Integration of the Nonlinear Schrödinger Equation with Periodic Boundary Conditions

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Summary. Consider 1D NLSE for the field $q(x,t)$:

$$i\dot{q} + \frac{1}{2}q_{xx} + |q|^2 q = 0$$

with periodic boundary condition in space. For Fourier harmonics:

$$i\dot{q}_n - \frac{1}{2}k_n^2 q_n + \sum_{n_1,n_2,n_3} q^*_{n_1} q_{n_2} q_{n_3} \delta_{n+n_1,n_2+n_3} = 0.$$

$$q(x,t) = \sum_n q_n(t) e^{ik_n t}, \quad q_n(t) = \frac{1}{L}\int_0^L q(x,t) e^{-ik_n t} dt, \text{ here } k_n = \frac{2\pi n}{L}.$$

In search of some canonical transformation

$$q_n \to b_n$$

Linear Schrodinger Equation | **New Hamiltonian**

$$i\dot{b}_n - \frac{1}{2}k_n^2 b_n + \Omega_n b_n = 0$$

$$H = \frac{1}{2}\sum_n k_n^2 |b_n|^2 - \frac{1}{2}\sum_{n,n_1} D_{nn_1} |b_n|^2 |b_{n_1}|^2$$

Ω_n **does** depend on n and initial conditions and **does not** depend on t

Resonant and Nonresonant Interactions

The nonlinear part of the Hamiltonian is divided into two parts,

Resonant and Nonresonant.

$$H = \frac{1}{2}\sum_{n} k_n^2 |q_n|^2 - \frac{1}{2}\sum_{n,n_1,n_2,n_3} q_n^* q_{n_1}^* q_{n_2} q_{n_3} \delta_{n+n_1,n_2+n_3}$$

Resonant R_4 is:

$$R_4 \Leftarrow \begin{cases} k_n^2 + k_{n_1}^2 - k_{n_2}^2 - k_{n_3}^2 = 0 \\ n + n_1 - n_2 - n_3 = 0. \end{cases}$$

$$\sum_{n,n_1,n_2,n_3} q_n^* q_{n_1}^* q_{n_2} q_{n_3} \delta_{n+n_1,n_2+n_3} = \sum_{n_i \in R^4} q_n^* q_{n_1}^* q_{n_2} q_{n_3} + \cancel{\sum_{n_i \notin R^4} q_n^* q_{n_1}^* q_{n_2} q_{n_3}}$$

Of course, the canonical transformation generates additional, multiple wave interactions terms in the Hamiltonian.

Auxiliary Hamiltonian

For Hamiltonian system with variable $b_n(t)$ the transformation

$$b_n(0) \rightarrow b_n(t)$$

is canonical. We will construct this transformation using an auxiliary Hamiltonian $\tilde{H}$(similar to Zakharov, Lvov and Falkovich, 1992)

$$\tilde{H} = \frac{i}{2} \sum_{n,n_1,n_2,n_3} B^{nn_1}_{n_2 n_3} b^*_n b^*_{n_1} b_{n_2} b_{n_3} \delta_{n+n_1,n_2+n_3} +$$
$$+ \frac{i}{3} \sum_{n,n_1,n_2,n_3,n_4,n_5} C^{nn_1 n_2}_{n_3 n_4 n_5} b^*_n b^*_{n_1} b^*_{n_2} b_{n_3} b_{n_4} b_{n_5} \delta_{n+n_1+n_2,n_3+n_4+n_5} + \dots$$

Coefficients in the Hamiltonian have the following properties:

1. Permutations of upper and lower indexes do not change their values
2. $$B^{nn_1}_{n_2 n_3} = -B^{n_2 n_3}_{nn_1} \qquad C^{nn_1 n_2}_{n_3 n_4 n_5} = -C^{n_3 n_4 n_5}_{nn_1 n_2}$$
3. $$B^{nn_1}_{nn_1} = 0 \qquad C^{nn_1 n_2}_{nn_1 n_2} = 0$$

Canonical Transformation

This Hamiltonian helps to calculate coefficients of the series of canonical transformation. One can express old variables q_n in terms of $b_n(0)$:

$$b_n(t) = b_n(0) + t\dot{b}_n(0) + \frac{t^2}{2}\ddot{b}_n(0) + \dots \qquad \dot{b}_n(0) = -i\frac{\delta \tilde{H}}{\delta b_n^*} \qquad \ddot{b}_n(0) = -i\frac{\partial}{\partial t}\frac{\delta \tilde{H}}{\delta b_n^*}$$

$$q_n(0) \quad \Rightarrow \quad b_n(0) \qquad\qquad t = 1$$

The general form of the canonical transformation in term of power series:

$$\begin{aligned}
q_n = b_n &+ \sum_{n_1,n_2,n_3} B^{nn_1}_{n_2n_3} b^*_{n_1} b_{n_2} b_{n_3} \delta_{n+n_1,n_2+n_3} + \\
&+ \frac{1}{2}\sum_{n,n_1,n_2,n_3,n_4,n_5} {}^2B^{nn_1n_2}_{n_3n_4n_5} b^*_{n_1} b^*_{n_2} b_{n_3} b_{n_4} b_{n_5} \delta_{n+n_1+n_2,n_3+n_4+n_5} + \\
&+ \sum_{n,n_1,n_2,n_3,n_4,n_5} C^{nn_1n_2}_{n_3n_4n_5} b^*_{n_1} b^*_{n_2} b_{n_3} b_{n_4} b_{n_5} \delta_{n+n_1+n_2,n_3+n_4+n_5} + \dots
\end{aligned}$$

We seek for the best choice for $B^{nn_1}_{n_2n_3}$, $C^{nn_1n_2}_{n_3n_4n_5}$, etc.

Resonant Manifolds

The Hamiltonian takes (after symmetrization) the following form:

$$
\begin{aligned}
H &= \frac{1}{2}\sum_n k_n^2|b_n|^2 - \frac{1}{2}\sum_{n,n_1,n_2,n_3} T_{n_2 n_3}^{n n_1} b_n^* b_{n_1}^* b_{n_2} b_{n_3} \underline{\delta_{n+n_1,n_2+n_3}} - \\
&- \frac{1}{6}\sum_{n,n_1,n_2,n_3,n_4,n_5} T_{n_3 n_4 n_5}^{n n_1 n_2} b_n^* b_{n_1}^* b_{n_2}^* b_{n_3} b_{n_4} b_{n_5} \underline{\delta_{n+n_1+n_2,n_3+n_4+n_5}} + \dots
\end{aligned}
$$

Here

$$
\begin{aligned}
T_{n_2 n_3}^{n n_1} &= 1 - \frac{1}{2}(k_n^2 + k_{n_1}^2 - k_{n_2}^2 - k_{n_3}^2) B_{n_2 n_3}^{n n_1} \\
T_{n_3 n_4 n_5}^{n n_1 n_2} &= R_{n_3 n_4 n_5}^{n n_1 n_2} - (k_n^2 + k_{n_1}^2 + k_{n_2}^2 - k_{n_3}^2 - k_{n_4}^2 - k_{n_5}^2) C_{n_3 n_4 n_5}^{n n_1 n_2}
\end{aligned}
$$

$$
R_4 \Leftarrow \begin{cases} k_n^2 + k_{n_1}^2 - k_{n_2}^2 - k_{n_3}^2 = 0 \\ n + n_1 - n_2 - n_3 = 0. \end{cases}
$$

$$
R_6 \Leftarrow \begin{cases} k_n^2 + k_{n_1}^2 + k_{n_2}^2 - k_{n_3}^2 - k_{n_4}^2 - k_{n_5}^2 = 0 \\ n + n_1 + n_2 - n_3 - n_4 - n_5 = 0. \end{cases}
$$

$T^{nn_1}_{n_2n_3} = 1 - \frac{1}{2}(k_n^2 + k_{n_1}^2 - k_{n_2}^2 - k_{n_3}^2)B^{nn_1}_{n_2n_3}$

$$\sum_{n,n_1,n_2,n_3} T^{nn_1}_{n_2n_3} b_n^* b_{n_1}^* b_{n_2} b_{n_3} = \sum_{n_i \in R^4} T^{nn_1}_{n_2n_3} b_n^* b_{n_1}^* b_{n_2} b_{n_3} + \cancel{\sum_{n_i \notin R^4} T^{nn_1}_{n_2n_3} b_n^* b_{n_1}^* b_{n_2} b_{n_3}}$$

The best choice for $B^{nn_1}_{n_2n_3}$ is:

$$B^{nn_1}_{n_2n_3} = \frac{2 \cdot 1}{k_n^2 + k_{n_1}^2 - k_{n_2}^2 - k_{n_3}^2} \qquad \text{if } n_i \notin R^4,$$
$$B^{nn_1}_{n_2n_3} = 0 \qquad \text{if } n_i \in R^4$$

R^4 has only trivial solutions:

$$n = n_2, \quad n_1 = n_3 \qquad\qquad n = n_3, \quad n_1 = n_2$$

$$T^{nn_1}_{n_2n_3} \quad \Rightarrow D_{nn_1} \quad = \begin{cases} 2 & \text{if } n \neq n_1 \leq 0, \\ 1 & \text{if } n = n_1. \end{cases}$$

Note: $B^{nn_1}_{n_2n_3}$ satisfies symmetry conditions and has no singularities.

$$T^{nn_1n_2}_{n_3n_4n_5} = R^{nn_1n_2}_{n_3n_4n_5} - (k_n^2 + k_{n_1}^2 + k_{n_2}^2 - k_{n_3}^2 - k_{n_4}^2 - k_{n_5}^2)C^{nn_1n_2}_{n_3n_4n_5}$$

$$\begin{aligned}\sum_{n_i} T^{nn_1n_2}_{n_3n_4n_5} b_n^* b_{n_1}^* b_{n_2}^* b_{n_3} b_{n_4} b_{n_5} &= \sum_{n_i \in R^6} T^{nn_1n_2}_{n_3n_4n_5} b_n^* b_{n_1}^* b_{n_2}^* b_{n_3} b_{n_4} b_{n_5} + \\ &+ \cancel{\sum_{n_i \notin R^6} T^{nn_1n_2}_{n_3n_4n_5} b_n^* b_{n_1}^* b_{n_2}^* b_{n_3} b_{n_4} b_{n_5}}\end{aligned}$$

The best choice for $C^{nn_1n_2}_{n_3n_4n_5}$ is:

$$\begin{aligned} C^{nn_1n_2}_{n_3n_4n_5} &= \frac{R^{nn_1n_2}_{n_3n_4n_5}}{k_n^2 + k_{n_1}^2 + k_{n_2}^2 - k_{n_3}^2 - k_{n_4}^2 - k_{n_5}^2} \qquad &\text{if } n_i \notin R^6, \\ C^{nn_1n_2}_{n_3n_4n_5} &= 0 &\text{if } n_i \in R^6 \end{aligned}$$

R^6 **has** nontrivial solutions.

$$T^{nn_1n_2}_{n_3n_4n_5} = \begin{cases} R^{nn_1n_2}_{n_3n_4n_5} & \text{if } n_i \in R^6, \\ 0 & \text{if } n_i \notin R^6. \end{cases}$$

Note: $C^{nn_1n_2}_{n_3n_4n_5}$ satisfies symmetry conditions and has no singularities.

New Hamiltonian

Six waves interactions

$$R^{nn_1 n_2}_{n_3 n_4 n_5} \equiv 0 \qquad \text{if } n_i \in R^6$$

(Dyachenko, Kachulin, Zakharov, JETP Letters, 2013))

$$H = \frac{1}{2}\sum_{n} k_n^2 |b_n|^2 - \frac{1}{2}\sum_{n,n_1} D_{nn_1}|b_n|^2|b_{n_1}|^2 - \cancel{\frac{1}{6}\sum_{n_i \in R^6} R^{nn_1 n_2}_{n_3 n_4 n_5} b^*_n b^*_{n_1} b^*_{n_2} b_{n_3} b_{n_4} b_{n_5}} + \ldots$$

Here

$$\sum_{n_1} D_{nn_1}|b_{n_1}|^2 = \Omega_n$$

does not depend on t and is defined by initial conditions only.

New Hamiltonian

$$H = \frac{1}{2}\sum_n k_n^2 |b_n|^2 - \frac{1}{2}\sum_{n,n_1} D_{nn_1} |b_n|^2 |b_{n_1}|^2$$

and the equation for b:

$$i\dot{b}_n - \frac{1}{2}k_n^2 b_n + \Big[\sum_{n_1} D_{nn_1} |b_{n_1}|^2\Big] b_n = 0$$

The solution of this equation is simple (action-angle):

$$b_n(t) = b_n(0) e^{i(\Omega_n - \frac{1}{2}k_n^2)t}$$

Applicability

Modulational instability

$$q = q_0 + \delta q$$

For Fourier harmonics:

$$\delta q_n \sim e^{\gamma_n t}$$

growth rate

$$\gamma_n = k_n \sqrt{2|q_0|^2 - k_n^2}$$

$$q_n^2 < \Delta k^2$$

$$i\dot{q} + \frac{1}{2} q_{xx} + |q|^2 q = 0$$

For Fourier harmonics:

$$i\dot{q}_n - \frac{1}{2} k_n^2 q_n + \sum_{n_1, n_2, n_3} q^*_{n_1} q_{n_2} q_{n_3} \delta_{n+n_1, n_2+n_3} = 0.$$

Canonical Transformation

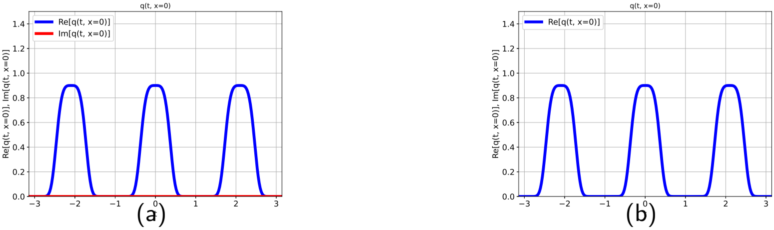
$$q_n = b_n + \sum_{k_n^2 + k_{n_1}^2 \neq k_{n_2}^2 + k_{n_3}^2} \frac{b^*_{n_1} b_{n_2} b_{n_3}}{(k_{n_1} - k_{n_2})(k_{n_1} - k_{n_3})} \delta_{n+n_1, n_2+n_3} \qquad \frac{b^2}{k^2} << 1$$

$$H = \frac{1}{2} \sum_n k_n^2 |b_n|^2 - \frac{1}{2} \sum_{n, n_1} D_{nn_1} |b_n|^2 |b_{n_1}|^2$$

$$i\dot{b}_n - \frac{1}{2} k_n^2 b_n + \Omega_n b_n = 0 \qquad b_n(t) = b_n(0) e^{i(\Omega_n - \frac{1}{2} k_n^2) t}$$

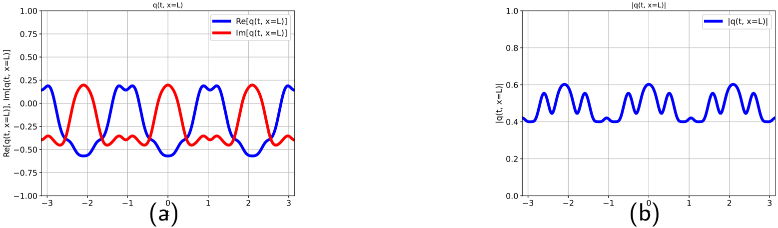
$\Omega_n = \left[\sum_{n_1} D_{nn_1} |b_{n_1}|^2 \right] =$ constant - is almost Number of Waves

Numerics (forward) $L_d/L_{nl} = 0.05$



(a) (b)

Figure: (a) – Re and Im of the signal, (b) – $|q|$



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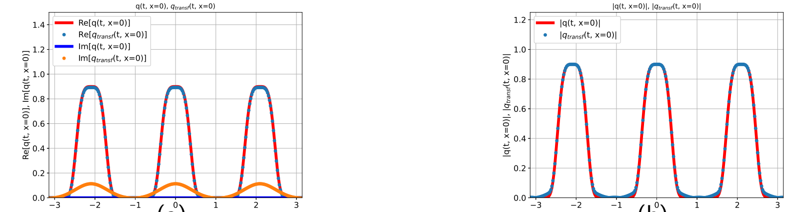
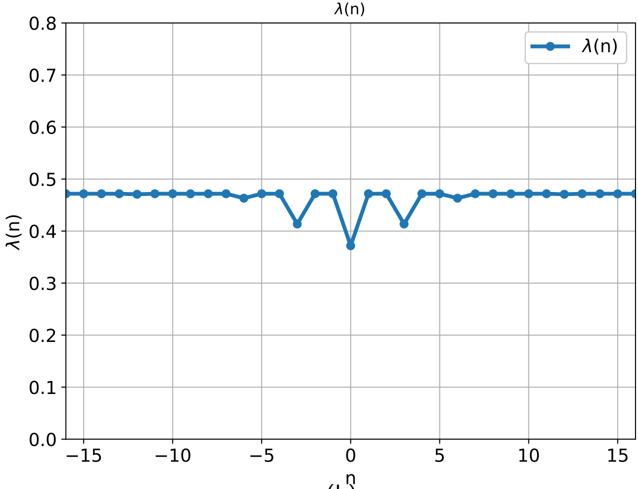


(a) (b)



(b)

Figure: Ω_n