Charge relaxation resistance in the cotunneling regime of multichannel Coulomb blockade: Violation of Korringa-Shiba relation

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We study the low-frequency admittance of a small metallic island coupled to a gate electrode and to a massive reservoir via a *multichannel* tunnel junction. The ac current is caused by a slowly oscillating gate voltage. We focus on the regime of inelastic cotunneling in which the dissipation of energy (the real part of the admittance) is determined by two-electron tunneling with creation of electron-hole pairs on the island. We demonstrate that at finite temperatures but low frequencies the energy dissipation is ohmic whereas at zero temperature it is superohmic. We find that (i) the charge relaxation resistance (extracted from the real part of the admittance) is strongly temperature dependent, and (ii) the imaginary and real parts of the admittance do not satisfy the Korringa-Shiba relation. At zero temperature the charge relaxation resistance vanishes in agreement with the recent zero-temperature analysis [M. Filippone and C. Mora, Phys. Rev. B **86**, 125311 (2012); P. Dutt, T. L. Schmidt, C. Mora, and K. Le Hur, Phys. Rev. B **87**, 155134 (2013)].

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I. INTRODUCTION

During the last decades Coulomb blockade has become a powerful tool for observation of interaction and quantum effects in single electron devices [1–6]. This phenomenon is widely observed in low-temperature electron transport through a single electron transistor. Another system in which low-temperature properties are affected by Coulomb blockade is a single electron box (SEB). It is schematically shown in Fig. 1. A small metallic island is coupled capacitively to the gate electrode with the voltage U_g. The number of electrons on the island is not conserved due to the tunneling in and out of an equilibrium electron reservoir. A time-dependent gate voltage $U_g(t)$ generates ac current through the device.

The equivalent electric circuit of a SEB (see Fig. 1) is characterized by two capacitances. The gate capacitance C_g controls the external (induced) charge q on the island, $q = C_g U_g$. The total capacitance C determines the so-called charging energy $E_c = e^2/2C$. It is the latter that is responsible for the Coulomb blockade effects. The tunnel junction is characterized by the dimensionless (in units e^2/h) conductance g. Throughout the paper we use a standard assumption that the Thouless energy of the island is the largest energy scale in the problem. This allows us to work in a zero-dimensional approximation neglecting spatial dependence of all quantities.

Since there is no dc transport through the SEB, an essential dynamic characteristic becomes the admittance which characterizes the response of ac current I_ω to the infinitely small ac part U_ω of the time-dependent gate voltage $U_g(t) = U_0 + U_\omega \cos\omega t$: $\mathcal{G}(\omega) = I_\omega/U_\omega$. Long ago it was demonstrated that the admittance of SEB is affected by Coulomb blockade at low temperatures $T \ll E_c$ [7]. However, since then the majority of works have addressed the so-called quantum capacitance: $C_{\text{eff}} = \partial Q/\partial U_g$, where Q is the average charge on the island, which determines the imaginary part of the admittance [8–14]. Classically, at high temperatures $T \gg E_c$ the effective capacitance coincides with C_g. As temperature decreases C_{eff} starts to deviate from C_g due to interaction and coherence effects. In seminal paper [15] it was suggested that the real and imaginary parts of the admittance in a SEB can be related in a universal way. After the paper [15] a SEB admittance has attracted significant theoretical interest [16–20]. The admittance in the quasistatic regime was measured in a single channel SEB constructed in two-dimensional (2D) electron gas [21]. At present, there exists a number of measurements of admittances for different realizations of a SEB performed with the help of radio-frequency reflectometry [22–25].

The classical electrodynamics of a SEB suggests the following expression for the admittance at low frequencies, $\omega \ll gE_c$:

$$\mathcal{G}(\omega) = -i\omega C_g + \omega^2 C_g C R, \tag{1}$$

where $R = h/(e^2 g)$ stands for the classical resistance of the tunnel junction. In Ref. [15] the following generalization of the classical result (1) has been proposed for the quantum coherent SEB with $C = C_g$:

$$\mathcal{G}(\omega) = -i\omega C_{\text{eff}} + \omega^2 C_{\text{eff}}^2 R_q, \tag{2}$$

where R_q was termed as the *charge relaxation resistance*. Treating the Coulomb interaction within the Hartree-Fock approximation, the authors of Ref. [15] demonstrate that for single channel tunnel junction the charge relaxation resistance in Eq. (2) becomes universal, $R_q = h/(2e^2)$. The full quantum-mechanical treatment of the charging energy in the case of a single-channel tunnel junction demonstrates that at zero temperature $R_q = h/(2e^2)$ ($R_q = h/e^2$) for frequencies $\omega \ll \delta$ ($\omega \gg \delta$) [26]. Here δ denotes the mean level spacing of single-particle states inside the island of a SEB. Both results follow from two observations: (i) the effective low-energy Hamiltonian of single-channel SEB is of Fermi-liquid type; (ii) the Korringa-Shiba relation [27,28] for the response function $i\mathcal{G}(\omega)/\omega$ holds within Fermi-liquid low-energy description [26]. Recently, the analysis of Ref. [26] has been generalized to the case of a SEB with a weak ($g \ll 1$) multichannel tunnel junction and a large island, $\delta \to 0$. It was found [29,30] that at

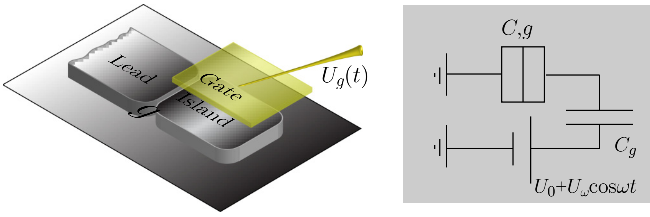

FIG. 1. (Color online) The setup: a SEB subjected to a time-dependent gate voltage $U_g(t)$ (left) and the equivalent electric circuit (right).

zero temperature the charge relaxation resistance is inversely proportional to the number of channels in a tunnel junction and is independent of the external charge; R_q vanishes in the limit of an infinite number of channels for any value of q.

Contrary to the admittance which is uniquely defined the charge relaxation resistance can be introduced in many ways. At finite temperatures the SEB with a multichannel tunnel junction in the limit of negligible mean level spacing, $\delta \to 0$, has been analyzed in Ref. [31]. In particular, it was demonstrated that in the limit of weak tunneling, $g \ll 1$, and near the charge degeneracy points the SEB admittance at low frequencies ($\omega \ll g \max\{|\Delta|,T\}$) can be set down in the following form:

$$\mathcal{G}(\omega) = -i\omega C_{\rm eff} + \omega^2 \frac{C}{\mathcal{C}_g}\mathcal{C}_g^2 \mathcal{R}_q. \tag{3}$$

Here Δ denotes the electrostatic energy due to one excess electron on the SEB island. It depends on the external charge q and satisfies the inequality $|\Delta| \ll E_c$ near a charge degeneracy point. The quantity $\mathcal{C}_g = \partial \mathcal{Q}/\partial U_0$ stands for the renormalized gate capacitance which measures the response of the effective charge $\mathcal{Q}$, introduced in Refs. [32,33] by one of us, to the static part of the gate voltage. Contrary to the average charge Q on the island, the effective charge is expected to be integer quantized at zero temperature [32–34]. This implies that $\mathcal{C}_g$ vanishes at $T=0$ contrary to $C_{\rm eff}$. The charge relaxation resistance in Eq. (3) is determined by the renormalized tunneling conductance $g(T)$, $\mathcal{R}_q = h/[e^2 g(T)] \gg h/e^2$. The very same conductance $g(T)$ determines the dc conductance of the single electron transistor (SET) under small bias between source and drain. We also note that Eq. (3) was proposed for the SEB with arbitrary relation between C and C_g. For the case of weak tunneling the treatment of Ref. [31] was restricted to the sequential tunneling approximation dressed by the renormalization due to virtual processes. The processes of inelastic cotunneling [35] which dominate the dc transport through the SET at low temperatures $T \ll T_{\rm in} \sim |\Delta|/\ln(1/g)$ were not taken into account. Therefore, the extrapolation of Eq. (3) down to the zero temperature and comparison with the result of Refs. [29,30] were not possible.

The real part of the admittance of a SEB with a multichannel tunnel junction in the regime of inelastic cotunneling has been studied in Ref. [7]. It was found that the real part of admittance is proportional to $g^2\omega^2 \max\{T^4,\omega^4\}/E_c^2$. This results implies the zero charge relaxation resistance at $T=0$ in agreement with the result of Refs. [29,30] extrapolated to the limit of the infinite number of channels in the tunnel junction. However, the analysis of Ref. [7] has been restricted to Coulomb valleys, i.e., to integer values of q.

In this paper we address the following question: How does the zero-temperature result for the charge relaxation resistance obtained in Refs. [29,30] cross over to the finite temperature result of Ref. [31]? To answer this question we performed a detailed study of the admittance of the multichannel SEB near the charge neutrality points in the low-temperature regime where the inelastic cotunneling processes dominate the dynamics. We found that the real part of admittance is proportional to $g^2\omega^2 \max\{T^2,\omega^2\}/\Delta^4$. The charge relaxation resistance [extracted from Eq. (2)] is strongly temperature dependent and small, $R_q \sim (h/e^2)(T/\Delta)^2 \ll h/e^2$. In agreement with Refs. [26,29], we obtained that R_q is independent of g and vanishes at zero temperature. Our explicit results demonstrate strong violation of Korringa-Shiba relation for the response function $i\mathcal{G}(\omega)/\omega$ and, consequently, support the non-Fermi-liquid behavior of the multichannel SEB near the charge degeneracy points.

The structure of the paper is as follows. In Sec. II we introduce the Hamiltonian and Kubo formula for admittance of a single electron box. The pseudofermion representation for the low-energy Hamiltonian valid in the cotunneling regime is presented in Sec. III. The results of the calculation of the admittance at low frequencies to the second order in the tunneling conductance g are given in Sec. IV. Finally, discussion of our results and conclusions are presented in Sec. V. The details of calculations are summarized in the Appendix. We use units with $\hbar = e = 1$ throughout the paper except for the final results.

II. FORMALISM

A. Hamiltonian

We start with the standard Hamiltonian describing Coulomb blockade in a SEB [36–38]:

$$H = H_l + H_d + H_c + H_t. \tag{4}$$

Here H_l (H_d) denotes the free-electron Hamiltonian in the lead (the island),

$$H_l = \sum_k \varepsilon_k^{(a)} a_k^\dagger a_k, \qquad H_d = \sum_\alpha \varepsilon_\alpha^{(d)} d_\alpha^\dagger d_\alpha, \tag{5}$$

where operators $a_k^\dagger$ ($d_\alpha^\dagger$) create an electron in the reservoir (the island). Energies $\varepsilon_k^{(a)}, \varepsilon_\alpha^{(d)}$ are counted from the chemical potential. The Hamiltonian

$$H_c = E_c(\hat{n}_d - q)^2, \quad \hat{n}_d = \sum_\alpha d_\alpha^\dagger d_\alpha, \tag{6}$$

takes into account the electrostatic energy due to the finite size of the island.

The Hamiltonian

$$H_t = \sum_{k,\alpha} t_{k\alpha} a_k^\dagger d_\alpha + \text{H.c.} \tag{7}$$

describes tunneling of electrons between the island and the reservoir. In order to characterize the tunnel junction, following

Ref. [31], we introduce the Hermitian matrix

$$\hat{g}_{\alpha\alpha'} = (2\pi)^2\big[\delta\big(\varepsilon_\alpha^{(d)}\big)\delta\big(\varepsilon_{\alpha'}^{(d)}\big)\big]^{1/2}\sum_k t^\dagger_{\alpha k}\delta\big(\varepsilon_k^{(a)}\big)t_{k\alpha'} \quad (8)$$

acting in the Hilbert space of the island's states. Here the δ functions are assumed to be smoothed on some intermediate scale between δ and $\min\{T,|\omega|\}$. The matrix $\hat{g}$ allows one to define the number of open channels $N_{\rm ch}$ and the effective channel conductance $g_{\rm ch}$:

$$N_{\rm ch} = \frac{(\operatorname{tr}\hat{g})^2}{\operatorname{tr}\hat{g}^2}, \quad g_{\rm ch} = \frac{\operatorname{tr}\hat{g}^2}{\operatorname{tr}\hat{g}}. \quad (9)$$

The dimensionless conductance g which characterizes the tunnel junction in classical electrodynamics is given as $g = g_{\rm ch}N_{\rm ch}$.

In the present paper we assume that $N_{\rm ch} \gg 1$ and $1/N_{\rm ch}^2 \ll g_{\rm ch} \ll 1$. Although within these assumptions the classical conductance g can be still large, in what follows we restrict our consideration to the case $1 \gg g \gg 1/N_{\rm ch}$. We are interested in temperatures much smaller than the charging energy but much larger than the mean level spacing, $E_c \gg T \gg \delta$.

B. Admittance and polarization operator

The admittance of a SEB being the linear response of the ac current to the ac part of the time-dependent gate voltage, $U_g(t) = U_0 + U_\omega \cos\omega t$, can be expressed as [31]

$$\mathcal{G}(\omega) = -i\omega C_g[1 + \Pi^R(\omega)/C], \quad (10)$$

where $\Pi^R(\omega)$ stands for the Fourier transform of the retarded polarization operator of electrons on the island:

$$\Pi^R(t) = i\Theta(t)\big\langle[\hat{n}_d(t),\hat{n}_d(0)]\big\rangle. \quad (11)$$

Here $\Theta(t)$ denotes the Heaviside step function. In the quasistatic regime $\omega \to 0$, the polarization operator $\Pi^R(\omega)$ can be expanded in regular series in ω:

$$\Pi^R(\omega) = \pi_0 + i\omega\pi_1 + \mathcal{O}(\omega^2), \quad (12)$$

where both π_0 and π_1 are real functions. We stress that we assume $|\omega| \gg \delta$ throughout the paper. The static part π_0 is fully determined by the average charge on the island Q:

$$\pi_0 = \frac{C}{C_g}C_{\rm eff} - C, \quad (13)$$

where we remind that $C_{\rm eff} = \partial Q/\partial U_0$ with $Q = \langle\hat{n}_d\rangle$. This result holds by virtue of the Ward identity which relates the static polarization operator and the compressibility [39]. We note that on the classical level $C_{\rm eff} = C_g$ and $\pi_0 = 0$. The classical electrodynamics result (1) implies $\pi_1 = 2\pi C^2/g$ on the classical level. In what follows we discuss how quantum effects due to inelastic cotunneling change naive classical expectations for π_0 and π_1.

III. WEAK-TUNNELING REGIME

A. Projected Hamiltonian

In what follows, contrary to Ref. [7], we confine our consideration to the vicinity of the one of the charge degeneracy points, i.e., points where the external charge $q = k + 1/2$ where k is an integer. At these points the gap $\Delta = 2E_c(k + 1/2 - q)$ between the ground and the first excited state of the charging Hamiltonian H_c vanishes. In the vicinity of the degeneracy point, the gap Δ is small in comparison with the charging energy, $|\Delta| \ll E_c$. The processes of inelastic cotunneling become important at low temperatures $T \ll |\Delta|$. At $|\Delta| \ll E_c$, one can truncate the Hilbert space of electrons on the isolated island to two charging states characterized by $Q = k$ and $Q = k + 1$ [8]. The projected Hamiltonian acquires the form of a 2×2 matrix acting in the isospin 1/2 space of these two charging states [8]:

$$\tilde{H} = H_l + H_d + \tilde{H}_t + \Delta S_z + \frac{\Delta^2}{4E_c} + \frac{E_c}{4}, \quad (14)$$

where $H_{l,d}$ are given by Eq. (5), and

$$\tilde{H}_t = \sum_{k,\alpha} t_{k\alpha}a_k^\dagger d_\alpha S^+ + \text{H.c.} \quad (15)$$

Here S^z, $S^\pm = S^x \pm iS^y$ stand for standard spin-1/2 operators. The Hamiltonian (14) is the Hamiltonian for the $N_{\rm ch}$ channel Kondo model. The gap Δ between the charging states plays the role of an effective magnetic field.

In the presence of ac component of the gate voltage, the energy detuning from the degeneracy point becomes time dependent: $\Delta(t) = \Delta - (C_g/C)U_\omega\cos\omega t$. Then, as follows from Eq. (14), the linear response of a SEB to ac gate voltage U_ω is determined by the longitudinal dynamical isospin susceptibility:

$$\Pi_s^R(t) = i\Theta(t)\big\langle[S_z(t),S_z(0)]\big\rangle. \quad (16)$$

In particular, the admittance $\mathcal{G}(\omega)$ is given as [31]

$$\mathcal{G}(\omega) = -i\omega\frac{C_g}{C}\Pi_s^R(\omega). \quad (17)$$

We note that the average charge on the island can be written as $Q = k + 1/2 - \langle S_z\rangle$, i.e., it is related to the isospin magnetization. As the consequence of Eq. (13), the dynamical isospin susceptibility should satisfy the relation $\Pi_s^R(0) = CC_{\rm eff}/C_g$.

B. Pseudofermion effective action

To deal with spin operators we employ the method of Abrikosov's pseudofermion operators. Following Ref. [40], we introduce fermion operators $\bar{\psi}_\alpha$, ψ_α such that

$$\boldsymbol{S} = \tfrac{1}{2}\bar{\psi}_\alpha\boldsymbol{\sigma}_{\alpha\beta}\psi_\beta. \quad (18)$$

Here $\boldsymbol{\sigma} = \{\sigma_x,\sigma_y,\sigma_z\}$ denotes standard Pauli matrices. To exclude redundant unphysical states (the states with $\sum_\alpha\bar{\psi}_\alpha\psi_\alpha > 1$) we add to the Hamiltonian $\tilde{H}$ an artificial chemical potential η for the pseudofermions. It is necessary to take the limit $\eta \to -\infty$ at the end of any calculation.

We remind that the physical partition function $\mathcal{Z}$ and correlation functions $\langle\mathcal{O}\rangle$ can be found from the pseudofermion ones with the help of the following rules:

$$\mathcal{Z} = \lim_{\eta\to-\infty}\frac{\partial}{\partial e^{\beta\eta}}\mathcal{Z}_{\rm pf},$$
$$\langle\mathcal{O}\rangle = \lim_{\eta\to-\infty}\left\{\langle\mathcal{O}\rangle_{\rm pf} + \frac{\mathcal{Z}_{\rm pf}}{\mathcal{Z}}\frac{\partial}{\partial e^{\beta\eta}}\langle\mathcal{O}\rangle_{\rm pf}\right\}. \quad (19)$$

In the case $N_{\rm ch} \gg 1$ and $1 \gg g \gg 1/N_{\rm ch}$, after the integration over electrons in the reservoir and the island the

Hamiltonian (14) can be transformed into the following imaginary-time effective action [31]:

$$S = \frac{\beta\Delta^2}{4E_c} + \int_0^\beta d\tau\,\bar{\psi}\Big(\partial_\tau + \frac{\sigma_z\Delta}{2} - \eta\Big)\psi + \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2 \times \alpha(\tau_{12})[\bar{\psi}(\tau_1)\sigma_-\psi(\tau_1)][\bar{\psi}(\tau_2)\sigma_+\psi(\tau_2)]. \tag{20}$$

Here $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ and the kernel

$$\alpha(\tau) = \frac{T}{\pi}\sum_{\omega_n} |\omega_n| e^{-i\omega_n\tau}, \tag{21}$$

where $\omega_n = 2\pi T n$ is the bosonic Matsubara frequency. We note that the action similar to Eq. (20) has been first analyzed by Larkin and Melnikov in Ref. [41]. Effective action (20) corresponds to the XY Bose-Kondo model for the spin 1/2 [42–44].

The dynamical spin susceptibility (16) is determined by the pseudofermion dynamical spin susceptibility:

$$\Pi_{s,\mathrm{pf}}(\tau) = \tfrac{1}{4}\langle \mathcal{T}_\tau[\bar{\psi}(\tau)\sigma_z\psi(\tau)][\bar{\psi}(0)\sigma_z\psi(0)]\rangle, \tag{22}$$

where $\mathcal{T}_\tau$ denotes the imaginary-time ordering and the average is taken with respect to the effective action (20). According to Eq. (19) we also need the expression for the physical partition function. It can be written as

$$\mathcal{Z} = \lim_{\eta\to 0} \mathcal{Z}_{\mathrm{pf}} e^{-\beta\eta} \sum_\sigma \mathcal{G}_\sigma(\tau)\big|_{\tau\to 0^-}, \tag{23}$$

where $\mathcal{G}_\sigma(\tau) = -\langle \mathcal{T}_\tau \psi_\sigma(\tau)\bar{\psi}_\sigma(0)\rangle$ stands for the exact imaginary-time pseudofermion Green's function.

As we discussed in the Introduction, in the Fermi liquid the imaginary and real parts of the dynamical spin susceptibility are related by the so-called Korringa-Shiba relation [27,28]. Taking into account that Δ in the effective action (20) plays the role of magnetic field, the Korring-Shiba relation for Π_s^R should have the following universal form:

$$\mathrm{Im}\,\Pi_s^R(\omega) \stackrel{?}{=} 2\pi\omega\big[\mathrm{Re}\,\Pi_s^R(0)\big]^2, \quad \omega\to 0. \tag{24}$$

If this relation were correct, the low-frequency admittance $\mathcal{G}(\omega) = -i\omega(C_g/C)(C_0 + i\omega C_0^2 R_q)$ would be characterized by the universal charge relaxation resistance, $R_q = h/e^2$, similar to the single-channel case. Here we introduce $C_0 = (C/C_g)C_{\mathrm{eff}}$. As we shall demonstrate below by direct calculation, the Korringa-Shiba relation (24) *does not hold* for the effective action (20).

IV. ADMITTANCE IN THE COTUNNELING REGIME

The effective action (20) is suitable for the perturbation theory in $g \ll 1$. Here we evaluate the imaginary part of the dynamical spin susceptibility $\Pi_s(\omega)$ to the second order in g. The Feynman rules for action (20) are shown in Fig. 2. The bare pseudofermion Matsubara Green's function is given as follows:

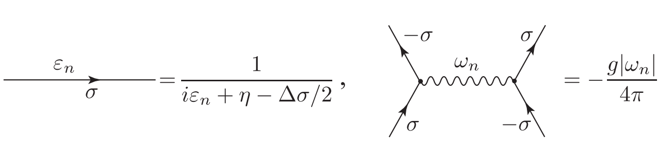

FIG. 2. Feynman rules for the pseudofermion action (20).

$$G_\sigma(i\varepsilon_n) = \frac{1}{i\varepsilon_n + \eta - \sigma\Delta/2}. \tag{25}$$

Thus from Eqs. (22) and (23) in the zeroth order in g we find

$$\mathcal{Z}_{\mathrm{pf}}^{(0)} = 1, \quad \mathcal{Z}^{(0)} = 2\cosh\frac{\beta\Delta}{2}, \quad \Pi_{s,\mathrm{pf}}^{(0)}(i\omega_n) = 0. \tag{26}$$

A. Perturbation theory in g: Sequential tunneling

Before discussing the inelastic cotunneling regime (second order in g) we remind briefly the result of Ref. [31] for the pseudofermion dynamical spin susceptibility in the regime of sequential tunneling. In Fig. 3 we present diagrams contributing to the pseudofermion dynamical spin susceptibility $\Pi_{s,\mathrm{pf}}(i\omega_n)$ in the first order in g. Their evaluation demonstrates that $\mathrm{Im}\,\Pi_s^R(\omega)$ suffers from singularity at $\omega\to 0$:

$$\mathrm{Im}\,\Pi_s^{R,(1)}(\omega) = \frac{g}{4\pi\omega}\frac{\beta\Delta}{\sinh(\beta\Delta)}. \tag{27}$$

This unphysical divergence stems from noncommutativity of the limits $\omega\to 0$ and $g\to 0$ in the structure of the $\mathrm{Im}\,\Pi_s^{R,(1)}(\omega)$. Summing the ladder-type diagrams and taking into account finite (the lowest order in g) broadening of the pseudofermion Green's function, we obtain the following expression [31]:

$$\Pi_s^{R,(\mathrm{lad})}(\omega) = \frac{g}{4\pi}\frac{\beta\Delta}{\sinh(\beta\Delta)}\left(-i\omega + \frac{g\Delta}{2\pi}\coth\frac{\beta\Delta}{2}\right)^{-1}. \tag{28}$$

We note that the broadening of the singularity at $\omega = 0$ in $\mathrm{Im}\,\Pi_{s,\mathrm{pf}}^R(\omega)$ is determined by the sum of in- and out-rate of single electron tunneling [3]. At low temperatures $T \ll |\Delta|$, the result (28) implies ($|\omega| \ll g|\Delta|/2\pi$)

$$\mathrm{Im}\,\Pi_s^{R,(\mathrm{lad})}(\omega) = \frac{2\pi\beta\omega}{g|\Delta|}e^{-\beta|\Delta|}. \tag{29}$$

Such Arrhenius-type dependence, $\exp(-|\Delta|/T)$, is characteristic of real processes in which an additional electron or hole remains on the island after each tunneling event. We remind that the SET conductance in the sequential tunneling approximation is also of the same Arrhenius-type form at low temperatures $T \ll |\Delta|$ [3,31]. As it is well known, at low temperatures SET conductance has also a power law (in $T/|\Delta|$) contribution of the second order in g due to the processes of inelastic cotunneling [35]. As we shall demonstrate in the next section there is a similar contribution to the admittance.

B. Perturbation theory in g: Inelastic cotunneling

The processes of inelastic cotunneling becomes important at temperatures $T \ll T_{\mathrm{in}} \ll |\Delta|$. However, these processes are of the second order in the tunneling conductance. Diagrams of the second order in g for the pseudofermion dynamical spin susceptibility are shown in Fig. 4. We remind that diagrams with pseudofermion loops vanish in the limit $\eta\to -\infty$. Evaluation of the ten diagrams in Fig. 4 yields the following result for the imaginary part of the low-frequency dynamical

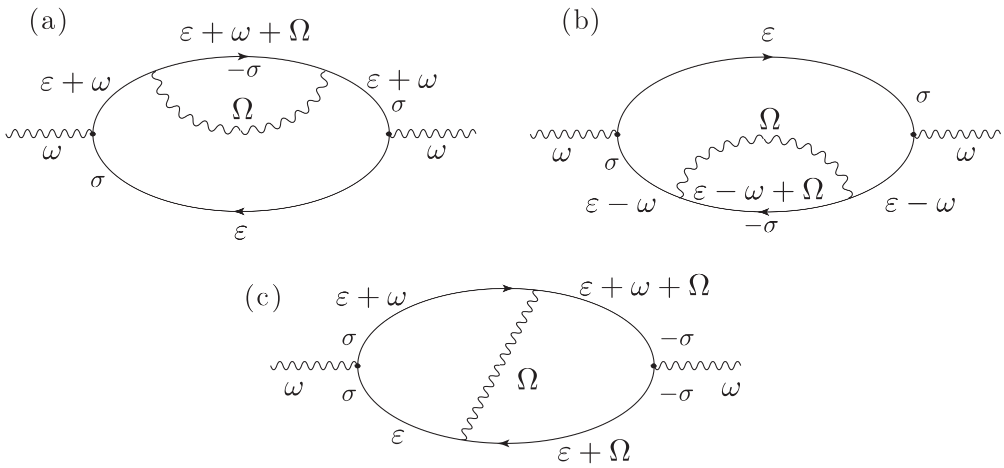

FIG. 3. Feynman diagrams for the pseudofermion dynamical spin susceptibility in the first order in g.

spin susceptibility (see the Appendix):

$$\mathrm{Im}\,\Pi_s^{R,(2)}(\omega)=\frac{g^2\omega}{24\pi\,\Delta^4}\left(T^2+\frac{\omega^2}{4\pi^2}\right),\quad |\omega|,T\ll|\Delta|. \tag{30}$$

This expression dominates over the contribution (29) due to sequential tunneling at low temperatures, $T\ll T_{\rm in}$.

In the discussion above we do not consider renormalization of the effective action (20) between the ultraviolet energy scale of the order of E_c and the infrared scale of the order of $\max\{|\Delta|,T\}$ [8,41,45]. The renormalized effective action can be obtained from Eq. (20) by the following substitutions $\psi\to\sqrt{Z}\psi$, $\bar\psi\to\sqrt{Z}\bar\psi$, $g\to\bar g=Z^2 g$, and $\Delta\to\bar\Delta=Z^2\Delta$, where the field renormalization factor Z is given within one-loop approximation as [41]

$$Z=\left[1+\frac{g}{2\pi^2}\ln\frac{E_c}{\max\{|\bar\Delta|,T\}}\right]^{-1/2}. \tag{31}$$

We note that the pseudospin operator S_z renormalizes according to $S_z\to Z^2 S_z$ [31]. Then from Eq. (30) we find

$$\mathrm{Im}\,\Pi_s^{R,(2)}(\omega)=\frac{Z^4\bar g^2\omega}{24\pi\,\bar\Delta^4}\left(T^2+\frac{\omega^2}{4\pi^2}\right). \tag{32}$$

Remarkably, the field renormalization parameter Z drops out from Eq. (32) such that it coincides with Eq. (30) in spite of the renormalization.

Taking into account one-loop renormalization of the effective action, one finds the following result for the average charge on the island [8]:

$$Q=k+\frac{1}{2}-\frac{Z^2}{2}\tanh\frac{\bar\Delta}{2T}. \tag{33}$$

In the case of low temperatures, $T\ll T_{\rm in}$, Eq. (33) can be simplified, and we obtain

$$\mathrm{Re}\,\Pi_s^{R,(1)}(0)=-\frac{\partial Q}{\partial\Delta}=\frac{Z^4\bar g}{4\pi^2|\bar\Delta|}. \tag{34}$$

We note that the factor Z^4 can be derived in the following way. Equation (31) determines the one-loop renormalization-group equation for the field renormalization factor. Then taking into account that (i) the relation $\partial\bar\Delta/\partial\Delta=Z^2$ holds within logarithmic accuracy and (ii) the renormalized conductance $\bar g$ corresponds to the energy scale $\bar\Delta$, one can obtain the result (34). Alternatively, the appearance of the factor Z^4 can be checked with the help of the expression for Q derived to the second order in g [33]. We emphasize that Eq. (34) implies that at $T\ll T_{\rm in}$ the effective capacitance becomes very different from C_g: $C_{\rm eff}=C_g Z^4\bar g E_c/(2\pi^2|\bar\Delta|)$.

Combining together Eqs. (30) and (34) we obtain the following result for the admittance of a SEB ($|\omega|\ll T\ll T_{\rm in}$):

$$\mathcal{G}(\omega)=-i\omega C_g\frac{Z^4\bar g E_c}{2\pi^2|\bar\Delta|}+\omega^2 C_g\frac{Z^4\bar g^2 T^2 E_c}{12\pi\,\bar\Delta^4}. \tag{35}$$

This is the main result of the present paper. Results (32) and (34) for the imaginary and real part of the dynamical isospin susceptibility implies *strong violation* of the Korringa-Shiba relation (24) at low temperatures when the processes of inelastic cotunneling dominate. Using the Korringa-Shiba relation, one overestimates erroneously $\mathrm{Im}\,\Pi_s^R(\omega)$ by a large factor $(\bar\Delta/T)^2\gg1$. The violation of the Korringa-Shiba relation signals that the Hamiltonian (14) and, consequently, the effective action (20), involves non-Fermi-liquid physics.

V. DISCUSSION AND CONCLUSIONS

According to Eq. (2), the result (35) for the admittance implies the following result for the charge relaxation resistance:

$$R_q=\frac{h}{e^2}\frac{\pi^2}{3}\left(\frac{T}{Z^2\bar\Delta}\right)^2,\quad T\ll T_{\rm in}\sim\frac{|\bar\Delta|}{\ln(1/\bar g)}. \tag{36}$$

We emphasize that due to Coulomb interaction the charge relaxation resistance is strongly temperature and gate voltage dependent. Moreover, R_q depends on the tunneling conductance g although through the field renormalization factor only. Therefore, the charge relaxation resistance depends in nontrivial way on the parameters of a SEB in contrast to the zero-temperature predictions of Refs. [29,30] and original ideas of Ref. [15]. Also the charge relaxation resistance is much smaller than the resistance quantum, $R_q\ll h/e^2$. At $T=0$ the charge relaxation resistance vanishes, $R_q=0$. This behavior is in agreement with the zero-temperature result of

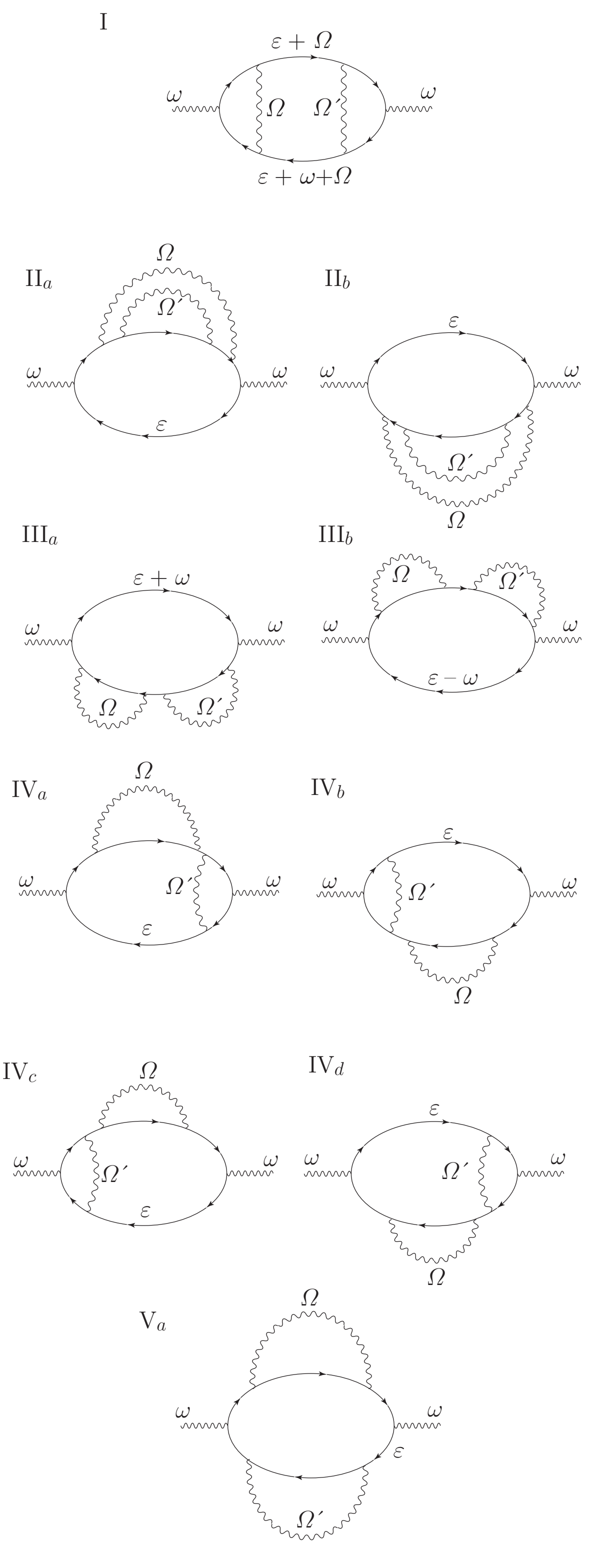

FIG. 4. Second order in g diagrams for the pseudofermion dynamical spin susceptibility.

Refs. [29,30], $R_q(T=0)=(h/e^2)/N_{\rm ch}$, which implies zero charge relaxation resistance at $T=0$ and $N_{\rm ch}\to\infty$.

The real part of the admittance determines the energy dissipation rate of a SEB, $\mathcal{W}=C_gE_c\,\mathrm{Re}\,\mathcal{G}(\omega)U_\omega^2$, which appears due to the time-dependent periodic gate voltage modulations. The result (35) leads to the ohmic dissipation at low frequencies:

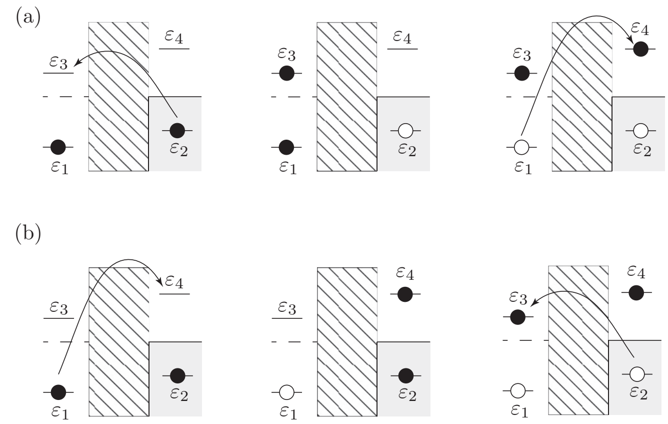

FIG. 5. The processes of inelastic cotunneling. The intermediate state has an additional electron (a) and hole (b).

$$\mathcal{W}=\omega^2\frac{Z^4\bar{g}^2T^2E_c^2}{3\pi^2\bar{\Delta}^4}C_g^2U_\omega^2,\quad |\omega|\ll T\ll T_{\rm in}. \tag{37}$$

This result for the dissipation rate has the following physical explanation. Let us estimate the rate $\Gamma_{\rm in}$ for the two-electron process in which one electron with energy ε_1 tunnels from the island into the reservoir and occupies the state with energy ε_4 whereas the other electron with energy ε_2 tunnels from the reservoir into the island occupying the state with energy ε_3 (see Fig. 5). For $\varepsilon_1\neq\varepsilon_3$ this process is inelastic and results in the electron-hole pair on the island at the end. It can go through two different intermediate states: with an additional electron and an additional hole on the island. The former costs the energy of the order of $2E_c$ whereas the latter costs the energy of the order of $|\Delta|$. In the considered case $|\Delta|\ll E_c$, we can neglect the contribution due to the intermediate state with an additional electron. Provided such transition is accompanied by a periodic perturbation which supplies the energy ω to the final state, we can estimate the corresponding rate as follows:

$$\Gamma_{\rm in}\sim\frac{g^2}{\Delta^2}\left(\prod_{j=1}^{4}\int_{-\infty}^{\infty}d\varepsilon_j\right)f_F(\varepsilon_1)[1-f_F(\varepsilon_3)]f_F(\varepsilon_2)\times[1-f_F(\varepsilon_4)]\delta(\varepsilon_3+\varepsilon_4-\varepsilon_1-\varepsilon_2-\omega). \tag{38}$$

Here we use the fact that typical electron or hole energies in the integral in Eq. (38) are of the order of $\max\{T,|\omega|\}\ll|\Delta|$. Then for $|\omega|\ll T\ll|\Delta|$ we find that the frequency-dependent part of the rate is estimated as $\Gamma_{\rm in}^{(\omega)}\sim g^2T^2\omega/\Delta^2$. In fact, this rate is similar to the rate derived for a SET biased by voltage in which case the role of ω is played by dc voltage [35]. The quadratic dependence of $\Gamma_{\rm in}^{(\omega)}$ on T is responsible for the T^2 factor in the expression (37) for the energy dissipation rate. Indeed, averaging $\Gamma_{\rm in}^{(\omega)}$ over time-dependent gate voltage, taking contribution proportional to U_ω^2 and multiplying the result by ω, one gets the estimate for the rate of energy dissipation, $\mathcal{W}\sim\omega(\partial^2\Gamma_{\rm in}^{(\omega)}/\partial\Delta^2)(C_gU_\omega/C)^2$, which, up to numerical factors, coincides with the result (37). At zero temperature the rate of inelastic cotunneling is given as $\Gamma_{\rm in}\sim g^2\omega^3/\Delta^2$. This suggests the following estimate for the energy dissipation rate: $\mathcal{W}\sim\omega^4g^2E_c^2C_gU_\omega/\Delta^4$, i.e., nonohmic dissipation of energy at zero temperature. Using the result (32) with $T = 0$, we obtain the following expression for the energy dissipation rate at zero temperature:

$$\mathcal{W} = \frac{Z^4 \bar{g}^2 \omega^4 E_c^2}{12\pi^4 \bar{\Delta}^4} C_g^2 U_\omega^2, \quad T \ll |\omega| \ll |\bar{\Delta}|. \tag{39}$$

It is worthwhile to discuss the result of Ref. [7] which is complementary to the result of the present paper. In Ref. [7] the real part of admittance was analyzed in the regime of inelastic cotunneling but precisely at $q = k$, i.e., at Coulomb valleys. It was found that $\operatorname{Re}\mathcal{G}(\omega) \propto g^2\omega^2 \max\{T^4,\omega^4\}/E_c^6$. We emphasize that the result of Ref. [7] is quite unexpected. For example, the contribution due to inelastic cotunneling is proportional to T^2/Δ^2 near the charge neutrality points and to T^2/E_c^2 at Coulomb valleys [35]. Therefore, on the basis of our result one could expect that at Coulomb valleys the energy dissipation rate is given by Eqs. (37) and (39) with Δ substituted by E_c. However, this naïve argument leads to overestimation of the energy dissipation rate by large factor $E_c^2/\max\{T^2,\omega^2\}$. The result of Ref. [7] comes from particular cancellation of terms proportional to $\omega^2 T^2/E_c^4$ and ω^4/E_c^4 in $\operatorname{Re}\mathcal{G}(\omega)$ (see comment at the end of the Appendix).

We mention that our result (35) is at odds with the expression (3) proposed by us in Ref. [31]. Since the effective charge $\mathcal{Q}$ is expected to be robustly integer quantized at zero temperature [32–34], the renormalized gate capacitance $\mathcal{C}_g$ is exponentially small at $T \ll |\bar{\Delta}|$ [33]. The renormalized conductance is known to be proportional to the temperature squared, $g(T) \sim \bar{g}^2 T^2/\bar{\Delta}^2$, in the regime of the inelastic cotunneling [33,35]. Therefore, Eq. (35) suggests the exponential suppression of the energy dissipation at $T \ll |\bar{\Delta}|$ contrary to the result (37). Thus for $g \ll 1$ the expression (3) works within the sequential tunneling approximation dressed by renormalization due to virtual processes only.

The effective action (20) predicts zero value of $\bar{g}$ under the renormalization in the infrared. The $N_{\rm ch}$ channel Kondo model (14) has the unstable fixed point at finite value of $\bar{g} = g_* \sim 1/N_{\rm ch}$ [8,46]. Therefore, our results obtained within the effective action (20) are applicable for the Hamiltonian (14) while $\bar{g} \gg g_*$. As follows from Eq. (31), this condition implies that our results hold not too close to the charge degeneracy point, $|\Delta| \gg (gE_c/\pi^2)\exp(-\pi^2/g_*)$. Since $g_* \sim 1/N_{\rm ch}$, the scale $(gE_c/\pi^2)\exp(-\pi^2/g_*)$ becomes extremely small already for not too very large values of $N_{\rm ch}$. Comparing our result (36) with the zero-temperature result of Refs. [29,30], we find that for the case of finite number of channels the charge relaxation resistance is given by Eq. (36) for temperatures $T \gg |\bar{\Delta}|/\sqrt{N_{\rm ch}}$.

To summarize, we have studied the low-frequency admittance of a multichannel SEB under a slowly oscillating gate voltage. Focusing on the regime of inelastic cotunneling, we have calculated the admittance $\mathcal{G}(\omega)$ [see Eq. (35)] near the charge degeneracy points. We found the following:

(i) At finite temperatures but low frequencies, $T_{\rm in} \gg T \gg |\omega|$, the energy dissipation rate (determined by the real part of the admittance) is ohmic and scales as the temperature squared, see Eq. (37), in agreement with qualitative arguments.

(ii) At zero temperature the energy dissipation rate is superohmic, $\sim \omega^4$, see Eq. (39), in agreement with qualitative estimates.

(iii) The imaginary and real parts of the response function $i\mathcal{G}(\omega)/\omega$ do not satisfy the Korringa-Shiba relation. This supports the non-Fermi-liquid behavior of the model near the charge degeneracy points.

(iv) The charge relaxation resistance R_q is strongly temperature dependent and small, $R_q \sim (h/e^2)(T/\bar{\Delta})^2 \ll h/e^2$. It vanishes at $T = 0$ in agreement with the recent zero-temperature analysis of Refs. [29,30].

(v) The relation between the real part of admittance and the effective charge $\mathcal{Q}$ conjected by us in Ref. [31] holds within the sequential tunneling approximation dressed by renormalization due to virtual processes only; it breaks down at low temperatures $T \ll T_{\rm in}$.

Finally, we mention that our result (35) for the admittance can be tested in a single electron box with small metallic island via radio-frequency reflectometry measurements [22,25]. Also we mention that following the approach of Refs. [47,48] our results can be extended to nonequlibrium conditions, e.g., different temperatures of the island and the reservoir.

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APPENDIX: COMPUTATION OF THE POLARIZATION OPERATOR: DIAGRAMS OF THE SECOND ORDER IN g

In this Appendix we present details of computation of the polarization operator within the second-order perturbation theory in g. There are contributions from the ten diagrams shown in Fig. 4. The task is simplified considerably by the fact that we only need the imaginary part of the retarded polarization operator. Each diagram consists of six pseudofermion Green's function lines and two interaction lines. Thus each diagram involves the summation over three internal energies: fermionic ε and two bosonic ones Ω,Ω'. As usual the fermionic sum is easily undertaken with the help of the following identity: $T\sum_\varepsilon f(\varepsilon) = (4\pi i)^{-1}\oint d\varepsilon \tanh(\varepsilon/2T) f(\varepsilon)$ where the contour of integration circles around all the poles of tanh.

1. Diagram I

The contribution from diagram I to the polarization operator can be written as

$$\begin{aligned}\Pi^{(2),\rm I}_{s,\rm pf}(i\omega_n) &= -\frac{T^3}{4}\sum_{\sigma=\pm}\sigma^2 \sum_{\varepsilon,\Omega,\Omega'} \tilde{\alpha}(i\Omega)\tilde{\alpha}(i\Omega')G_\sigma(i\varepsilon)\\ &\quad\times G_\sigma(i\varepsilon+i\omega_n)G_{-\sigma}(i\varepsilon+i\Omega)G_{-\sigma}(i\varepsilon+i\Omega+i\omega_n)\\ &\quad\times G_\sigma(i\varepsilon+i\Omega+i\Omega')G_\sigma(i\varepsilon+i\Omega+i\Omega'+i\omega_n),\end{aligned} \tag{A1}$$

where we introduce the kernel $\tilde{\alpha}(i\Omega) = g\alpha(i\Omega)/4 = g|\Omega|/(4\pi)$. Evaluating the sum over the fermionic energy ε and performing analytic continuation, $i\omega_n \to \omega + i0$, we obtain

$$\operatorname{Im}\Pi^{R,(2),\mathrm{I}}_{s,\mathrm{pf}}(\omega) = -\frac{T}{4\omega}\sum_{\sigma=\pm}(f'_\sigma + f'_{-\sigma})\operatorname{Im}K^{R,(2,1)}_\sigma(\omega) + \sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{2\omega}\operatorname{Im}\left[K^{R,(1,1)}_\sigma(\omega)\right]^2. \quad \text{(A2)}$$

Here $f_\sigma = f_F(-\eta + \Delta\sigma/2)$, $f'_\sigma = \partial f_F(\varepsilon)/\partial\varepsilon|_{\varepsilon=-\eta+\Delta\sigma/2}$, and $f_F(\varepsilon) = 1/[1+\exp(\varepsilon/T)]$ denotes the Fermi-Dirac distribution function. The functions $K^{R,(n,m)}_\sigma(\omega)$ are retarded function corresponding to the following Matsubara function

$$K^{(n,m)}_\sigma(i\omega) = T\sum_\Omega \frac{[\tilde\alpha(i\Omega + i\omega) - \tilde\alpha(i\Omega)]^n}{(i\omega)^n(i\Omega + \Delta\sigma)^m}. \quad \text{(A3)}$$

2. Diagrams IIa and IIb

The contribution from diagram IIa to the polarization operator can be written as

$$\Pi^{(2),\mathrm{IIa}}_{s,\mathrm{pf}}(i\omega_n) = -\frac{T^3}{4}\sum_{\sigma=\pm}\sigma^2\sum_{\varepsilon,\Omega,\Omega'}\tilde\alpha(i\Omega)\tilde\alpha(i\Omega') \times G^2_\sigma(i\varepsilon)G_\sigma(i\varepsilon + i\omega_n)G^2_{-\sigma}(i\varepsilon + i\Omega) \times G_\sigma(i\varepsilon + i\Omega + i\Omega'). \quad \text{(A4)}$$

The contribution from diagram IIb can be found from the expression above by reverting the sign of ω_n: $\Pi^{(2),\mathrm{IIb}}_{s,\mathrm{pf}}(i\omega_n) = \Pi^{(2),\mathrm{IIa}}_{s,\mathrm{pf}}(-i\omega_n)$. Evaluating the sum over the fermionic energy ε, combining two contributions together, and performing analytic continuation, we find

$$\operatorname{Im}\Pi^{R,(2),\mathrm{II}}_{s,\mathrm{pf}}(\omega) = -\partial_\Delta\sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{2\omega\sigma}Y_\sigma\operatorname{Im}K^{R,(1,1)}_\sigma(\omega) + \frac{T}{8}\sum_{\sigma=\pm}(f'_\sigma + f'_{-\sigma})\operatorname{Im}K^{R,(2,2)}_\sigma(\omega). \quad \text{(A5)}$$

Here we introduce the following function:

$$Y_\sigma = T\sum_\Omega\frac{\tilde\alpha(i\Omega)}{i\Omega + \Delta\sigma}. \quad \text{(A6)}$$

Strictly speaking the sum defining Y_σ is divergent. The summation is truncated at $\Omega = \Omega_{\max} \sim E_c$ which is the model cutoff. The function Y_σ is therefore cutoff dependent. It is also important to note that when evaluating summation over boson frequencies Ω' and coming across divergent expressions we assume symmetric limits $-\Omega_{\max} < \Omega' < \Omega_{\max}$. Truncated symmetric sums allows us to repeatedly shift a summation variable safely.

3. Diagrams IIIa and IIIb

The contribution from diagram IIIa to the polarization operator can be written as

$$\Pi^{(2),\mathrm{IIIa}}_{s,\mathrm{pf}}(i\omega_n) = -\frac{T^3}{4}\sum_{\sigma=\pm}\sigma^2\sum_{\varepsilon,\Omega,\Omega'}\tilde\alpha(i\Omega)\tilde\alpha(i\Omega') \times G^3_\sigma(i\varepsilon)G_\sigma(i\varepsilon + i\omega_n)G_{-\sigma}(i\varepsilon + i\Omega) \times G_{-\sigma}(i\varepsilon + i\Omega'). \quad \text{(A7)}$$

The contribution from diagram IIIb can be found from the expression above by reverting the sign of ω_n: $\Pi^{(2),\mathrm{IIIb}}_{s,\mathrm{pf}}(i\omega_n) = \Pi^{(2),\mathrm{IIIa}}_{s,\mathrm{pf}}(-i\omega_n)$. Evaluating the sum over the fermionic energy ε, combining two contributions together, and performing analytic continuation, we find

$$\operatorname{Im}\Pi^{R,(2),\mathrm{III}}_{s,\mathrm{pf}}(\omega) = \sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{2\omega}\operatorname{Im}\left[K^{R,(1,1)}_\sigma(\omega)\right]^2 + \sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{\omega^2}Y_\sigma\operatorname{Im}K^{R,(1,1)}_\sigma(\omega). \quad \text{(A8)}$$

4. Diagrams IVa, IVb, IVc, and IVd

The contribution from diagram IVa to the polarization operator can be written as

$$\Pi^{(2),\mathrm{IVa}}_{s,\mathrm{pf}}(i\omega_n) = -\frac{T^3}{4}\sum_{\sigma=\pm}(-\sigma^2)\sum_{\varepsilon,\Omega,\Omega'}\tilde\alpha(i\Omega)\tilde\alpha(i\Omega') \times G^2_\sigma(i\varepsilon)G_\sigma(i\varepsilon + i\omega_n)G_{-\sigma}(i\varepsilon + i\Omega') \times G_{-\sigma}(i\varepsilon + i\Omega)G_{-\sigma}(i\varepsilon + i\Omega + i\omega_n). \quad \text{(A9)}$$

The contribution from the other three diagrams can be found from the expression above by reverting the sign of ω_n and the sign of σ and the summation sign: $\Pi^{(2),\mathrm{IVb}}_{s,\mathrm{pf}}(i\omega_n,\sigma) = \Pi^{(2),\mathrm{IVa}}_{s,\mathrm{pf}}(-i\omega_n,-\sigma)$, $\Pi^{(2),\mathrm{IVc}}_{s,\mathrm{pf}}(i\omega_n,\sigma) = \Pi^{(2),\mathrm{IVa}}_{s,\mathrm{pf}}(i\omega_n,-\sigma)$, and $\Pi^{(2),\mathrm{IVd}}_{s,\mathrm{pf}}(i\omega_n,\sigma) = \Pi^{(2),\mathrm{IVa}}_{s,\mathrm{pf}}(-i\omega_n,\sigma)$. Evaluating the sum over the fermionic energy ε, combining all four contributions together, and performing analytic continuation, we find

$$\operatorname{Im}\Pi^{R,(2),\mathrm{IV}}_{s,\mathrm{pf}}(\omega) = -\partial_\Delta\sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{\omega\sigma}Y_\sigma\operatorname{Im}K^{R,(1,1)}_\sigma(\omega) + \sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{\omega}\operatorname{Im}\left[K^{R,(1,1)}_\sigma(\omega)\right]^2 - T\sum_{\sigma=\pm}\frac{f'_\sigma + f'_{-\sigma}}{4\omega}\operatorname{Im}\left[2K^{R,(2,1)}_\sigma(\omega) - \omega K^{R,(2,2)}_\sigma(\omega)\right]. \quad \text{(A10)}$$

5. Diagram V

The contribution from diagram V to the polarization operator can be written as

$$\Pi^{(2),\mathrm{V}}_{s,\mathrm{pf}}(i\omega_n) = -\frac{T^3}{4}\sum_{\sigma=\pm}\sigma^2\sum_{\varepsilon,\Omega,\Omega'}\tilde\alpha(i\Omega)\tilde\alpha(i\Omega')G^2_\sigma(i\varepsilon) \times G^2_\sigma(i\varepsilon + i\omega_n)G_{-\sigma}(i\varepsilon + i\Omega')G_{-\sigma}(i\varepsilon + i\Omega). \quad \text{(A11)}$$

Evaluating the sum over the fermionic energy ε and performing analytic continuation, we find

$$\begin{aligned}\mathrm{Im}\,\Pi^{R,(2),\mathrm{V}}_{s,\mathrm{pf}}(\omega)\\ &= -\sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{\omega^2}Y_\sigma\,\mathrm{Im}\,K^{R,(1,1)}_\sigma(\omega)\\ &\quad -\partial_\Delta\sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{2\omega\sigma}Y_\sigma\,\mathrm{Im}\,K^{R,(1,1)}_\sigma(\omega)\\ &\quad - T\sum_{\sigma=\pm}\frac{f'_\sigma + f'_{-\sigma}}{8\omega}\,\mathrm{Im}\left[2K^{R,(2,1)}_\sigma(\omega) - \omega K^{R,(2,2)}_\sigma(\omega)\right].\end{aligned} \tag{A12}$$

6. Result for $\mathrm{Im}\,\Pi^R_{s,\mathrm{pf}}(\omega)$ in the second order in g

Combining all contributions, Eqs. (A2), (A5), (A8), (A10), and (A12), together, we find

$$\begin{aligned}\mathrm{Im}\,\Pi^{R,(2)}_{s,\mathrm{pf}}(\omega) &= \frac{T}{2}\sum_\sigma (f'_\sigma + f'_{-\sigma})\,\mathrm{Im}\,K^{R,(2,2)}_\sigma(\omega)\\ &\quad - 2\partial_\Delta\sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{\omega\sigma}Y_\sigma\,\mathrm{Im}\,K^{R,(1,1)}_\sigma(\omega)\\ &\quad - T\sum_\sigma\frac{f'_\sigma + f'_{-\sigma}}{\omega}\,\mathrm{Im}\,K^{R,(2,1)}_\sigma(\omega)\\ &\quad + 2\sum_{\sigma=\pm}\frac{f_\sigma - f_{-\sigma}}{\omega}\,\mathrm{Im}\left[K^{R,(1,1)}_\sigma(\omega)\right]^2.\end{aligned} \tag{A13}$$

The following comment is in order here. Alternatively, one can compute the admittance by means of the current-current correlation function [7]. The latter consists of two operators which are of the first and second order in g [49]. Then the first and second lines in Eq. (A14) come from the renormalization of the operator of the first order in g whereas the third and fourth lines correspond to the contributions from the operator of the second order in g. Using Eq. (26), we find

$$\begin{aligned}\mathrm{Im}\,\Pi^{R,(2)}_s(\omega)\\ &= -2\tanh\frac{\beta\Delta}{2}\sum_{\sigma=\pm}\frac{\sigma}{\omega}\,\mathrm{Im}\left[K^{R,(1,1)}_\sigma(\omega)\right]^2\\ &\quad + \frac{2}{\cosh(\beta\Delta/2)}\partial_\Delta\sum_{\sigma=\pm}\frac{\sinh(\beta\Delta/2)}{\omega}Y_\sigma\,\mathrm{Im}\,K^{R,(1,1)}_\sigma(\omega)\\ &\quad + \sum_\sigma \mathrm{Im}\left[\frac{1}{2}K^{R,(2,2)}_\sigma - \frac{1}{\omega}K^{R,(2,1)}_\sigma(\omega)\right].\end{aligned} \tag{A14}$$

This expression is convenient for analysis at $|\omega|,T \ll |\Delta|$. Converting the sum over Matsubara frequencies in Eq. (A3) into the integral and performing analytic continuation, $i\omega \to \omega + i0$, we obtain

$$\begin{aligned}\mathrm{Im}\,K^{R,(1,1)}_\sigma(\omega) &= \int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi\omega}(\mathcal{B}_\varepsilon - \mathcal{B}_{\varepsilon+\omega})\,\mathrm{Im}\,D^R_\sigma(\varepsilon)\\ &\quad \times \mathrm{Im}\,\tilde{\alpha}^R(\varepsilon+\omega).\end{aligned} \tag{A15}$$

Here $\mathcal{B}_\varepsilon = \coth(\varepsilon/2T)$, $\tilde{\alpha}^R(\omega) = -\tilde{\alpha}^A(\omega) = -ig\omega/(4\pi)$, and $D^R_\sigma(\varepsilon) = [\varepsilon + \Delta\sigma + i0]^{-1}$. Since $\mathrm{Im}\,D^R_\sigma(\varepsilon) = -\pi\delta(\varepsilon + \Delta\sigma)$, the imaginary part of $K^{R,(1,1)}_\sigma$ is exponentially small, $\sim \exp(-|\Delta|/T)$. We note that $K^{R,(1,1)}_\sigma$ is defined by the divergent Matsubara sum. It should be understood as a finite sum truncated at $\Omega = \Omega_{\max} \sim E_c$. Then the analytical continuation is possible. Moreover the imaginary part of $K^{R,(1,1)}_\sigma$ is E_c independent.

The terms proportional to $K^{R,(1,1)}_\sigma$ are responsible for the renormalization of the first-order perturbative result (27). Therefore, only the last line in Eq. (A14) contributes to $\mathrm{Im}\,\Pi^{R,(2)}_s(\omega)$ in the regime $|\omega|,T \ll |\Delta|$.

Again converting the sun over Matsubara frequencies in Eq. (A3) into the integral and performing analytic continuation, $i\omega \to \omega + i0$, we find

$$\begin{aligned}\mathrm{Im}\,K^{R,(2,1)}_\sigma(\omega) &= \int_{-\infty}^{\infty}\frac{d\varepsilon}{\pi\omega^2}(\mathcal{B}_{\varepsilon+\omega} - \mathcal{B}_\varepsilon)\,\mathrm{Re}\,D^R_\sigma(\varepsilon)\,\mathrm{Im}\,\tilde{\alpha}^R(\varepsilon)\\ &\quad \times \mathrm{Im}\,\tilde{\alpha}^R(\varepsilon+\omega) = \left(\frac{g}{4\pi}\right)^2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}I_k}{\pi\omega^2(\Delta\sigma)^k}.\end{aligned} \tag{A16}$$

Here we perform expansion in series in $1/\Delta$. The functions I_k are defined as follows:

$$I_k = \int_{-\infty}^{\infty} d\varepsilon\,\varepsilon^k(\varepsilon+\omega)(\mathcal{B}_{\varepsilon+\omega} - \mathcal{B}_\varepsilon). \tag{A17}$$

We list several first functions I_k (for $k = 1,2,3,4$) in Table I. We note that the very same functions I_k determine the imaginary part of $K^{R,(2,2)}_\sigma(\omega)$:

TABLE I. The first few integrals I_k.

$I_1 = -\omega(\omega^2 + 4\pi^2T^2)/3$
$I_2 = \omega^2(\omega^2 + 4\pi^2T^2)/6$
$I_3 = -\omega(3\omega^4 + 20\pi^2T^2\omega^2 + 32\pi^4T^4)/30$
$I_4 = \omega^2(\omega^4 + 10\pi^2T^2\omega^2 + 24\pi^4T^4)/15$

$$\begin{aligned}\mathrm{Im}\,K^{R,(2,2)}_\sigma(\omega) &= -\sigma\,\partial_\Delta\,\mathrm{Im}\,K^{R,(2,2)}_\sigma(\omega)\\ &= \left(\frac{g}{4\pi}\right)^2\sum_{k=1}^{\infty}\frac{k(-1)^{k-1}I_k}{\pi\omega^2(\Delta\sigma)^{k+1}}.\end{aligned} \tag{A18}$$

Using the functions I_k from Table I, from Eq. (A14) we obtain the result (30) in the main text.

Finally, we note that for the case of $q = k$, i.e., at the Coulomb peaks, one cannot adopt the pseudofermion technique and has to take into account all charging states. Then, the contribution to $\mathrm{Re}\,\mathcal{G}(\omega)$ due to the inelastic cotunneling is given by the last line of Eq. (A14) multiplied by a factor of 2 and with the following substitutions: $\Delta \to E_c$ and

$$\frac{1}{\omega} \to \frac{1}{\omega}\left(1 + \frac{\omega^2}{2E_c(2E_c - \omega\sigma)}\right). \tag{A19}$$

It is due to this additional term the contribution of the order of $1/E_c^4$ cancels and the admittance becomes proportional to $1/E_c^6$ [7].

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