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Analytical description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping) [★]

A.I. Dyachenko ^{a,1}, E.A. Kuznetsov ^a, M.D. Spector ^b, V.E. Zakharov ^{a,c}^a Landau Institute for Theoretical Physics, 2 Kosygin Street, Moscow 117334, Russian Federation²^b Department of Fluid Mechanics and Heat Transfer, Tel Aviv University, Tel Aviv 69978, Israel^c Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

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Abstract

Using the combination of the canonical formalism for free-surface hydrodynamics and conformal mapping to the half-plane we obtain a simple system of pseudo-differential equations for the surface shape and hydrodynamic potential. The system is well-adjusted for a numerical simulation. Some typical results of such a simulation are presented.

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1. Introduction

The analytical description of the free-surface hydrodynamics is a classical problem. For two-dimensional geometry the traditional approach uses the conformal mapping. This method is well-known for the study of stationary surface waves. The first important results date from the middle of the last century and were due to Stokes [1]. Since the classical works of Nekrasov [2] and Levi-Civita [3] performed in the 1920s many publications have been devoted to this subject (see, for instance, the beautiful book by Stoker [4], and ref-

erences therein). The mathematical aspects of these works gave a powerful pulse to the development of some branches of the theory of integral equations and functional analysis.

For nonstationary surface phenomena in the sixties and later the Lagrangian description was more common [5–7]. Some authors (see Ref. [8], and references therein) tried to perform an analytical continuation with respect to the Lagrangian coordinates. But this continuation inevitably faces a singularity in both halves of the complex plane (or inside and outside of the unit circle) and so far does not allow one to obtain actually effective results.

Recently, Tanveer [9,10] suggested to use the conformal mapping for the nonstationary problem directly in the Euler description. He applied to the deep periodic water case the mapping of the fluid region into the inside of the unique circle. The equations obtained turn out to be quite complicated and therefore sufficiently

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¹ E-mail: alexd@cpd.landau.ac.ru.

² Address for correspondence.

difficult for both analytical and numerical analysis.

In this Letter we offer a simpler description of the nonstationary problem based on the combination of canonical formalism known since Ref. [11] and conformal mapping. The main idea is to perform the conformal mapping to the complex half-plane in action and therefore the equations we obtain are Hamiltonian ones from the very beginning. We present two equivalent versions of the equations. The first follow from the variational principle and include in the absence of capillarity the quadratic nonlinearity only.

The equations, given in this form, are very convenient for analytical consideration. In particular, they contain the quadratic equation for stationary gravity waves found by Longuet-Higgins in 1978 (see Ref. [12]). It will be shown in the next article [13] that these equations are the natural basis for the analytical description of wave breaking and for construction of integrable models in the theory of deep water. However, these equations are unresolved with respect to time derivatives and are not so good for a numerical simulation.

Another form of the equations can be derived either from the previous system or directly from the Bernoulli equation and from the kinematic boundary conditions. These equations are much simpler than Tanveer's. They are resolved with respect to time derivatives of the velocity potential and the free surface shape and include differentiation, taking the Hilbert transform on the whole axis, and rational nonlinearities. Therefore they are naturally adjusted for numerical simulations by using the well-developed spectral methods. We develop the corresponding algorithm for the case including both gravity and capillarity and present in this article the numerical results of its implementation. We believe that the elaborated algorithm is a very effective tool for the numerical simulation of wave breaking.

2. Lagrangian for ideal fluid in conformal variables

Let us consider an ideal fluid of infinite depth occupying on the plane (x, y) the domain $y < \eta(x, t)$. Let the motion of the fluid be a potential one ($\mathbf{v} = \nabla\Phi$), and the fluid be incompressible ($\text{div } \mathbf{v} = 0$). Hence, the potential obeys the Laplace equation

$$\nabla^2 \Phi(x, y, t) = 0. \quad (2.1)$$

The boundary conditions are

$$\frac{\partial \eta}{\partial t} = (\Phi_y - \eta_x \Phi_x)|_{y=\eta} = v_n \sqrt{1 + (\eta_x)^2}, \quad (2.2)$$

$$\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 \right)_{y=\eta} + gy - \sigma \frac{\partial}{\partial x} \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right) = 0. \quad (2.3)$$

Here g is the gravity acceleration and σ is the surface tension. If we introduce

$$\psi(x, t) = \Phi(x, \eta(x, t), t), \quad (2.4)$$

then, as it was shown in Ref. [11], the system is a Hamiltonian one and the boundary conditions (2.2) and (2.3) are equivalent to the canonical equations

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad (2.5)$$

where H is the total energy of the system

$$H = H_{\text{kin}} + H_{\text{pot}}, \quad (2.6)$$

$$H_{\text{kin}} = \frac{1}{2} \int dx \int_{-\infty}^{\eta} (\nabla \Phi)^2 dy, \quad (2.7)$$

$$H_{\text{pot}} = \frac{1}{2} g \int \eta^2 dx + \sigma \int (\sqrt{1 + \eta_x^2} - 1) dx. \quad (2.8)$$

It is impossible to express H explicitly in terms of η and ψ . One can use the expansion in powers of nonlinearity $k\eta_k$ (see, for instance, Ref. [14]).

Eqs. (2.5) realize an extremum of the action

$$S = \int L dt, \quad (2.9)$$

with the Lagrangian

$$L = \int \Psi \frac{\partial \eta}{\partial t} dx - H. \quad (2.10)$$

Let us perform a conformal mapping of the domain $y < \eta$ to the lower half-plane of the complex variable $w = u + iv$. In conformal variables the shape of the surface is given by the parametric representation

$$y = y(u, t), \quad x = x(u, t) = u + \tilde{x}(u, t). \quad (2.11)$$

We assume here and further that $y \rightarrow 0$, $\psi \rightarrow 0$ at $|u| \rightarrow \infty$. In this case y and \tilde{x} are related by the Hilbert transform

$$y = \hat{H}\tilde{x}, \quad \tilde{x} = -\hat{H}y, \quad (2.12)$$

where by definition

$$\begin{aligned} \hat{H}f(u) &= \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{f(u')}{u' - u} du', \\ \hat{H}^2 f(u) &= -f(u). \end{aligned} \quad (2.13)$$

In the conformal variables

$$\eta_t dx = (y_t x_u - x_t y_u) du, \quad (2.14)$$

and the kinetic and potential energy are

$$\begin{aligned} H_{\text{kin}} &= -\frac{1}{2} \int \psi \hat{H}\psi_u du, \\ H_{\text{pot}} &= \frac{1}{2} g \int_{-\infty}^{+\infty} y^2 x_u du + \sigma \int_{-\infty}^{+\infty} (|z_u| - x_u) du. \end{aligned} \quad (2.15)$$

Here and further

$$z = x + iy, \quad |z_u|^2 = x_u^2 + y_u^2. \quad (2.16)$$

Finally, the Lagrangian takes the form

$$\begin{aligned} L &= \int \Psi (y_t x_u - x_t y_u) du + \frac{1}{2} \int \Psi \hat{H}\Psi_u \\ &\quad - \frac{1}{2} g \int y^2 x_u du - \sigma \int (|z_u| - x_u) du. \end{aligned} \quad (2.17)$$

To take into account relations (2.12) one must replace

$$L \rightarrow \tilde{L} = L + \int_{-\infty}^{+\infty} f \times (y - \hat{H}\tilde{x}) du. \quad (2.18)$$

3. Equations of motion (implicit form)

To obtain the equations of motion one has to put the variational derivatives of the action S equal to zero. The condition $\delta S / \delta \Psi = 0$ gives the equation

$$y_t x_u - x_t y_u = -\hat{H}\Psi_u, \quad (3.1)$$

or

$$y_t(1 + \tilde{x}_u) - \tilde{x}_t y_u = -\hat{H}\Psi_u. \quad (3.2)$$

This is nothing but the kinematic boundary condition (2.2) written in the conformal variables.

Integrating Eq. (3.2) over the real axis $-\infty < u < +\infty$ gives the mass conservation law,

$$\frac{d}{dt} \int_{-\infty}^{+\infty} y(1 + \tilde{x}_u) du = \frac{d}{dt} \int_{-\infty}^{+\infty} \eta dx = 0. \quad (3.3)$$

The conditions $\delta S / \delta x = 0$ and $\delta S / \delta y = 0$ give

$$y_u \Psi_t - y_t \Psi_u + g y y_u - \sigma \frac{\partial}{\partial u} \left(\frac{x_u}{|z_u|} \right) = \hat{H}f, \quad (3.4)$$

$$-x_u \Psi_t + x_t \Psi_u - g y x_u - \sigma \frac{\partial}{\partial u} \left(\frac{y_u}{|z_u|} \right) = f. \quad (3.5)$$

The Lagrangian factor f can be easily excluded. For simplicity we present the result only in the case of $\sigma = 0$. Now the resulting equation is quadratic,

$$x_u \Psi_t - x_t \Psi_u + \hat{H}(-y_u \Psi_t + y_t \Psi_u) + g(y x_u - \hat{H}y y_u) = 0 \quad (3.6)$$

or

$$\begin{aligned} (1 + \tilde{x}_u) \Psi_t - x_t \Psi_u + \hat{H}(-y_u \Psi_t + y_t \Psi_u) \\ + g(y + y \tilde{x}_u - \hat{H}y y_u) = 0. \end{aligned} \quad (3.7)$$

Eqs. (3.2) and (3.7) are especially simple for stationary waves. In this case

$$\frac{\partial}{\partial t} = c \frac{\partial}{\partial u} \quad (3.8)$$

and from (3.2) one can find

$$\Psi = c \hat{H}y. \quad (3.9)$$

Eq. (3.7) now takes the form

$$c^2 \hat{H}y_u + g(y - y \hat{H}y_u - \hat{H}y y_u) = 0, \quad (3.10)$$

which is exactly the quadratic equation, obtained in 1978 by Longuet-Higgins [12].

In the absence of gravity Eq. (3.7) is

$$(1 + \tilde{x}_u) \Psi_t - x_t \Psi_u + \hat{H}(-y_u \Psi_t + y_t \Psi_u) = 0. \quad (3.11)$$

The system (3.2), (3.11) allows an interesting class of exact solutions. Let us put

$$\begin{aligned}\Psi &= \Psi_0(u) + \Psi_1(u)t, \quad y = y_0(u) + y_1(u)t, \\ \tilde{x} &= \tilde{x}_0(u) + \tilde{x}_1(u)t.\end{aligned}\quad (3.12)$$

The coefficients in (3.12) satisfy the following close system,

$$\begin{aligned}y_1 \tilde{x}_{1u} - \tilde{x}_1 y_{1u} &= -\hat{H} \Psi_{1u}, \\ \Psi_1 \tilde{x}_{1u} - \tilde{x}_1 \Psi_{1u} &= \hat{H}(\Psi_1 y_{1u} - y_1 \Psi_{1u}),\end{aligned}\quad (3.13)$$

$$\begin{aligned}y_1(1 + \tilde{x}_{0u}) - \tilde{x}_1 y_{0u} &= -\hat{H} \Psi_{0u}, \\ \Psi_1(1 + \tilde{x}_{0u}) - \tilde{x}_1 \Psi_{0u} &= \hat{H}(\Psi_1 y_{0u} - y_1 \Psi_{0u}).\end{aligned}\quad (3.14)$$

A discussion of the solutions of the system (3.13), (3.14) is beyond the scope of this article.

One can express Ψ from (3.2)

$$\Psi = \partial_u^{-1} \hat{H}[y_t(1 + \tilde{x}_u) - \tilde{x}_t y_u] \quad (3.15)$$

and substitute the result to the Lagrangian (2.17). After a simple calculation we find [12]³

$$\begin{aligned}L &= H_{\text{kin}} - H_{\text{pot}}, \\ H_{\text{kin}} &= -\frac{1}{2} \int_{-\infty}^{+\infty} (y_t x_u - x_t y_u) \hat{H} \partial_u^{-1} (y_t x_u - x_t y_u) du, \\ H_{\text{pot}} &= \frac{1}{2} g \int_{-\infty}^{+\infty} y^2 x_u du + \sigma \int_{-\infty}^{+\infty} (|z_u| - x_u) du.\end{aligned}\quad (3.16)$$

In particular, in the case $g = 0$, $\sigma = 0$

$$L = H. \quad (3.17)$$

This is a general relation between a Hamiltonian and a Lagrangian for the case when H is a quadratic functional of the momenta.

It is important that in the absence of capillarity L is a quartic functional of the coordinate $y(u, t)$.

After integrating Eq. (3.11) along the real axis we find

$$\frac{d}{dt} \int_{-\infty}^{+\infty} \Psi(1 + \tilde{x}_u) du = 0. \quad (3.18)$$

After applying to (3.11) the Hilbert transformation and integrating along the real axis we obtain

$$\frac{d}{dt} \int_{-\infty}^{+\infty} \Psi y_u du = 0. \quad (3.19)$$

The identities (3.18) and (3.19) are the conservation laws for the vertical and horizontal momentum of the fluid.

We mention also that the stationary equation (3.10) can be obtained by minimization of the following action functional,

$$S = \frac{1}{2} c^2 \int_{-\infty}^{+\infty} y \hat{H} y_u du + \frac{1}{2} g \int_{-\infty}^{+\infty} y^2 (1 + \tilde{x}_u) du, \quad (3.20)$$

which is nothing but the action (2.9) calculated for stationary waves.

4. Equations of motion (explicit form)

It is important to resolve Eqs. (3.1), (3.4) and (3.5) with respect to y_t , x_t and Ψ_t . Eq. (3.1) can be rewritten now as follows,

$$z_t z_u^* - z_t^* z_u = -2i \hat{H} \Psi \quad (4.1)$$

or

$$\frac{z_t}{z_u} - \frac{z_t^*}{z_u^*} = -2i \frac{\hat{H} \Psi}{|z_u|^2}. \quad (4.2)$$

Let us introduce the projection operators

$$\hat{P}^{\pm} = \frac{1}{2} (1 \mp i \hat{H}), \quad (P^{\pm})^2 = P^{\pm}. \quad (4.3)$$

The function z_t/z_u can be analytically continued to the lower half-plane, while the function z_t^*/z_u^* can be continued to the upper half-plane. Hence

$$\hat{P}^- \left(\frac{z_t^*}{z_u^*} \right) = \frac{1}{2} (1 + i \hat{H}) \frac{z_t^*}{z_u^*} = 0$$

and

$$z_t = z_u (\hat{H} - i) \frac{\hat{H} \Psi_u}{|z_u|^2}, \quad (4.4)$$

or

$$y_t = (y_u \hat{H} - x_u) \frac{\hat{H} \Psi_u}{|z_u|^2}, \quad (4.5)$$

³ We are grateful to Dr. A. Balk, who directed our attention to the possibility of representing the Lagrangian in the form (3.16).

$$x_t = (x_u \hat{H} + y_u) \frac{\hat{H}\Psi_u}{|z_u|^2}. \quad (4.6)$$

The simplest way to find the equation for Ψ_t is by rewriting the dynamic boundary condition (2.3) in conformal variables. For $\Psi(u, t) = \Phi(x(u, t), y(u, t), t)$,

$$\Phi_t|_{y=\eta} = \Psi_t - \Phi_x x_t - \Phi_y y_t. \quad (4.7)$$

Using the parametric representation (2.11) and the Cauchy–Riemann conditions,

$$x_u = y_v, \quad x_v = -y_u, \quad (4.8)$$

and the identity $\Phi_v|_{v=0} = -\hat{H}\Psi_u|_{v=0}$ one can easily see that at $v = 0$

$$\begin{aligned} \Phi_x &= (\Psi_u x_u + \hat{H}\Psi_u y_u) \frac{1}{|z_u|^2}, \\ \Phi_y &= (-\hat{H}\Psi_u x_u + \Psi_u y_u) \frac{1}{|z_u|^2}. \end{aligned} \quad (4.9)$$

Substituting (4.9) and (4.7) into (2.3) and using Eqs. (4.5) and (4.6) one can find after elementary calculations

$$\Psi_t = \frac{\hat{H}(\Psi_u \hat{H}\Psi_u)}{|z_u|^2} + \Psi_u \hat{H} \left(\frac{\hat{H}\Psi_u}{|z_u|^2} \right) - gy + \sigma \frac{1}{x_u} \frac{\partial}{\partial u} \frac{y_u}{|z_u|}. \quad (4.10)$$

One can recall that Ψ and y are no longer canonically conjugated variables. The momentum conjugated to y_t was found by the authors (E.K. and A.D.) and was used in Ref. [15] to calculate high-order terms in the Hamiltonian expansion.

The equation for stationary waves, obtained from the system (4.5), (4.6) and (4.10), has the following form,

$$\frac{c^2}{2|z_u|^2} + gy - \sigma \frac{1}{x_u} \frac{\partial}{\partial u} \frac{y_u}{|z_u|} = \frac{c^2}{2}, \quad (4.11)$$

and it is cubic if $\sigma = 0$.

5. Numerical simulation

We performed the numerical integration of the system (4.5), (4.10) imposing the periodic boundary conditions, $\Phi(w) = \Phi(w + 2\pi)$, $z(w) = z(w + 2\pi)$,

$-\pi \leq u \leq \pi$. In this case the Hilbert transformation (2.13) has to be replaced as follows,

$$\hat{H}f = \frac{1}{2\pi} \text{P} \int_{-\pi}^{\pi} f(u') \cot\left[\frac{1}{2}(u' - u)\right] du'.$$

Its spectral (Fourier) eigenvalues, as for the infinite domain, are nothing but $i \operatorname{sgn}(k)$, so that

$$(\hat{H}f)_k = i \operatorname{sgn}(k) f_k.$$

Therefore, using the pseudo-spectral method is the most suitable way to integrate (4.5) and (4.10). For this system we developed an implicit difference scheme where all operations in space (including Hilbert transformation) were performed using the pseudo-spectral method. The difference approximation to the equations is

$$\begin{aligned} \frac{y^{n+1} - y^n}{\tau} &= y_u^\sigma \hat{H} \frac{\hat{H}\Psi_u^\sigma}{J^\sigma} - x_u^\sigma \frac{\hat{H}\Psi_u^\sigma}{J^\sigma}, \\ \frac{\Psi^{n+1} - \Psi^n}{\tau} &= \Psi_u^\sigma \hat{H} \frac{\hat{H}\Psi_u^\sigma}{J^\sigma} + \frac{(\hat{H}\Psi_u^\sigma)^2 - (\Psi_u^\sigma)^2}{2J^\sigma}. \end{aligned} \quad (5.1)$$

The index σ means that the value for y_u was chosen as

$$y_u^\sigma = \sigma y_u^{n+1} + (1 - \sigma) y_u^n,$$

and $J = |z_u|^2$ is the Jacobian of the conformal transformation. For $\sigma = \frac{1}{2}$ scheme (5.1) conserves all the integrals of motion of the original equations, namely: energy, amount of fluid, and both momenta. Unfortunately, for $\sigma = \frac{1}{2}$ there was observed a numerical instability in (5.1) and we were forced to use $\sigma = \frac{2}{3}$. Calculations were performed with quartic accuracy (approximately 30 decimal digits).

We considered a periodic problem in the infinite half-strip with width 2π (both in the real space and after conformal transformation) in the absence of gravity and surface tension. Initial conditions were chosen symmetric with respect to x : $\eta(x) = \eta(-x)$. We used this symmetry in the computer simulation, performing integration in the domain $0 \leq x \leq \pi$. Calculations were stopped when the accuracy to resolve the peak required more than 2^{15} grid points in the whole domain. The dimensionless time at that moment was usually about 1.0.

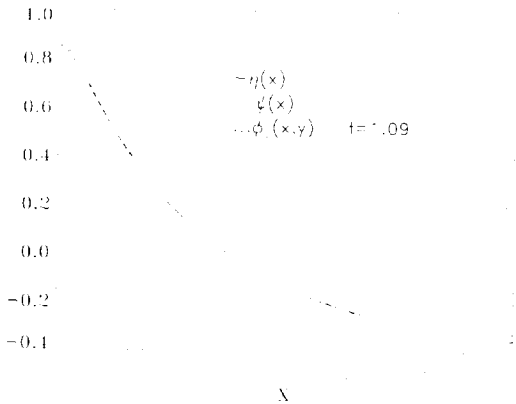


Fig. 1. Surface profile $\eta(x)$ (solid line), potential on the surface $\psi(x)$ (dashed line) and vertical velocity on the surface $\phi_y(x, \eta)$ (dotted line) at $t = 1.09$.

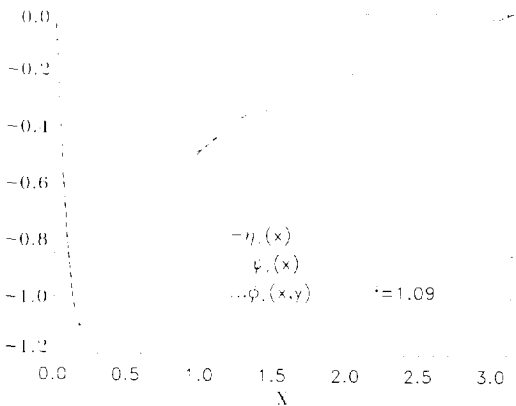


Fig. 2. Steepness η_x (solid line), ψ_x (dashed line) and horizontal velocity $\phi_x(x, \eta)$ (dotted line) at $t = 1.09$.

Below we present a typical example of the numerical integration. Initial conditions for this run were chosen in the form

$$z = u, \quad \Phi = -A \log[1 - \exp(-iu - 1)],$$

$$A = 0.2 \sinh(e). \quad (5.2)$$

Fig. 1 displays the surface profile $\eta(x)$, the potential on the surface $\psi(x)$ and the vertical velocity on the surface $\phi_y(x, \eta)$. The surface profile has a jet form, which is typical for such problems. In Fig. 2 the steepness η_x , ψ_x and the horizontal velocity $\phi_x(x, \eta)$ are given.

The behavior of the real part of the conformal mapping on the real axis $x(u)$ in the vicinity of the origin

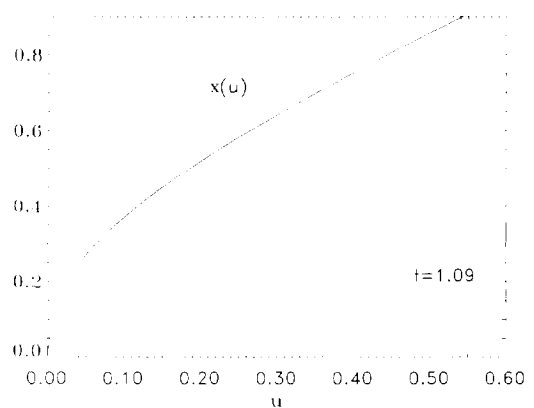


Fig. 3. The real part of the conformal mapping on the real axis $x(u)$ in the vicinity of the origin at $t = 1.09$.

is shown in Fig. 3. It should be mentioned here that the surface profile in conformal variables is much steeper than in the real variables, $x_u \rightarrow \infty$ in the origin. This fact is an obstacle for simulation at large t .

It is important to compare our numerical results with recent theoretical predictions of the formation of singularities on the surface of deep fluid [16,17]. In these articles we studied an “almost flat” free surface in the absence of gravity and capillarity. The first two terms in the expansion of the Hamiltonian in “natural” (not in conformal!) variables give the integrable equation

$$\frac{\partial \Psi^\pm}{\partial t} + 2 \left(\frac{\partial \Psi^\pm}{\partial x} \right)^2 = 0. \quad (5.3)$$

(Here $\Psi^\pm = \hat{P}^\pm \Psi$.) This equation describes the formation of “weak” singularities $\eta \approx |x|^{3/2}$, so that the curvature $\eta_{xx} \approx |x|^{-1/2}$ becomes infinite as $x \rightarrow 0$. So far we did not observe a definite tendency to the creation of weak singularities in the numerical simulation. We instead observed the formation of growing peaks of the finger or jet type. A qualitative explanation of this fact is offered in Ref. [13]. Here we can only mention that the reduced model (5.3) acquires instability, which disappears when taking into account the next term in the expansion of the Hamiltonian.

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