \sum_{s \ldots s_q} \int G_{k \ldots k_q}^{s \ldots s_q} a_k^{s} \ldots a_{k_q}^{s_q} \delta(P_q)\, dk \ldots dk_q \,. \tag{2.5.1}$$

Here,

$$P_q = sk + s_1 k_1 + \ldots + s_q k_q \,,$$

and g_k is some function of wave numbers. Substituting (2.5.1) into (2.1.1), it is clear that the functions $G^{s \ldots s_q}_{k \ldots k_q}$ are expressed from the recurrent formulae

$$G^{s s_1 s_2}_{k k_1 k_2} = \frac{s g_k + s_1 g_1 + s_2 g_2}{E^{s s_1 s_2}_{k k_1 k_2}} V^{s s_1 s_2}_{k k_1 k_2} \tag{2.5.2}$$

$$G^{s \ldots s_q}_{k \ldots k_q} = \frac{\mathcal{P}^{s \ldots s_q}_{k \ldots k_q}}{E^{s \ldots s_q}_{k \ldots k_q}} \,. \tag{2.5.3}$$

In these formulae,

$$E^{s \ldots s_q}_{k \ldots k_q} = s\omega_k + s_1 \omega_{k_1} + \ldots + s_q \omega_{k_2}$$

and the function $\mathcal{P}^{s \ldots s_q}_{k \ldots k_q}$ is linearly expressed via $G^{s \ldots s_{q-1}}_{k \ldots k_{q-1}}$. It is not necessary for us to write out this dependence.

It follows from (2.5.2, 3) that the coefficient functions in the integrals possess singularities on all possible resonance manifolds of the form

$$E_n = E^{s \ldots s_n}_{k \ldots k_n} = 0 \,, \quad P_n = 0 \,. \tag{2.5.4}$$

We may continue as follows: Let the wave field $a(r)$ in a physical space be a rapidly decreasing function. Then its Fourier transformation – the field a_k – is a smooth function. This makes it possible to peform the regularization in the expression (2.5.1), but not in a unique way. For example, in all denominators one can perform the substitution:

$$E^{s \ldots s_q}_{k \ldots k_q} \to E^{+ s \ldots s_q}_{k \ldots k_q} = E^{s \ldots s_q}_{k \ldots k_q} + \mathrm{i}0 \,, \tag{2.5.5}$$

or the substitution

$$E^{s \ldots s_q}_{k \ldots k_q} \to E^{- s \ldots s_q}_{k \ldots k_q} = E^{s \ldots s_q}_{k \ldots k_q} - \mathrm{i}0 \,. \tag{2.5.6}$$

Generally speaking, in this case we obtain different integrals of motion; let us designate them as $G^{\pm}$. Any linear combination of these integrals may be the integral of motion; particularly, the difference $(1/2\pi \mathrm{i}) G^0 = G^+ - G^-$. The integral G^0 does not have a quadratic part; its expansion in powers of a^s_k starts from the term:

$$G^0 = \sum_{s, s_i} \int \left(s g_k + s_1 g_{k_1} + s_2 g_{k_2}\right) \delta\left(s\omega_k + s_1 \omega_{k_1} + s_2 \omega_{k_2}\right) \times \delta\left(sk + s_1 k_1 + s_2 k_2\right) V^{s s_1 s_2}_{k k_1 k_2} a^s_k a^{s_1}_{k_1} a^{s_2}_{k_2} dk \, dk_1 dk_2 \,. \tag{2.5.7}$$

The integral G^0 can be called an essentially nonlinear one. It is one of a large number of such integrals. The linear equation

$$\dot{a}^s_k + \mathrm{i}s\omega_k a^s_k = 0$$

allows an essentially nonlinear integral of the form:

$$I = \int \Phi^{s \ldots s_q}_{k \ldots k_q} \delta\left(E^{s \ldots s_q}_{k \ldots k_q}\right) \delta\left(P_q\right) a^s_k \ldots a^{s_q}_{k_q} dk \ldots dk_q \,. \tag{2.5.8}$$

Here, q is an arbitrary integer; $\Phi^{s \ldots s_q}_{k \ldots k_q}$ is an arbitrary function. In the nonlinear system (2.1.1), one can search for the integral in the form of the integropower series in a^s_k, the first term of which is the expression (2.5.8). In this case the regularization problem of the denominators of the form $E^{s \ldots s_r}_{k \ldots k_r}$, $r > q$ again exists, and cannot be solved uniquely. The different integrals obtained will differ by the essentially nonlinear integrals of higher orders.

One can attach a simple physical sense to the integrals $G^{\pm}$ occurring as a result of the regularizations (2.5.5) and (2.5.6). It is easy to see that

$$G^{\pm} = \int g_k \left|a^{\pm}_k\right|^2 dk \,. \tag{2.5.9}$$

Here, $a^{\pm}_k$ are asymptotic states of the wave field at $t \to \pm\infty$. Formula (2.5.9) shows that an arbitrary system (2.1.1) in the rapidly decreasing case is completely integrable. Actually, the change $a^s_k(t)$ in time is a canonical transformation, so the variables $a^{s\pm}_k(t) = c^{\pm}_k \exp(-\mathrm{i}s\omega_k t)$ are canonical. It is now evident that the variables

$$I^{\pm}_k = \left|a^{\pm}_k\right| \quad \text{and} \quad \varphi^{\pm}_k = \arg a^{\pm}_k$$

are the action-angle variables for the system (2.1.1), irrespective of the form of its Hamiltonian. This rather impressive statement is based on a rapid decrease of the function $a(r)$ and, respectively, on the smoothness of the function $a(k)$. In the periodic case, when the function $a(k)$ represents a set of δ-functions,

$$a(k) = \sum_n a_n \delta(k - nk_0) \tag{2.5.10}$$

(k_0 being the vector of the reverse lattice and n a multiindex); integrals (2.5.1) in a general position make no sense (become infinite) and as a rule, integrability vanishes. In the periodic case only those integrals still make sense, the coefficient functions of which remain finite on all resonance manifolds, i.e., where reduction of singularities occurs. For further discussion of the periodic case, see Sect. 2.6.

To observe the singularities, let us introduce the operators $R^{\pm}$, inverse with respect to the operator of the transition (2.1.6), taken for simplicity at $t < 0$:

$$a^{\pm}_k = R^{\pm}_k [a_k] \tag{2.5.11}$$

$$a^{\pm s}_k = a^s_k + \sum_q \sum_{s \ldots s_q} \int R^{\pm\, -s s_1 \ldots s_q}_{\varepsilon\, k k_1 \ldots k_q} a^{s_1}_{k_1} \ldots a^{s_q}_{k_q} \delta\left(P_q\right) dk_1 \ldots dk_q \,.$$

The coefficients $R^{\pm s \dots s_q}_{\varepsilon k \dots k_q}$ at $\varepsilon = 0$ do not depend on time. They have singularities on all possible resonance surfaces $E_q = 0$. Let us put

$$\lim_{\varepsilon\to 0} R^{\pm s s_1 \dots s_q}_{\varepsilon k k_1 \dots k_q} = \frac{\tilde{R}^{\pm s s_1 \dots s_q}_{k k_1 \dots k_q}}{E_q \pm \mathrm{i}0} \,. \tag{2.5.12}$$

Expression $\tilde{R}^{\pm s s_1 \dots s_q}_{k k_1 \dots k_q}$ is regular on the resonance manifold $E_q = 0$, $P_q = 0$, but it can possess singularities on various "junior" resonance manifolds.

Let us consider the operator $R^{\pm}$ and let $t \to -\infty$ in (2.5.11). In this case, $a_k \to a_k^-$, and operator R^+ is to be transformed into a classical scattering matrix. That means that on the resonance surface $E_q = 0$, $P_q = 0$, the numerator in (2.5.12) coincides with the corresponding element in the scattering matrix,

$$\tilde{R}^{+s \dots s_q}_{k \dots k_q} = S^{s \dots s_q}_{k \dots k_q} \,. \tag{2.5.13}$$

Now let us represent the integral of the motion G^+ in the form

$$G^+ = \int g_k a_k a_k^* dk + \int g_k a_k^* \left(a_k^+ - a_k\right) dk + \int g_k a_k \left(a_k^{*+} - a_k^*\right) dk + \int g_k \left(a_k^+ - a_k\right)\left(a_k^{*+} - a_k^*\right) dk \,, \tag{2.5.14}$$

and substitute (2.5.11) into (2.5.14). We collect the terms in (2.5.14) having the singularity on the whole resonant manifold (2.5.14) and having a complete power q; such terms are only contained in the second and third terms in (2.5.14). After symmetrization they are reduced to the form

$$\frac{1}{N} \int \frac{L^{s \dots s_q}_{k \dots k_q}}{E^{s \dots s_q}_{k \dots k_q}} \tilde{R}^{+s \dots s_q}_{k \dots k_q} a^s_k \dots a^{s_q}_{k_q} \delta\left(P_q\right) dk \dots dk_q \,. \tag{2.5.15}$$

N is some integer,

$$L^{s \dots s_q}_{k \dots k_q} = s g_k + \dots + s_q g_{k_q} \,. \tag{2.5.16}$$

Comparing (2.5.14) with (2.5.3), it is clear that $\mathcal{P}^{s \dots s_q}_{k \dots k_q}$ can be represented as follows:

$$\mathcal{P}^{s \dots s_q}_{k \dots k_q} = \frac{1}{N} L^{s \dots s_q}_{k \dots k_q} \tilde{R}^{s \dots s_q}_{k \dots k_q} + A^{s \dots s_q}_{k \dots k_q} E^{s \dots s_q}_{k \dots k_q} \,, \tag{2.5.17}$$

where $A^{s \dots s_q}_{k \dots k_q}$ is regular on $E_q = 0$, although it probably has singularities on the "junior" resonant surfaces.

Let the dispersion law $\omega(k)$ be nondegenerative and the system (2.1.1) have an additional integral of motion with continuous coefficients. As we have already seen, this leads to the triviality of the scattering matrix and the coincidence of asymptotic states $a_k^{\pm}$. Now on the resonance manifold $E_q = 0$, $P_q = 0$ the matrix element $\tilde{R}^{s \dots s_q}_{k \dots k_q} = 0$. This means that on the resonance surface $E_q = 0$, $P_q = 0$, the singularity in the motion integral is cancelled. It can be seen directly from (2.5.2) that the singularity is cancelled in the junior term of expansion (2.5.1) as well.

Now applying induction, we observe that generally all the singularities are cancelled. Thus, in the case under consideration, one can use an arbitrary function g_k in order to construct the motion invariants of the system (2.1.1). Roughly speaking, in this case there are as many integrals with continuous coefficients having a quadratic part as there are in the linear problem. All these integrals are conserved in the periodic case as well; i.e., the periodic system (2.1.1) is quite integrable. In particular, the periodic equation KP-2 is integrable. *Krichever* has recently come to this conclusion on the basis of his algebrogeometric approach [13]. We should keep in mind that our results have been obtained on the level of a formal series, the convergency of which we still do not know.

Now let the dispersion law be degenerative. We restrict ourselves to a case in the form of (2.3.15) at $d = 2$. Now the scattering matrix is different from unity, $S^{s \dots s_q}_{k \dots k_q} \neq 0$. However, the nonvanishing scattering matrix is concentrated on the minimal manifold $\Gamma^{n,m}_M$, when all the scattering occurs with the participation of real intermediate waves only.

Now in the expression (2.5.17), $\tilde{R}^{s \dots s_q}_{k \dots k_q} \neq 0$ and, generally speaking, the integral of the form (2.5.1) is singular. The only way out of this situation is to require the vanishing of the expression $L^{s \dots s_q}_{k \dots k_q}$. It is possible to do this on the manifold $\Gamma^{n,m}_M$ by requiring $g(k) = f(k)$; i.e., the function itself should represent the degenerative dispersion law, permitting parametrization:

$$p = \xi_1 - \xi_2 \,, \qquad q = a(\xi_1) - a(\xi_2)$$
$$\omega = b(\xi_1) - b(\xi_2) \,, \quad g = c(\xi_1) - c(\xi_2) \,.$$

Here, the function $c(\xi)$ is arbitrary.

Thus, in the given case, system (2.1.1) also has an infinite set of integrals of motion with continuous coefficients, but this set is sufficiently narrower than in the previous case. Instead of an arbitrary function of two variables at our disposal, we have only an arbitrary function of one variable. This is not quite enough for the integrability in the periodic case. So the systems with a degenerative dispersion law under periodic boundary conditions are nonintegrable [14], although they might possess an infinite set of integrals of motion. In the following, we study the periodic boundary conditions and search for the action-angle variables.

2.6 The Integrability Problem in the Periodic Case. Action-Angle Variables [5,7]

2.6.1 Canonical Transformations The formulation of the problem of integrability in the periodic case differs from its analogue discussed above, because of the discreteness of all wave vectors and the absence of asymptotic states and a scattering matrix. Therefore we have to find the appropriate language with which to study it. This language does exist and is the infinite-dimensional analogue of the Birkhoff method of finding canonical transformations to the normal form.

We shall see that the Hamiltonian wave system with an additional integral and nondegenerate dispersion law can be reduced by such a transformation to the form of the infinite-dimensional Birkhoff chain.

In spite of all these differences we can get some useful information from the case of smooth a_k. Consider system (2.1.1) with one type of wave and then equations (2.1.14–17). We have used them to represent the "current" fields b_k via asymptotic fields c_k^- and to find the transition matrix operator.

Now we go from interaction representation to the usual fields $a_k = b_k e^{-i\omega_k t}$ and $a_k^- = c_k^- e^{-i\omega_k t}$. We see that as $\epsilon \to 0$, the explicit dependence on time variable t in (2.1.14, 17) disappears and the transformation between $a_k(t)$ and $a_k^-(t)$ becomes a time-independent formal canonical transformation. (The formal scattering marix defines the formal transformation from a_k^- to a_k^+.)

Certainly, these transformations are generally divergent, due to the classical problem of resonances. In each order the corresponding terms in this transformation have the structure (2.1.24). If our system has an additional integral of motion (and we shall take this for granted in what follows), Theorem 2.2.1 holds. So, if the dispersion law is degenerative with respect to decays, our canonical transformation has unequivocally the resonance in the first order and does not exist. Naturally, it does not exist in the periodic case either.

If the dispersion law is nondegenerative, all resonances vanish and the canonical transformations $a_k \to a_k^-$ and $a_k^- \to a_k$ exist; the first of them map equation (2.1.1) to its linear part.

2.6.2 Small Denominators. Let us try to find a periodic analogue to the abovementioned canonical transformations in the nondegenerative case. All wave vectors belong to the lattice

$$k = k_n = (2\pi\nu_1/l_1, \dots, 2\pi\nu_d/l_d) , \tag{2.6.1}$$

where ν_i, $i = 1, \dots, d$ are integrals, l_i are space periods, and $n = (\nu_1, \dots, \nu_d)$ is an integer-valued vector. Sometimes we shall write n for k_n to simplify the formulae. In our notation (Sect. 2.2) the Hamiltonian of (2.2.1) takes the form

$$H^{(3)} = \frac{1}{3!}\sum_{1,2,3} \hat{U}_{1,2,3} a_1 a_2 a_3 . \tag{2.6.2}$$

Consider the canonical transformation $a_k \to a_k^-$:

$$a_k^-(t) = a_k(t) + \sum_{p\geq 2}\sum_{0,1,\dots,p} \hat{\Psi}_{-0,1,\dots,p} a_1 \dots a_p \tag{2.6.3}$$

with all fields defined on the lattice (2.6.1). Coefficients $\Psi_{-0,1,\dots,p}$ are coefficients of the inverse of (2.1.8), rewritten via $a_k(t)$, $a_k^-(t)$. This transformation, as we have seen, generates a motion invariant $I = \sum_n f_n |a_n^-|^2$, which is the same integral as in Sect. 2.5, restricted on the lattice (2.6.1).

If we now consider lattice values of $k_1, \dots, k_p$ in (2.6.3) obeying

$$s_1 k_1 + \dots + s_p k_p = 0 , \tag{2.6.4a}$$

they generally do not belong to the resonant surface:

$$s_1\omega_{k_1} + \dots + s_p\omega_{k_p} = 0 . \tag{2.6.4b}$$

However there are special values which always belong to (2.6.4) and correspond to trivial billiard scattering with $p = 2q$, $s_1, \dots, s_q = 1$, $s_{q+1}, \dots, s_{2q} = -1$ where the set $(k_1, \dots, k_q)$ is a transposition of the set $(k_{q+1}, \dots, k_{2q})$. We shall see that these billiard scattering processes play an important role in the construction of the normal form.

Regardless of the special valuès of periods l_i, $i = 1, \dots, d$, at large $|k_i|$, $i = 1, \dots, p$, the corresponding set $(k_1, \dots, k_p)$ can satisfy (2.6.4) with great accuracy, and we come to the problem of small denominators. However, in our case, Theorem 2.2.1 guarantees that coefficients of (2.6.3) are finite at these points; thus we only have to deal with trivial scattering.

2.6.3 Trivial Scattering and the Normal Form

Theorem 2.6.1. Let the space dimensionality be $d \geq 2$ and let $H_{\rm int}$ be defined by the formulae (2.6.2); furthermore, let the corresponding system (2.6.3) have one more motion invariant (in addition to H and P) of the form

$$I_g = \sum_n g_n |a_n|^2 + \sum_{p\geq 2}\sum_{0\dots p} {}^g I_{01\dots p} a_0 a_1 \dots a_p , \tag{2.6.5}$$

where $g_n \neq$ const, and all coefficient functions ${}^g I_{01\dots p}$ (referred to in the following simply as "coefficients") are finite [7]. Then:

1) If the dispersion law ω_k is degenerative with respect to decays (2.3.1) (so that $d = 2$), then for any f_k satisfying (2.3.14) on (2.3.1) an integral of the motion I_f for the system (2.6.3) exists. The I_f can be obtained by substituting f for g in all terms of the series I_g (2.6.4), and all coefficients ${}^f I_{01\dots p}$ of I_f are finite. [We have learned that such f_k should have the form of (2.3.15)]. However the action-angle variables analytic in a_k, a_k^* do not exist in the periodic case.

2) If ω_k is nondegenerative and has a zero limit as $|k| \to 0$, then there exist integrals I_f with any continuous $f_k \to 0$ as $|k| \to 0$. If in addition $U_{nn0}^{-111} = 0$ is true, then there exists a canonical transformation,

$$\gamma_{n_0}^{s_0} = a_{n_0}^{s_0} + \sum_{p\geq 2}\sum_{1\dots p} \hat{\Gamma}_{-01\dots p} a_1 \dots a_p , \tag{2.6.6a}$$

mapping the system (2.6.1) to the form

$$is\dot{\gamma}_n^s = \Omega_n \gamma_n^s , \quad \Omega_n = \Omega_n\left[|\gamma|^2\right] \tag{2.6.6b}$$

and its Hamiltonian to the normal form

$$H = \sum_n h_n\left[|\gamma|^2\right] = \sum_n \omega_n |\gamma_n|^2 + \dots , \tag{2.6.7}$$

and $\Omega_n = \partial H / \partial |\gamma_n|^2$. The quantities $\Gamma_{-01\ldots p}$ can be obtained by recurrence or with the aid of the diagram technique and are finite at any p so that the zero denominators are absent in the canonic transformation (2.6.6).

3) If ω_k has a singularity near $|k| \to 0$, then the results are the same when imposing additional constraints. For example, for the KP-2 equation (1.1.3) with $\alpha^2 = -i$, it is necessary to impose a condition $a_n = 0$ at $n = (0, \nu_2)$; this constraint is compartible with the equation.

4) Let $H_{\text{int}} = H^{(4)}$;

$$H^{(4)} = \frac{1}{4} \sum_{1,2,3,4} \hat{T}^{+1+1-1-1}_{1234} a_1 a_2 a_3^* a_4^* . \tag{2.6.8}$$

then the system (2.1.1) having an additional motion invariant I_g analytic in a, a^* also has the additional integrals I_f with any continuous f_k, under the assumption that there exist limits of ω_k as $|k| \to 0$ and of T_{1234} as $k_1 \to k_3$ or $k_1 \to k_4$. Under these conditions there exists a canonical transformation mapping this system to the form (2.6.6) and its Hamiltonian to the normal form (2.6.7). The canonical transformation can be constructed in full analogy with (2.6.6).

5) If T_{1234} does not have such a limit (as an example one can think of the Davey-Stewartson equation (1.1.4) [15]), then these singularities are to be analysed separately; for the Davey-Stewartson equation, all the results of statement 4) are true [7].

Proof. Statement 1) is actually proved in Sect. 2.6a; therefore let us go to the nondegenerative case and suppose for convenience that ω_k is nondecaying, nonsingular at $|k| = 0$ and $\omega(0) = 0$. Then Theorem 2.2.1 does not imply any restriction in the first order on th quantity $U_{i,jl}$. Actually, in the space (k_1, k_2), the surface

$$\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2) \tag{2.6.9}$$

consists of two planes $k_1 = 0$, $k_2 = 0$, and each function f_k with $f(0) = 0$ is degenerative on them.

We shall seek coefficients of the transformation (2.6.9) immediately from (2.6.6, 7). We see that $\gamma_k[a]$ differs from $a_k^-[a]$, given by (2.6.3), only due to the nonlinear frequency shift:

$$\Omega_k = \omega_k + \delta\Omega_k \left[|\gamma|^2\right]$$

$$\delta\Omega_{k_0} = \sum_1 \Omega_{k_0 k_1} |\gamma_{k_1}|^2 + \sum_{p \geq 2} \sum_{1\ldots p} \Omega_{01\ldots p} |\gamma_1|^2 \ldots |\gamma|^2 .$$

This means that in the first order, (2.6.3) and (2.6.6) coincide, i.e.,

$$E_{012} \hat{\Gamma}_{012} = E_{012} \hat{\Psi}_{012} = \frac{-s_0}{2} \hat{U}_{012} . \tag{2.6.10}$$

So, we see that for (2.6.6) to exist, we have to require that

$$U^{1-1\pm1}_{nn0} = 0 , \tag{2.6.11a}$$

and (2.6.2) becomes

$$H^{(3)} = \frac{1}{3!} \sum_{1,2,3}{}' \hat{U}_{123} a_1 a_2 a_3 , \tag{2.6.2a}$$

where the prime indicates that terms with zero n_1 or n_2 or n_3 are absent. If this is true, one can choose

$$\Gamma^{1-1\pm1}_{nn0} = 0 , \tag{2.6.11b}$$

and (2.6.6) does not have small denominators in the first order. In the second order we have

$$E^{-1-111}_{0123} \Gamma^{-1-111}_{0123} = S^{-1-111}_{0123} - \Omega_{n_0 n_1} \delta(T_2) , \tag{2.6.12}$$

where T_2 is a surface of trivial scattering of the second order:

$$T_2 : \begin{array}{c} n_0 = n_3 \\ n_1 = n_2 \end{array} \quad \text{or} \quad \begin{array}{c} n_0 = n_2 \\ n_1 = n_3 . \end{array} \tag{2.6.13}$$

$\delta(T_2) = 1$ if $(n_0 \ldots n_3) \in T_2$, and $\delta(T_2) = 0$ otherwise, and ($\tilde{\sigma}$ is the symmetrization operator)

$$S^{-1-111}_{0123} = E^{-1-111}_{0123} \Psi^{-1-111}_{0123} = -\frac{1}{3} \hat{\sigma}_{0123} \sum_{s'n'} \Psi^{s'}_{-01n'} \hat{U}^{-s's_2s_3}_{n'n_2n_3} .$$

We see that Γ_{0123} equals Ψ_{0123} outside T_2 and differs from it on T_2, where Ψ_{0123} does not exist. Outside T_2, Ψ_{0123} exists, due to Theorem 2.2.1. From (2.6.12) we find $\Omega_{n_0 n_1}$ as

$$\Omega_{n_0 n_1} = S^{-1-111}_{0123} \big|_{T_2} . \tag{2.6.14}$$

Now on T_2 the Γ^{-1-111}_{0123} is undetermined and should be obtained from the canonicity conditions

$$\{\gamma_p, \gamma_q\} = 0 = \{\gamma_p^*, \gamma_q^*\} , \quad \{\gamma_p, \gamma_q^*\} = i\delta_{pq} , \tag{2.6.15}$$

where

$$\{\alpha, \beta\} = \mathrm{i} \sum_n \left(\frac{\partial \alpha}{\partial a_n} \frac{\partial \beta}{\partial a_n^*} - \frac{\partial \alpha}{\partial a_n^*} \frac{\partial \beta}{\partial a_n} \right) .$$

This gives, for example,

$$\Gamma^{-1-111}_{pqpq} = \frac{1}{2} \left| \Gamma^{-1-11}_{pqp+q} \right|^2 . \tag{2.6.16}$$

We shall consider the higher orders by induction. It is more convenient to do this for the inverse of (2.6.6),

$$a_{n_0}^{s_0} = A_{n_0}^{s_0}[\gamma] = \gamma_{n_0}^{s_0} + \sum_{p\geq 2}\sum_{1\ldots p} \hat{A}_{-01\ldots p}\gamma_1 \ldots \gamma_p . \tag{2.6.17}$$

The recurrence condition for coefficients of A can be written down in more compact form if we introduce the notation

$$\hat{\Pi}_{01\ldots q} = \Pi^{(q)} , \quad \hat{U} = \hat{U}_{ijl} , \quad \Omega_{n_\alpha}^{(p)} = \Omega_{n_\alpha 1\ldots p}$$

and

$$\Pi^{(q)} * X^{(r)} = \sum_{n' s'} \hat{\Pi}_{01\ldots q-1}{}_{n'}^{s'} \hat{X}_{n' q+1\ldots q+r}^{-s'}$$

for any functions. Then we have

$$E^{(2p)} A^{(2p)} = -\hat{\sigma}\left\{ -s_0 \hat{U} * A^{(2p-1)} - \frac{s_0}{2}\sum_{q=2}^{2p-2} \hat{U} * A^{(q)} * A^{(2p-q)} + \sum_{q=1}^{p-1} A^{(2p-2q)} \sum_{\alpha=1}^{2p-2q} s_\alpha \Omega_{n_\alpha}^{(q)} \delta\left(\mathcal{T}_q\right)\right\} \tag{2.6.18a}$$

$$E^{(2p+1)} A^{(2p+1)} = -\hat{\sigma}\left\{ -s_0 \hat{U} * A^{(2p)} - \frac{s_0}{2}\sum_{q=2}^{2p-1} \hat{U} * A^{(q)} * A^{(2p-q+1)} + \Omega_{n_0}^{(p)} \delta\left(\mathcal{T}_{p+1}\right) + \sum_{q=2}^{p} A^{(2p-2q+1)} \sum_{\alpha=1}^{2p-2q+1} s_\alpha \Omega_{n_\alpha} \delta\left(\mathcal{T}_q\right)\right\} ; \tag{2.6.18b}$$

$A^{(1)} = 1$, and outside the resonant surface $E_{ijl} = 0$, one has

$$A^{(2)} = \hat{A}_{ijl} = -\hat{\Gamma}_{ijl} = \frac{s_i}{2}\frac{\hat{U}_{ijl}}{E_{ijl}} . \tag{2.6.18c}$$

On the resonant surface, if $E_{ijl} = 0$, we put (as above for $\Gamma^{(2)}$)

$$\hat{A}_{ijl} = 0 .$$

After this, the right-hand side of (2.6.18b) for $A^{(3)}$ up to the sign coincides with the right-hand side of (2.6.12), and we obtain Ω_{01} in the $\delta\Omega$ of the form (2.6.14). On the resonant surface

$$E_{0123} = 0 , \quad P_{0123} = 0 .$$

Let us define $A^{(3)} \sim \delta(\mathcal{T}_2)$; the coefficient of proportionality must be obtained from canonicity conditions (2.6.15).

When going to higher orders $p \geq 4$, we suppose that $A^{(2q)}$; $2q < p$ equals zero on the resonant surfaces

$$E^{(2q)} = 0 , \quad P^{(2q)} = 0 , \tag{2.6.19a}$$

while $A^{(2q+1)}$, $2q+1 < p$, on the manifold

$$E^{(2q+1)} = 0 , \quad P^{(2q+1)} = 0 \tag{2.6.19b}$$

may be nonzero only on the trivial scattering submanifolds $\mathcal{T}_q$ of (2.6.19b) indicated above. Their values on $\mathcal{T}_q$ should be defined from (2.6.15).

Now consider the inversion procedure used to obtain (2.6.17) from (2.6.6). If we write (2.6.6) symbolically as

$$\gamma = a + \Gamma^{(2)}aa + \Gamma^{(3)}aaa + \Gamma^{(4)}aaaa + \ldots , \tag{2.6.20}$$

then

$$\begin{aligned} a = &\gamma - \Gamma^{(2)}\gamma\gamma - \Gamma^{(3)}\gamma\gamma\gamma + 2\Gamma^{(2)} * \Gamma^{(2)}\gamma\gamma\gamma - \big[\Gamma^{(4)} \\ &+ 3\Gamma^{(3)} * \Gamma^{(2)} + 2\Gamma^{(2)}\Gamma^{(3)} - \Gamma^{(2)} * \Gamma^{(2)} * \Gamma^{(2)} \\ &-6\Gamma^{(2)}\Gamma^{(2)} * \Gamma^{(2)}\big]\,\gamma\gamma\gamma\gamma + \ldots . \end{aligned} \tag{2.6.21}$$

So we see that

$$A^{(p)} = -\Gamma^{(p)} + \sum \alpha_q \Gamma^{(q)} * \Gamma^{(p-q+1)} + \ldots ,$$

and that

$$E^{(p)} A^{(p)} \underset{E^{(p)}\to 0}{\longrightarrow} - E^{(p)}\Gamma^{(p)}$$

in points of a general position on (2.6.19). Therefore we can apply the considerations in Sect. 2.6a based on Theorem 2.2.1, and prove the solvability of (2.6.18) in points of a general position on (2.6.19). According to the induction hypothesis, in other points (of a special position) on (2.6.19), the terms which do not contain Ω can be nonzero only on a submanifold $\mathcal{T}_q$ of (2.6.19b) with $q = p$, where Ω-containing terms are nonzero only.

We consider the term $U * A^{(q)} * A^{(r)}$ on the resonant surface $E^{(q+r)} = 0$ in special points. This means that $E^{(q)} = 0$ and $E^{(r)} = 0$. According to the induction hypothesis, $A^{(r)} \neq 0$ only on the trivial scattering submanifold if r is odd. As for $U * A^q$ on $E^{(q)} = 0$, we have already seen that due to Theorem 2.2.1, it can be nonzero only in special points of $E^{(q)} = 0$, i.e., on the trivial scattering submanifold of $E^{(q)} = 0$ only (if at all). The $\Omega^{(q)}$ with $q < p$ are already known from junior orders, while $\Omega^{(p)}$ in (2.6.18b) is not known and should be chosen so as to eliminate the right-hand side of (2.6.18b) on $\mathcal{T}_p$.

Now (2.6.18) are solvable, but $A^{(2p+1)}$ are undetermined on corresponding resonant surfaces. We set $A^{(2p)}|_{E^{(2p)}=0} = 0$ while $A^{(2p+1)} \sim \delta(\mathcal{T}_p)$, and the coefficient of proportionality is to be found from (2.6.15). We can come to the next order and Theorem 2.6.1 is proven. (For the self-contained proof of this theorem, which does not apply to the rapidly decreasing case, see [7].)

Consider now the singular dispersion laws. The typical example is the Kadomtsev-Petviashvili equation (1.1.3) with $\omega_k = P^3 + \alpha^2 Q^2/P$, $k = (P, Q)$. This equation is known to have an infinite number of motion invariants both at $\alpha^2 = 1$ (KP-1), when the dispersion law is degenerative, and at $\alpha^2 = -1$ (KP-2), when it is nondecaying and nondegenerative. In the Hamiltonian description it corresponds to the equation with $H_{\text{int}} = H^{(3)}$, and

$$V_{123}^{111} = 0\ , \quad V_{123}^{-111} = \sqrt{P_1 P_2 P_3}\,\theta(P_1)\theta(P_2)\theta(P_3)\ , \tag{2.6.22}$$

where θ is a Heaviside function. From this form of H_{int} it follows that $(d/dt)|a_{0Q}|^2 = 0$. But $\omega_k \to \infty$ as $P \to 0$, $Q \neq 0$ so that the KP-1 equation is senseless with resepct to a_{0Q}. The complete determination of this equation for a_{0Q} in the case of rapidly decreasing initial conditions leads to infinite numbers of constraints [16] except in the periodic case, where there is only one constraint, which can be easily obtained from the description of the KP equation in the form of a system,

$$\begin{aligned} &u_t + u u_x + u_{xxx} + 3\alpha w_y = 0 \\ &w_x = \alpha u_y \end{aligned} \tag{2.6.23}$$

Let us consider the Fourier-image of (2.6.23) with boundary conditions periodic in x and y, and particularly the dynamics of components with $P = 0$. One can see that for the solvability of the second equation with respect to w, it is necessary to impose a constraint $u_n = 0$ at $n = (0, Q)$, and the requirement of its invariance means that w_{0Q}. The latter is equivalent to introducing an integration constant,

$$\frac{1}{l}\int_0^l dx \int_0^x u_{yy} dx'\ ,$$

when reducing (2.6.23) to a single equation (l is a period in x). This additional term was obtained in [17] from a consideration of the Hamiltonian structure of the KP equation as generated by the Lie-Berezin-Kirillov bracket on orbits of a coadjoint action of the gauge group. In the form of (2.6.3) this leads to the correct form of the periodic KP-1 equation $[n_j = (P_j, Q_j)]$:

$$\begin{aligned} &\dot{a}_{0Q}^{s} = 0 \\ &\mathrm{i} s_0 \dot{a}_0 = \omega_{n_0} a_0 + \sum_{1,2}{}' \hat{U}_{-012} a_1 a_2\ , \quad P_0 \neq 0\ , \end{aligned} \tag{2.6.24}$$

where the prime near the summation sign indicates the absence of terms with $P_1 = 0$, $P_2 = 0$.

Starting from (2.6.24) and imposing the constraint $a_{0Q} = 0$, one can perform all the above procedures and see that if $\alpha^2 = -1$, the canonical transformation (2.6.6) exists and thus the nondegenerative KP equation is completely integrable.

If $\alpha^2 = 1$, the canonical transformation (2.6.6) does not exist and actions analytic in field variables are absent.

It should be noted that if we consider the case of rapidly decreasing boundary conditions, the distinctions between degenerative and nondegenerative equations disappear (we have pointed out this fact in Sect. 2.5 already). This is the reason for thinking of both KP equations as completely integrable systems [18]. The analogical distinction between two KP equations was recently obtained by *Krichever*, using his algebrogeometric appraoch [13].

Consider now the singular four-particle interaction when T_{1234} in (2.6.8) has a singularity on T_2 [(2.6.20) and above]. The important example for physical applications is the Davey-Stewartson equation (1.1.4) having T_{1234} of the form [19]:

$$\begin{aligned} T_{1234} = &\frac{(P_1 - P_3)^2 - (Q_1 - Q_3)^2}{(P_1 - P_3)^2 + (Q_1 - Q_3)^2} \\ &+ \frac{(P_1 - P_4)^2 - (Q_1 - Q_4)^2}{(P_1 - P_4)^2 + (Q_1 - Q_4)^2} = \frac{\kappa_1^2\kappa_2^2 - \mu_1^2\mu_2^2}{(\kappa_1^2 + \mu_1^2)(\kappa_2^2 + \mu_2^2)}\ , \end{aligned} \tag{2.6.25}$$

where κ_1, κ_2, μ_1, μ_2 and P_0, Q_0 are the coordinates parametrizing the resonant surface:

$$\begin{aligned} &P_1 = P_0 + \tfrac{1}{2}(\kappa_1 + \kappa_2)\ , \quad P_2 = P_0 - \tfrac{1}{2}(\kappa_1 + \kappa_2) \\ &P_3 = P_0 + \tfrac{1}{2}(\kappa_1 - \kappa_2)\ , \quad P_4 = P_1 - \tfrac{1}{2}(\kappa_1 - \kappa_2) \\ &Q_1 = Q_0 + \tfrac{1}{2}(\mu_1 + \mu_2)\ , \quad Q_2 = Q_0 - \tfrac{1}{2}(\mu_1 - \mu_2) \\ &Q_3 = Q_0 + \tfrac{1}{2}(\mu_1 - \mu_2)\ , \quad Q_4 = Q_0 + \tfrac{1}{2}(\mu_1 + \mu_2)\ , \end{aligned} \tag{2.6.26}$$

where the resonance condition $E_{1234} = 0$ takes the form

$$\kappa_1\kappa_2 - \mu_1\mu_2 = 0\ . \tag{2.6.27}$$

In points of a general position we see that $T_{1234} = 0$, in accordance with Theorem 2.6.1. Those points are singular where

$$\kappa_1 = \mu_1 = 0 \quad \text{or} \quad \kappa_2 = \mu_2 = 0 \tag{2.6.28}$$

In points of a general position for the transformation (2.6.6), we have

$$\Gamma_{1234} = \frac{\kappa_1\kappa_2 + \mu_1\mu_2}{(\kappa_1^2 + \mu_1^2)(\kappa_2^2 + \mu_2^2)}\ .$$

In the periodic case we see from (1.4) that on the manifolds (2.6.28), say, $\kappa_2 = \mu_2 = 0$, one has to put

$$\begin{aligned} &T = \frac{\kappa_1^2 - \mu_1^2}{\kappa_1^2 + \mu_1^2} \qquad \text{if } \kappa_1^2 + \mu_1^2 \neq 0\ , \quad \kappa_2 = \mu_2 = 0 \\ &T = 0 \qquad \text{if } \kappa_1 = \kappa_2 = \mu_1 = \mu_2 = 0\ . \end{aligned}$$

As a result we come to the nonsingular vertex T_{1234}; furthermore, we may construct the transformation (2.6.6) and prove the absence of zero denominators in it (using the existence of an additional integral) via the scheme outlined above.

3. Applications to Particular Systems

3.1 The Derivation of Universal Models

The present volume contains a paper by F. Calogero devoted to the derivation of universal models for nonlinear wave interactions from rather general types of differential equations. Many of these models appear to be exactly solvable; in fact even the more interesting situations hold true.

Let us take as a starting point some particular physical wave system, anywhere from solid state physics to astrophysics; sometimes this model can be stated in terms of differential equations. Then let us perform the asymptotic expansion procedure on it. In doing so we single out the essential kernel of the physical phenomenon under consideration. The resulting model will prove universal and applicable to many physical problems at once. It is very likely that it will appear to be exactly solvable. In that case the model itself represents an important mathematical object. In order to study it, we may have to use advanced mathematics, like Lie group theory or algebraic geometry.

The occurrence of such wonderful things seems incomprehensible and an explanation may lie in the field of philosophy rather than in science. Here it is worth recalling the well-known paper by E. Wiggner, "On the Incomprehensible Effectivity of Mathematics in Natural Science" [20]. All of the above concerns both conservative and dissipative systems. We do not have a sufficiently general language for describing dissipative systems, but for conservative ones, we do: it is the language of Hamiltonian mechanics. This language takes its most simple form in the case of translationally invariant systems; i.e., when considering phenomena occurring in homogeneous space. Then it is possible to introduce canonical variables (amplitudes of progressive waves) $a_{\boldsymbol{k}}$ and take a Hamiltonian of the system in the form of a functional power series in $a_{\boldsymbol{k}}$, $a_{\boldsymbol{k}}^*$ as a starting point.

We have considered the form of a Hamiltonian in the beginning of the present paper; dealing with such a Hamiltonian, it is easy to construct different universal models. A detailed description of the procedure may be found in papers [21–23]; we now consider several particular examples.

Let the Hamiltonian of the system have the form (our notation is the same is in Chap. 2)

$$H = \int \omega_{\boldsymbol{k}} |a_{\boldsymbol{k}}|^2 \, d\boldsymbol{k} + \frac{1}{3!} \int \hat{V}_{012} a_0 a_1 a_2 \, d\boldsymbol{k}_0 d\boldsymbol{k}_1 d\boldsymbol{k}_2 \, . \qquad (3.1.1)$$

Let the oscillations in the medium with the wave vectors lying near three values $\boldsymbol{\xi}_0$, $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ be excited; then the resonance conditions are fulfilled:

$$\boldsymbol{\xi}_0 = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 \, , \quad \omega_{\xi_0} = \omega_{\xi_1} + \omega_{\xi_2} \, . \qquad (3.1.2)$$

Suppose that the domains in $\boldsymbol{k}$-space occupied by these three packets do not overlap. Then one can introduce three fields, $a_0(\boldsymbol{k})$, $a_1(\boldsymbol{k})$, $a_2(\boldsymbol{k})$, to describe the behaviour of the system at times shorter than the time of the next order interaction (when one can neglect the higher nonlinear processes). Thus

$$a_{\boldsymbol{k}} = a_0(\boldsymbol{k}) + a_1(\boldsymbol{k}) + a_2(\boldsymbol{k}) \, ,$$

and in the Hamiltonian (3.1.1), one can make a substitution:

$$V_{012} a_0 a_1 a_2 \cong V^{s_0 - s_0 - s_0}_{\xi_0 \xi_1 \xi_2} a_0^{s_0}(\boldsymbol{k}_0) a_1^{-s_0}(\boldsymbol{k}_1) a_2^{-s_0}(\boldsymbol{k}_2)$$

$$H \approx \sum_{j=0} \int \left[\omega_{\xi_j} + (\boldsymbol{v}_j, \boldsymbol{k} - \boldsymbol{\xi}_j) \right] |a_j(\boldsymbol{k})|^2 \, ,$$

where $\boldsymbol{v}_j = \nabla_{\boldsymbol{k}} w(\boldsymbol{k} = \boldsymbol{\xi}_j)$. Now coming to the envelope fields,

$$A_j(\boldsymbol{r}, t) = \frac{\exp(\mathrm{i}\omega_{\xi_j} t)}{(2\pi)^{d/2}} \int a_j(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} \, ,$$

we obtain the well-known three-wave system, q is a constant:

$$\begin{aligned} \dot{A}_0 + (v_0 \nabla) A_0 &= q A_1 A_2 \\ \dot{A}_1 + (v_1 \nabla) A_1 &= q^* A_0 A_2^* \\ \dot{A}_2 + (v_2 \nabla) A_2 &= q^* A_0 A_1^* \, . \end{aligned} \qquad (3.1.3)$$

Now we show how the Hamiltonian (3.1.3) arises in physics, using the example of waves in media with weak dispersion [23] in which

$$\omega_{\boldsymbol{k}}^2 = c^2 k^2 \left(1 + \lambda k^2\right) \, . \qquad (3.1.4)$$

Such a dispersion law is characteristic for waves on the surface of shallow water or for ion-acoustic waves in plasma. Media with a weak dispersion are described by the hydrodynamiclike equations with an additional term [22]:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \varrho \nabla \phi = 0 \qquad (3.1.5)$$

$$\frac{\partial \phi}{\partial t} + \frac{(\Delta \phi)^2}{2} = -\frac{c^2}{\varrho_0} \left(\delta\varrho + \frac{3}{2} - g \frac{\delta \varrho^2}{\varrho_0} - 2\lambda \Delta \delta\varrho \right) \, .$$

Here, ϕ is a hydrodynamic potential, $\delta\varrho$ is a quantity canonically conjugated to it, which can be called the "denstiy"; g is a constant. Introducing new variables $a_{\boldsymbol{k}}$ by formulae ($d = 3$)

$$\delta\varrho = \frac{1}{(2\pi)^{3/2}} \int \frac{\varrho_0^{1/2} k^{1/2}}{\sqrt{2} c^{1/2}} \left(a_{\boldsymbol{k}} + a^*_{-\boldsymbol{k}}\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} \tag{3.1.6}$$

$$\nabla\phi = \frac{-i}{(2\pi)^{3/2}} \int \frac{\boldsymbol{k}}{k^{1/2}} \frac{c^{1/2}}{\sqrt{2}\varrho_0^{1/2}} \left(a_{\boldsymbol{k}} - a^*_{-\boldsymbol{k}}\right) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} ,$$

we obtain an interaction Hamiltonian (3.1.1) with

$$V^{s_0-s_0-s_0}_{\boldsymbol{k}_0\boldsymbol{k}_1\boldsymbol{k}_2} = V^{s_0s_0s_0}_{\boldsymbol{k}_0\boldsymbol{k}_1\boldsymbol{k}_2} = \frac{c^{1/2}}{16(\pi^3\varrho_0)^{1/2}} \left\{ \frac{(\boldsymbol{k}_0\cdot\boldsymbol{k}_1)k_2^{1/2}}{k_0^{1/2}k_1^{1/2}} + \frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)k_0^{1/2}}{k_1^{1/2}k_2^{1/2}} + \frac{(\boldsymbol{k}_2\cdot\boldsymbol{k}_0)k_1^{1/2}}{k_2^{1/2}k_0^{1/2}} + 3g\,(k_0k_1k_2)^{1/2} \right\} . \tag{3.1.7}$$

The corresponding equations are

$$\frac{\partial a_{\boldsymbol{k}_0}}{\partial t} = -i\frac{\delta H}{\delta a^*_{\boldsymbol{k}_0}} = -i\omega_{\boldsymbol{k}_0} a_{\boldsymbol{k}_0} - i\Big\{2\int \Big[2V^*_{\boldsymbol{k}_0\boldsymbol{k}_1\boldsymbol{k}_2} a_1 a_2^* \delta_{\boldsymbol{k}_0+\boldsymbol{k}_2-\boldsymbol{k}_1} + V_{\boldsymbol{k}_0\boldsymbol{k}_1\boldsymbol{k}_2} a_1 a_2 \delta_{\boldsymbol{k}_0-\boldsymbol{k}_1-\boldsymbol{k}_2} + V^*_{\boldsymbol{k}_0\boldsymbol{k}_1\boldsymbol{k}_2} a_1^* a_2^* \delta_{\boldsymbol{k}_0+\boldsymbol{k}_1+\boldsymbol{k}_2}\Big] d\boldsymbol{k}_1 d\boldsymbol{k}_2 \Big\} . \tag{3.1.8}$$

They describe weakly nonlinear waves which are close to sinusoidal if the nonlinear correction to the frequency is much less than the dispersion correction. In essence this is a validity condition for the approximation (3.1.7–8).

Now let us convince ourselves that weakly nonlinear and weakly dispersive waves in the system (3.1.7–8) are described by the KP equation. Remember that in the original equations (3.1.5) we supposed the long wave approximation. Because the nonlinearity and dependence of a transverse coordinate are small, in the interaction Hamiltonian this transverse coordinate can be omitted. It should be taken into account only in the linear part of (3.1.8). As a result we obtained from (3.1.6)

$$V^{-s_0s_1s_2}_{\boldsymbol{k}_0\boldsymbol{k}_1\boldsymbol{k}_2} = \frac{c^{1/2}(3g+3)}{16(\pi^3\varrho_0)^{1/2}} (p_0p_1p_2)^{1/2} ,$$

where $\boldsymbol{k}_i = (p_i, q_i)$, $i = 0,1,2$ and $q_i \ll p_i$. In the linear term one has to expand $\omega_{\boldsymbol{k}}$ in a power series of the small q at finite p to obtain ($\lambda p^2 \ll 1$, $q^2 \ll p^2$):

$$\omega_{\boldsymbol{k}} = c\sqrt{p^2+q^2+\lambda(p^2+q^2)^2} \approx cp\sqrt{1+q^2/p^2+\lambda p^2} \approx c\left(p + \frac{\lambda}{2}p^3 + \frac{q^2}{2p}\right) .$$

Up to some coefficients which can be removed by scaling transformations and the term cp in $\omega_{\boldsymbol{k}}$, which can be removed by transformation to the movable reference system, we obtain the KP Hamiltonian in normal coordinates $a_{\boldsymbol{k}}(t)$, related with the original variable $u(x,y,t)$ via a formula like (3.1.6a).

Now consider the general case of an interaction of high-frequency (short) and low-frequency (long) waves [23] in the conservative medium with the Hamiltonian H. We introduce normal coordinates $a_{\boldsymbol{k}}$ for short waves and $b_{\boldsymbol{k}}$ for long waves. In these coordinates the quadratic part of the Hamiltonian has the form

$$H_0 = \int \omega_{\boldsymbol{k}} a_{\boldsymbol{k}} a^*_{\boldsymbol{k}} d\boldsymbol{k} + \int \Omega_{\boldsymbol{k}} b_{\boldsymbol{k}} b^*_{\boldsymbol{k}} d\boldsymbol{k} , \tag{3.1.9}$$

where $\omega_{\boldsymbol{k}}$ and $\Omega_{\boldsymbol{k}}$ are the dispersion laws of high-frequency and low-frequency waves. The interaction Hamiltonian can be represented in the form

$$H_{\text{int}} = H_1 + H_2 + H_3 ,$$

where H_1 describes the mutual interaction of the short waves, H_2 describes their interaction with the long waves and H_3 describes the mutual interaction of long waves. The motion equations have the standard form

$$\dot a_{\boldsymbol{k}} = -i\frac{\delta H}{\delta a^*_{\boldsymbol{k}}} , \quad \dot b_{\boldsymbol{k}} = -i\frac{\delta H}{\delta b^*_{\boldsymbol{k}}} . \tag{3.1.10}$$

In what follows we shall suppose that the b-amplitudes are small ($b_{\boldsymbol{k}} \ll a_{\boldsymbol{k}}$), and neglect the H_3. In the H_3 we keep only terms linear in b and of the lowest order in $a_{\boldsymbol{k}}$ which do not disappear when averaging over the long-wave period. These requirements enable us to find the Hamiltonian

$$H_2 = \int \left[h_{\boldsymbol{k}_0\boldsymbol{k}_1\boldsymbol{k}_2} b_{\boldsymbol{k}_0} a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_1} + (^*)\right] \delta_{\boldsymbol{k}_0-\boldsymbol{k}_1-\boldsymbol{k}_2} d\boldsymbol{k}_0 d\boldsymbol{k}_1 d\boldsymbol{k}_2 .$$

Here, (*) indicates the complex conjugated expression. The theory is valid when

$$H_2 \gg \int \omega_{\boldsymbol{k}} |b_{\boldsymbol{k}}|^2 d\boldsymbol{k}$$

and the low-frequency waves are strongly rearranged by the action of the high-frequency waves. We choose the Hamiltonian H_1 as

$$H_1 = \tfrac{1}{2} \int W_{\boldsymbol{k}_0\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} a_0^* a_1^* a_2 a_3 \delta_{\boldsymbol{k}_0+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3} d\boldsymbol{k}_0 \ldots d\boldsymbol{k}_3 .$$

This structure of H_1 is characteristic for a medium with cubic nonlinearity and in some cases, for a medium with quadratic nonlinearity and in some cases, for a medium with quadratic nonlinearity when cubic terms in the Hamiltonian may be removed by canonical transformation [22].

The interaction Hamiltonian is greatly simplified when the high-frequency waves form a narrow packet in the $\boldsymbol{k}$-space near $\boldsymbol{k} \approx \boldsymbol{k}_0$. Then one can put

$$W_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} \approx W_{\boldsymbol{k}_0\boldsymbol{k}_0\boldsymbol{k}_0\boldsymbol{k}_0} = q \tag{3.1.11a}$$

$$h_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2} \approx h_{\boldsymbol{k}\boldsymbol{k}_0\boldsymbol{k}_0} = f(\boldsymbol{k}, \boldsymbol{k}_0) \tag{3.1.11b}$$

$$\omega_k = \omega_{k_0} + \frac{\partial\omega}{\partial k}\delta k + \frac{1}{2}\frac{\partial^2\omega}{\partial k_\alpha \partial k_\beta}\delta k_\alpha \delta k_\beta . \tag{3.1.11c}$$

If the low-frequency waves are acoustic waves, $\Omega = ck$, it is possible to calculate $f(\boldsymbol{k}, \boldsymbol{k}_0)$ and the Hamiltonian of the system explicitly. To do so, let us replace a_k by the new variable (envelope field),

$$\Psi(\boldsymbol{r}, t) = \frac{1}{(2\pi)^{3/2}} \int a_k \exp\{i\omega(\boldsymbol{k}_0)t + i(\boldsymbol{k} - \boldsymbol{k}_0)\cdot \boldsymbol{r}\}\, d\boldsymbol{k} , \tag{3.1.12}$$

and b_k by two scalar functions: the density variation $\delta\varrho$ and the medium velocity $\boldsymbol{v}$ defined by formula (3.1.6) (with b_k standing for a_k). The energy of the narrow packet in the $\boldsymbol{k}$-space is $\omega(\boldsymbol{k}_0)|\Psi(\boldsymbol{r},t)|^2$. In the presence of the sound wave the quantity $\omega(\boldsymbol{k}_0)$ acquires a variation,

$$\delta\omega(\boldsymbol{k}_0) = \frac{\partial\omega(\boldsymbol{k}_0)}{\partial\varrho}\delta\varrho + \frac{\partial\omega(\boldsymbol{k}_0)}{\partial \boldsymbol{v}}\boldsymbol{v} ,$$

and the corresponding variation of the high-frequency wave energy is

$$\delta\varepsilon = \int |\Psi|^2 \left(\frac{\partial\omega}{\partial\varrho}\delta\varrho + \frac{\partial\omega}{\partial\boldsymbol{v}}\boldsymbol{v}\right) d\boldsymbol{r} . \tag{3.1.13}$$

The quantity $\delta\varepsilon$ obviusly coincides with H_2. In the isotropic medium,

$$\frac{\partial\omega(\boldsymbol{k}_0)}{\partial\boldsymbol{v}} = \alpha\frac{\boldsymbol{k}_0}{k_0} .$$

Let us introduce the notation

$$\frac{\partial\omega(\boldsymbol{k}_0)}{\partial\varrho} = \beta , \quad \boldsymbol{v} = \nabla\Phi ,$$

where Φ is the hydrodynamic potential. As one can see from (3.1.12), the quantity Ψ is a canonical transformation of a_k and therefore

$$i\frac{\partial\Psi}{\partial t} = iv_g\frac{\partial\Psi}{\partial z} - \frac{1}{2}\omega_k''\frac{\partial^2\Psi}{\partial z^2} - \frac{v_g}{2k_0}\Delta_\perp\Psi + \Psi\left(q|\Psi|^2 + \beta\delta\varrho + \alpha\frac{\partial\Phi}{\partial z}\right) = \frac{\delta(H - \int\omega_0|\Psi|^2 d\boldsymbol{r})}{\delta\Psi^*} . \tag{3.1.14}$$

The variables Φ and $\delta\varrho$ are canonically conjugated and obey the equations

$$\frac{\partial\delta\varrho}{\partial t} = -\varrho_0\Delta\Phi - \alpha\frac{\partial}{\partial z}|\Psi|^2 = \frac{\delta H}{\delta\Phi} \tag{3.1.15}$$

$$\frac{\partial\Phi}{\partial t} = -c^2\frac{\delta\varrho}{\varrho_0} - \beta|\Psi|^2 = -\frac{\delta H}{\delta\varrho} . \tag{3.1.16}$$

Inserting (3.1.12) and formulae like (3.1.16) (expressing $\delta\varrho$ and v in terms of b_k) into (3.1.13), we find for $f(\boldsymbol{k}, \boldsymbol{k}_0)$ the expression

$$f(\boldsymbol{k}, \boldsymbol{k}_0) = \left(\frac{k}{16\pi c\varrho_0}\right)^{1/2} \left(\beta\varrho_0 + \alpha c\frac{(\boldsymbol{k}, \boldsymbol{k}_0)}{k k_0}\right) .$$

Equations (3.1.14–16) describe the interaction of high-frequency waves of any nature with waves of the acoustic type. In many physical problems the dependence of a high-frequency wave dispersion law on the medium velocity may be neglected, and one may put $\alpha = 0$. Then (3.1.15, 16) may be reduced to one equation:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\delta\varrho = +\beta\varrho_0\Delta|\Psi|^2 . \tag{3.1.17}$$

Let $v_g \neq c$ and the amplitude of high-frequency waves be sufficiently small. Then one may consider the low-frequency waves as purely forced and replace $\partial/\partial t$ by $v_g\partial/\partial z$. System (3.1.14–16) is reduced now to the form

$$\begin{aligned} i\Psi_t &= iv_g\Psi_z - \frac{1}{2}\omega''\Psi_{zz} - \frac{v_g}{2k}\Delta_\perp\Psi + u\Psi \\ L_1 u &= L_2|\Psi|^2 \\ u &= q|\Psi|^2 + \beta\delta\varrho + \alpha\Phi_z , \end{aligned} \tag{3.1.18}$$

where L_1 and L_2 are second-order homogeneous partial differential operators:

$$L_n = C_{ij}^{(n)}\frac{\partial^2}{\partial\kappa_i\partial\kappa_j} . \tag{3.1.19}$$

The system (3.1.18) is universal for the description of small-amplitude, high-frequency waves with acoustic-type waves.

3.2 Kadomtsev-Petviashvili and Veselov-Novikov Equations

Let us apply the results obtained in the Chap. 2 to the KP equation. Let us begin with KP-2. The dispersion law of this equation,

$$\omega = p^3 - \frac{3q^2}{p} ,$$

is nondecaying. Therefore, from the results of Chap. 2 it follows that states asymptotic as $t \to \pm\infty$ coincide for KP-2 with rapidly decreasing boundary conditions [see (2.4.3) and [5], [6], [9] also]. It also follows that amplitudes of the classical scattering matrix become zero on the corresponding resonant surfaces in points of general position. In the first order this fact is trivial: the dispersion law for KP-2 is nondecaying, $V_{012} = (p_0p_1p_2)^{1/2}$, and (2.6.9) has in this case solutions $\boldsymbol{k}_1 = 0$ or $\boldsymbol{k}_2 = 0$ only. The analogous identity for second-order amplitudes was verified in [4].

The phenomenon of coincidence of asymptotic states for KP-2 was obtained independently in [18] via the inverse scattering technique. With periodic boundary conditions the KP-2 may be transformed to the normal form (2.6.6) and is completely integrable [7]. The hypothesis of complete integrability of periodic KP-2 was proposed for the first time in [5].

We have already mentioned in Chap. 2 that the dispersion law of the KP-1 equation is degenerative:

$$p = \xi_1 - \xi_2 \ , \quad q = \xi_1^2 - \xi_2^2 \ , \quad \omega_k = 4\left(\xi_1^3 - \xi_2^3\right) \ .$$

On the resonant surface the coefficient function in the Hamiltonian is

$$V_{kk_1k_2} = \sqrt{pp_1p_2} = [(\xi_1 - \xi_2)(\xi_2 - \xi_3)(\xi_1 - \xi_3)]^{1/2} \neq 0 \ .$$

Therefore the KP-1 equation describes a nontrivial scattering. The states asymptotic as $t \to \pm\infty$ do not coincide and are related by the formula

$$\begin{aligned} C^{*}_{\xi_1\xi_2} - C^{-}_{\xi_1\xi_2} = & \frac{\pi \mathrm{i}}{12(\xi_1 - \xi_2)} \int_{\xi_2}^{\xi_1} [(\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_3 - \xi_2)]^{1/2} \\ & \times \left(C^{+}_{\xi_1\xi_3} + C^{-}_{\xi_1\xi_3}\right)\left(C^{+}_{\xi_3\xi_2} + C^{-}_{\xi_3\xi_2}\right) d\xi_3 \\ & + \int_{-\infty}^{\xi_2} [(\xi_1 - \xi_2)(\xi_1 - \xi_3)(\xi_2 - \xi_3)]^{1/2} \\ & \times \left(C^{*+}_{\xi_2\xi_3} + C^{*-}_{\xi_2\xi_3}\right)\left(C^{+}_{\xi_1\xi_3} + C^{-}_{\xi_1\xi_3}\right) d\xi_3 \ , \end{aligned}$$

where $C_{\xi_1\xi_j} = C(p, q)$, $p = \xi_i - \xi_j$, $q = \xi_i^2 - \xi_j^2$. In the periodic case (and in any case in which boundary conditions are vanishing the KP-1 equation proves to be a nonintegrable system.

Recently, the Veselov-Novikov equation [24]

$$v_t = \partial^3 v + \bar{\partial}^3 v + \partial(uv) + \bar{\partial}(\bar{u}v) \tag{3.2.1}$$

$$\bar{\partial} u = -3\partial v \ , \quad v = \bar{v}$$

has been considered. Here, $\partial = \partial_z = \partial_x - \mathrm{i}\partial_y$, $z = x + \mathrm{i}y$, and the bar indicates complex conjugation. The solutions independent of y for this equation are reduced to the solution of the Korteveg–de Vries equation. The Equation (3.2.1) can be solved via the inverse scattering transform method [24] and allows a L-A-B triad representation,

$$\frac{\partial L}{\partial t} + [L, A + \bar{A}] = fL \ ,$$

where

$$L = -\Delta + v(z, \bar{z}) \ , \quad A = \partial^3 + u\partial \ \text{ and } \ f = \partial u + \bar{\partial} u \ .$$

The properties of (3.2.1) depend strongly on conditions asymptotic as $z \to \infty$. If $v \to 0$ as $z \to \infty$, then the Veselov-Novikov equation has a dispersion law,

$$\omega_k = 2\left(p^3 - 3q^2\right) \ , \quad \boldsymbol{k} = (p, q) \ , \tag{3.2.2}$$

which is nondegenerative (because it is analytic in p and q), like the KP-2 ($\alpha^2 = -1$) dispersion law. Hence (3.2.2) must possess all the properties of the KP-2 equation. Since the dispersion law (3.2.2) is nondecaying, we have to verify that the first-order scattering amplitude becomes zero on the resonant surface. The resonant manifold is determined in the space (p_1, p_2, q_1, q_2) by the equation

$$\begin{aligned} &(p_1p_2 - q_1q_2)(q_1 + q_2) + (p_1q_2 + p_2q_1)(p_1 + p_2) \\ &\quad = p_1p_2(p_1 + p_2) + p_1q_2^2 + p_1q_2^2 + p_2q_1^2 = 0 \ . \end{aligned} \tag{3.2.3}$$

Now let us calculate the first-order scattering amplitude. For this we make the Fourier transformation in (3.2.1) via x and y:

$$v(x, y, t) = \frac{1}{2\pi} \int \left(v_k + v^*_{-k}\right) \mathrm{e}^{\mathrm{i}(px+qy)} dx\, dy \ .$$

Now the relation between u and v takes the form of ($\boldsymbol{k} = (p, q)$):

$$u_k = -3\frac{\kappa^*}{\kappa}\left(v_k + v^*_{-k}\right) \ , \quad \kappa = p + \mathrm{i}q \ . \tag{3.2.4}$$

Substituting (3.2.4) into (3.2.1), we obtain in the nonlinear term the expression (up to the coefficients unessential for us)

$$\int \left[\kappa\left(\frac{\kappa_1^*}{\kappa_1} + \frac{\kappa_2^*}{\kappa_2}\right) + \kappa^*\left(\frac{\kappa_1}{\kappa_1^*} + \frac{\kappa_2}{\kappa_2^*}\right)\right] v_{k_1}v_{k_2}\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) d\boldsymbol{k}_1 d\boldsymbol{k}_2 \tag{3.2.5}$$

and other terms containing v^*v, which we need not write down because we know beforehand that the (3.2.1) is a Hamiltonian equation. The squared bracket in (3.2.5), after making the substitution $\kappa = p + \mathrm{i}q$, taking into account the δ-function and making some algebraic transformations, becomes

$$\frac{[p_1p_2 + q_1q_2][p_1p_2(p_1 + p_2) + p_1q_2^2 + p_2q_1^2]}{(p_1^2 + q_1^2)(p_2^2 + q_2^2)} \ . \tag{3.2.6}$$

Now making the replacement

$$v_k = a_k(p^2 + q^2)^{1/2} \ ,$$

and rewriting (3.2.1) in terms of a_k, we have the interaction Hamiltonian

$$H_{\mathrm{int}} = \int V_{k_1+k_2,k_1k_2}\left(a^*_{k_1+k_2}a_{k_1}a_{k_2} + (^*)\right) d\boldsymbol{k}_1 d\boldsymbol{k}_2 \ ,$$

where

$$V_{k_1+k_2,k_1k_2} = \text{const}\frac{(p_1p_2+q_1q_2)(p_1p_2(q_1+q_2)+p_1q_2^2+p_2q_1^2)}{[[(p_1+p_2)^2+(q_1+q_2)^2](p_1^2+q_1^2)(p_2^2+q_2^2)]^{1/2}} .$$

Comparing this expression with (3.2.3), we see that $V_{k_1+k_2,k_1k_2}$ contains the energy denominator $E = \omega_{k_1+k_2} - \omega_{k_1} - \omega_{k_2}$ as a factor and becomes zero simultaneously with it, in agreement with Theorem 2.2.1. For the Veselov-Novikov equation, other statements concerning $KP-2$ are also true, namely the coincidence of asymptotics as $t \to \pm\infty$, the triviality of scattering and the existence of a transformation to the normal form in the periodic case.

3.3 Davey-Stewartson-Type Equations. The Universality of the Davey-Stewartson Equation in the Scope of Solvable Models

As we have seen in Sect. 3.1, the problem of the interaction of small-amplitude, quasimonochromatic wave packets with acoustic waves leads in an natural way to equations which we shall call Davey-Stewartson-type equations:

$$\begin{aligned} &i\Psi_t + L_1\Psi + u\Psi = 0 \\ &L_2 u = L_3|\Psi|^2 . \end{aligned} \tag{3.3.1}$$

Here, $u(\boldsymbol{r},t)$ is a real function indicating a mean field while $\Psi(\boldsymbol{r},t)$ is a complex function representing the envelope, $\boldsymbol{r} = (x_1, \dots, x_d)$, $d = 2,3$ and

$$L_n = \sum_{i,j}^{d} C_{ij}^{(n)} \frac{\partial^2}{\partial x_i \partial x_j} , \quad n = 1,2,3 . \tag{3.3.2}$$

The Davey-Stewartson equation itself is written via operators (3.3.2) of the form

$$L_1 = L_3 = \frac{\partial}{\partial x^2} \pm \frac{\partial^2}{\partial y^2}$$

$$L_2 = \pm\left(\frac{\partial^2}{\partial x^2} \mp \frac{\partial^2}{\partial y^2}\right) .$$

It arises when applying the multiscale expansion technique to the KP equation [25, 26] and in the theory of two-dimensional long waves over finite depth liquids [15].

To study the system (3.3.1) it is convenient to rewrite it in the explicitly Hamiltonian form

$$i\dot{\Psi}_{k_0} + L_1(k_0)\Psi_{k_0} + \int \hat{T}_{0123}\Psi^*_{k_1}\Psi_{k_2}\Psi_{k_3}dk_1dk_2dk_3 = 0 , \tag{3.3.3}$$

where $L(\boldsymbol{k})$ are symbols of the operators (3.3.2). The vertex $\hat{T}_{0123} = T_{0123}\delta(P_{0123})$ has the form

$$T_{0123} = \frac{L_3(k_0-k_2)}{L_2(k_0-k_2)} + \frac{L_3(k_0-k_3)}{L_2(k_0-k_3)} \tag{3.3.4}$$

and is defined on the surface $P_{0123} \equiv \boldsymbol{k}_0 + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3$. The Hamiltonian of the equation (3.3.1) has the form

$$\begin{aligned} H = &\int L_1(\boldsymbol{k})|\Psi|^2 d\boldsymbol{k} \\ &+ \tfrac{1}{2}\int \hat{T}_{0123}\Psi^*_{k_0}\Psi^*_{k_1}\Psi_{k_3}\Psi_{k_3}dk_0 \dots dk_3 . \end{aligned} \tag{3.3.5}$$

The quadratic form $L_1(\boldsymbol{k})$ may be transformed to the diagonal form via the non-degenerative map. After doing so, the new coefficients $\tilde{C}_{ij}^{(2)}$, $\tilde{C}_{ij}^{(3)}$ arise; we shall designate α_{ij}, β_{ij}, correspondingly. The dispersion law $w = L_1(\boldsymbol{k})$ is degenerative only when $L_1(\boldsymbol{k}) = k_1^2$, $\boldsymbol{k} = (k_1, \dots, k_d)$; we shall not take this case into account. In all other cases the system (3.3.1) may have additional motion invariants only if T becomes zero on the resonant surface (2.3.32). One should note that if T becomes zero at some L_2, L_3, then it also becomes zero upon interchanging L_2 and L_3. To distinguish between these systems, one has to analyse the second-order vertex [19], This analysis leads to the following results at $d = 2$, when

$$w_k = k_1^2 + \sigma k_2^2 , \quad \sigma = \pm 1 .$$

Let $\sigma = 1$. Then the solvable system is

$$\begin{aligned} &\beta_{12} = 0 , \quad \beta_{11} = \beta_{22} = \beta , \quad \alpha_{11} = -\alpha_{22} = \alpha , \\ &i\Psi_t + \Delta\Psi + u\Psi = 0 \\ &\left[\alpha\left(\partial_x^2 - \partial_y^2\right) + 2\alpha_{12}\partial_x\partial_y\right] u = \beta\Delta|\Psi|^2 . \end{aligned} \tag{3.3.6}$$

By a change of variables the last equation could be transformed to the form

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) u = \Delta|\Psi|^2 .$$

If $\sigma = -1$, we have the counterpart of (3.3.6): $\beta_{11} = \beta_{22} = \beta$, $\alpha_{11} = -\alpha_{22} = \alpha$, $\alpha_{12} = 0$. As in (3.3.6), in diagonal form we obtain:

$$\begin{aligned} &i\Psi_t + \left(\partial_x^2 - \partial_y^2\right)\Psi + u\Psi = 0 \\ &\alpha\Delta u = \beta\left(\partial_x^2 - \partial_y^2\right)|\Psi|^2 \end{aligned} \tag{3.3.7}$$

and also the system

$$N\left(\partial_x \pm \partial_y\right)\left[\left(\partial_x \pm \partial_y\right)u + \left(\partial_x \mp \partial_y\right)|\Psi|^2\right] = 0$$

$$i\Psi_t + \left(\partial_x^2 - \partial_y^2\right)\Psi + u\Psi = 0 .$$

The latter in coordinates $x_1 - x_2 = \xi$, $x_1 + x_2 = \eta$ becomes

$$i\Psi_t + \Psi_{\xi\eta} + u\Psi = 0 , \quad u_\xi = |\Psi|^2_\eta . \tag{3.3.8}$$

The system (3.3.8) and its L-A pair has been presented in [9]. Equations (3.3.6, 7) are also integrable via the inverse scattering; more detailed information can be found in [19]. At $d = 3$, analogous but much more extensive analysis shows that the system (3.3.1) does not have any additional invariants.

It is useful to keep in mind the following fact. If one takes some two-dimensional, exactly solvable model and considers the initial conditions, like rapidly oscillating waves with slowly varying amplitude, then after the averaging procedure (or multiple scale expansion), one obtains the envelope equation in the form of one of the Davey-Stewartson equations (with one of the two admissible combinations of L_2 and L_3), the so-called DS-1 and DS-2. Specific examples can be found in [26].

3.4 Applications to One-Dimensional Equations

The ideas developed above can be explored in the one-dimensional case using the results contained in Sect. 3.3a). We present here the results for: a) the two coupled nonlinear Schrödinger equation system [12]; b) the systems describing the long-acoustic and short-wave interaction (first neglecting [11] and then taking into account [27] the effects of eigen nonlinearity and the dispersion of long waves); and c) the system describing the interaction of two counter-directed wave packets in the cubic medium [28].

The system of two coupled nonlinear Schrödinger equations arises in nonlinear optics [29] and has the form

$$\begin{aligned} i\Psi_{1t} &= C_1\Psi_{1xx} + 2\alpha|\Psi_1|^2\Psi_1 + 2\beta|\Psi_2|^2\Psi_1 \\ i\Psi_{2t} &= C_2\Psi_{2xx} + 2\gamma|\Psi_2|^2\Psi_2 + 2\beta|\Psi_1|^2\Psi_2 . \end{aligned} \tag{3.4.1}$$

It is a Hamiltonian system:

$$\begin{aligned} H = \int \Big\{ & C_1|\Psi_{1x}|^2 + C_2|\Psi_{2x}|^2 + \alpha|\Psi_1|^2 \\ & + 2\beta|\Psi_1|^2|\Psi_2|^2 + \gamma|\Psi_2|^2 \Big\} . \end{aligned} \tag{3.4.2}$$

The exact solvability of (3.4.1) with $C_1 = C_2$, $\alpha = \beta = \gamma$ has been shown in [30]. To study the system (3.4.1) in the general case we have to first determine whether the set of dispersion laws

$$w_1(k) = C_1k^2 , \quad w_2(k) = C_2k^2 \tag{3.4.3}$$

is degenerative to the process (2.3.8). As we have already seen, at $\varrho = C_1/C_2 \neq \pm 1$ the set (3.4.3) is nondegenerative to the process (2.3.8). Because the amplitude of the process (2.3.8) is a constant in all k-space and equal to $2\beta \neq 0$, the system (3.4.1) cannot have an additional integral at $\varrho \neq \pm 1$. At $\varrho = \pm 1$, one has

to calculate the second-order amplitude corresponding to the next nonlinear process. One may calculate, for example, the amplitude of the process (2.3.10). The corresponding manifold in the space $(k_1, \ldots, k_0)$ is quadratic and has a rational parametrization [12]. Using it, one may show that the set (3.4.3) is nondegenerative to (2.3.10). The amplitude of the process (2.3.10) is rather complicated; it is important that this amplitude become zero in two cases:

$$\varrho = 1, , \quad \alpha = \beta \text{ and } \varrho = -1 , \quad \alpha = -\beta . \tag{3.4.4}$$

Analogously one may obtain $\beta = \gamma$ at $\varrho = 1$ and $\beta = -\gamma$ at $\varrho = -1$. Therefore except for the "vector Schrödinger equation" $\varrho = 1$, the equations (3.4.1) with

$$\varrho = -1 , \quad \alpha = -\beta = \gamma \tag{3.4.5}$$

may be integrable also. The system (3.4.1) with coefficients (3.4.5) is indeed integrable. In fact, in [31] it has been shown that the inverse scattering method is applicable to the system

$$\begin{aligned} i\Psi_t &= \Psi_{xx} + \Psi X\Psi \\ -iX_t &= X_{xx} + X\Psi X , \end{aligned} \tag{3.4.6}$$

Where X and Ψ are matrices. We choose

$$\Psi = (\Psi_1, \ldots, \Psi_n) , \quad X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

and consider the reduction $X = A\Psi^+$, where A is a Hermitean matrix. Then we have

$$i\Psi_{mt} = \Psi_{mxx} + u\Psi_m , \quad m = 1, \ldots, n , \tag{3.4.7}$$

where $u = \Psi A\Psi^+$ is a real function. By a unitary transformation, the matrix A may be transformed to the diagonal form $A \to \alpha_i\delta_{ij}$. Therefore if $n = 2$, there are only two possibilities, namely the vector Schrödinger case [30] and the system with coefficients (3.4.5). This integrable system was obtained independently in [12] and [32].

The nonintegrability of the system describing the resonant interaction of long acoustic waves and short waves derived in [33] may be proved in an analogous way [11]:

$$\begin{aligned} & i\Psi_t + \Psi_{xx} + u\Psi = 0 \\ & u_{tt} + c^2u_{xx} = 2\left(|\Psi|^2\right)_{xx} ; \end{aligned} \tag{3.4.8}$$

as well as the nonintegrability of the system [27]

$$\begin{aligned} & u_t + \left(u^2 + \alpha|\Psi|^2 + u_{xx}\right)_x = 0 \\ & i\Psi_t + \Psi_{xx} + u\Psi = 0 , \end{aligned} \tag{3.4.9}$$

generalizing the system (3.4.8). In nonlinear optics a system also arises [34]:

$$\frac{\partial S^+}{\partial \xi} = S^+ \times IS^- + S^+ \times I^+ S^+$$
$$\frac{\partial S^+}{\partial \xi} = S^- \times IS^+ + S^- \times I^- S^- , \tag{3.4.10}$$

where

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial t} - v\frac{\partial}{\partial x} , \quad \frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} ,$$

and I, I^+, I^- are diagonal matrices. If $I^+ = I^- = 0$, the system (3.4.10) coincides with the assymetric chiral field equations [35] and is integrable. In [28] it is shown that this case exhausts all the possibilities of integrability of (3.4.10). The proof uses Theorem 2.3.1 and the lemma from paper [36] concerning the system (3.4.10) with $\partial/\partial x = 0$. Let us discuss this point in more detail.

Lemma. For a reduced system (3.4.9) with $\partial/\partial x = 0$ to be integrable, it is necessary that the system

$$\frac{\partial S^+}{\partial t} = S^+ \times I^+ S^+$$
$$\frac{\partial S^-}{\partial t} = S^- \times IS^+ \tag{3.4.11}$$

possess an additional integral to $I = (S^+ J^+ S^+)$ linear in S^- and of the degree 1 in S^+, l is an integer.

From this lemma it follows that if matrices

$$J^+ = \text{diag}\,(J_1^+, J_2^+, J_3^+) \ , \quad J_1^+ \neq J_2^+ \neq J_3^+$$

are nondegenerative, then the equality

$$(J_1^+ - J_2^+)(J_3^+)^2 + (J_2^+ - J_3^+)(J_1^+)^2 + (J_3^+ - J_1^+)(J_2^+)^2$$
$$+ k^2 (J_1^+ - J_2^+)(J_2^+ - J_3^+)(J_3^+ - J_1^+) = 0 , \quad k \in \mathbf{N} , \quad k \neq 0$$

is a necessary condition for the system (3.4.11) to be integrable. Even if at $\partial/\partial\xi = \partial/\partial\eta = \partial/\partial t$ this condition is fulfilled, under other reductions,

$$\frac{\partial}{\partial \xi} \to m\frac{\partial}{\partial t} , \quad \frac{\partial}{\partial \eta} \to m^2\frac{\partial}{\partial t} ,$$

it is not fulfilled. This means that for (3.4.10) to be integrable, it is necessary that two entries J_j^+, $j = 1,2,3$ coincide. Becuase we may add to J^+ any diagonal matrix, one can set $J_1^+ = J_2^+ = 0$, $J_3^+ \neq 0$. Then the first equation in (3.4.11) can be solved easily and

$$S^+ = \begin{pmatrix} M_1 \cos\left(J_3^+ M_0 t + \varphi_0\right) \\ -M_1 \sin\left(J_3^+ M_0 t + \varphi_0\right) \\ M_0 \end{pmatrix} .$$

Here M_0, M_1, φ_0 are arbitrary constants. Further, let an integral exist,

$$Z = \sum S_i^- P_{l_i}^-(S^+) , \tag{3.4.12}$$

where $P_l^i(S^+)$ are polynomials of S^+ of degree l, $i = 1,2,3$. As has been shown in [36], from the existence of an additional integral of the reduced system (3.4.11), it follows that $J_1 = J_2$.

Because all the aforesaid also applies to J^-, we conclude that there are only two possibilities for the system (3.4.10) to be integrable:

$$J = \text{diag}\,(J_1, J_1, J_3) \ , \quad J^{\pm} = \text{diag}\,\left(0,0,J_3^{\pm}\right)$$

and

$$J = \text{diag}\,(J_1, J_1, J_1) \ , \quad J^+ = \text{diag}\,\left(0,J_2^+,0\right) \ , \quad J^- = \text{diag}\,\left(0,0,J_3^+\right) .$$

Now one has to use the Holdstein-Primakov variables,

$$S_1^+ + \mathrm{i}S_2^+ = a\sqrt{2M^+ - |a|^2} , \quad M^+ = |S^+| ,$$
$$S_1^- + \mathrm{i}S_2^- = b\sqrt{2M^- - |b|^2} , \quad M^- = |S^-| ,$$

by which the system (3.4.10) acquires the standard form (2.1.1) with $\alpha = 2$, $a^{(1)} = a$, $a^{(2)} = b$ and dispersion laws

$$\omega_k^{(1),(2)} = \omega_k^{\pm} = c_1 \pm c_2\sqrt{c_3 k^2 + 1} , \quad c_i = \text{const} .$$
$$c_i = \text{const} .$$

The set $\{\omega^+\}$ is nondegenerative to the six-particle processes. By calculating the second-order vertex and checking that it is nonzero on the resonant surface at $J^{\pm} \neq 0$, we obtain the required statement.

The system generalizing (3.4.1) with the Hamiltonian has also been studied [37]

$$H = \int \Big\{ c_1 |\Psi_{1x}|^2 + c_2 |\Psi_{2x}|^2 + \alpha |\Psi_1|^4 + 2\beta |\Psi_1|^2 |\Psi_2|^2 + \gamma |\Psi_2|^4$$
$$+\delta\left(\Psi_1^2 \Psi_2^{*2} + \Psi_1^{*2}\Psi_2^2\right)\Big\}\, dx .$$

Quite analogously to (3.4.2), when $\beta \neq 0$, one obtains $c_1 = \pm c_2$. At $\delta \neq 0$, only the possibility $c_1 = c_2$ remains (one has to consider the process $\Psi_1 + \Psi_1 + \Psi_2 \to \Psi_2 + \Psi_2 + \Psi_2$). The result is that the integrable cases are already known and can be found in [38].

Appendix I [6]

Proofs of the Local Theorems (of Uniqueness and Others from Sect. 2.3)

We are seeking the functions $\omega(p,q)$ and $f(p,q)$ determined parametrically in the form

$$\begin{aligned} &p=\xi_1-\xi_2 \quad q=a(\xi_1)-a(\xi_2)\\ &\omega=b(\xi_1)-b(\xi_2)+\sum_{n=1}^{\infty}\varepsilon^n\omega_n(\xi_1,\xi_2)\\ &f=c(\xi_1)-c(\xi_2)+\sum_{n=1}^{\infty}\varepsilon^n f_n(\xi_1,\xi_2)\,. \end{aligned} \tag{A.1.1}$$

Here, ε is a small denominator. It is convenient to set the three-dimensional resonance manifold parametrically in the form

$$\begin{aligned} &p_1=\xi_1-\xi_2\,,\quad q_1=a(\xi_1+\eta)-a(\xi_2+\eta)\\ &p_2=\xi_3-\xi_2\,,\quad q_2=a(\xi_3+\nu)-a(\xi_2+\nu)\,, \end{aligned} \tag{A.1.2}$$

requiring additionally that

$$\begin{aligned} q&=q_1+q_2=a(\xi_1)-a(\xi_2)\\ &=a(\xi_1+\eta)-a(\xi_3+\eta)+a(\xi_3+\nu)-a(\xi_2+\nu)\,. \end{aligned} \tag{A.1.3}$$

Now conditions (2.3.13, 14) together with (A.1.3) will impose three equations upon five parameters ξ_1, ξ_2, ξ_3, η, ν. This system of equations must define η and ν in the form of a series in ε:

$$\eta=\sum_{n=1}^{\infty}\varepsilon^n\eta_n(\xi_1,\xi_2,\xi_3)\,,\quad \nu=\sum_{n=1}^{\infty}\varepsilon^n\nu_n(\xi_1,\xi_2,\xi_3)\,.$$

We have a linear overdetermined system in the first order in ε :

$$\begin{aligned} &[a'(\xi_1)-a'(\xi_3)]\,\eta_1+[a'(\xi_3)-a'(\xi_2)]\,\nu_1=0\\ &[b'(\xi_1)-b'(\xi_3)]\,\eta_1+[b'(\xi_3)-b'(\xi_2)]\,\nu_2=\Omega_1\\ &[c'(\xi_1)-c'(\xi_3)]\,\eta_1+[c'(\xi_3)-c'(\xi_2)]\,\nu_3=F_1\,. \end{aligned} \tag{A.1.4}$$

Here

$$\begin{aligned} &\Omega_1=\omega_1(\xi_1,\xi_2)-\omega_1(\xi_1,\xi_3)-\omega_1(\xi_3,\xi_2)\\ &F_1=f_1(\xi_1,\xi_2)-f_1(\xi_1,\xi_3)-f_1(\xi_3,\xi_2)\,. \end{aligned} \tag{A.1.5}$$

The consistency condition of the system (A.1.4) has the form

$$\Omega_1B=F_1A\,, \tag{A.1.6}$$

where

$$\begin{aligned} &A(\xi_1,\xi_2,\xi_3)=\Delta_{ab}=\alpha(\xi_1,\xi_2)+\alpha(\xi_2,\xi_3)+\alpha(\xi_3,\xi_1)\\ &B(\xi_1,\xi_2,\xi_3)=\Delta_{ac}=\beta(\xi_1,\xi_2)+\beta(\xi_2,\xi_3)+\beta(\xi_3,\xi_1) \end{aligned} \tag{A.1.7}$$

$$\begin{aligned} &\alpha(\xi_1,\xi_2)=b'(\xi_1)a'(\xi_2)-b'(\xi_2)a'(\xi_1)\\ &\beta(\xi_1,\xi_2)=c'(\xi_1)a'(\xi_2)-c'(\xi_2)a'(\xi_1)\,. \end{aligned} \tag{A.1.8}$$

Functions $A(\xi_1,\xi_2,\xi_3)$ and $B(\xi_1,\xi_2,\xi_3)$ are antisymmetric relative to all argument permutations. By interchanging ξ_1 and ξ_3 in (A.1.6) and summing up the results, we can convince ourselves that functions $\omega_1(\xi_1,\xi)$ and $f_1(f_1,f_2)$ are antisymmetric:

$$\omega_1(\xi_1,\xi_2)=-\omega_1(\xi_2,\xi_1)\quad f_1(\xi_1,\xi_2)=-f_1(\xi_2,\xi_1)\,.$$

Thus, we may put

$$\begin{aligned} &\Omega_1=\omega_1(\xi_1,\xi_2)+\omega_1(\xi_2,\xi_3)+\omega_1(\xi_3,\xi_1)\\ &F_1=f_1(\xi_1,\xi_2)+f_1(\xi_2,\xi_3)+f_1(\xi_3,\xi_1)\,. \end{aligned} \tag{A.1.9}$$

So our problem is to solve the functional equation (A.1.6). It is easy to check that (A.1.6) has the following solution:

$$\omega_1(\xi_1,\xi_2)=\frac{b'(\xi_1)-b'(\xi_2)}{a'(\xi_1)-a'(\xi_2)}(l(\xi_1)-l(\xi_2)) \tag{A.1.10}$$

$$f_1(\xi_1,\xi_2)=\frac{c'(\xi_1)-c'(\xi_2)}{a'(\xi_1)-a'(\xi_2)}(l(\xi_1)-l(\xi_2))\,. \tag{A.1.11}$$

Here, $l(\xi)$ is any function. This solution does not result in a new dispersion law, but represents the result of reparametrization in (A.1.1).

Let us put

$$\begin{aligned} &\xi_1-\xi_2=\eta_1-\eta_2\\ &a(\xi_1)-a(\xi_2)=a(\eta_1)-a(\eta_2)+\varepsilon\,[l(\eta_1)-l(\eta_2)]\\ &b(\xi_1)-b(\xi_2)=b(\eta_1)-b(\eta_2)+\varepsilon\omega(\eta_1,\eta_2)\,. \end{aligned} \tag{A.1.12}$$

The $\omega(\eta_1,\eta_2)$ represents in itself a series in powers of ε, the first term of which is given by formulae (A.1.10, 11). One more trivial solution of (A.1.12) is

$$\omega_1=p(\xi_1)-p(\xi_2)\,,\quad f_1=q(\xi_1)-q(\xi_2)\,;$$

($p(\xi)$ and $q(\xi)$ are any functions, representing variations of $b(\xi)$ and $c(\xi)$).

It is important to note that (A.1.6) possesses one more solution as well. Let us assume

$$\begin{aligned} &\omega_1(\xi_1,\xi_2)=\alpha(\xi_1,\xi_2)S(\xi_1,\xi_2)\\ &f_1(\xi_1,\xi_2)=\beta(\xi_1,\xi_2)S(\xi_1,\xi_2)\,. \end{aligned} \tag{A.1.13}$$

After substitution of (A.1.13) into (A.1.6) we can be convinced that $S(\xi_1, \xi_2)$ satisfies the surprisingly simple equation,

$$S(\xi_1,\xi_2)\left[a'(\xi_1)-a'(\xi_2)\right]+S(\xi_2,\xi_3)\left[a'(\xi_2)-a'(\xi_3)\right]+S(\xi_3,\xi_1)\left[a'(\xi_3)-a'(\xi_1)\right]=0 \tag{A.1.14}$$

$$S(\xi_1,\xi_2)=\frac{r(\xi_1)-r(\xi_2)}{a'(\xi_1)-a'(\xi_2)}\,. \tag{A.1.15}$$

Here, $r(\xi)$ is an arbitrary function again. The solution (A.1.15) is also a trivial one and results from reparametrization of a dispersion law of the form

$$p=\xi_1-\xi_2+\varepsilon\left[r(\xi_1)-r(\xi_2)\right]\,,$$

$$q=a(\xi_1)-a(\xi_2)\,,\quad \omega=b(\xi_1)-b(\xi_2)\,,$$

which is to the first order in ε equivalent to (2.3.15) with a modified function $a(\xi)$. To obtain given $a(\xi)$, one needs to make a change of variables of the form

$$\xi_1=\eta_1+\varepsilon a'(\eta_2)\frac{r(\eta_1)-r(\eta_2)}{a'(\eta_1)-a'(\eta_2)}\,;$$

$$\xi_2=\eta_2+\varepsilon a'(\eta_1)\frac{r(\eta_1)-r(\eta_2)}{a'(\eta_1)-a'(\eta_2)}\,.$$

Substituting new variables into the expression for ω, and expanding in ε, we go to expression (A.1.1) with the term linear in ε being of the form (A.1.13, 15).

We shall consider (A.1.6) as a system of linear algebraic equations relative to the unknown functions $\omega(\xi_1,\xi_2)$ and $f(\xi_1,\xi_2)$. Let variable ξ_3 take two arbitrary values $\xi_3=\sigma_1$ and $\xi_3=\sigma_2$. Let us write:

$$A_{1,2}=A_{1,2}(\xi_1,\xi_2)=A|_{\xi_3=\sigma_{1,2}}$$
$$B_{1,2}=B_{1,2}(\xi_1,\xi_2)=B|_{\xi_3=\sigma_{1,2}} \tag{A.1.16}$$

$$f(\xi,\alpha_i)=g_i(\xi)\,,\quad \omega(\xi,\alpha_i)=h_i(\xi)\,,\quad i=1,2\,. \tag{A.1.17}$$

We can see from (A.1.17) that in the most general case, the solution of (A.1.6) may depend on not more than four functions of one variable $g_{1,2}(\xi)$ and $h_{1,2}(\xi)$.

Our solution depends upon these very four functions, $l(\xi)$, $p(\xi)$, $q(\xi)$ and $r(\xi)$. Solving (A.1.6) at $\xi_3=\sigma_{1,2}$ and making an elementary analysis of the solution, we can be convinced that we have constructed a general solution of the functional equation (A.1.6). The result obtained can be considered as the local uniqueness theorem for degenerative dispersion laws. This theorem without a complete proof was presented in [9]. The global uniqueness theorem appears in Appendix II.

Let $\omega(p,q)$ be a differentiable function, and $\omega(0,0)=0$. Let $\omega(p,q)$ satisfy one more condition,

$$\frac{|\omega(p,q)|}{R}\underset{R\to 0}{\longrightarrow}0\quad R=\left|p^2=q^2\right|^{1/2}\,. \tag{A.1.18}$$

Then the dispersion law $\omega(p,q)$ is decaying. There is a manifold $\Gamma^{1,2}$, because it contains a two-dimensional plane $p_2=q_2=0$ and a vicinity of this plane, given by the following equation

$$\frac{\partial\omega}{\partial p}(p_1,q_1)p_2+\frac{\partial\omega}{\partial q}(p_1,q_1)q_2=0 \tag{A.1.19}$$

Putting $p_2=q_2=0$ in (2.3.14), we get $f(0,0)=0$; moreover,

$$\lim_{R\to 0}\left[\frac{f(R,\vartheta)}{R}\right]=f_0(\theta)<\infty\quad\text{at all }\theta\,.$$

Here, $\vartheta=\operatorname{arctg}(q_2/p_2)$.

Thus, in the vicinity of zero, $f(p,q)$ may tend asymptotically to the homogeneous function of the first order. But we assumed that this function is analytic. Thus, $f_0(\vartheta)=0$ and function f also submit to condition (A.1.19). Now in the vicinity of $p_2=q_2=0$ we have, from (2.3.13):

$$\frac{\partial f}{\partial p}(p_1,q_1)p_2+\frac{\partial f}{\partial q}(q_1,p_1)q_2=0\,.$$

This means that the Jacobian between functions f and ω is equal to zero, and the latter are functionally dependent,

$$f(p,q)=F\left[\omega(p,q)\right]\,.$$

Now we have from (2.3.13, 14):

$$F\left[\omega(p_1,q_1)+\omega(p_2,q_2)\right]=F\left[\omega(p_1,q_1)\right]+F\left[\omega(p_2,q_2)\right]\,,$$

from which we conclude $F(\xi)=\lambda\xi$, λ is a constant. The important consequence of this result is Theorem 2.3.3.

Let us designate a wave number corresponding to a new space dimension via "r", and consider the dispersion law, which becomes (2.3.15) at $r=0$. The proof of the theorem 2.3.5 [6]:

Let the degenerative law $\omega(p,q,r)$ be parametrized in the vicinity of $r=0$ as follows:

$$p=\xi_1-\xi_2\quad q=a(\xi_1)-a(\xi_2)$$
$$\omega(p,q,r)=b(\xi_1)-b(\xi_2)+r\sum_{n=0}^{\infty}r^n\omega_n(\xi_1,\xi_2)\,, \tag{A.1.20}$$

and let manifold $\Gamma^{1,2}$ have dimensionality 5. Then $\omega_0=\text{const}$, $\omega_n=0$, $n>0$. Then the resonance manifold $\Gamma^{1,2}$ for the dispersion law (A.1.20) may be given in the form

$$a(\xi_1)-a(\xi_2)=a(\xi_1+\eta)-a(\xi_3-\eta)+a(\xi_3+\nu)-a(\xi_2+\nu)$$

$$\sum_{k=0}^{\infty}(r_1+r_2)^{k+1}\omega_k(\xi_1,\xi_2) = -b(\xi_1)+b(\xi_2)+b(\xi_1+\eta)$$
$$+b(\xi_3+\eta)+b(\xi_3+\nu)-b(\xi_2+\nu)+\sum_{n=0}^{\infty}\left[r_1^{n+1}\omega_n(\xi_1+\eta,\xi_3\right.$$
$$\left.+\eta)+r_2^{n+1}\omega_n(\xi_3+\nu,\xi_2+\nu)\right] . \qquad \text{(A.1.21)}$$

Let us choose ξ_1, ξ_2, ξ_3, r_1 and r_2 as independent variables and then consider ν and η as their functions, analytical in r_1 and r_2.

The degeneracy condition can be written in its usual form,

$$f(p,q,r_1+r_2) = f(p_1,q_1,r_1)+f(p_2,q_2,r_2) . \qquad \text{(A.1.22)}$$

The solution of (A.1.22) may be found in the form

$$f(p,q,r) = c(\xi_1)-c(\xi_2)+r\sum_{n=0}^{\infty}r^n f_n(\xi_1,\xi_2)$$
$$\eta = \sum_{m+n=1}^{\infty}\eta_{mn}r_1^m r_2^n , \quad \nu = \sum_{m+n=1}^{\infty}\nu_{mn}r_1^m r_2^n . \qquad \text{(A.1.23)}$$

Considering terms linear in r_1 and r_2 in (A.1.21, 22), and marking $\eta_0 = \eta_{10}r_1 + \eta_{01}r_2$, $\nu_0 = \nu_{10}r_1+\nu_{01}r_2$, we obtain

$$\eta_0\left[a'(\xi_1)-a'(\xi_3)\right]+\nu_0\left[a'(\xi_3)-a'(\xi_2)\right]=0$$
$$(r_1+r_2)\omega_0(\xi_1,\xi_2) = r_1\omega_0(\xi_1,\xi_3)+r_2\omega_0(\xi_3,\xi_2)$$
$$+\eta_0\left[b'(\xi_1)-b'(\xi_3)\right]+\nu_0\left[b'(\xi_3)-b'(\xi_2)\right] \qquad \text{(A.1.24)}$$
$$(r_1+r_2)f_0(\xi_1,\xi_2) = r_1 f_0(\xi_1,\xi_3)+r_2 f_0(\xi_3,\xi_2)$$
$$+\eta_0\left[c'(\xi_1)-c'(\xi_3)\right]+\nu_0\left[c'(\xi_3)-c'(\xi_2)\right] .$$

Setting coefficients equal in (A.1.24) at r_1, r_2 separately, we obtain overdetermined the system of equations for η_{10}, ν_{10}, η_{01} and ν_{01}. Their consistency conditions are

$$\left[\omega_0(\xi_1,\xi_2)-\omega_0(\xi_1,\xi_3)\right]B$$
$$= \left[f_0(\xi_1,\xi_2)-f_0(\xi_1,\xi_3)\right]A \qquad \text{(A.1.25)}$$
$$\left[\omega_0(\xi_1,\xi_2)-\omega_0(\xi_2,\xi_2)\right]B$$
$$= \left[f_0(\xi_1,\xi_2)-f_0(\xi_3,\xi_2)\right]A . \qquad \text{(A.1.26)}$$

Here, A and B are given by formulae (A.1.7, 8).

In contrast to (A.1.6), (A.1.25, 26) do not possess nontrivial solutions. To convince ourselves of this, let us differentiate (A.1.25) in ξ_3 and then apply operator $\partial^3/\partial\xi_3^3 - \partial^3/\partial\xi_3^2\partial\xi_2$ for the same equation, further putting $\xi_3 = \xi_2$. We obtain the system of the two homogeneous equations for $\partial\omega/\partial\xi_2$, $\partial f/\partial\xi_2$, having a nonzero determinant. So, $\partial\omega/\partial\xi_2 = 0$, $\partial f/\partial\xi_2 = 0$. Similarly, we get $\partial\omega/\partial\xi_1 = 0$, $\partial f/\partial\xi_1 = 0$ from (A.1.26). Thus, the unique solution of (A.1.25) is

$\omega_0 = \text{const}$, $f_0 = \text{const}$, $\nu_0 = \eta_0 = 0$. We can further prove this fact via induction. Let ν_k, η_k, be the sums of the sequence terms in (A.1.23), for which $m+n=k$. Let $\nu_q = \eta_q = 0$ at $q<k$. Collecting in (A.1.21, 22) terms of degree k, we have

$$(r_1+r_2)^k\omega_{k-1}(\xi_1,\xi_2) = r_1^k\omega_{k-1}(\xi_1,\xi_3)+r_2^k\omega_{k-1}(\xi_3,\xi_2)$$
$$+\eta_k\left[b'(\xi_1)-b'(\xi_3)\right]+\nu_k\left[b'(\xi_3)-b'(\xi_2)\right]=0 \qquad \text{(A.1.27)}$$
$$\eta_k\left[a'(\xi_1)-a'(\xi_3)\right]+\nu_k\left[a'(\xi_3)-a'(\xi_2)\right]=0 ,$$

and an analogous equation for f. Taking the mixed derivative in r_1, r_2 of the k-th order $\partial^k/\partial r_1^{k-1}\partial r_2$, we get

$$k!\omega_k(\xi_1,\xi_2) = \left[b'(\xi_1)-b'(\xi_3)\right]\frac{\partial^k\eta_k}{\partial r_1^{k-1}\partial r_2}$$
$$+\left[b'(\xi_3)-b'(\xi_2)\right]\frac{\partial^k\nu_k}{\partial r_1^{k-1}\partial r_2}$$
$$k!f_k(\xi_1,\xi_2) = \left[c'(\xi_1)-c'(\xi_3)\right]\frac{\partial^k\eta_k}{\partial r_1^{k-1}\partial r_2}$$
$$+\left[c'(\xi_3)-c'(\xi_2)\right]\frac{\partial^k\nu_k}{\partial r_1^{k-1}\partial r_2} .$$

Consistency of these equations with (A.1.27) results in the equation of the form (A.1.6),

$$\omega_k(\xi_1,\xi_2)\Delta_{ac} = f_k(\xi_1,\xi_2)\Delta_{ba} ,$$

whihc is not fulfilled, as Δ_{ac}/Δ_{ba} is a function of ξ_1, ξ_2, ξ_3.

Actually, Δ_{ac} and Δ_{ba} are totally antisymmetric functions, so their ratio is a totally symmetric function of ξ_1, ξ_2 and ξ_3, and is not equal to the constant, as b and c are different functions. The theorem is proven.

On the basis of this theorem, one may suggest the hypothesis that at $d>2$ and under the condition of maximal dimensionality of $\Gamma^{1,2}$, no dispersion laws exist which are degenerate with respect to the process $1 \leftrightarrow 2$. Requirement of maximum dimensionality of $\Gamma^{1,2}$ is essential, indeed, at any $d \geq 2$, the linear dispersion law $\omega = |\boldsymbol{k}|\varphi(\boldsymbol{k}/|\boldsymbol{k}|)$ is degenerative. However, manifold $\Gamma^{1,2}$ is given by the parallelism condition on $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}$ and so has dimensionality 4, less than maximum.

Let us now consider the scattering process of two interacting waves. The manifold $\Gamma^{2,2}$ is given by the equations (2.3.32). The dispersion law $\omega(k)$ is nondegenerative relative to this process, if in some region of the manifold $\Gamma^{2,2}$, functional equation (2.3.33) has a nontrivial solution. Apparently, manifold $\Gamma^{2,2}$ includes two hypersurfaces, set by conditions

$$\boldsymbol{k}=\boldsymbol{k}_2 , \quad \boldsymbol{k}_1=\boldsymbol{k}_3 \quad \text{or} \quad \boldsymbol{k}=\boldsymbol{k}_3 , \quad \boldsymbol{k}_1=\boldsymbol{k}_2 ,$$

crossing each other via $k = k_1 = k_2 = k_3$. On this submanifold, the $\tilde{\Gamma}^{2,2}$ equation is fulfilled at any $f(k)$. At $d=1$, $\Gamma^{2,2} = \tilde{\Gamma}^{2,2}$, and any dispersion law is deneneratіve.

Theorem 2.3.4 is the evident consequence of the following lemma:

Lemma 1. The quadratic dispersion law with any signature is nondegenerative with respect to (2.3.32) at $d \geq 2$.

Proof. Let us reduce the quadratic form (2.3.31) to a diagonal form via coordinate system rotation; then ($k = (k^{(1)}, \ldots, k^{(d)})$):

$$\omega(k) = k^{(1)^2} + \sigma_2 k^{(2)^2} + \ldots + \sigma_d k^{(d)^2}$$

$$\sigma_i = \pm 1 , \quad i = 2, \ldots, d . \tag{A.1.28}$$

All signs in (A.1.28) are independent. With the dispersion law (A.1.28) the manifold $\Gamma^{2,2}$ has a rational parametrization,

$$\begin{aligned} k^{(1)} &= P_1 + \tfrac{1}{2}\mu(1-Q) \qquad & k_1^{(1)} &= P_1 - \tfrac{1}{2}\mu(1-Q) \\ k_2^{(1)} &= P_1 - \tfrac{1}{2}\mu(1+Q) \qquad & k_3^{(1)} &= P_1 + \tfrac{1}{2}\mu(1+Q) \\ k^{(i)} &= P_i + \tfrac{1}{2}\mu(\tau_i + s_i) \qquad & k_3^{(i)} &= P_i - \tfrac{1}{2}\mu(\tau_i + s_i) \\ k_2^{(i)} &= P_i + \tfrac{1}{2}\mu(\tau_i - s_i) \qquad & k_3^{(i)} &= P_i - \tfrac{1}{2}\mu(\tau_i - s_i) \end{aligned} \tag{A.1.29}$$

$$i = 2, \ldots, d ,$$

where

$$Q = \sum_{n=2}^{d} \sigma_n \tau_n s_n ,$$

and $P_1, \ldots, P_d$, μ, τ_i, s_i are independent coordinates on resonance surface (2.3.32). Let us put parametrization (A.1.29) into the functional equation

$$\begin{aligned} & f\left(P_1 + \tfrac{1}{2}\mu(1-Q), P_2 + \tfrac{1}{2}\mu(\tau_2 - s_2), \ldots\right) \\ &\quad + f\left(P_1 - \tfrac{1}{2}\mu(1-Q), P_2 - \tfrac{1}{2}\mu(\tau_2 + s_2), \ldots\right) \\ &\quad = f\left(P_1 - \tfrac{1}{2}\mu(1+Q), P_2 + \tfrac{1}{2}\mu(\tau_2 - s_2), \ldots\right) \\ &\quad + f\left(P_1 + \tfrac{1}{2}\mu(1+Q), P_2 - \tfrac{1}{2}\mu(\tau_2 - s_2), \ldots\right) . \end{aligned} \tag{A.1.30}$$

Differentiating (A.1.30) in τ_i, s_i, supposing $\tau_i = s_i$, subtracting one from the other, differentiating in τ and supposing $\mu = 0$, we find

$$\partial^2 f(P_1, \ldots, P_d)/\partial P_1 \partial P_i = 0 , \quad i = 2, \ldots, d ,$$

from which

$$f = F_1\left(k^{(1)}\right) + \Phi\left(k^{(2)}, \ldots, k^{(d)}\right) . \tag{A.1.31}$$

Substituting (A.1.31) into (A.1.30), writing down the equations obtained via differentiation in τ_i, τ_j, s_i, s_j and supposing all τ, s to be equal to zero, after simple transformations, we obtain that $\partial^2\Phi/\partial P_i \partial P_j = 0$ or

$$f = F_1\left(k^{(1)}\right) + \ldots + F_d\left(k^{(d)}\right) . \tag{A.1.32}$$

Let us substitute (A.1.32) into (A.1.30) and differentiate in P_1. We obtain

$$\begin{aligned} & F_1'\left(P_1 + \tfrac{1}{2}\mu(1-Q)\right) + F_1'\left(P_1 - \tfrac{1}{2}\mu(1-Q)\right) \\ &\quad = F_1'\left(P_1 - \tfrac{1}{2}\mu(1+Q)\right) + F_1'\left(P_1 + \tfrac{1}{2}\mu(1-Q)\right) . \end{aligned}$$

Differentiating in Q and μ, we get two equations on F_1'', whose consistency condition is written in the form of the equation (at $Q = 0$)

$$F_1''\left(P_1 - \mu/2\right) = F_1''\left(P_1 + \mu/2\right) .$$

On account of the arbitrariness of P_1 and μ we obtain that $F'' = \text{const}$. Exactly in the same way, differentiating (A.1.30) in P_i and then in τ_i, s_i, subtracting one from the other and supposing $\tau_i = -s_i$, we obtain

$$F_i''(P_i + \mu\tau_i) = F_i''(P_i - \mu\tau_i) ,$$

from which, on account of the arbitrariness of P_i, μ, τ_i, we conclude that $F_i'' = \text{const}$.

Thus, $F_i = c_i k^{(i)^2} + B_i k^{(i)} + D_i$. It is easy to see that $c_i = \sigma_i c$ from (A.1.30) that proves nondegeneracy. It follows from this that dispersion laws which are completely degenerative relative to process (2.3.32) do not exist. Besides theorem 2.3.4, the statement which follows suggests that it is doubtful that even partially degenerative dispersion laws exist relative to process (2.3.32).

Let the dispersion law $\omega(k)$ be decaying. Then manifold $\Gamma^{2,2}$ of codimensionality one is given by the system of equations

$$\begin{aligned} & k + k_1 = k_2 + k_3 = q \\ & \omega(k) + \omega(k_1) = \omega(k_2) + \omega(k_3) = \omega(q) . \end{aligned} \tag{A.1.33}$$

If the dispersion law is degenerative relative to the process "one into two", then on manifold $\Gamma_M^{2,2}$, function $f(k)$ is sure to satisfy the following equation:

$$f(k) + f(k_1) = f(k_2) + f(k_3) = f(q) , \tag{A.1.34}$$

which, of course, does not mean even partial degeneracy of the dispersion law $\omega(k)$. For degeneracy to occur, it is necessary to fulfill (2.3.33) on $\Gamma^{2,2}$ in the vicinity of just one point of manifold (2.3.32).

Let us study this possibility in the simplest case $d = 2$, when the dispersion law belongs to the class (2.3.15) we are considering. Now manifold $\Gamma_M^{2,2}$ (A.1.33) is parametrized as follows (at $d = 2$ its dimensionality is equal to four):

$$\begin{aligned} p &= \xi_1 - \xi_2 , \quad p_1 = \xi_2 - \xi_3 , \quad p_2 = \xi_1 - \xi_4 , \quad p_3 = \xi_4 - \xi_3 \\ q &= a(\xi_1) - a(\xi_2) , \qquad q_1 = a(\xi_2) - a(\xi_3) , \\ q_2 &= a(\xi_1) - a(\xi_4) , \qquad q_3 = a(\xi_4) - a(\xi_3) . \end{aligned} \tag{A.1.35}$$

Let us consider the vicinity of a point on $\Gamma^{2,2}$, given on $\Gamma_M^{2,2}$ via coordinates ξ_1, ξ_2, ξ_3, ξ_4. We may set it, having retained expression (A.1.35) for p_i and defined

$$\begin{aligned} q &= a(\xi_1) - a(\xi_2) , \qquad q_1 = a(\xi_2 + \nu_1) - a(\xi_3 + \nu_1) , \\ q_2 &= a(\xi_1 + \nu_2) - a(\xi_4 + \nu_2) , \quad q_3 = a(\xi_4 + \nu_3) - a(\xi_3 + \nu_3) . \end{aligned}$$

Similarly we can define ω_i. Resonance conditions impose two conditions upon ν_i:

$$\begin{aligned} \left[a'(\xi_2) - a'(\xi_3)\right] \nu_1 &= \left[a'(\xi_1) - a'(\xi_3)\right] \nu_2 \\ &+ \left[a'(\xi_4) - a'(\xi_3)\right] \nu_3 \end{aligned}$$

$$\begin{aligned} \left[b'(\xi_2) - b'(\xi_3)\right] \nu_1 &= \left[b'(\xi_1) - b'(\xi_3)\right] \nu_2 \\ &+ \left[b'(\xi_4) - b'(\xi_3)\right] \nu_3 . \end{aligned}$$

Degeneracy condition yields one more equation:

$$\begin{aligned} \left[c'(\xi_2) - c'(\xi_3)\right] \nu_1 &= \left[c'(\xi_1) - c'(\xi_3)\right] \nu_2 \\ &+ \left[c'(\xi_4) - c'(\xi_3)\right] \nu_3 . \end{aligned}$$

If functions a, b, c are linearly independent, these equations possess zero solutions only. Thus, submanifold $\Gamma_M^{2,2}$ cannot be locally enlarged while retaining degeneracy.

Appendix II

Proof of the Global Theorem for Degenerative Dispersion Laws [40]

Consider the $d = 2$ case. Our goal is to find the resonant manifold Γ itself instead of the dispersion law $\omega(p, q)$. the latter is defined by

$$\omega(p_1 + p_2, q_1 + q_2) = \omega(p_1, q_1) + \omega(p_2, q_3) . \tag{A.2.1}$$

Due to the degeneracy of $\omega(p, q)$, functions $f_i(p, q)$ $(i = 1, 2, 3)$ exist, satisfying the same equation on Γ :

$$f_i(p_1 + p_2, q_1 + q_2) = f_i(p_1, q_1) + f_i(p_2, q_3) .$$

Consider the function

$$\tilde{\omega}(p, q) = \omega(p, q) + ap + bq + \sum_{i=1}^{3} c_i f_i(p, q) .$$

Here a, b, c_i are some constants. The function $\tilde{\omega}$ satisfies the same equation (A.2.1) on Γ, and we shall think of ω in (A.2.1) as containing five arbitrary constants.

Let ξ_1, ξ_2, ξ_3, be the coordinates on Γ described by functions $p_1(\xi_i)$, $p_2(\xi_i)$, $q_1(\xi_i)$, $q_2(\xi_i)$. Let us fix a point $\boldsymbol{\xi}$ on Γ and differentiate (A.2.1) via ξ_i. In what follows we designate

$$\omega_1 = \frac{\partial \omega}{\partial p} , \quad \omega_2 = \frac{\partial \omega}{\partial q} , \quad \omega_{20} = \frac{\partial^2 \omega}{\partial p^2} , \quad \omega_{11} = \frac{\partial^2 \omega}{\partial p \partial q} , \quad \omega_{02} = \frac{\partial^2 \omega}{\partial q^2} , \dots .$$

We have

$$\begin{aligned} &\tilde{\omega}_1(p_1, q_1)\frac{\partial p_1}{\partial \xi_i} + \tilde{\omega}_2(p_1, q_1)\frac{\partial q_1}{\partial \xi_i} + \tilde{\omega}_1(p_2, q_2)\frac{\partial p_2}{\partial \xi_i} \\ &+ \tilde{\omega}_2(p_2, q_2)\frac{\partial q_2}{\partial \xi_i} = F_i \equiv \tilde{\omega}_1(p_1 + p_2, q_1 + q_2) \\ &\times \left(\frac{\partial p_1}{\partial \xi_i} + \frac{\partial p_2}{\partial \xi_i}\right) + \tilde{\omega}_2(p_1 + p_2, q_1 + q_2)\left(\frac{\partial q_1}{\partial \xi_i} + \frac{\partial q_2}{\partial \xi_i}\right) . \end{aligned} \tag{A.2.2}$$

We also adopt the following notation for the Jacobi determinant of three functions $A(\xi)$, $B(\xi)$, $C(\xi)$:

$$\{A, B, C\} = \begin{vmatrix} \frac{\partial A}{\partial \xi_1} & \frac{\partial B}{\partial \xi_1} & \frac{\partial C}{\partial \xi_1} \\ \frac{\partial A}{\partial \xi_2} & \frac{\partial B}{\partial \xi_2} & \frac{\partial C}{\partial \xi_2} \\ \frac{\partial A}{\partial \xi_3} & \frac{\partial B}{\partial \xi_3} & \frac{\partial C}{\partial \xi_3} \end{vmatrix} ,$$

and also set

$$w_1 = \{p_1, p_2, q_1\} ; \quad w_2 = \{p_1, p_2, q_2\}$$

$$v_1 = \{q_1, q_2, p_1\} ; \quad v_2 = \{q_1, q_2, p_2\} .$$

From (A.2.2) we obtain

$$w_1\omega_1(p_2, q_2) + v_1\omega_2(p_2, q_2) = R , \tag{A.2.3}$$

where

$$R = \begin{vmatrix} \frac{\partial p_1}{\partial \xi_1} & \frac{\partial q_1}{\partial \xi_1} & F_1 \\ \frac{\partial p_1}{\partial \xi_2} & \frac{\partial q_1}{\partial \xi_2} & F_2 \\ \frac{\partial p_1}{\partial \xi_3} & \frac{\partial q_1}{\partial \xi_3} & F_3 \end{vmatrix} .$$

Differentiating (A.2.3) in ξ_i one obtains

$$w_1 \frac{\partial p_2}{\partial \xi_i}\omega_{20}(p_2,q_2) + \left(w_1\frac{\partial q_2}{\partial \xi_i} + v_1\frac{\partial p_2}{\partial \xi_i}\right)\omega_{11}(p_2,q_2)$$
$$+ v_1\frac{\partial q_2}{\partial \xi_i}\omega_{02}(p_2,q_2) + \frac{\partial w_1}{\partial \xi_i}\omega_1(p_2,q_2) \quad (A.2.4)$$
$$+ \frac{\partial v_1}{\partial \xi_i}\omega_2(p_2,q_2) = \frac{\partial R}{\partial \xi_i} .$$

From (A.2.4) we have

$$\{p_2,q_2,w_1\}\omega_1(p_2,q_2) + \{p_2,q_2,v_1\}\omega_2(p_2,q_2) = \{p_2,q_2,R\} .$$

Now by choosing special values of a, b, c_i let us achieve that, in the given point ξ,

$$\omega_1(p_1+p_2,q_1+q_2) = 0 , \quad \omega_2(p_1+p_2,q_1+q_2) = 0 , \quad \omega_{ij}(p_1+p_2,q_1+q_2) = 0$$

so that $R = 0$, and also $\{p_2,q_2,R\} = 0$. From the compartibility condition of (A.2.3) ($R = 0$) and the latter equality

$$\{p_2,q_2,w_1\}\omega_1(p_2,q_2) + \{p_2,q_2,v_1\}\omega_2(p_2,q_2) = 0$$

we obtain

$$v_1\{p_2,q_2,w_1\} = w_1\{p_2,q_2,v_1\} . \quad (A.2.5a)$$

Because all of the expression is symmetric with respect to the permutation of indices 1 and 2, we also have

$$v_2\{p_1,q_1,w_2\} = w_2\{p_1,q_1,v_2\} . \quad (A.2.5b)$$

Analogously choosing the constants a, b, c in another way, it is easy to obtain the relations

$$\begin{aligned}
&(v_1+v_2)\{p_2,q_2,w_1+w_2\} = (w_1+w_2)\{p_2,q_2,v_1+v_2\}\\
&v_2\{p_1+p_2,q_1+q_2,w_2\} = w_2\{p_1+p_2,q_1+q_2,v_2\}\\
&(v_1+v_2)\{p_1,q_1,w_1+w_2\} = (w_1+w_2)\{p_1,q_1,v_1+v_2\}\\
&v_1\{p_1+p_2,q_1+q_2,w_1\} = w_1\{p_1+p_2,q_1+q_2,v_1\} .
\end{aligned} \quad (A.2.6)$$

Now we consider new functions α, β, γ such that

$$v_1 = \alpha w_1 , \quad v_2 = \beta w_2 , \quad v_1 + v_2 = \gamma(w_1+w_2) .$$

From (A.2.5, 6) we have

$$\{p_2,q_2,\alpha\} = 0 ; \quad \{p_1,q_1,\beta\} = 0 ;$$
$$\{p_2,q_2,\gamma\} = 0 ; \quad \{p_1,q_1,\gamma\} = 0 ;$$
$$\{p_1+p_2,(q_1+q_2),\beta\} = 0 , \quad \{p_1+p_2,(q_1+q_2),\alpha\} = 0 . \quad (A.2.7)$$

It follows from (A.2.7) that

$$p_2 = P_2(\alpha,\gamma) , \quad q_2 = Q_2(\alpha,\gamma) , \quad p_1 + p_2 = A(\alpha,\beta)$$
$$p_1 = P_1(\beta,\gamma) , \quad q_1 = Q_1(\beta,\gamma) , \quad q_1 + q_2 = B(\alpha,\beta) .$$

Then we obviously obtain

$$\begin{aligned}
&P_1(\beta,\gamma) + P_2(\alpha,\gamma) = A(\alpha,\beta)\\
&Q_1(\beta,\gamma) + Q_2(\alpha,\gamma) = B(\alpha,\beta) .
\end{aligned} \quad (A.2.8)$$

Functional equations (A.2.8) can be solved easily, leading to

$$P_1 = a_1(\beta) - a_2(\gamma) , \quad P_2 = a_2(\gamma) - a_3(\alpha)$$
$$A = a_1(\beta) - a_3(\alpha) , \quad Q_1 = b_1(\beta) - b_2(\gamma)$$
$$Q_2 = b_2(\gamma) - b_3(\alpha) , \quad B = b_1(\beta) - b_3(\alpha) .$$

Here, a_i, b_i, $i = 1,2,3$ are arbitrary functions of one variable. The result obtained leads to dispersion laws of the form (2.3.15, 26). In the above it has been supposed that functions α, β, γ are functionally independent. This is really true in the general case. Special cases should be obtained by some limiting procedure. Obviously the unique possibility is to obtain the homogeneous functional of degree one.

Conclusion

Let us summarize. In the present paper we have aimed at showing that a method like Poincaré's analysis of the integrability of dynamical systems, based on the study of the perturbation theory series, proves to be very effective. Earlier, an analgous method proved the nonexistence of a strong recursive operator for multidimensional systems [41]; we can only hope that this does not exhaust its capacities. However, it has recently been shown [42, 43] that, by generalizing the recursion operator concept, it is possible to construct both recursion operators and bi-Hamiltonian structures for multidimensional solvable equations. Interesting examples include the KP and DS systems. One can not exclude a priori the possibility that only essentially nonlinear integrals exist for some systems (2.1.1).

With regard to the systems considered in this paper, i.e., those containing integrals which are quadratic in the main part, certain questions have been answered since our paper [6] was published in 1987: namely, the question of action-angle variables in nondegenerative systems with periodic boundary conditions ([7]; Sect. 2.6) and that of a global description of the degenerative dispersion laws ([40]; Appendix II).

Nevertheless some questions remain unanswered; for example: Can the resonant manifold for decays $1 \to 2$ always be described via only one parametrization (i.e. consisting of two parts) corresponding to the replacements $\xi_i \to \xi_i$,

$a(\xi_i) \to -a(-\xi_i)$, $b(\xi_i) \to -b(-\xi_i)$, as in the KP-1 equation? KP-1-like equations with degenerative dispersion laws are especially interesting. Although they are exactly solvable by the inverse scattering technique, current methods still cannot provide solutions which are not rapidly decreasing and are in general position. In contrast to soliton and finite gap solutions which in the space of all solutions of such equations are not dense, these types of solutions of a general position possess stochastic properties and must be studied statistically. The study of these solutions (which are generally not weakly nonlinear) is rather important from the viewpoint of understanding the turbulent nature of dynamical systems. A weakly nonlinear solution of these equations may be studied by the kinetic equation technique (see [20]), which is particularly interesting and was first considered in [14].

Finally, we wish to point out that the integrals of the two-dimensional systems we have considered do not exhaust the algebra of integrals; and it is only its commutative subalgebra. It corresponds to commutative symmetries. Symmetries and integrals, explicitly dependent of space-time variables exist, which comprise a noncommutative algebra. Corresponding equations are also solvable; see, for example [44,45].

Note added in proof. The existence condition for the *three* additional functions f_i in Theorem 2.3.4 cannot be relaxed. Let us consider the equations

$$\omega(p,q)+\omega(p_1,q_1)+\omega(p_2,q_2)=0$$

$$p+p_1+p_2=0$$

$$q+q_1+q_2=0\,.$$

They are satisfied on the manifold

$$2p_1p_2(q_1+q_2)+p_1^2q_2+p_2^2q_1=0 \tag{A}$$

for three linearly independent functions

$$\omega_1(p,q)=qp^2\quad \omega_2(p,q)=\frac{q^3}{p^2}\quad \omega_3(p,q)=\frac{q}{p}\,. \tag{B}$$

This fact, which is easily directly verified, is important for the weakly turbulent theory of drift waves in plasmas and Rossby waves in geophysics. It was established by *Balk, Nazarenko* and *Zakharov* [46] who also found that the number of functions $\omega_i(p,q)$ can not be increased. The two functions $\omega_1(p,q)$ and $\omega_2(p,q)$ are odd and also satisfy the relations

$$\begin{aligned}\omega(p,q)&=\omega(p_1,q_1)+\omega(p_2,q_2)\\ p&=p_1+p_2\\ q&=q_1+q_2\,.\end{aligned} \tag{C}$$

The function $\omega(p,q)$ is analytic. In accordance with Theorem 2.3.2, the function $\omega_2(p,q)$ is not analytic, but it is unique. This fact is generic for any analytic dispersion law (*Schulman, Tsakaya,* [47]). It is interesting that the function

$$\omega(p,q)=qp^2+\frac{q^3}{p^2}=\omega_1(p,q)+\omega_2(p,q)$$

is a degenerative dispersion law belonging to the class (2.3.15). In this case one has

$$a(\xi)=\xi^2 \qquad b(\xi)=\tfrac{1}{4}\xi^4\,.$$

This is nothing but the dispersion law in the "KP-hierarchy" which follows after KP-1. In this case the resonant manifold C is a sum of three disconnected parts. One of them is given by (A), the two others by (2.3.20).

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What Is an Integrable Mapping?

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Introduction

Rational mappings of $\mathbb{C}P^1$ and dynamic properties of their iterations once again attract the attention of mathematicians. The dynamic theory of such mappings has been developed in the classical works of G. Julia and P. Fatou. The recent investigations of Sullivan, Thurston, Douady and Hubbard throw new light upon this problem and uncover deep connections with the theory of Kleinian groups and Teichmüller space [1]. It is a very surprising fact that the notion of the integrability for such mappings is not discussed in these papers.

The first part of the present paper is devoted to such discussion. As the basis of the definition of the integrability, we place the existence of commuting mapping with suitable properties. Such a definition is motivated by the classical results of *Julia, Fatou* and *Ritt* [2–4] and by modern soliton theory, more precisely, the theory of finite-gap operators [5] and the theory of symmetries of the partial differential equations (PDE) [6] (see also the paper by Mikhailov, Shabat and Sokolov in this book). The most interesting result which we propose is the intriguing connection of such integrable polynomial mappings of $\mathbb{C}^n$ with the theory of Lie algebras. The construction, discovered in [7], allows us to match every simple complex Lie algebra of rank n to the family of the integrable polynomial (rational) mappings of $\mathbb{C}^n(\mathbb{C}P^n)$. We discuss also the analogous construction for the correspondences in $\mathbb{C}^n \times \mathbb{C}^n$ (or $\mathbb{C}P^n \times \mathbb{C}P^n$) and its relation with the Yang-Baxter equation. A separate section is devoted to the polynomial Cremona mappings of $\mathbb{C}^2$.

In the second part we consider the discrete analogs of the integrable systems of classical mechanics, following in the main the author's paper [8]. The corresponding class of mappings contains the following Lagrangean systems with discrete time.

Let M^n be any smooth manifold, $\mathcal{L}$ be the function on $M^n \times M^n$. Let us consider the problem of the extremum of the functional $S(q)$, $q = (q_i)$, $q_i \in M^n$, $i \in \mathbb{Z}$:

$$S(q) = \sum_{k \in \mathbb{Z}} \mathcal{L}(q_k, q_{k+1}) . \quad (1)$$

In a coordinate system (x^i, y^i) on $Q = M^n \times M^n$, which is induced by the coordinates u^i on M^n, we have

$$\delta S = 0 \iff \frac{\partial \mathcal{L}}{\partial x^i}(q_k, q_{k+1}) + \frac{\partial \mathcal{L}}{\partial y^i}(q_{k-1}, q_k) = 0 . \quad (2)$$