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Envelope Equation for Water Waves

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Abstract:	Water waves have long been a subject of attention of both mathematicians and physicists. The formulation of the problem is simple enough to be considered fundamental, but as of yet many questions still remain unanswered and many phenomena associated with wind-driven turbulence remain puzzling. %This article is mainly devoted to the one them - numerical simulation of wave breaking. We consider a ``unidirectional'' motion of weakly nonlinear gravity waves, i.e. we assume that the spectrum of the free surface contains only nonnegative wavenumbers. We use remarkably simple form of the water wave equation %{\color{red} Reference here.}, that we named ``the super compact equation''. This new equation includes a nonlinear wave term (\`{a} la NLSE) together with an advection term, that can describe the initial stage of wave-breaking. This equation has also very important property. It allows to introduce exact envelope for waves without assumption of narrowness bandwidth.
Corresponding Author:	Alexander I Dyachenko, Doctor of Science Landau Institute for Theoretical Physics Chernogolovka, Moscow Region RUSSIAN FEDERATION
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Landau Institute for Theoretical Physics
Corresponding Author's Secondary Institution:	
First Author:	Alexander I Dyachenko, Doctor of Science
First Author Secondary Information:	
Order of Authors:	Alexander I Dyachenko, Doctor of Science
	Dmitry I Kachulin, PhD
	Vladimir E Zakharov, Prof.
Order of Authors Secondary Information:	
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Envelope equation for water waves

Soliton turbulence and wavebreaking

A.I. Dyachenko · D.I. Kachulin · V.E. Zakharov

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Abstract Water waves have long been a subject of attention of both mathematicians and physicists. The formulation of the problem is simple enough to be considered fundamental, but as of yet many questions still remain unan-

A.I Dyachenko

Landau Institute for Theoretical Physics, 142432, Chernogolovka, Russia

Novosibirsk State University, 630090, Novosibirsk-90, Russia

E-mail: alexd@itp.ac.ru

D.I. Kachulin

Novosibirsk State University, 630090, Novosibirsk-90, Russia

V.E. Zakharov

Lebedev Physical Institute RAS, 119991, Moscow, Russia

Novosibirsk State University, 630090, Novosibirsk-90, Russia Landau Institute for Theoretical Physics, 142432, Chernogolovka, Russia

Department of Mathematics, University of Arizona, Tucson, AZ, 857201, USA

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We consider a "unidirectional" motion of weakly nonlinear gravity waves, i.e. we assume that the spectrum of the free surface contains only nonnegative wavenumbers. We use remarkably simple form of the water wave equation that we named "the super compact equation". This new equation includes a nonlinear wave term (à la NLSE) together with an advection term, that can describe the initial stage of wave-breaking. This equation has also very important property. It allows to introduce exact envelope for waves without assumption of narrowness bandwidth.

Keywords Wave breaking · Hamiltonian formalism · modulational instability · envelope equation

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1 Introduction

A potential flow of an ideal incompressible fluid with free surface in a gravity field is described [Zakharov (1968)] by the following Hamiltonian system:

$$\frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta} \qquad\qquad \frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}. \tag{1}$$

Hereafter we study only the case of one horizontal direction, unidirectional waves. Now

$\eta = \eta(x,t)$ – shape of the surface,

$$\psi = \psi(x,t) = \phi(x, \eta(x,t), t) \text{ – potential on the surface,}$$

$$\phi(x,z,t) \text{ – potential inside the fluid.} \tag{2}$$

The Hamiltonian H is

$$H = \frac{1}{2}\int dx \int_{-\infty}^{\eta} |\nabla\phi|^2 dz + \frac{g}{2}\int \eta^2 dx \tag{3}$$

The potential $\phi(x,z,t)$ satisfies the Laplace equation:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

with the asymptotic boundary conditions:

$$\frac{\partial\phi}{\partial z} \to 0, \qquad \text{at } z \to -\infty.$$

If the steepness of surface is small, $|\eta_x| << 1$, the Hamiltonian can be presented by the infinite series

$$\begin{aligned}
H &= H_2 + H_3 + H_4 + \dots \\
H_2 &= \frac{1}{2}\int (g\eta^2 + \psi\hat{k}\psi)dx, \\
H_3 &= -\frac{1}{2}\int \{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx, \\
H_4 &= \frac{1}{2}\int \{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx
\end{aligned} \tag{4}$$

where $\hat{k}\psi$ means multiplication by $|k|$ in k-space ($\hat{k} = \sqrt{-\frac{\partial^2}{\partial x^2}}$).

Equations (1), although truncated according to (4), even for the full 3-D case, can be efficiently used for numerical simulations of water wave dynamics (see, for instance [Korotkevich et al. (2008)]). However, they are not convenient for analytic study because $\eta(x,t)$ and $\psi(x,t)$ are not"optimal" canonical variables. One can choose better Hamiltonian variables by performing a proper canonical transformation. This transformation is obtained in [Dyachenko et al. (2015)] and cancels nonresonant third and fourth order terms and simplifies the only remaining resonant fourth-order term in the Hamiltonian. This is possible due to an unexpected cancellation [Dyachenko and Zakharov (1994)] of nontrivial four-wave interactions. What we obtain as a result of this transformation is the so called “compact equation” [Dyachenko and Zakharov (2011), Dyachenko and Zakharov (2012)].

This equation was intensively used as a base for both numerical simulations [Fedele and Dutykh (2012a), Fedele and Dutykh (2012b), Dyachenko et al. (2013a), Dyachenko et al. (2014), Fedele (2014a), Fedele (2014b), Dyachenko et al. (2015), Dyachenko et al. (2015), Dyachenko et al. (2016a)] and analytical proof on non-integrability of Zakharov equation [Dyachenko et al. (2013b)].

However the most optimal (in our opinion) version of the compact equation which we call “the super compact equation” for water waves was presented in [Dyachenko et al. (2016b)].

Also the derivation of the spatial version of “the super compact equation” becomes remarkably straight-forward [Dyachenko and Zakharov (2016)].

2 Super Compact Equation for normal complex variable

So, instead of Hamiltonian (4) with pair of classical physical hamiltonian variables $\eta(x,t)$ and $\psi(x,t)$ we will use normal complex variable $c(x,t)$. Canonical transformation

$$\eta(x,t), \psi(x,t) \quad \Rightarrow \quad c(x,t) \tag{5}$$

is describe in detail in [Dyachenko et al. (2015), Dyachenko et al. (2016b)]. The canonical transformation (5) in $k-$ space is power series of c_k up the third order for η_k and ψ_k:

$$\eta_k = \eta_k^{(1)} + \eta_k^{(2)} + \eta_k^{(3)}, \qquad \psi_k = \psi_k^{(1)} + \psi_k^{(2)} + \psi_k^{(3)}. \tag{6}$$

Here we present only the linear and second order terms. All of these terms can be written in $k-$ space in a compact form.

$$\eta_k^{(1)} = \frac{1}{\sqrt{2\omega_k}}[c_k + c_{-k}^*], \qquad \psi_k^{(1)} = -i\sqrt{\frac{g}{2k\omega_k}}[c_k - c_{-k}^*]. \tag{7}$$

$$\begin{aligned}
\eta_k^{(2)} &= \frac{|k|}{4\sqrt{2g\pi}}\left[\int k_1^{-\frac{1}{4}} c_{k_1} k_2^{-\frac{1}{4}} c_{k_2} \delta_{k-k_1-k_2} dk_1 dk_2 \right.\\
&\left. + \int k_1^{-\frac{1}{4}} c_{k_1}^* k_2^{-\frac{1}{4}} c_{k_2}^* \delta_{k+k_1+k_2} dk_1 dk_2 - 2\int k_1^{-\frac{1}{4}} c_{k_1}^* k_2^{-\frac{1}{4}} c_{k_2} \delta_{k+k_1-k_2} dk_1 dk_2 \right],\\
\psi_k^{(2)} &= -\frac{i}{4\sqrt{2\pi}}\left[\int (\sqrt{k_1}+\sqrt{k_2}) k_1^{-\frac{1}{4}} c_{k_1} k_2^{-\frac{1}{4}} c_{k_2} \delta_{k-k_1-k_2} dk_1 dk_2 - \right.\\
&- \int (\sqrt{k_1}+\sqrt{k_2}) k_1^{-\frac{1}{4}} c_{k_1}^* k_2^{-\frac{1}{4}} c_{k_2}^* \delta_{k+k_1+k_2} dk_1 dk_2\\
&\left. - 2\mathbf{sign}(k)\int (\sqrt{k_1}+\sqrt{k_2}) k_1^{-\frac{1}{4}} c_{k_1}^* k_2^{-\frac{1}{4}} c_{k_2} \delta_{k+k_1-k_2} dk_1 dk_2 \right].
\end{aligned} \tag{8}$$

This is an important property that allows one to recover physical values without multidimensional integrals.

$$\eta^{(1)}(x) = \frac{1}{\sqrt{2}g^{\frac{1}{4}}}(\hat{k}^{-\frac{1}{4}}c(x) + \hat{k}^{-\frac{1}{4}}c(x)^*), \qquad \psi^{(1)}(x) = -i\frac{g^{\frac{1}{4}}}{\sqrt{2}}(\hat{k}^{-\frac{3}{4}}c(x) - \hat{k}^{-\frac{3}{4}}c(x)^*). \tag{9}$$

The operators $\hat{k}^\alpha$ act in the Fourier space as multiplication by $|k|^\alpha$.

$$\begin{aligned}
\eta^{(2)}(x) &= \frac{\hat{k}}{4\sqrt{g}}\left[\hat{k}^{-\frac{1}{4}}c(x) - \hat{k}^{-\frac{1}{4}}c(x)^*\right]^2,\\
\psi^{(2)}(x) &= \frac{i}{2}\left[\hat{k}^{-\frac{1}{4}}c(x)^*\hat{k}^{\frac{1}{4}}c(x)^* - \hat{k}^{-\frac{1}{4}}c(x)\hat{k}^{\frac{1}{4}}c(x)\right] +\\
&+ \frac{1}{2}\hat{H}\left[\hat{k}^{-\frac{1}{4}}c(x)\hat{k}^{\frac{1}{4}}c(x)^* + \hat{k}^{-\frac{1}{4}}c(x)^*\hat{k}^{\frac{1}{4}}c(x)\right],
\end{aligned} \tag{10}$$

Here, $\hat{H}$ is the Hilbert transformation with eigenvalue $i\mathbf{sign}(k)$. This accuracy (second-order power series) is sufficient to compare numerical data with the data in a flume.

As a result of the transformation two equations (1) with Hamiltonian (4) turn into one complex equation [Dyachenko et al. (2017)]

$$\frac{\partial c}{\partial t} + i\hat{\omega}_k c - i\partial_x^+ \left(|c|^2 \frac{\partial c}{\partial x} \right) = \partial_x^+(\mathcal{U}c). \tag{11}$$

Here

$$\begin{aligned}
\hat{\omega}_k &\to \sqrt{gk} \text{ in } k\text{-space},\\
\mathcal{U} &= \hat{k}|c|^2 \text{ advection velocity},\\
\hat{k} &\to |k| \text{ in } k\text{-space},\\
\partial_x^+ &\to ik\theta(k) \text{ in } k\text{-space, here } \theta(k) \text{ is the step function.}
\end{aligned}$$

Operator ∂_x^+ is the manifestation of unidirectional waves. Thus $c(x,t)$ is analytic function in the upper half-plane $(x+iy)$. Its Fourier series does not have negative harmonics. Equation (11) is dimensional one, $|c|^2$ has dimension of potential. The equation is equivalent to equations (1-4) with the same accuracy (due to truncation).

The Hamiltonian of the equation be written in x-space:

$$H = \int c^* \hat{V}_k c\, dx + \frac{1}{2}\int \left[\frac{i}{4}(c^2 \frac{\partial}{\partial x} c^{*2} - c^{*2} \frac{\partial}{\partial x} c^2) - |c|^2 \hat{k}(|c|^2) \right] dx \tag{12}$$

Here the operator $\hat{V}_k$ acts in k-space, so that $V_k = \frac{\omega_k}{k}$. Motion equation then is the following:

$$\frac{\partial c}{\partial t} + \partial_x^+ \frac{\delta H}{\delta c^*} = 0. \tag{13}$$

3 Envelope equation for water waves

Let us suppose that Fourier spectrum of waves has a maximum at $k = k_0$. Specific type of cubic nonlinearity in the super compact equation (11) allows to introduce envelope for waves, capital $C(x,t)$:

$$c(x,t) = C(x,t)\mathrm{e}^{i(k_0 x - \omega_{k_0} t)} \tag{14}$$

Actually it is canonical transformation. All nonlinear terms in the equation (11) contain $|c|^2$ only, then

$$|c|^2 = |C|^2$$

Fourier harmonics c_k and C_k are related as follows

$$C_k = c_{k_0+k} \qquad -k_0 < k < \infty$$

For the envelope (14) one can easily derive the *exact* equation without assumption of narrow bandwidth for C:

$$\frac{\partial C}{\partial t} + i\left[\hat{\omega}_{k_0+k} - \omega_{k_0} - \frac{\partial \omega_{k_0}}{\partial k_0}\hat{k}\right] C - i\frac{\partial^{k_0}}{\partial x}\left(|C|^2 \frac{\partial^{k_0}}{\partial x} C\right) = \frac{\partial^{k_0}}{\partial x}\left(\mathcal{U}C\right) \tag{15}$$

$$\frac{\partial^{k_0}}{\partial x} = (ik_0 + \frac{\partial}{\partial x})\hat{\theta}_{k_0+k} \qquad \Rightarrow \qquad i(k_0+k)\theta(k_0+k) \qquad \text{in } k\text{-space}$$

This equation is written in the framework that moves with group velocity $\frac{\partial \omega_{k_0}}{\partial k_0}$. This is why the last term in square brackets has appeared. One can use special notation for it:

$$\hat{\mathcal{D}}_k^{(2)} = \left[\hat{\omega}_{k_0+k} - \omega_{k_0} - \frac{\partial \omega_{k_0}}{\partial k_0}\hat{k}\right] \tag{16}$$

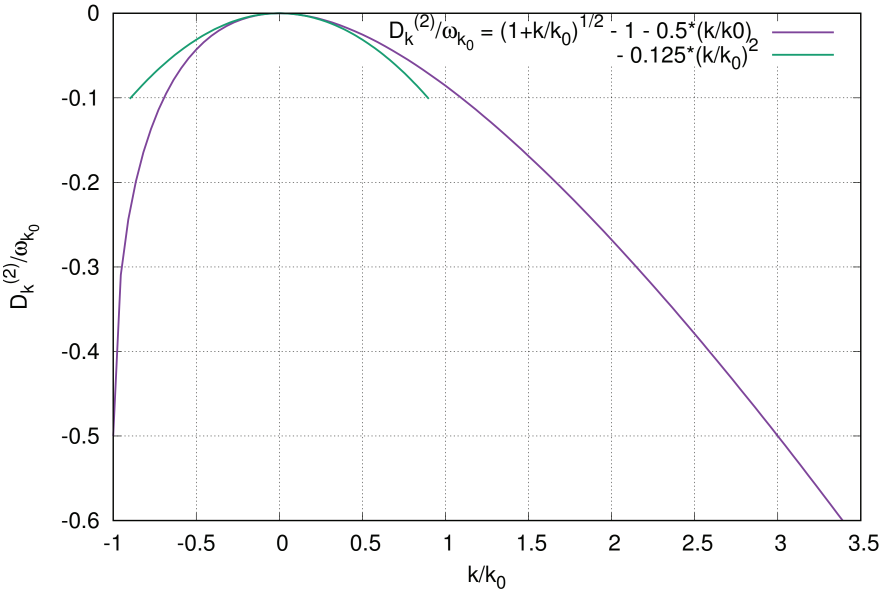

Fig. 1 $\frac{\mathcal{D}_k^{(2)}}{\omega_{k_0}}$(purple curve) versus $-\frac{k^2}{8k_0^2}$ (green curve) dependence on dimensionless wavenumber k/k_0

Being expanded in a Taylor series it is

$$\hat{\mathcal{D}}_k^{(2)} = -\frac{\omega_{k_0}}{8k_0^2}\hat{k}^2 + \ldots$$

This $\hat{\mathcal{D}}_k^{(2)}$ coincides with the corresponding term in the limit of Nonlinear Schrodinger Equation. The difference between the NLSE and (16) is shown in the Figure 1. The equation for envelope (15) is Hamiltonian, and the Hamiltonian is:

$$\mathcal{H} = \int C^* \hat{V}_k C\, dx + \frac{1}{2}\int |C|^2 \left[k_0|C|^2 + \frac{i}{2}(CC'^* - C^*C') - \hat{k}|C|^2\right] dx$$
$$V_k = \frac{\mathcal{D}_k^{(2)}}{k_0 + k}$$

Corresponding equation of motion is the following

$$\begin{aligned}&\frac{\partial C}{\partial t} + \frac{\partial^{k_0}}{\partial x}\frac{\delta \mathcal{H}}{\delta C^*} = 0\\ &\frac{\delta \mathcal{H}}{\delta C^*} = \hat{V}C + k_0|C|^2C - i|C|^2C' - \mathcal{U}C \qquad \mathcal{U} = \hat{k}|C|^2.\end{aligned}$$

This is exact equation for envelope of water waves. Along with the Hamiltonian it conserves number of waves and momentum. One can extract the NLSE and the Dysthe [Dysthe (1979)] equations from (15). Indeed let us write it the following way:

$$\begin{aligned}\frac{\partial C}{\partial t} &+ i\hat{\mathcal{D}}_k^{(2)}C + ik_0^2\hat{\theta}_{k_0+k}\left[|C|^2C\right] + && \text{NLSE}\\ &+ k_0\hat{\theta}_{k_0+k}\left[C\frac{\partial}{\partial x}|C|^2 + 2|C|^2\frac{\partial C}{\partial x} - i\mathcal{U}C\right] - && \text{Dysthe}\\ &- \frac{\partial}{\partial x}\left[\mathcal{U}C + i|C|^2\frac{\partial C}{\partial x}\right] = 0 && \text{exact} \qquad (17)\end{aligned}$$

Equation (15) or (17) describes two phenomena – nonlinear waves and advection. Advection velocity is equal to

$$\mathcal{V} = \left[3k_0|C|^2 - \mathcal{U}\right] \qquad \text{(this is the coefficient for } C'\text{)} \qquad (18)$$

Not being a constant this velocity may result in wave breaking.

We want to stress that equation (17) although is the envelope equation, is valid without usual narrow-banded spectrum approximation.

In addition to the Hamiltonian equation (17) conserves two quantities

$$N = \int_{-k_0}^{\infty} \frac{|C_k|^2}{k_0 + k}dk \qquad \text{number of waves}$$

$$P = \int_{-k_0}^{\infty} |C_k|^2 dk \qquad \text{which has dimension of momentum.}$$

4 Numerical studying of pre-breaking wave

In this section we study in detail initial stage of wave-breaking in the framework of equation (17). The initial conditions were the following:

- 10km of periodic domain
- uniform wave train with wavelength 100m (slightly perturbed)
- steepness of waves $\mu \sim 0.1$
- group velocity - 6.24 m/sec.

Fourier spectrum C_k has a sharp maximum at $k = 0$.

Modulational instability results in appearing of extreme (pre-breaking) wave.

Fourier spectrum of this wave is shown in the Figure 2. One can see exponential behavior at large k. Slope of the exponent corresponds to the distance from real axis x to the nearest singularity in the lower half-plane $x + iy$. This singularity moves closer and closer to the axis during pre-breaking stage. At this moment

$$C_k \sim e^{-ak} \quad a \simeq 3m.$$

Breaking of wave is defined by advection velocity (18). Being non constant it results in crossing of characteristics in pure advection equation. Although equation (17) is more complicated, it has advection part. So, that sharp gradient of advection velocity $\mathcal{V}$ indicates on pre-breaking stage. Breaking criteria based on advection velocity has been used in [Bjorkavas et al. (2011)] for

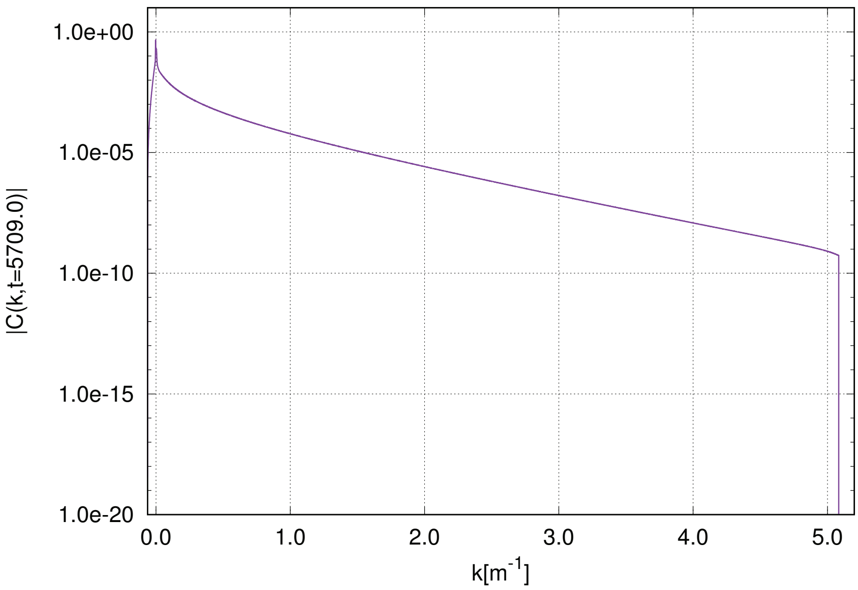

Fig. 2 Fourier spectrum of pre-breaking wave.

Boussinesq models and shown good qualitative agreement with the experiments.

Nonlocal advection velocity $\mathcal{U}$ and $3k_0|C|^2$ are shown in the Figure 3. Maximal value of $\mathcal{U}$ becomes large than half of group velocity.

Fourier spectrum $|C_k|$ is show in the Figure 4. One can see that right satellite (grown up due to modulational instability) is almost absent. It is this satellite has produced exponential "tail" in the Figure 2. *During pre-breaking stage part of low k component of energy (or approaching singularity) is transferred to high wave numbers. And later it dissipates. It dissipates indirectly.*

At this moment we simulate dissipation due to wave breaking in the following way:

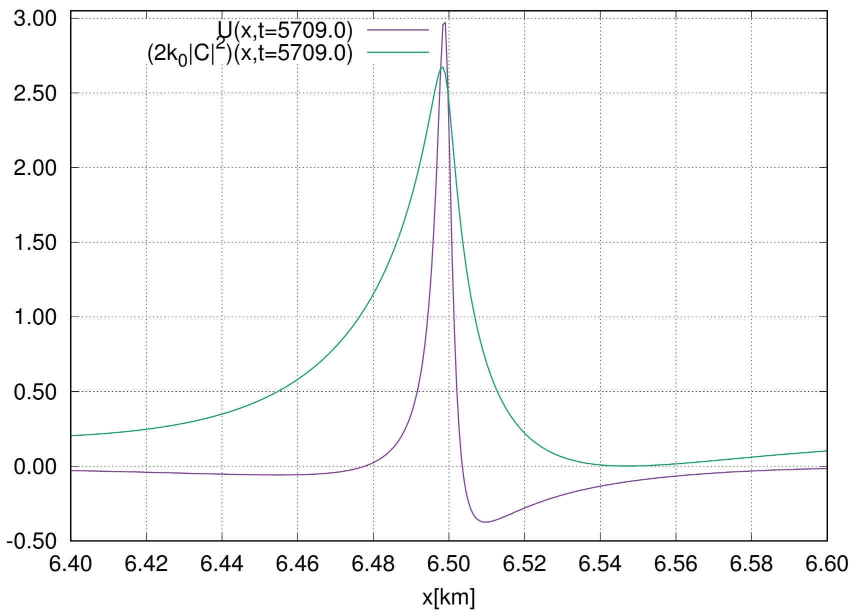

Fig. 3 Nonlocal advection velocity and envelope before breaking. y-axis shows velocity (m/sec)

– Evolution of Fourier spectrum obeys the equation $\frac{\partial}{\partial\tau}C_k + \gamma_k C_k = 0$

– $\gamma_k \simeq vk$ at large k and close to zero at small k

Such choice of dissipation provides damping of high Fourier harmonics of C_k

$$C_k \sim e^{-ak} \Rightarrow \sim e^{-(\underline{a+v\tau})k}$$

In the Figure 5 one can see result of such damping for $v\tau = 40m$. Detailed view of this spectra in the energy containing region (small k) is shown in the Figure 6. As a result peak of the spectrum is shifted to the smaller wavenumbers. Proposed model for damping "remove" only pre-breaking wave leaving unchanged other part of the surface. It is seen in the Figure 7 and Figure 8, where real physical surface η is shown before and after damping.

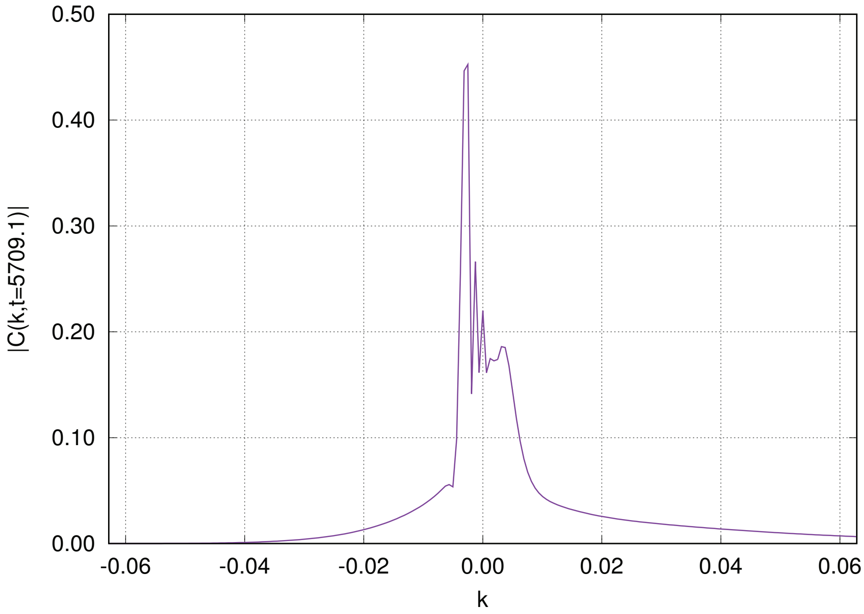

Fig. 4 Fourier spectrum $|C_k|$ before breaking

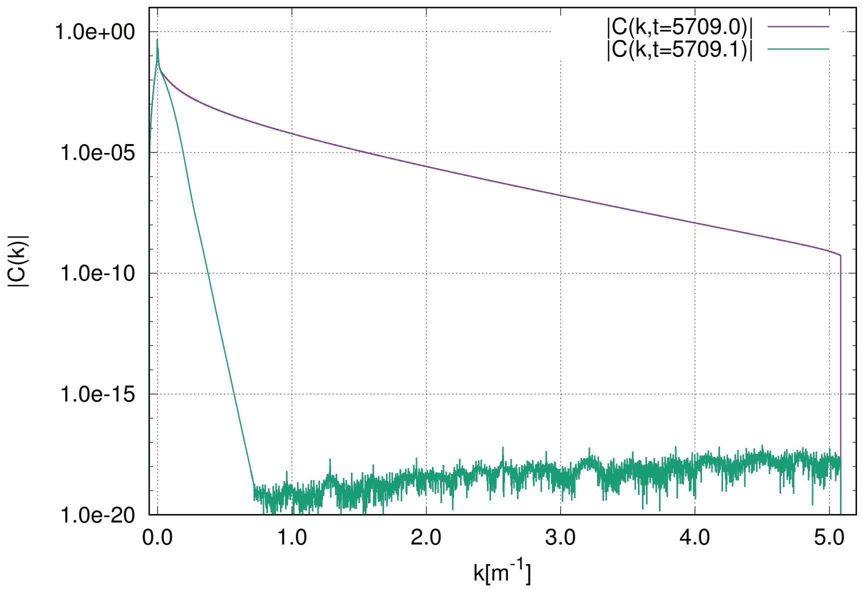

Fig. 5 Fourier spectra before and after damping

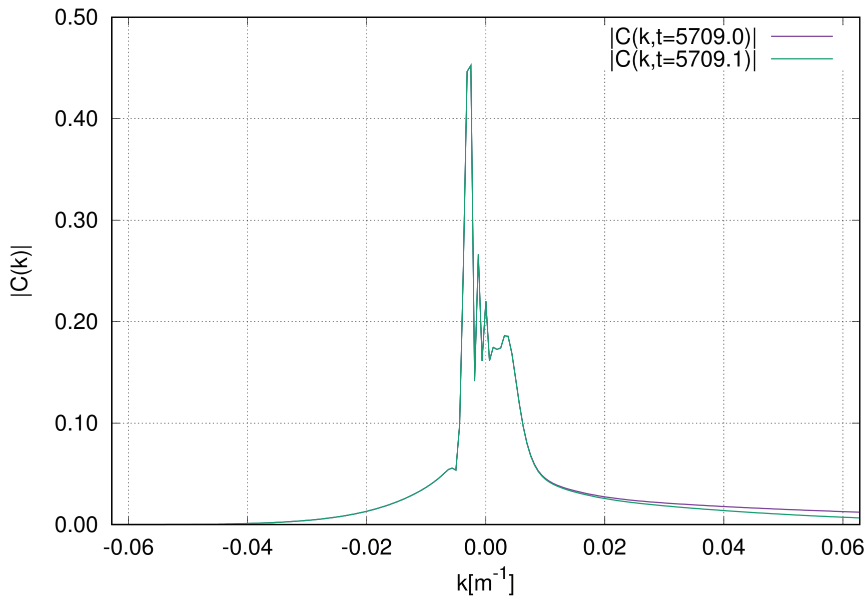

Fig. 6 Fourier spectra before (violet) and after damping (green). Energy containing domain.

We want to emphasize again that energy from low wave numbers was already transferred to high k. And at this moment model damping is "switched on". This damping provides regularization of the envelope equation. It eliminates breaking wave (or singularity at the surface). No other dissipative terms were used in the equation.

5 Numerical simulation of soliton turbulence

Using equation (17) we perform numerical simulation of long time evolution with the initial conditions given in the beginning of the Section 4. Initially almost homogeneous wave train after ~ 3.5 hours splits into few dozens of solitons. All of them have approximately the same width, see Figure 9. Each

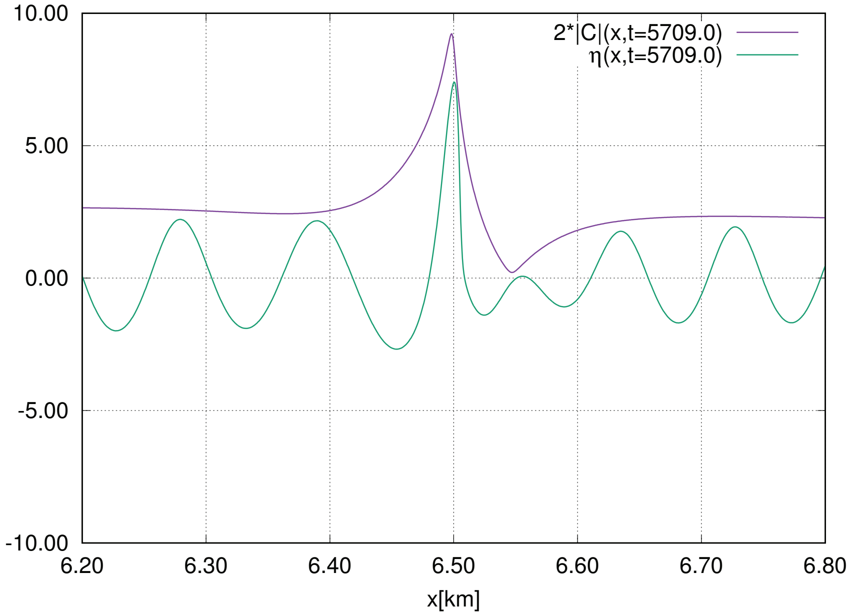

Fig. 7 $|C|$ and η before damping. y-axis shows height (m)

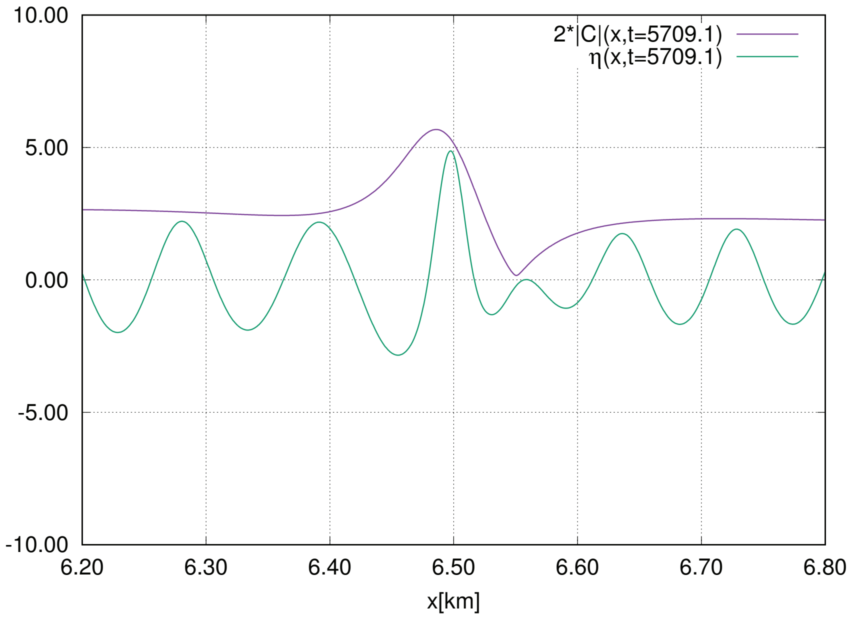

Fig. 8 $|C|$ and η after damping. y-axis shows height (m)

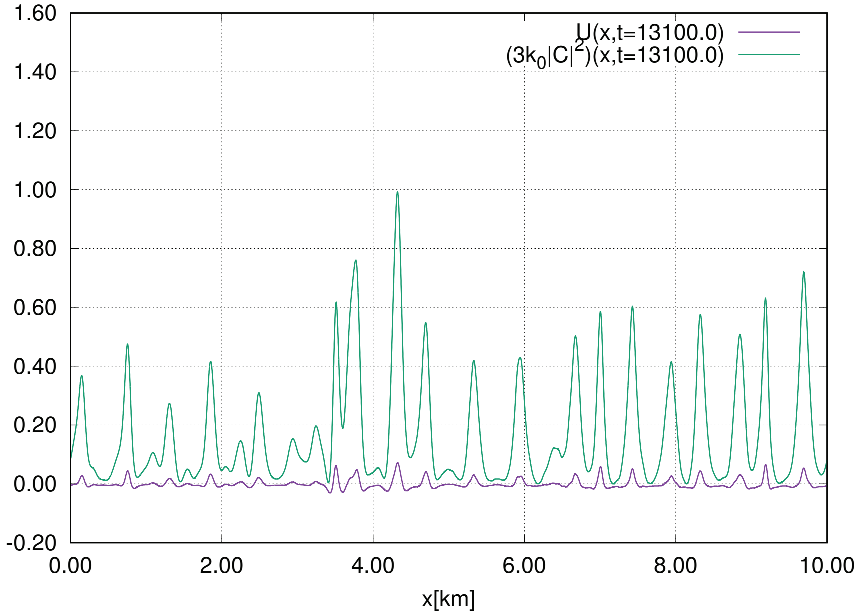

Fig. 9 Wave train splits into set of solitons. Nonlocal advection velocity is also shown.y-axis shows velocity (m/sec)

soliton has about 3 real waves under the envelope (30 solitons and 100 waves). It corresponds well-known phenomenon 3 sisters.

Solitons have different velocities, they collide exchanging the energy. After a long time, ~ 1.5 days, picture of turbulence looks different. One can observed in the Figure 10 four large solitons which obviously took energy from the weaker ones. One of this soliton, near point 3.3 km, is shown in the Figure 11. Here real physical surface $\eta(x)$ is shown. Obviously this soliton consist of 3 waves, and the highest crest is about 5m. During this time, 1.5 days, a dozen of wave breaking took place. The time evolution of integrals of motion demonstrates this. In the Figure 12 one can see evolution of "wave action" or "number of par-

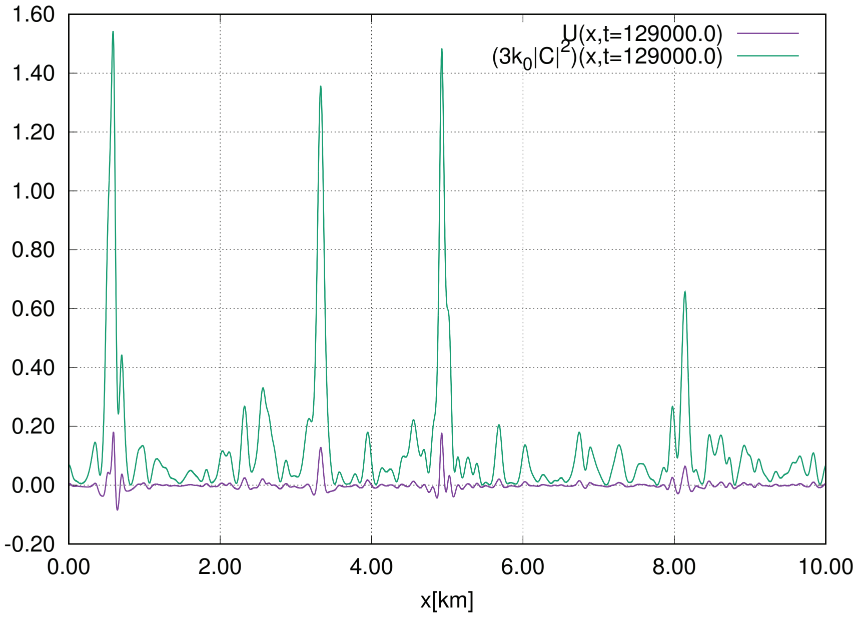

Fig. 10 Few large solitons. Nonlocal advection velocity is also shown. y-axis shows velocity (m/sec)

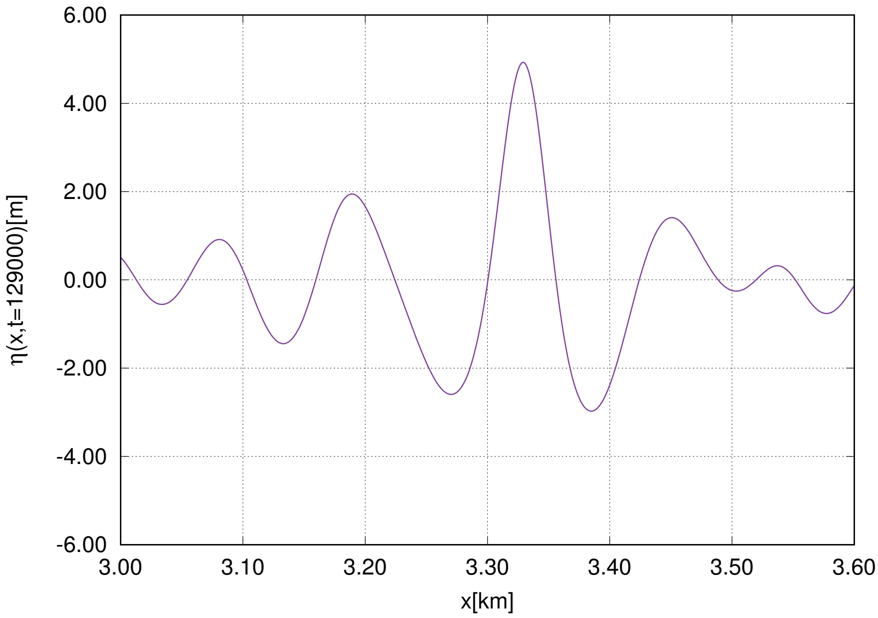

Fig. 11 Three large waves in the soliton - 3 sisters.

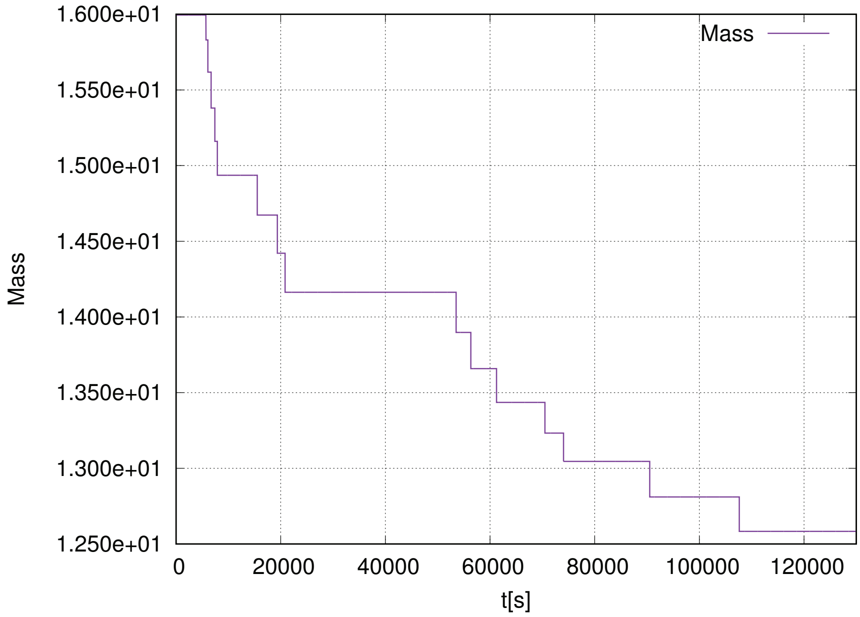

Fig. 12 Evolution of "wave action" in soliton turbulence.

ticles". It is seen sharp drops of it when wave breaking takes place. Full movie of the soliton turbulence is available at http://alexd.itp.ac.ru/2k0C2U.avi

6 Conclusion

In this work we have attempted to improve the understanding of the water waves on the surface of ideal fluid.

First we derived exact new equation for envelope without narrow band assumption. This is the main difference between envelope equation for water waves derived in [Craig et al (2010)] where Hamiltonian approach was also used. *It can describe very narrow solitons or breathers along with pre-breaking stage. At the same time numerical simulations in the framework of this equa-*

tion is so simple as for Nonlinear Schrodinger or Dysthe equations. Another words, computational efficiency of envelope equation (17) is the same for equations mentioned above. Using this equation we studied initial stage of wave breaking in detail.

Being unavoidable phenomenon on the surface of the ocean, wave breaking is very difficult to describe or simulate. Wave breaking change the topology of the surface of the ocean. Here, in this article, we propose simple model of dissipation of such waves. This dissipation acts only in the point of breaking and does not affect other part of water surface. Numerical simulation shows reasonable behavior of integrals of motion, they drop by fixed value. After each event of this model breaking the surface again becomes smooth.

Numerical simulation of soliton turbulence have shown decay of homogeneous wave train into set of solitons. These solitons are very narrow, there are about three waves under the envelope (three sisters). Probably it is the limiting size of soliton on the deep water, when the solitary wave solution does not exist.

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