Theory of Freak Waves and Possible Integrability of the Hydrodynamics with Free Surface

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New Year wave and Black Sea wave

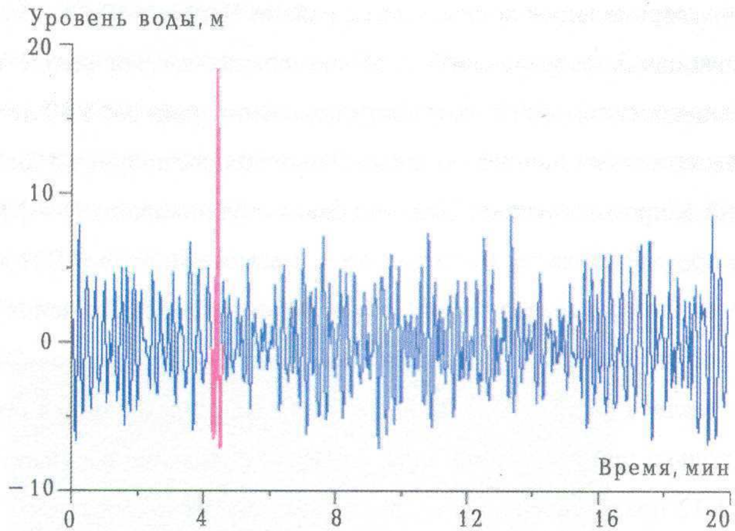


Рис. 1.9. «Новогодняя волна», зарегистрированная в Северном море 1 января 1995 г. [106]

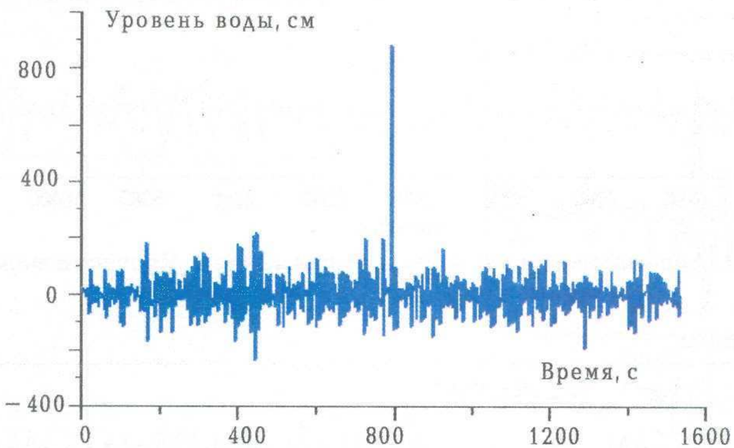
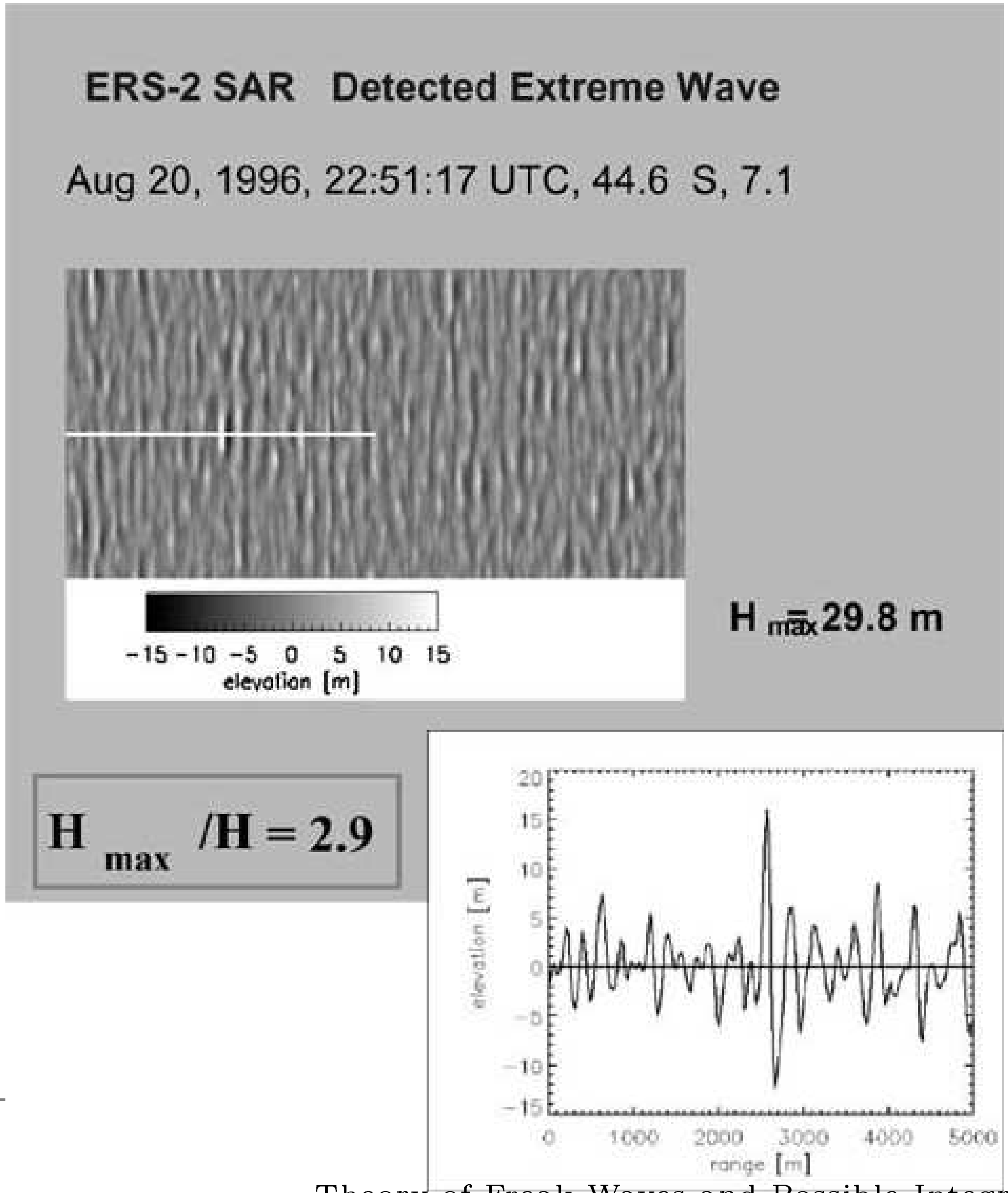


Рис. 1.10. Аномальная волна, зарегистрированная с буя

Satellite View



NLSE approximation

From the equation for potential flow

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\phi_x^2 + g\eta = -\frac{P}{\rho} \qquad \text{at } z = \eta,$$

$$(1) \qquad \frac{\partial \eta}{\partial t} + \eta_x \phi_x = \phi_z \qquad \text{at } z = \eta.$$

one can derive nonlinear Shredinger equation:

$$(2) \qquad i\left(\frac{\partial A}{\partial t} + C_g A_x\right) - \frac{\omega_0}{8k_0^2} A_{xx} - \frac{1}{2}\omega_0 k_0^2 |A|^2 A = 0.$$

A is the envelope of the surface elevation $\eta(x,t)$, so that

$$(3) \qquad \eta(x,t) = \frac{1}{2}\left(A(x,t)e^{i(\omega_0 t - k_0 x)} + c.c.\right)$$

NLSE Soliton

Soliton solution for $A(x,t)$ is

$$A(x,t) = e^{-i\Lambda^2 t}\frac{\lambda}{\sqrt{2}k_0^2}\frac{\cos(k_0(x - V_{phase}t))}{\cosh(\lambda(x - C_g t))} \tag{4}$$

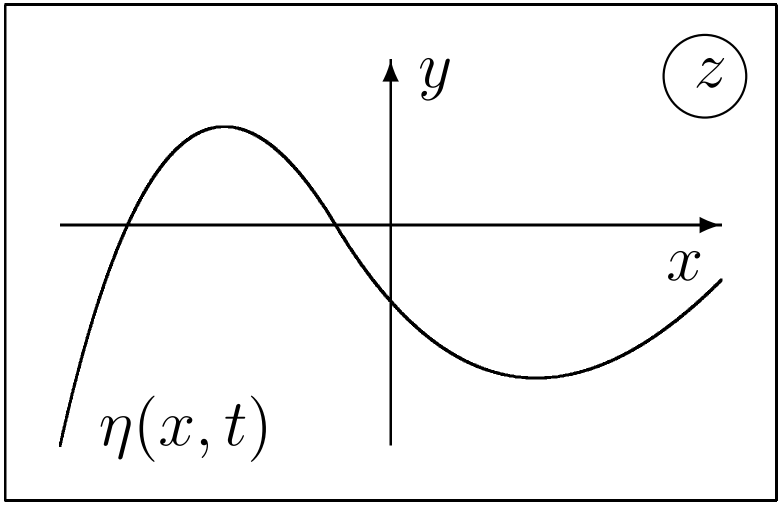
$$\Lambda^2 = \frac{\omega_0 \lambda^2}{8k_0^2}.$$

Wavetrain of the amplitude a with wavenumber k_0 is unstable with respect to large scale modulation δk. Growth rate of the instability γ is

$$\gamma = \frac{\omega_0}{2}\left(\left(\frac{\delta k}{k_0}\right)^2 (ak_0)^2 - \frac{1}{4}\left(\frac{\delta k}{k_0}\right)^4\right)^{\frac{1}{2}}. \tag{5}$$

Here $\omega_0 = \sqrt{gk_0}$.

From Physical to Conformal Equations...



irrotational

$$\Delta\phi(x,y,t) = 0$$

Boundary conditions: $\left[\begin{array}{l} \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = \frac{P}{\rho}, \\ \frac{\partial\eta}{\partial t} + \eta_x\phi_x = \phi_y \end{array}\right|$ at $y = \eta(x,t)$.

$$\begin{aligned} \frac{\partial\phi}{\partial y} &= 0, y \to -\infty, \\ \frac{\partial\phi}{\partial x} &= 0, |x| \to \infty, \text{or periodic} \end{aligned}$$

Conformal mapping

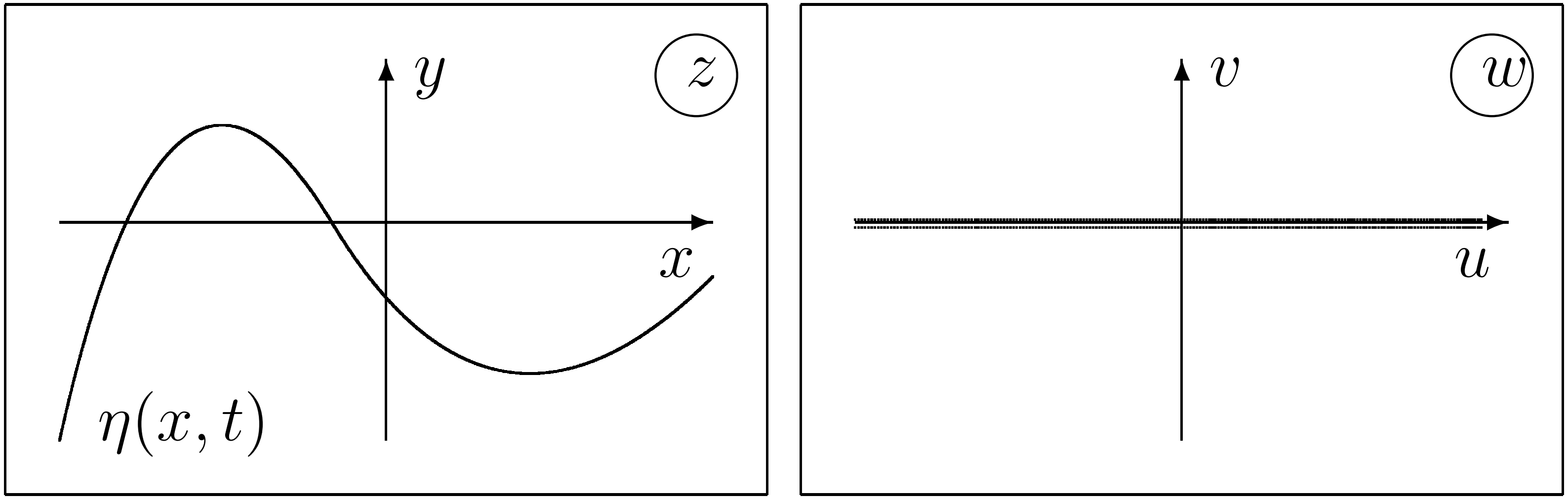
Domain on Z-plane $Z = x + iy$,

$$-\infty < x < \infty, \quad -\infty < y \le \eta(x,t)\,,$$

to the lower half-plane,

$$-\infty < u < \infty, \quad -\infty < v \le 0\,,$$

W-plane $W = u + iv$.



Equations for Z and Φ

If *conformal mapping* has been applied then it is naturally introduce complex analytic functions

$$Z = x + iy, \quad \text{and complex velocity potential} \quad \Phi = \Psi + i\hat{H}\Psi.$$

$$\begin{aligned} Z_t &= iUZ_u, \\ \Phi_t &= iU\Phi_u - \hat{P}\left(\frac{|\Phi_u|^2}{|Z_u|^2}\right) + ig(Z - u). \end{aligned}$$

U is a complex transport velocity:

$$U = \hat{P}\left(\frac{-\hat{H}\Psi_u}{|Z_u|^2}\right). \qquad u \to w$$

Projector operator $\hat{P}(f) = \frac{1}{2}(1 + i\hat{H})(f)$.

Cubic equations for R and V

Surface dynamics (and the fluid bulk!) is described by two analytic functions, $R(w,t)$ and $V(w,t)$. They are related to conformal mapping Z and complex velocity potential:

$$R = \frac{1}{Z_w}, \qquad \Phi_w = -iVZ_w.$$

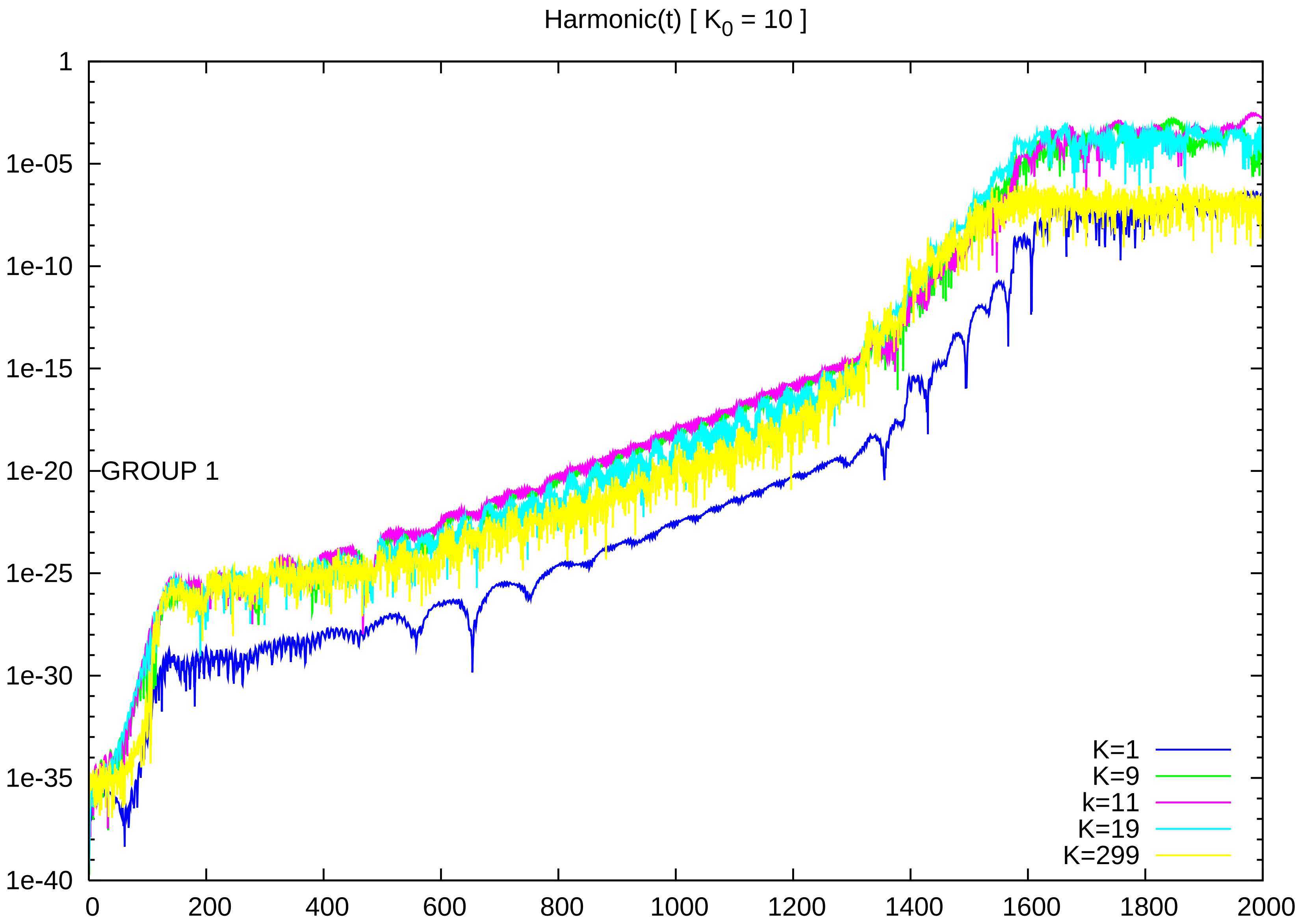
For R and V dynamic equations have the simplest form:

$$\begin{aligned} R_t &= i\left[UR' - U'R\right], \\ V_t &= i\left[UV' - B'R\right] + g(R-1). \end{aligned}$$

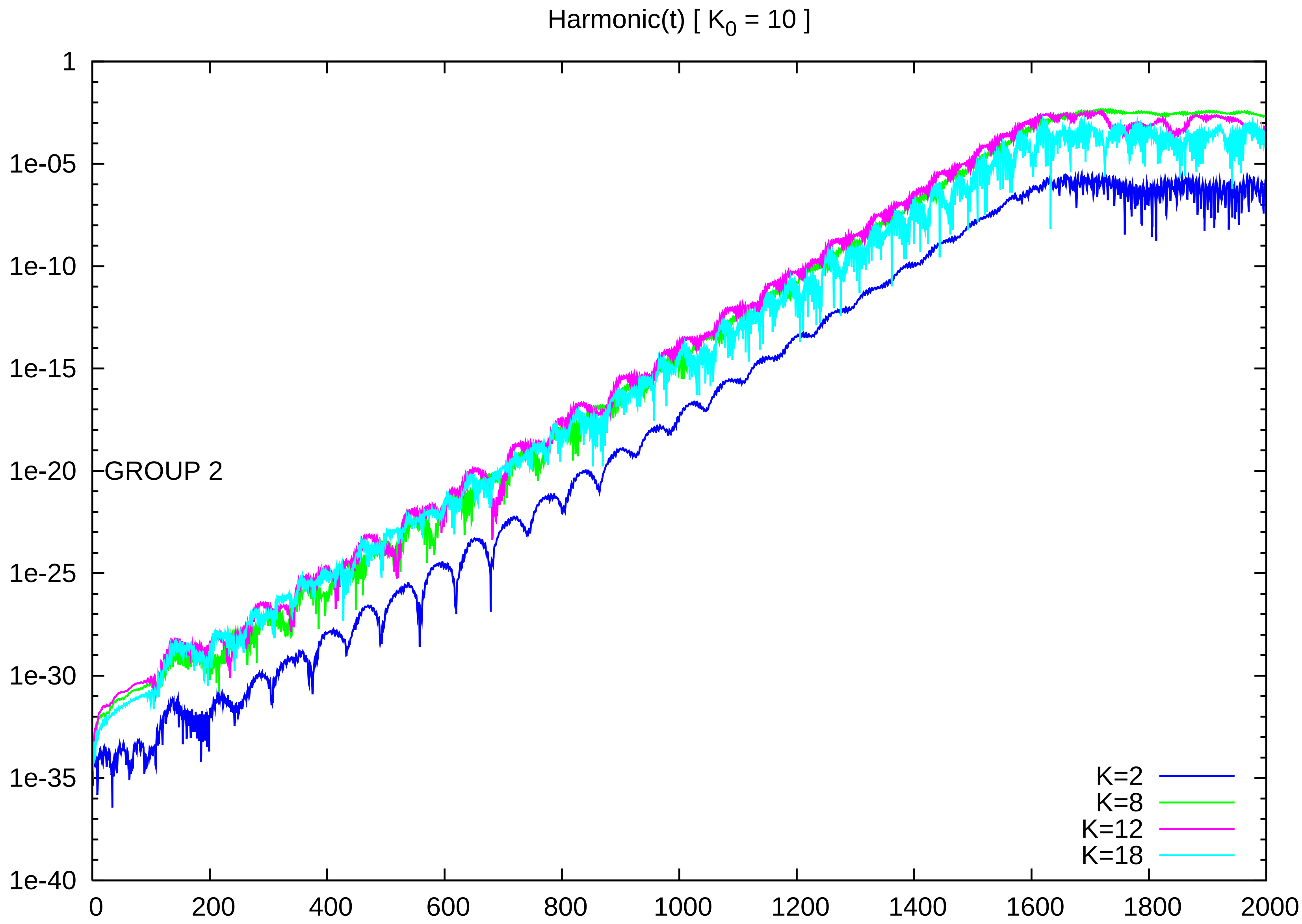
Complex transport velocity U is defined as

$$U = \hat{P}(V\bar{R} + \bar{V}R), \qquad \text{and} \qquad B = \hat{P}(V\bar{V}).$$

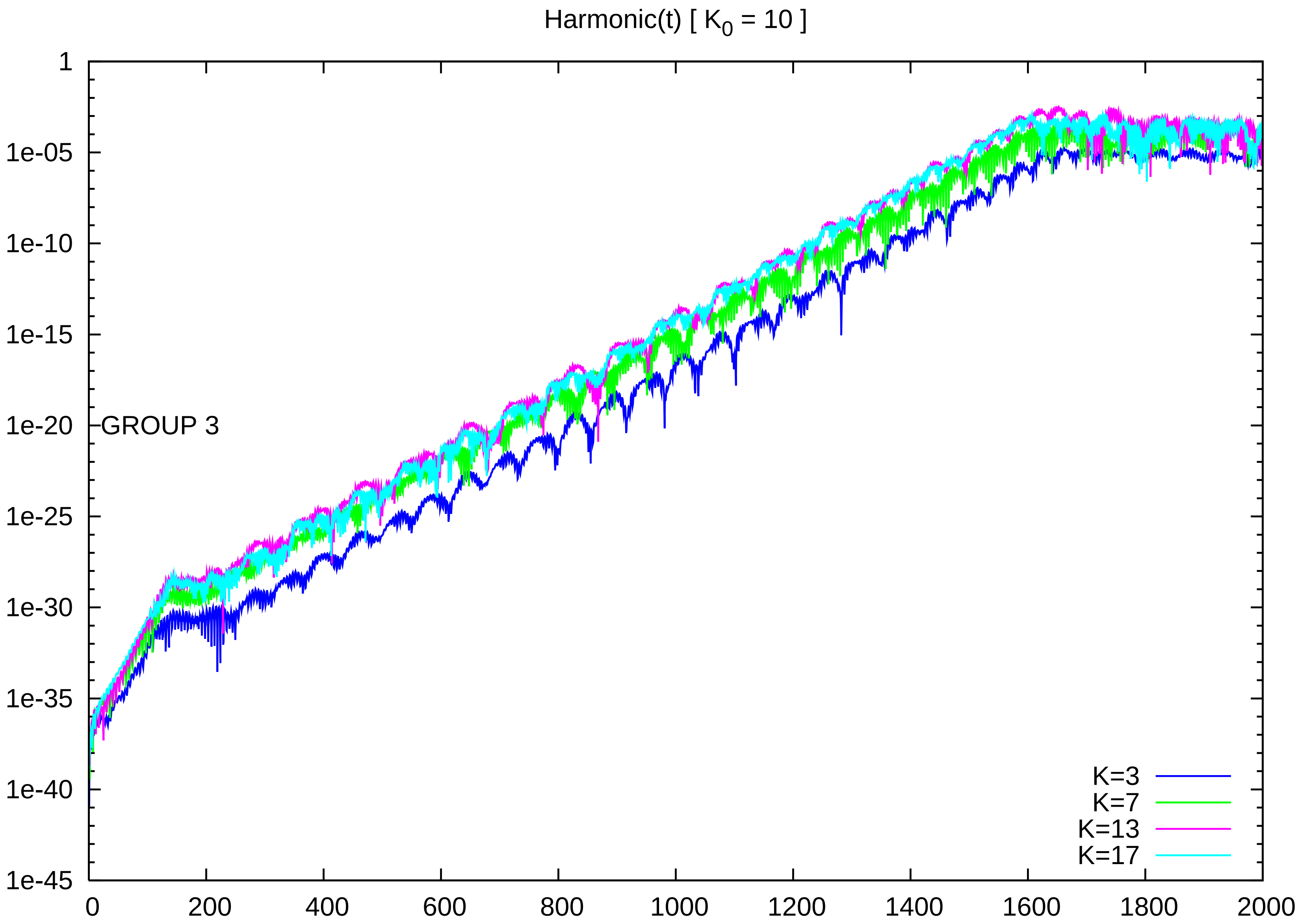
Modulation Instability, Group 1



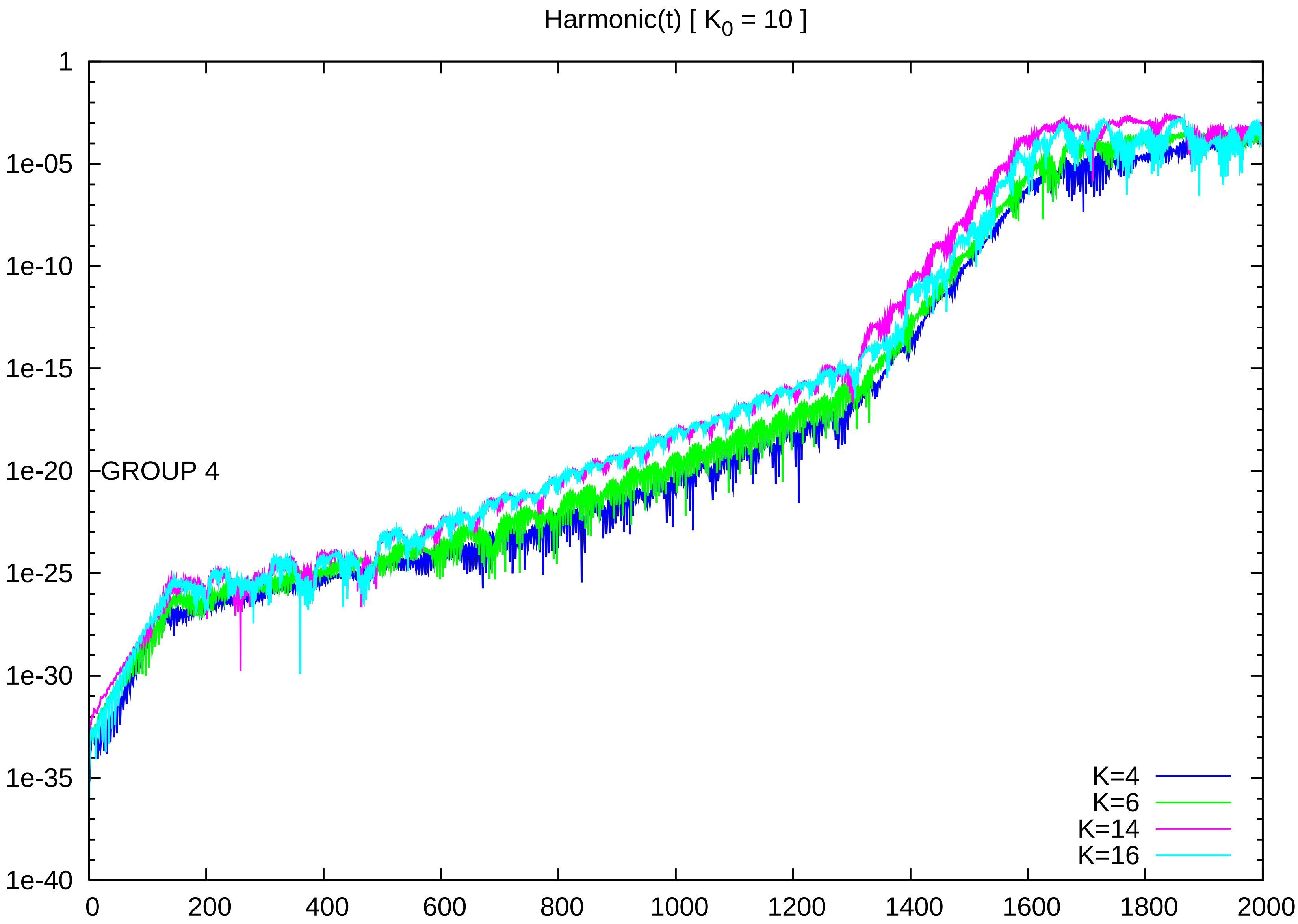
Modulation Instability, Group 2



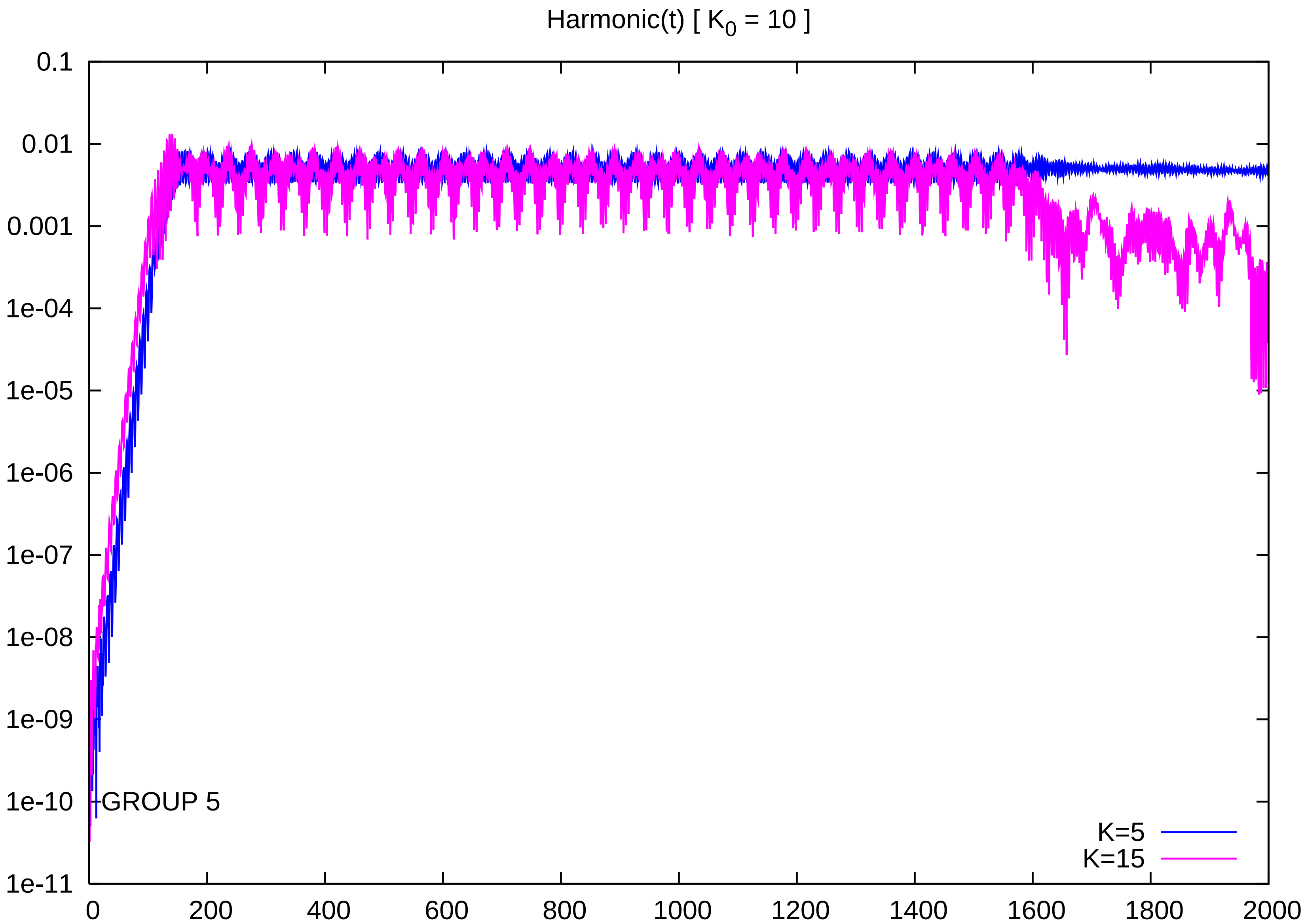
Modulation Instability, Group 3



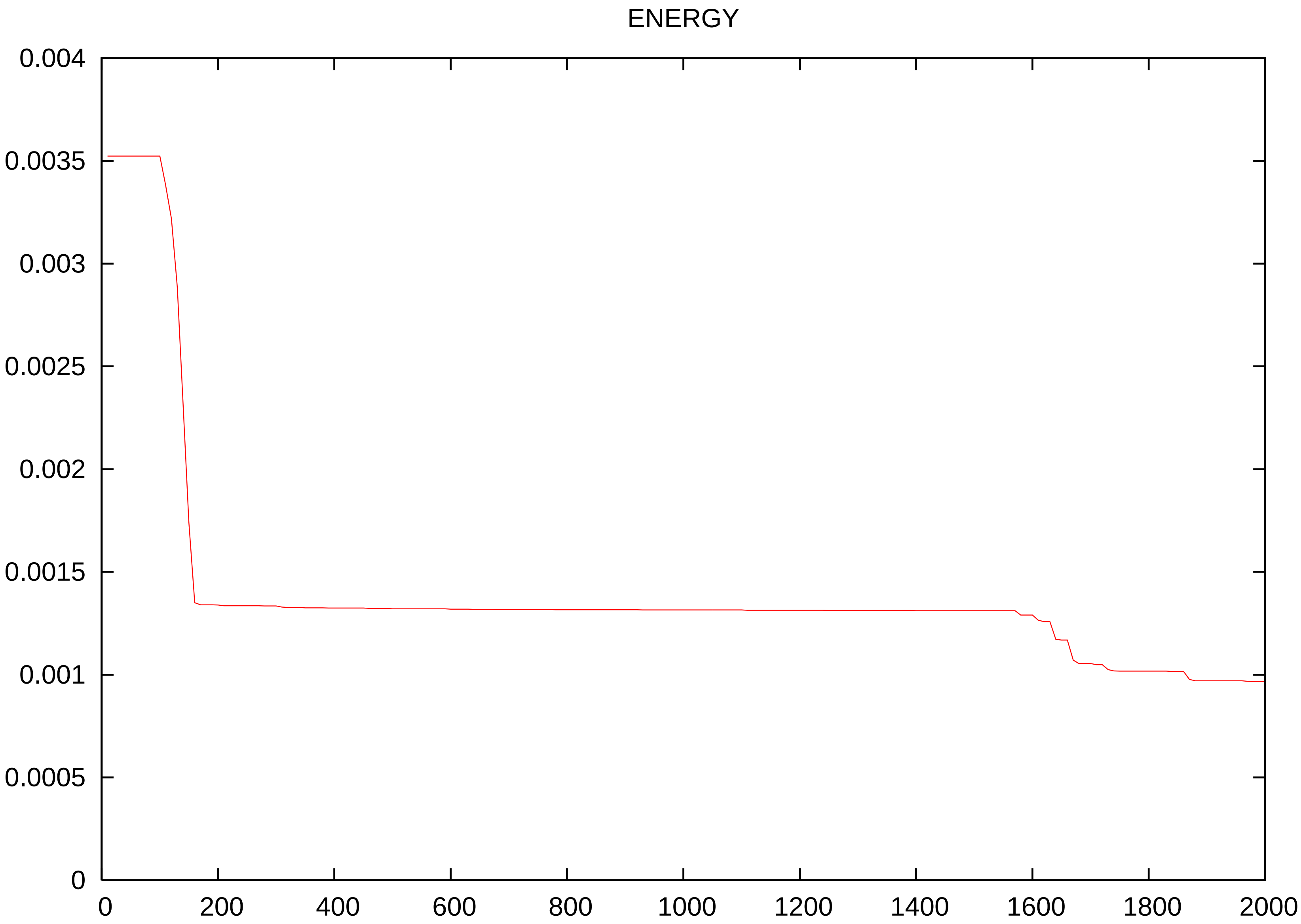
Modulation Instability, Group 4



Modulation Instability, Group 5



Energy dissipation



NLSE Solition

$$\eta(x) = \frac{\lambda}{\sqrt{2}k_0^2}\frac{\cos(k_0 x)}{\cosh(\lambda x)} \tag{6}$$

$$\eta(x,t) = \frac{1}{2}(A(x,t)e^{i(\omega_0 t - k_0 x)} + c.c.)$$

A is the envelope of the surface elevation $\eta(x,t)$.

Example - soliton with local steepness $\mu \simeq \frac{\lambda}{k_0} \simeq 0.1$

Giant Breather

$$\eta(x) = \frac{\lambda}{\sqrt{2}k_0^2}\frac{\cos(k_0 x)}{\cosh(\lambda x)} \tag{7}$$

Initial condition - soliton with local steepness $\mu \simeq \frac{\lambda}{k_0} \simeq 0.6$

Breather is clerly observed after radiation goes away.

k-ω spectrum

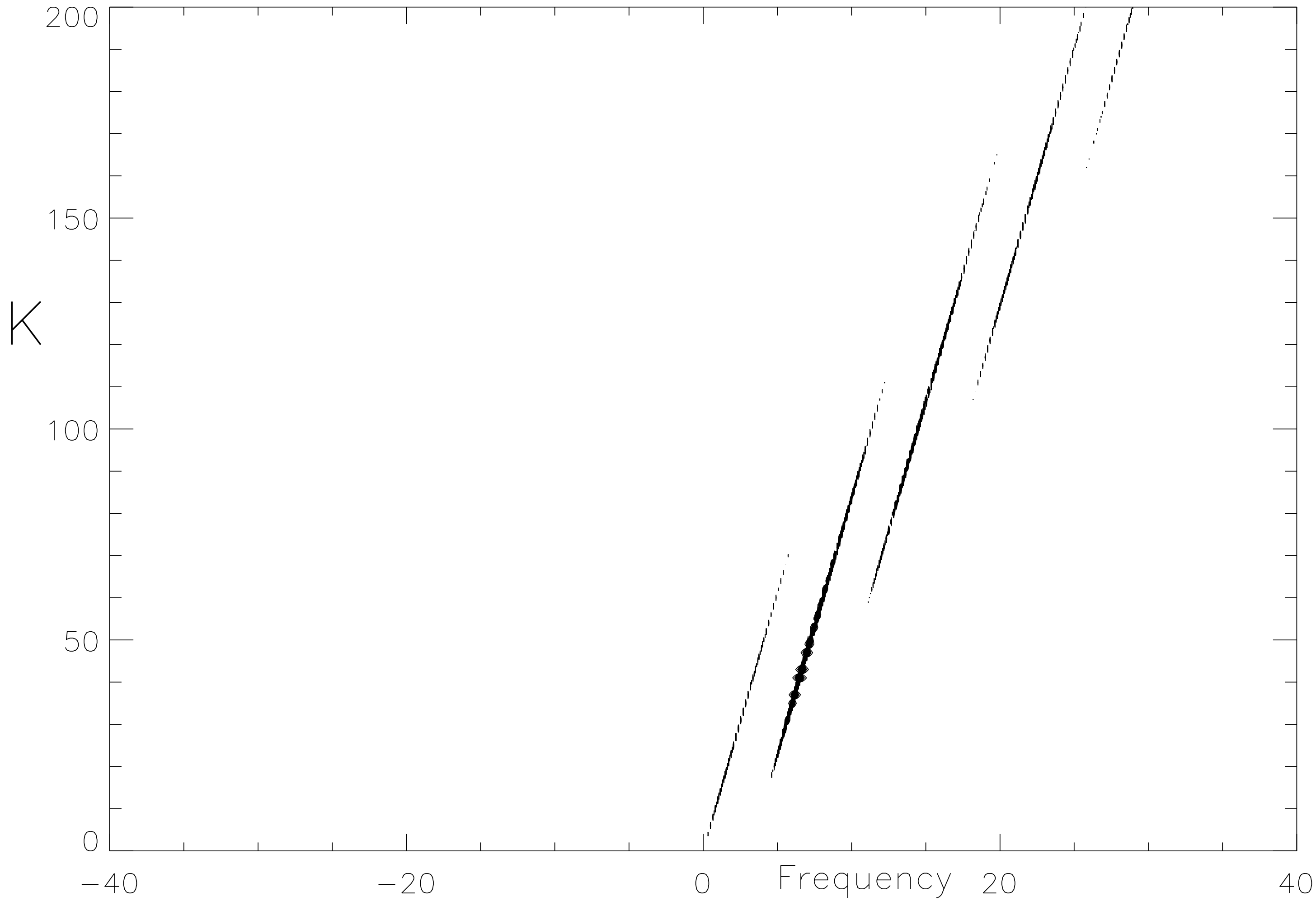


Figure 1. Negative frequency is absent!.

$X' - 1$ and Ψ