Is the free surface hydrodynamics an integrable system?

A.I.Dyachenko and V.E.Zakharov

Department of Mathematics, University of Arizona, Tucson, AZ, 85721, USA

Landau Institute for Theoretical Physics, Russia, 117334, Moscow, 2 Kosygina st.

PHYSYCS LETTERS A 190 (1994) 144-148

Abstract

A strong argument is found in support of the integrability of free-surface hydrodynamics in one-dimensional case. It is shown that the first term in the series of perturbation in powers of nonlinearity is identically equal to zero, the consequences of which are discussed as well.

1. It is well known that the equations describing the ideal fluid with a free surface in a gravity field are completely integrable in several important limiting cases. Integrability occurs for long waves in shallow water (KdV[1] and KP[2] equations, the Boussinesq approximation[3], Kaup's approximation[4], Holm-Camassa's approximation[5]) and for spectrally narrow wave trains in the fluid of arbitrary depth (nonlinear Schrödinger equation[6]). The weakly nonlinear motion of the fluid in the absence of a gravity field is integrable as well[7].

It is very natural to formulate a conjecture that an arbitrary one–dimensional motion of the ideal fluid in a gravity field is integrable. In this article we give arguments in support of this conjecture. We will consider weakly nonlinear waves on a surface of a fluid of infinite depth and study their simplest resonant interactions, and we will show that the amplitude of this process is zero.

Given the current stage of the mathematical physics there are no effective general methods for checking and proving integrability for the nonlinear wave Hamiltonian systems. Proving nonintegrability is a much easily problem. Following Poincare, one can do that by analysing the series of perturbation in powers of nonlinearity [8]. Terms of this series being limited on their resonant manifolds are identified with the "amplitudes of the nonlinear interactions" in the wave system.

Nonintegrability is quite evident fact. To have nonintegrability, it is enough to prove that at least one of these amplitudes is nonzero. As the complexity of calculations increases significantly with the order of nonlinearity, much information can be extracted from the consideration of the first (lowest order) nontrivial nonlinear process. For instance, nonintegrability of the nonlinear Schrödinger equation for $d \geq 2$ is a trivial fact due to the nonzero amplitude for the process $2 \to 2$ wave scattering. This scattering is trivial for the integrable case $d = 1$. One may verify (albeit with much blood, sweat and tears) that the amplitude of the first nontrivial scattering $3 \to 3$ is identically zero in this case.

2. A one-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid is described by the following set of equations:

$$\phi_{xx} + \phi_{zz} = 0 \qquad (\phi_z \to 0, z \to -\infty),$$

$$
\begin{aligned}
\eta_t + \eta_x \phi_x &= \phi_z \Big|_{z=\eta} \\
\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta &= 0\Big|_{z=\eta};
\end{aligned} \tag{1}
$$

here $\eta(x,t)$ - is the shape of a surface, $\phi(x,z,t)$ - is a potential function of the flow and g - is a gravitational constant. As was shown in[9], the variables $\eta(x,t)$ and $\psi(x,t) = \phi(x,z,t)\Big|_{z=\eta}$ are canonically conjugated, and their Fourier transforms satisfy the equations

$$
\frac{\partial \psi_k}{\partial t} = -\frac{\delta H}{\delta \eta_k^*} \qquad\qquad \frac{\partial \eta_k}{\partial t} = \frac{\delta H}{\delta \psi_k^*}.
$$

Here $H = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms:

$$
K = \frac{1}{2}\int dx \int_{-\infty}^{\eta} v^2 dz \qquad U = \frac{g}{2}\int \eta^2 dx
$$

A Hamiltonian can be expanded in an infinite series in powers of a characteristic wave steepness $k\eta_k << 1$ (see[9, 10])

$$
H = H_0 + H_1 + H_2 + \dots \tag{2}
$$

It is convenient to introduce a normal complex variable a_k:

$$
\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a_{-k}^*) \quad \psi_k = -i\sqrt{\frac{2g}{\omega_k}}(a_k - a_{-k}^*) \tag{3}
$$

here $\omega_k = \sqrt{gk}$ -is the dispersion law for the gravity waves. This variable satisfies the equation

$$
\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a_k^*} = 0, \tag{4}
$$

where

$$
\begin{aligned}
H_0 &= \int \omega_k a_k a_k^* dk, \\
H_1 &= \int V_{kk_1k_2}\{a_k^* a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^*\}\delta_{k-k_1-k_2} dk dk_1 dk_2
\end{aligned}
$$

$$+\quad \frac{1}{3}\int U_{kk_1k_2}\{a_k a_{k_1} a_{k_2} + a^*_k a^*_{k_1} a^*_{k_2}\}\delta_{k+k_1+k_2} dk dk_1 dk_2 \tag{5}$$

$$\begin{aligned} V_{kk_1k_2} &= \frac{g^{\frac{1}{4}}}{8\pi\sqrt{2}}\left(\left(\frac{k}{k_1k_2}\right)^{\frac{1}{4}} L_{k_1k_2} - \left(\frac{k_2}{kk_1}\right)^{\frac{1}{4}} L_{-kk_1} - \left(\frac{k_1}{kk_2}\right)^{\frac{1}{4}} L_{-kk_2}\right) \\ U_{kk_1k_2} &= \frac{g^{\frac{1}{4}}}{8\pi\sqrt{2}}\left(\left(\frac{k}{k_1k_2}\right)^{\frac{1}{4}} L_{k_1k_2} + \left(\frac{k_2}{kk_1}\right)^{\frac{1}{4}} L_{kk_1} + \left(\frac{k_1}{kk_2}\right)^{\frac{1}{4}} L_{kk_2}\right) \end{aligned} \tag{6}$$

$$L_{kk_1} = (\vec{k}\vec{k_1}) + |k||k_1| \tag{7}$$

Among the various components of H_2 only one is important:

$$H_2 = \frac{1}{2}\int W_{kk_1k_2k_3} a^*_k a^*_{k_1} a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 \tag{8}$$

$$\begin{aligned} W_{k_1k_2k_3k_4} = -\frac{1}{64\pi^2}\{ & M_{-k_1-k_2k_3k_3} + M_{k_3k_4-k_1-k_2} - M_{-k_1k_3-k_2k_4} \\ & -M_{-k_1k_4-k_2k_3} - M_{-k_2k_3-k_1k_4} - M_{-k_2k_4-k_1k_3}\} \end{aligned} \tag{9}$$

$$M_{k_1k_2k_3k_4} = |k_1k_2|\left(\frac{k_3k_4}{k_1k_2}\right)^{\frac{1}{4}}(|k_1+k_3|+|k_1+k_4|+|k_2+k_3|+|k_2+k_4|-2|k_1|-2|k_2|).$$

The variable a_k is not appropriate for gravity waves because of the Hamiltoniat contains cubic terms, while there are no three–wave frequence resonances. The following canonical (up to $O(a^5)$) transformation from a_k to b_k (see [9, 11]) (Note: in[9] there is a misprint - the negative sign is missing for W_{kk_1,k_2k_3})

$$\begin{aligned} a_k = b_k &+ \int \Gamma^{(1)}_{kk_1k_2} b_{k_1} b_{k_2} \delta_{k-k_1-k_2} - 2\int \Gamma^{(1)}_{k_2kk_1} b^*_{k_1} b_{k_2} \delta_{k+k_1-k_2} + \\ &+ \int \Gamma^{(2)}_{kk_1k_2} b^*_{k_1} b^*_{k_2} \delta_{k+k_1+k_2} + \int B_{kk_1k_2k_3} b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} \\ B_{kk_1k_2k_3} &= \Gamma^{(1)}_{k_1k_2k_1-k_2}\Gamma^{(1)}_{k_3kk_3-k} + \Gamma^{(1)}_{k_1k_3k_1-k_3}\Gamma^{(1)}_{k_2kk_2-k} - \\ &-\Gamma^{(1)}_{kk_2k-k_2}\Gamma^{(1)}_{k_3k_1k_3-k_1} - \Gamma^{(1)}_{k_1k_3k_1-k_3}\Gamma^{(1)}_{k_2k_1k_2-k_1} - \\ &-\Gamma^{(1)}_{k+k_1kk_1}\Gamma^{(1)}_{k_2+k_3k_2k_3} + \Gamma^{(2)}_{-k-k_1kk_1}\Gamma^{(2)}_{-k_2-k_3k_2k_3} \end{aligned} \tag{10}$$

$$
\begin{aligned}
\Gamma^{(1)}_{kk_1k_2} &= -\frac{V_{kk_1k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}} \\
\Gamma^{(2)}_{kk_1k_2} &= -\frac{U_{kk_1k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}
\end{aligned}
$$

transforms the Hamiltonian to a form not containing cubic terms:

$$
H = \int \omega_k b_k b_k^* dk + \frac{1}{2}\int T_{kk_1,k_2k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \ldots; \quad (11)
$$

here T_{kk_1,k_2k_3} satisfies the symmetry conditions:

$$
T_{kk_1,k_2k_3} = T_{k_1k,k_2k_3} = T_{kk_1,k_3k_2} = T_{k_2k_3kk_1}
$$

and has the form:

$$
\begin{aligned}
T_{kk_1,k_2k_3} = W_{k_1k,k_2k_3} &- \\
&-V_{kk_2k-k_2}V_{k_3k_1k_3-k_1}\left[\frac{1}{\omega_{k_2}+\omega_{k-k_2}-\omega_k}+\frac{1}{\omega_{k_1}+\omega_{k_3-k_1}-\omega_{k_3}}\right]- \\
&-V_{k_1k_2k_1-k_2}V_{k_3kk_3-k}\left[\frac{1}{\omega_{k_2}+\omega_{k_1-k_2}-\omega_{k_1}}+\frac{1}{\omega_{k}+\omega_{k_3-k}-\omega_{k_3}}\right]- \\
&-V_{kk_3k-k_3}V_{k_2k_1k_2-k_1}\left[\frac{1}{\omega_{k_3}+\omega_{k-k_3}-\omega_k}+\frac{1}{\omega_{k_1}+\omega_{k_2-k_1}-\omega_{k_2}}\right]- \\
&-V_{k_1k_3k_1-k_3}V_{k_2kk_2-k}\left[\frac{1}{\omega_{k_3}+\omega_{k_1-k_3}-\omega_{k_1}}+\frac{1}{\omega_{k}+\omega_{k_2-k}-\omega_{k_2}}\right]- \\
&-V_{k+k_1kk_1}V_{k_2+k_3k_2k_3}\left[\frac{1}{\omega_{k+k_1}-\omega_k-\omega_{k_1}}+\frac{1}{\omega_{k_2+k_3}-\omega_{k_2}-\omega_{k_3}}\right]- \\
&-U_{-k-k_1kk_1}U_{-k_2-k_3k_2k_3}\left[\frac{1}{\omega_{k+k_1}+\omega_k+\omega_{k_1}}+\frac{1}{\omega_{k_2+k_3}+\omega_{k_2}+\omega_{k_3}}\right] \quad (12)
\end{aligned}
$$

The first nontrivial process is four-wave scattering, which is governed by the following resonant conditions (k_i are one-dimensional)

$$
k + k_1 = k_2 + k_3,
$$

$$\omega_k + \omega_{k_1} \;=\; \omega_{k_2} + \omega_{k_3}, \tag{13}$$

and all frequencies ω_{k_i} are positive here. The system (13) describes a certain two-dimensional manifold in four-dimensional space (k, k_1, k_2, k_3). This manifold has a trivial component

$$k_2 = k_1, \quad k_3 = k, \quad \textit{or} \quad k_2 = k, \quad k_3 = k_1, \tag{14}$$

but it has also a nontrivial part. Let $k, k_1, k_3 > 0, k_2 < 0$. Now (13) describes the rational manifold, which can be parameterized in the following way:

$$\begin{aligned} k &= a(1+\zeta)^2, \\ k_1 &= a(1+\zeta)^2\zeta^2, \\ k_2 &= -a\zeta^2, \\ k_3 &= a(1+\zeta+\zeta^2)^2; \end{aligned} \tag{15}$$

here $0 < \zeta < 1$ and $a > 0$. It is easy to see that these two manifolds, (14) and (15), represent the general solution for resonant interaction (except trivial permutations).

The main result of this article can be summed up as follows: **the amplitude T_{kk_1,k_2k_3} is identically equal to zero on the resonant surface (15)**. This fact can be checked by direct calculation of T_{kk_1,k_2k_3} using expression (12) (we did it with the help of 'Mathematica' [12]). The cancellation of dozens of terms in T_{kk_1,k_2k_3} apparently is not accidental; the cancellation would take place naturally if the system (1) were integrable, or had at least an additional integral of motion [8]. Of course, this is not a strict proof of integrability (we have no way of checking all higher order amplitudes in (11)), but there are other evidences, mentioned above, supporting our conjecture. Additionally, the integrability of gravity waves in the fluid of finite depth that can be checked by calculation of the appropriate T_{kk_1,k_2k_3} as well. Also, the recently published numerical simulation [13] of the evolution of a set of waves has shown that the wave number frequency spectrum is concentrated in **discrete** points near the curves $\omega_k = \sqrt{nk}$, n=1,2,3,.... In addition, the discrete spectrum is the direct consequence of the integrability of the system (1).

However, we can obtain the full proof of integrability by developing a new method of integration (e.g., inverse scattering method), or by finding an $L - A$ pair, etc.

The vanishing T_{kk_1,k_2k_3} on the resonant surface leaves in effect only trivial interaction (14), which corresponds to the nonlinear frequency shift of separate modes:

$$\widetilde{\omega_k} = \omega_k + \int T_{kk_1}|b_{k_1}|^2 dk_1,$$

where

$$T_{kk_1} = \frac{1}{4\pi^2}(\vec{k}\vec{k}_1)min(|k|, |k_1|),$$

and the Hamiltonian can be written (using new canonical variables c_k) as

$$H = \int \omega c_k {c_k}^* dk + \frac{1}{2}\int T_{kk_1}|c_k|^2|c_{k_1}|^2 dk dk_1 + O({c_k}^5).$$

Therefore, integrability occurs at least up to the fifth order of c_k (or steepness $k\eta_k$). Furthermore, any quantity of the form

$$I_4 = \int f(k)|c_k|^2 dk$$

is also the integral of motion up to the fifth order.

3. The integrability of the one-dimensional free-surface hydrodynamics results in a rather different view of the problem of sea waving. It is well known that the well-developed the surface wave turbulent spectrum is very narrow in an angle (indeed, it is almost one-dimensional). Thus, there is a natural small parameter $k_\perp/k_\parallel$ permitting the development of a perturbation theory for sea waves. Wave breaking, which is also a nerly one-dimensional phenomenon, can be described by an integrable set of equations as well.

This work is supported by the ONR Grant N00 14–92–J–1343 and Russian Basic Research Foundation Grant N00 94-01-00898.

References

[1] Gardner,C.S., Greene,J.M., Kruskal,M.D. and Miura,R.M., Phys. Rev. Lett., **19**, (1967), 1095

[2] Zakharov,V.E., Shabat,A.B., Funct. Anal. Appl., **8**, (1974), 226

[3] Zakharov,V.E., Sov.Phys.JETP, **38**, (1974), 108

[4] Kaup,D.J., Prog. Theor. Physics (Japan), **54**, (1975), 396

[5] Holm,D., Camassa,R., Phys. Rev. Lett., **71**, (1993), 1671

[6] Zakharov,V.E. and Shabat,A.B., Sov. Phys. JETP , **34**, (1972), 62

[7] Kuznetsov,E.A., Spector,M.D. and Zakharov,V.E., Phys. Rev. E, (accepted for publication).

[8] Zakharov,V.E., Shulman,E., in 'What is Integrability?', Edtr. V.E.Zakharov, Springer-Verlag, (1991), 190

[9] Zakharov,V.E., Prikl.Mekh.Tekh.Fiz.(in Russian), **2**, (1968), 190

[10] Crawford,D.E., Yuen,H.G. and Saffman,P.G., Wave Motion, **2**, (1980), 1

[11] Krasitskii,V.P., Sov. Phys. JETP, **71**,(1990), 921

[12] Wolfram,V.P., Mathematica, Addison-Wesley Pub.Comp., (1988)

[13] Sheinin,D., Chalikov,D., Proc. ONR Ocean Waves Workshop, (1994)