On the dynamics of an ideal fluid with the free surface

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Abstract

Exact cubic equations are derived for two-dimensional flows of fluid with the free surface. The equations involve inverse derivative of conformal mapping of the domain occupied by fluid to the lower half-plane, and complex velocity of the fluid. They are hydrodynamic-type equations and describe transport of singularities of conformal mapping in the upper half-plane.

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1 Physical (natural) variables

The irrotational motion of an incompressible inviscid fluid with a free surface is a very basic problem in fluid dynamics. The equations describing the motion were first formulated by Stokes in 1845. In this article two-dimensional flows on the plane are considered, that allows to apply conformal mapping theory.

The velocity potential ϕ of the fluid satisfies the equation

$$\triangle\phi = 0$$

in the domain occupied by fluid. The domain is bounded by the free surface $y = \eta(x,t)$ with the boundary conditions for velocity potential and moving free surface:

$$\begin{aligned}\frac{\partial\phi}{\partial t} &+ \frac{1}{2}|\nabla\phi|^2 + g\eta = P,\\ \frac{\partial\eta}{\partial t} &+ \eta_x\phi_x = \phi_y\end{aligned} \tag{1.1}$$

at $y = \eta(x,t)$ and

$$\frac{\partial\phi}{\partial y} = 0, y \to -\infty,$$

$$\frac{\partial \phi}{\partial x} = 0, |x| \to \infty. \tag{1.2}$$

Here g is the gravity acceleration and P - constant pressure at the surface (Let $P = 0$). The system (1.1) and (1.2) is a Hamiltonian one with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\eta(x,t)} |\nabla\phi|^2 dy \ + \ \frac{1}{2}\int_{-\infty}^{\infty} g\eta^2(x,t)dx \tag{1.3}$$

and canonically conjugated variables $\psi(x,t) = \phi(x, \eta(x,t), t)$ and $\eta(x,t)$ (see [1]):

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{\partial \mathcal{H}}{\partial \psi}, \\ \frac{\partial \psi}{\partial t} &= -\frac{\partial \mathcal{H}}{\partial \eta}. \end{aligned} \tag{1.4}$$

Along with the Hamiltonian there are three more integrals of motion, amount of fluid:

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \eta(x,t)dx = 0, \tag{1.5}$$

and vertical and horizontal momenta:

$$\begin{aligned} \frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\eta(x,t)} \phi_y dy &= 0, \\ \frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\eta(x,t)} \phi_x dy &= 0. \end{aligned} \tag{1.6}$$

The equations (1.1) and (1.2) are functionally nonlinear and can be hardly studied. The most important known solutions were derived by Dirichlet in 1960, and they are ellipsoid (or ellipse in 2D case), hyperbola and parabola. Dirichlet solutions are described in detail in[2].

2 Conformal variables

To simplify the equations (1.1) and (1.2) let us perform conformal mapping (see[3]) of the domain in z-plane occupied by fluid to the lower half-plane of a complex variable $w = u + iv$

$$-\infty < u < \infty, \qquad -\infty < v < 0.$$

Now the shape of the surface is given parametrically by real and imaginary parts of conformal mapping given on real axis:

$$y = y(u,t), \quad x = x(u,t) = u + \tilde{x}(u,t), \tag{2.7}$$

here $\tilde{x}(u,t)$ and $y(u,t)$ are related through Hilbert Transformation

$$y = \hat{H}\tilde{x}, \qquad \tilde{x} = -\hat{H}y, \qquad \hat{H}^2 = -1$$

and

$$\hat{H}(f(u)) = P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u')du'}{u'-u}.$$

After conformal mapping $\phi(x,y,t) \to \phi(u,v,t)$. Let $\Psi(u,t) = \phi(u,0,t)$, the potential on the surface. It was shown in[3] that $y(u,t)$ and $\Psi(u,t)$ obey the following system of equations:

$$y_t = (y_u\hat{H} - x_u)\frac{\hat{H}\Psi_u}{J} \tag{2.8}$$

$$\Psi_t = -\frac{\Psi_u^2 + \hat{H}\Psi_u^2}{2J} + \hat{H}\left(\frac{\hat{H}\Psi_u}{J}\right)\Psi_u + \frac{\hat{H}\Psi_u}{J}\hat{H}\Psi_u - gy \tag{2.9}$$

here g is gravity acceleration, and

$$J = x_u^2 + y_u^2 = 1 + 2\tilde{x}_u + \tilde{x}_u^2 + y_u^2. \tag{2.10}$$

Another form of the equation (2.8) and (2.9) (see[3]) be

$$\begin{aligned} y_t x_u - x_t y_u &= -\hat{H}\Psi_u, \\ \Psi_t x_u - x_t\Psi_u + gyx_u &= \hat{H}\left(\Psi_t y_u - y_t\Psi_u + gyy_u\right) \end{aligned} \tag{2.11}$$

The integrals of motion acquire the following form:

$$\begin{aligned} \mathcal{H} &= -\int_{-\infty}^{\infty}\Psi\hat{H}\Psi_u du + \frac{g}{2}\int_{-\infty}^{\infty}y^2x_u du, \\ M &= \int_{-\infty}^{\infty}yx_u du, \\ P_y &= \int_{-\infty}^{\infty}\Psi x_u du, \\ P_x &= \int_{-\infty}^{\infty}\Psi y_u du. \end{aligned} \tag{2.12}$$

When dealing with conformal mapping $z(w,t)$ and velocity potential it is convenient to write down instead the real equations (2.8) and (2.9) the complex equations for $z(w,t)$ and complex velocity potential $\Phi(w,t)$. One can get these new equations applying projector operator $\hat{P}$

$$\hat{P}(f) = \frac{1}{2}(1 + i\hat{H})(f)$$

to the equations (2.8) and (2.9):

$$\begin{aligned} z_t &= iUz_u, \\ \Phi_t &= iU\Phi_u - \hat{P}\left(\frac{|\Phi_u|^2}{|z_u|^2}\right) + ig(z-u). \end{aligned} \tag{2.13}$$

Here U is a complex transport velocity:

$$U = \hat{P}\left(\frac{-\hat{H}\Psi_u}{|z_u|^2}\right).$$

It occurs that equations (2.13) can be simplified just by changing variables. Indeed, let us introduce instead of $z(w,t)$ and $\Phi(w,t)$ another functions $R(w,t)$ and $V(w,t)$ in a following way

$$
\begin{aligned}
R &= \frac{1}{z_w}, \\
\Phi_w &= -iVz_w.
\end{aligned}
\tag{2.14}
$$

(V is just $i\frac{\partial\Phi}{\partial z}$, i.e. complex velocity). Note, that because of $z(w,t)$ is conformal mapping, its derivative does exist in lower half-plane and does not have zeroes in there. Thus function $R(w,t)$ is analytic in the lower half-plane and has the following boundary condition:

$$R(w,t) \to 1, \quad |w| \to \infty, \quad Im(w) \le 0.$$

It is obvious that boundary condition for V is:

$$V(w,t) \to 0, \quad |w| \to \infty, \quad Im(w) \le 0.$$

Then for these analytic functions equations acquire very nice form:

$$
\begin{aligned}
R_t &= i(UR' - U'R), \\
V_t &= i(UV' - R\hat{P}'(V\bar{V})) + g(R-1).
\end{aligned}
\tag{2.15}
$$

Here

$$U = \hat{P}(V\bar{R} + \bar{V}R).$$

The equations are cubically nonlinear and include linear integral operator. They are similar to hydrodynamics type equations, but given in the complex plane. Important role there plays complex transport velocity U. It conveys zeroes of $R(w,t)$, which are singular points of conformal mapping $z(w,t)$. Possible types of zeroes need to be study in the next article. Here it should be just mentioned that in the numerical simulations only $\sqrt{w}$ branch points are observed.

It is easy to include surface tension in the equations (2.15), one has just to replace the second equation by

$$V_t = i(UV' - R\hat{P}'(V\bar{V}) + g(R-1)) - 2\sigma R\hat{P}'(Q'\bar{Q} - \bar{Q}'Q),$$

where σ is surface tension coefficient, and $Q = \sqrt{R}$.

Equations (2.15) keep the same form for other boundary conditions. Instead of boundary conditions (1.2) one can consider periodic boundaries as well as fluid of the finite depth.

Note that equations include now only derivatives of conformal mapping and complex velocity potential. As regards integrals of motion they acquire more complicated form. But if one restores complex velocity potential

$$\Phi = -i\int \frac{V}{R}dw$$

then Hamiltonian is equal to

$$\mathcal{H} = -\int_{-\infty}^{\infty} Re(\Phi)Im(\Phi')$$

and momenta

$$P_y = \int_{-\infty}^{\infty} Re(\Phi)Re(\frac{1}{R})du, \quad P_x = \int_{-\infty}^{\infty} Re(\Phi)Im(\frac{1}{R})du.$$

3 Conclusions

The main statement of the article is the following. The well-known equations for flows of fluid with the free surface (1.1) are equivalent to the cubic equations (2.15) written for the inverse derivative of the conformal mapping $R(w,t) = 1/z'(w,t)$ and fluid velocity $V(w,t)$. These equations are of hydrodynamic-type and describe transport of singularities of conformal mapping in the upper half-plane. And moving singularities define the shape of the surface.

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