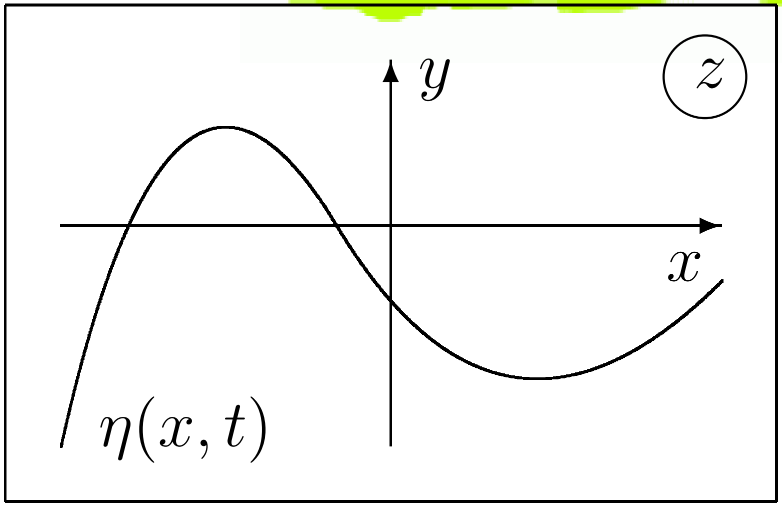
Breaking of Progressive Gravity Surface Wave

Dyachenko A.I.

Landau Institute for Theoretical Physics RAS

Equations



potential irrotational flow

$$\triangle\phi(x,y,t)=0$$

Boundary conditions: $$\left[\begin{array}{l}\dfrac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta=\dfrac{P}{\rho},\\ \dfrac{\partial\eta}{\partial t}+\eta_x\phi_x=\phi_y\end{array}\right| \text{ at } y=\eta(x,t).$$

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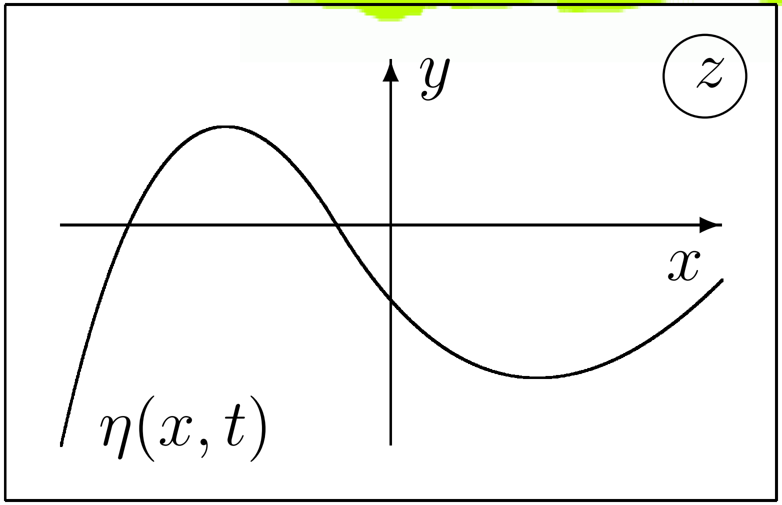

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$$\begin{aligned}\frac{\partial\phi}{\partial y} &= 0, y\to-\infty,\\ \frac{\partial\phi}{\partial x} &= 0, |x|\to\infty.\end{aligned}$$

Hamiltonian and Lagrangian

Hamiltonian H is the total energy of the fluid $H = T + U$

$$T = \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\eta}^{\eta} (\nabla\Phi)^2 dy\,,$$

$$U = \frac{g}{2}\int \eta^2 dx\,.$$

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi}\,,$$

$$\frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta}\,,$$

$$\Psi(x,t) = \Phi(x,y,t)\big|_{y=\eta}$$

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$$

Action $S = \int L dt$

$$\delta S = 0,$$

Lagrangian

$$L = \int \Psi \frac{\partial\eta}{\partial t} dx - H\,.$$

Conformal Mapping

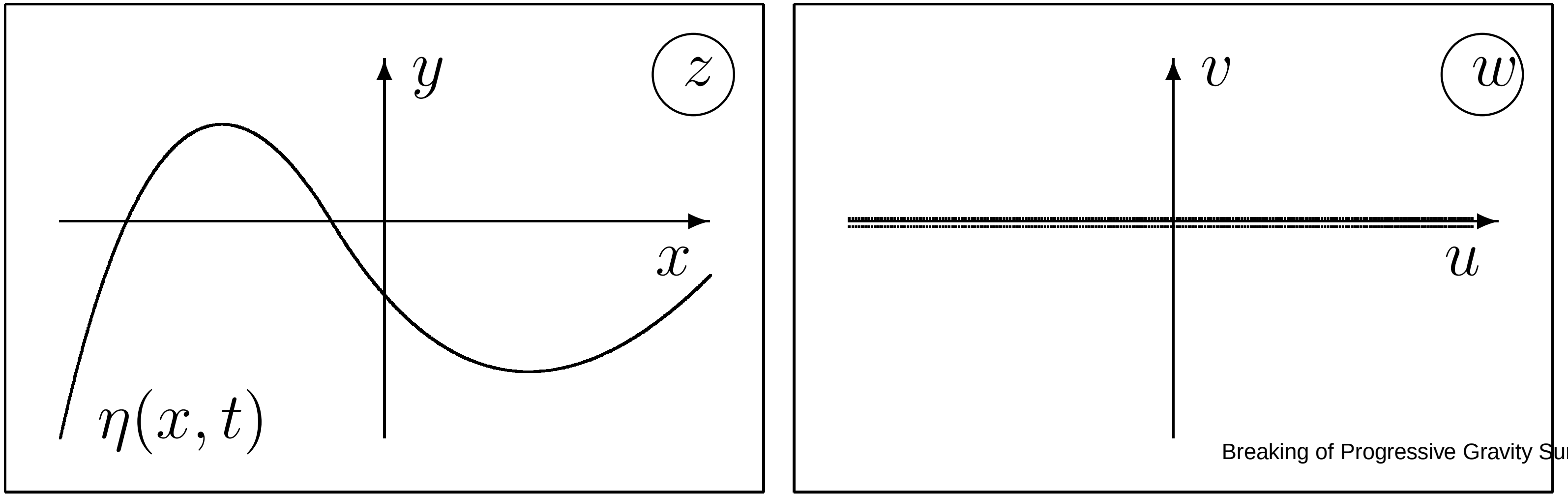
Let us apply the conformal mapping of the domain on the plane $z = x + iy$,

$$-\infty < x < \infty, \quad -\infty < y \leq \eta(x,t)\,,$$

to the lower half-plane,

$$-\infty < u < \infty, \quad -\infty < v \leq 0\,,$$

on the plane $\omega = u + iv$.



Conformal Mapping

After this transformation, the shape of the surface is given parametrically by

$$y = y(u,t), \quad x = u + \tilde{x}(u,t)\,.$$

Functions y and $\tilde{x}$ are coupled by the relations:

$$y = \hat{H}\tilde{x} \qquad\qquad \tilde{x} = -\hat{H}y\,.$$

Here $\hat{H}$ is the Hilbert transformation,

$$\hat{H}f(u) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(u')}{(u'-u)} du'\,.$$

For Fourier harmonics $y_k = i\text{sign}(k)x_k$.

Implicit Equations

Lagrangian can be expressed as follows,

$$L = \int_{-\infty}^{\infty} \Psi(y_t x_u - x_t y_u) du + \frac{1}{2}\int_{-\infty}^{\infty} \left(\Psi \hat{H} \Psi_u du - g y^2 x_u \right) du + \int_{-\infty}^{\infty} f(y - \hat{H}\tilde{x})$$

Here f is the Lagrange multiplier. Hamilton's principle,

$$\frac{\delta S}{\delta \Psi} = 0\,, \quad \frac{\delta S}{\delta y} = 0, \quad \text{and} \quad \frac{\delta S}{\delta x} = 0\,.$$

gives the following equations,

$$\begin{aligned} y_t x_u - x_t y_u &= -\hat{H}\Psi_u\,, \\ \Psi_t x_u - \Psi_u x_t + g y x_u &= \hat{H}(\Psi_t y_u - \Psi_u y_t + g y y_u)\,. \end{aligned}$$

Ovsyannikov, 1973

Explicit Equations

If *conformal mapping* has been applied then it is naturally introduce complex analytic functions

$z = x + iy,$ and complex velocity potential $\Phi = \Psi + \hat{H}\Psi.$

$$\begin{aligned} z_t &= iU z_u, \\ \Phi_t &= iU\Phi_u - \hat{P}\left(\frac{|\Phi_u|^2}{|z_u|^2}\right) + ig(z - u). \end{aligned}$$

U is a complex transport velocity:

$$U = \hat{P}\left(\frac{-\hat{H}\Psi_u}{|z_u|^2}\right). \qquad u \to w$$

Projector operator $\hat{P}(f) = \frac{1}{2}(1 + i\hat{H})(f)$.

Classical variables Ψ, η

$$H = \frac{1}{2}\int_{-\infty}^{\infty} \Psi \hat{G}(\eta)\Psi dx \quad + \quad \frac{g}{2}\int_{-\infty}^{\infty} \eta^2 dx$$

Normal complex variable a_k:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}) \qquad \psi_k = -i\sqrt{\frac{2g}{\omega_k}}(a_k - a^*_{-k}) \qquad \omega_k = \sqrt{gk}$$

a_k satisfies the equation

$$\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a_k^*} = 0,$$

$a_k \rightarrow b_k,$ excluding nonresonant terms.

Classical variables b_k

Hamiltonian without cubic terms:

$$H = \int \omega_k b_k b_k^* dk + \frac{1}{2} \int T_{kk_1}^{k_2 k_3} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \dots$$

$$k + k_1 = k_2 + k_3, \qquad \omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3},$$

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$$T_{k_4 k_5}^{k_1 k_2 k_3} = \frac{2}{g^{1/2} \pi^{3/2}} \sqrt{\frac{\omega_{k_1} \omega_{k_2} \omega_{k_3}}{\omega_{k_4} \omega_{k_5}}} \frac{k_1 k_2 k_3 k_4 k_5}{max(k_1, k_2, k_3)}$$

Cubic Equations

It turned out, that the equations can be simplified just by changing variables. Introduce instead of $z(w,t)$ and $\Phi(w,t)$ another analytic functions $R(w,t)$ and $V(w,t)$

$$R = \frac{1}{z_w}, \qquad \Phi_w = -iVz_w.$$

$$\begin{aligned} R_t &= i\left[UR' - U'R\right], \\ V_t &= i\left[UV' - R\hat{P}(V\bar{V})'\right] + g(R-1). \end{aligned}$$

Complex transport velocity U is defined via $\hat{P}$

$$U = \hat{P}(V\bar{R} + \bar{V}R).$$

Setup the problem

- The shape of stationary progressive wave is given by:

$$y = \frac{c^2}{2g}(1 - \frac{1}{|z_w|^2}),$$

while Φ is related to the surface as

$$\Phi = -c(z - w), \qquad V = ic(R - 1).$$

The amplitude of the wave $\frac{h}{L} \simeq 0.088$,

For the sharp peaked limiting wave $\frac{h}{L} \simeq 0.141$.

Setup the problem

- Put 10 such waves in the periodic domain of 2π.
- Add long scale perturbation to this wave train

$$V_k = ic(R_k - 1) + \epsilon\delta_{1,k}.$$

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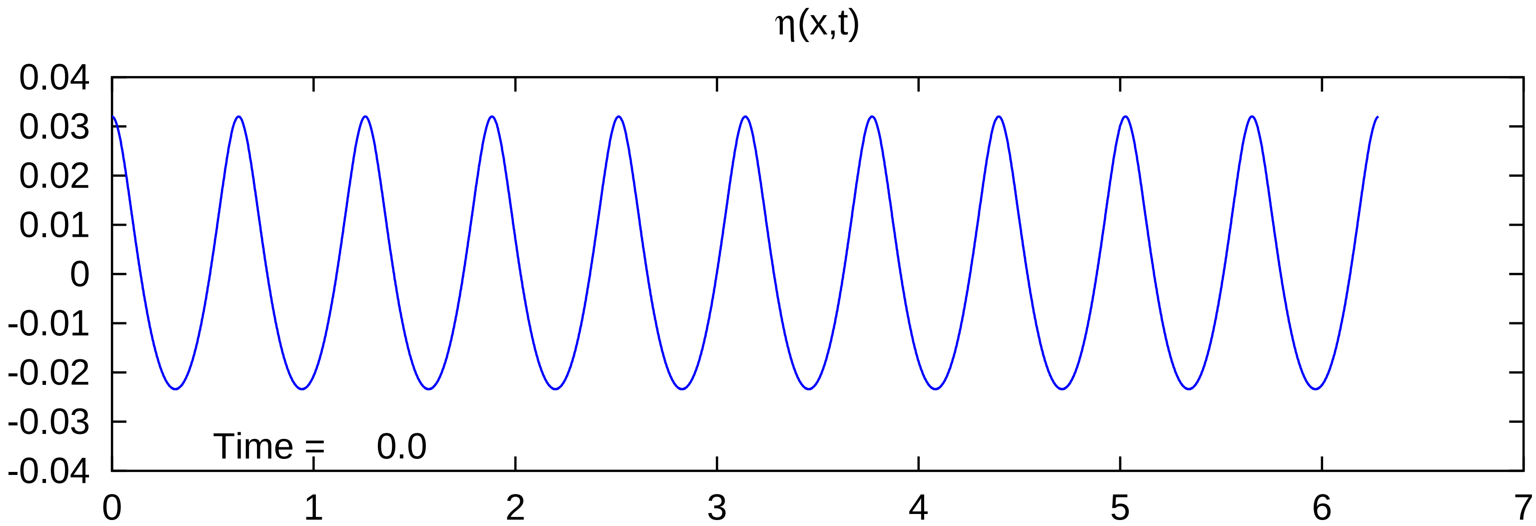


Figure 2: Initial profile of the wave train

Simulation

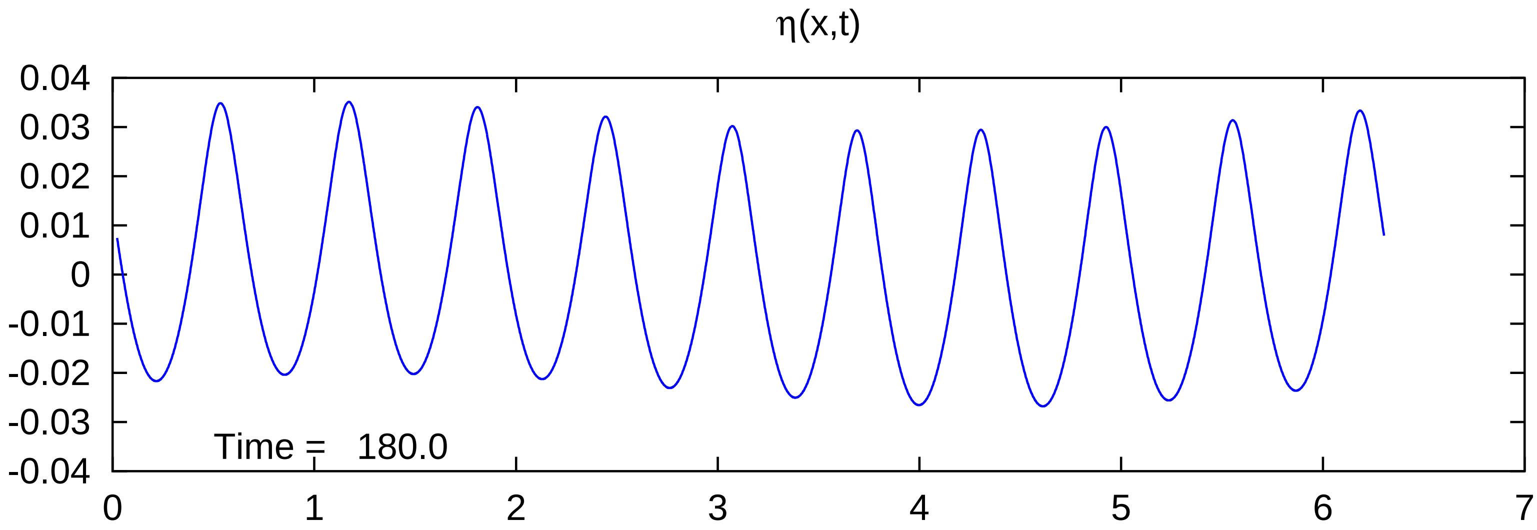
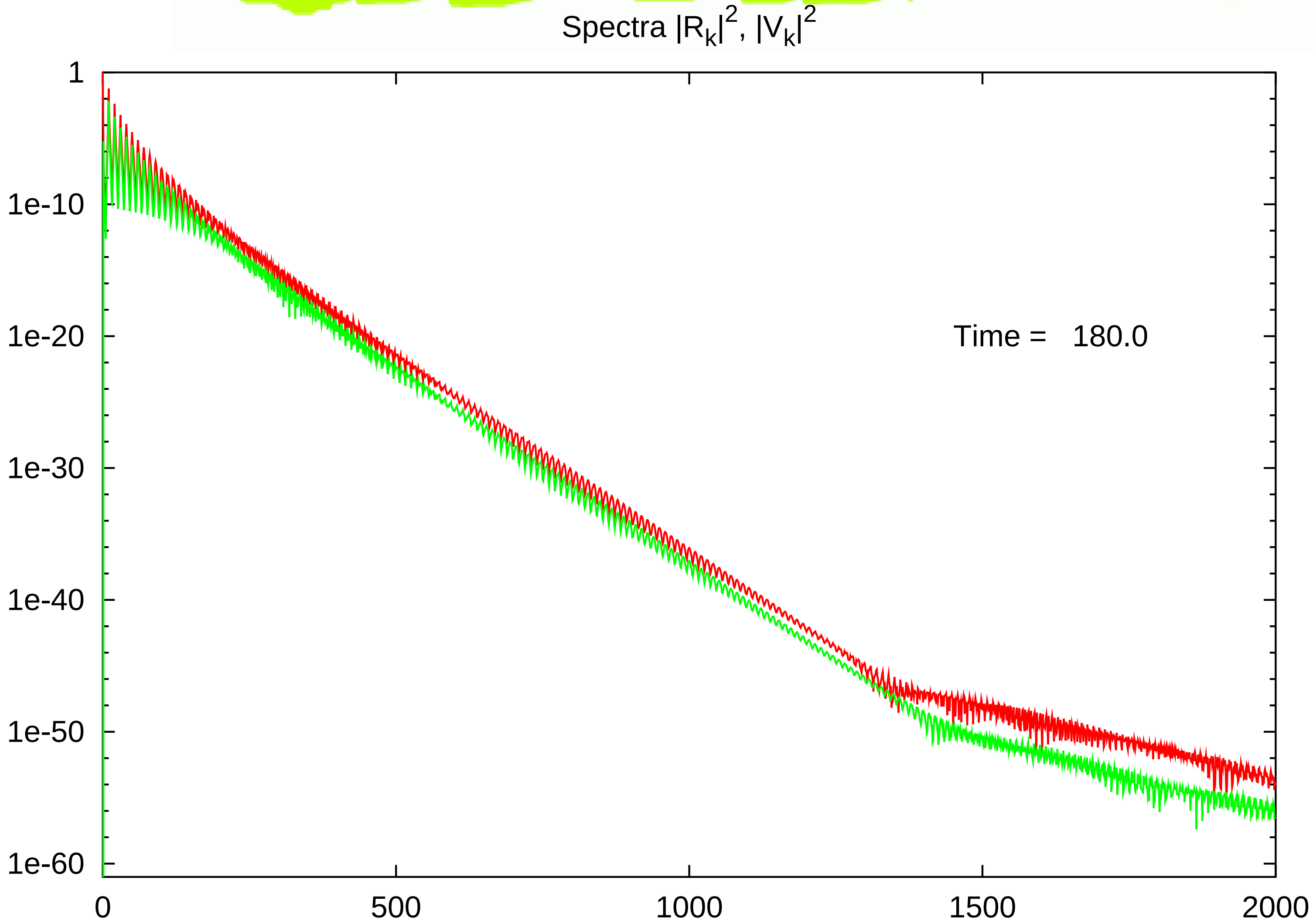


Figure 3: Typical profile of the wave train

Simulation



Spectra $|R_k|^2$, $|V_k|^2$
1
1e-10
1e-20
1e-30
1e-40
1e-50
1e-60
0
500
1000
1500
2000
Time = 180.0

Wave breaks

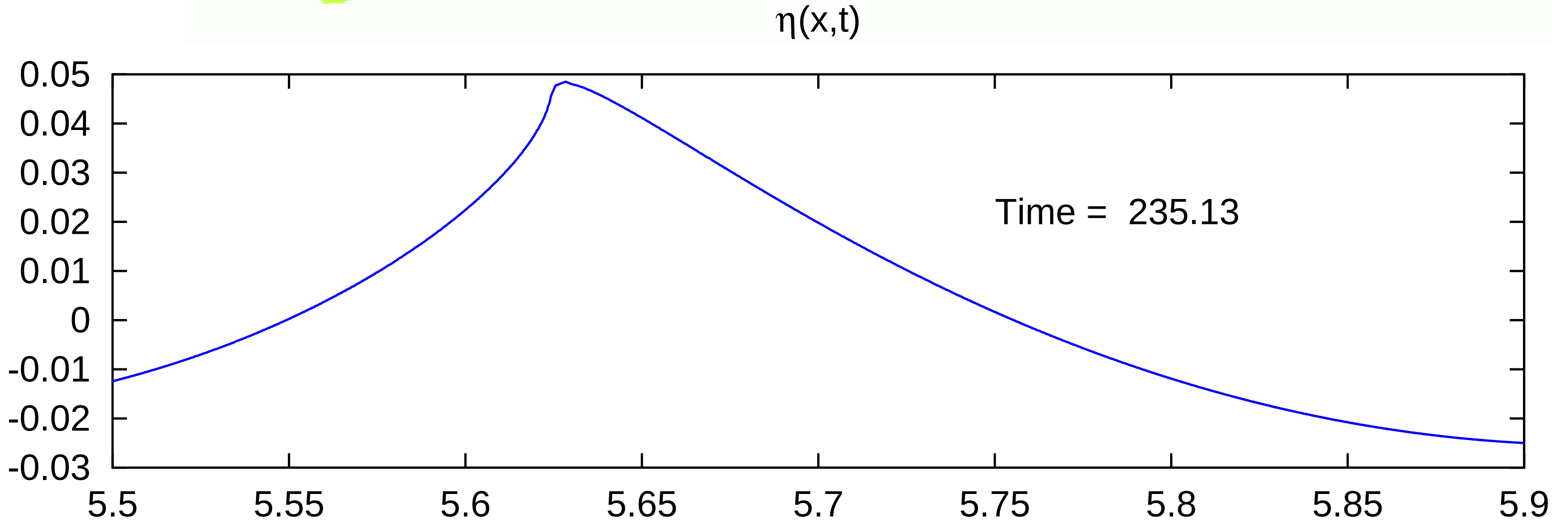


Figure 5: Wave breaks.

Slope of the surface

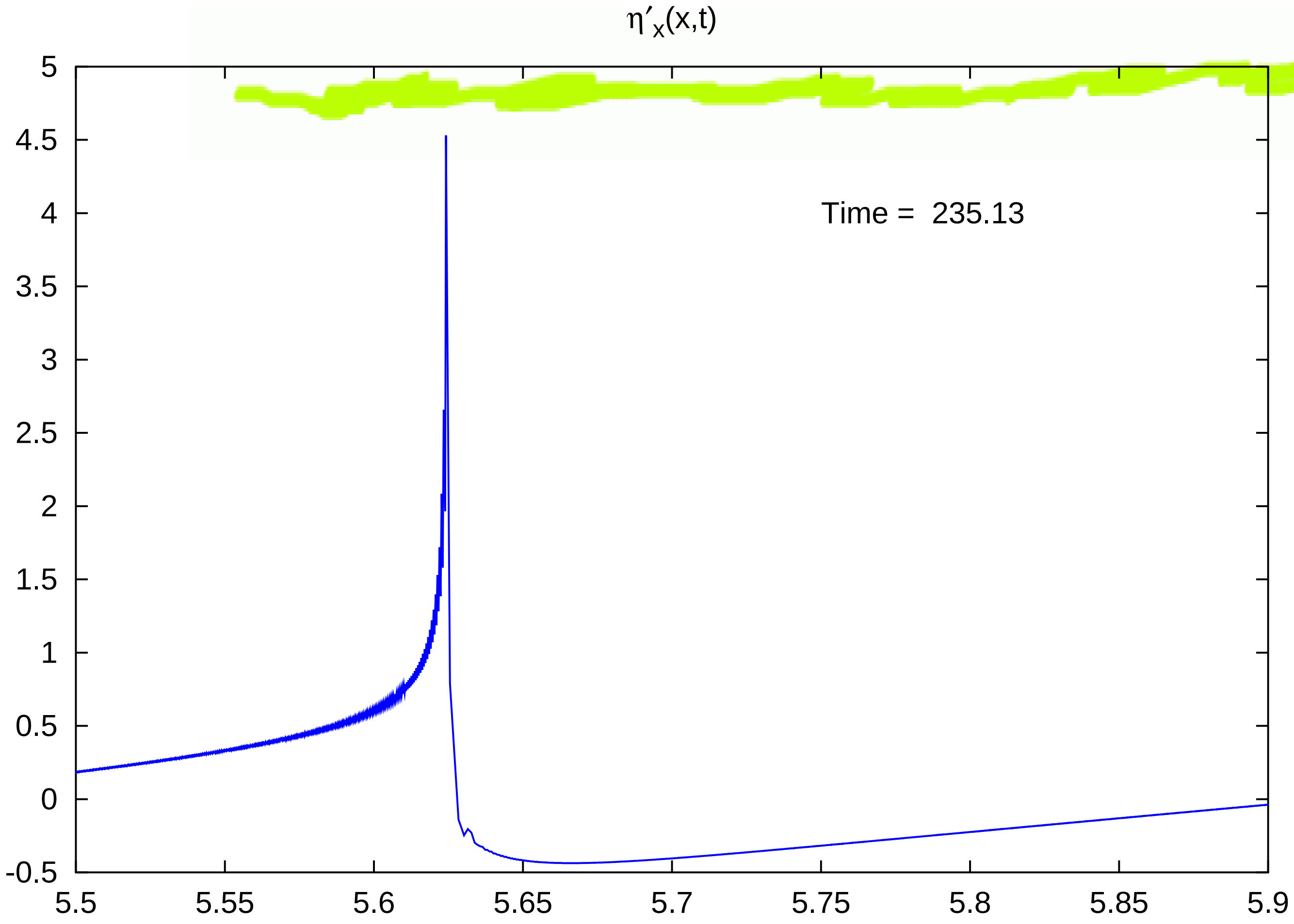


Figure 6: Sharp wedge with the angle $\dfrac{\pi}{3}$

Power spectrum

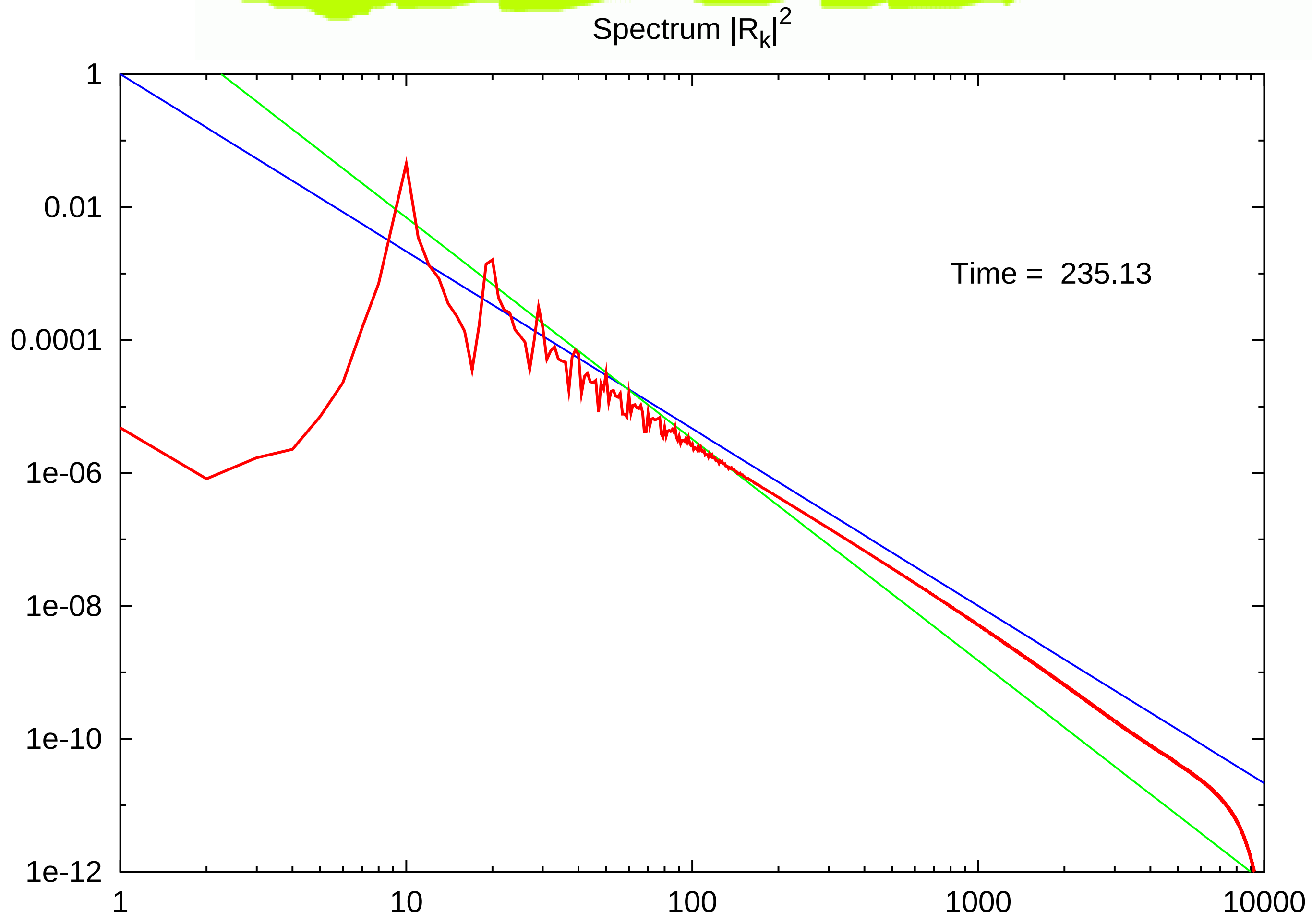


Figure 7: Spectrum of $|R_k|^2$ and $k^{-\frac{8}{3}}$ and $k^{-\frac{10}{3}}$.

? ? ?

Sharp peaked stationary gravity wave at the crest

$$z \sim i\frac{c^2}{2g} + \left(\frac{9}{4}\frac{c^2}{g}\right)^{\frac{1}{3}} [-iw^2]^{\frac{1}{3}} + \epsilon w^{\alpha} + \ldots \qquad (\sqrt[3]{-i} = e^{-i\frac{\pi}{6}})$$

This is well known result of Stokes with the angle - $\frac{2\pi}{3}$.
For breaking wave numerics shows - $\frac{\pi}{3}$ wedge.

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$$z \sim iA(t) + B(t)[-i(w - a(t))(w - b(t))]^{\frac{1}{3}} + \ldots$$

Root branch that corresponds to the numerics is

$$z = \begin{cases} -i|u|^{1/3}, & \textit{missing real part} \quad u \leq 0; \\ |u|^{1/3}(\frac{\sqrt{3}}{2} - \frac{i}{2}), & u > 0, \end{cases}$$

? ? ?

Wave profile in (x, y) coordinates:

$$\begin{cases} y^3 = x, & x \leq 0; \\ y = \frac{1}{\sqrt{3}}x, & x > 0. \end{cases}$$

"Overturning" takes place only near the wave top, main shape of the wave looks like Stokes wave. Distance between branch points a and b is small with respect to wavelength.