Residual bulk viscosity of a disordered two-dimensional electron gas

V. A. Zakharov ⊙[1] and I. S. Burmistrov[2,3]

[1]*Skolkovo Institute of Science and Technology, Moscow 121205, Russia*
[2]*L. D. Landau Institute for Theoretical Physics, Acad. Semenova av. 1-a, 142432 Chernogolovka, Russia*
[3]*Laboratory for Condensed Matter Physics, HSE University, 101000 Moscow, Russia*

Nonzero bulk viscosity signals breaking of the scale invariance. We demonstrate that disorder in a two-dimensional noninteracting electron gas in a perpendicular magnetic field results in nonzero disorder-averaged bulk viscosity. We derive an analytic expression for the bulk viscosity within the self-consistent Born approximation. This residual bulk viscosity provides the lower bound for the bulk viscosity of 2D interacting electrons at low enough temperatures.

I. INTRODUCTION

Hydrodynamic description of a viscous electron flow has a long history [1]. Progress in this field was detained by a lack of experiments (see, however, Ref. [2]). After experimental realization of graphene, there was a revival in theoretical [3–13] and experimental [14–24] research on the hydrodynamic description of an electron transport in two spatial dimensions.

In the presence of the rotational symmetry, the viscosity tensor of a two-dimensional (2D) system can be parametrized by three parameters only:

$$\eta_{jk,ls} = (\zeta - \eta_s)\delta_{jk}\delta_{ls} + \eta_s(\delta_{jl}\delta_{ks} + \delta_{js}\delta_{kl})$$
$$+ (\eta_H/2)(\epsilon_{jl}\delta_{ks} + \epsilon_{js}\delta_{kl} + \epsilon_{kl}\delta_{js} + \epsilon_{ks}\delta_{jl}). \quad (1)$$

Here ζ stands for the bulk viscosity. The shear viscosity is denoted as η_s. The second line of Eq. (1) appears if the time-reversal symmetry is broken, e.g., by a perpendicular magnetic field B. Similar to the Hall conductivity, the Hall viscosity, η_H, describes the nondissipative part of the viscosity tensor. We mention that the existence of the Hall viscosity was well appreciated a long time ago in the field of high-temperature magnetized plasma [25–29].

Although it is frequently said that viscosity exists only in the context of hydrodynamics, in fact, it has an implication on its own: as a linear response that characterizes a change of the stress tensor under time-dependent deformations [30]. In electron systems, microscopic calculation of the viscosity tensor has been traditionally performed for the shear and Hall components only [30–45]. The latter attracted much interest due to its relation to the geometrical response [32–38] and quantization for translationally and rotationally invariant gapped quantum systems [36].

It is well-known that for a monoatomic gas, the Boltzmann kinetic equation predicts zero value for the bulk viscosity [46,47]. Zero bulk viscosity implies that the system is scale invariant and can expand isotropically without dissipation. One more example of such a system is the unitary Fermi gas [48]. However, generically, interaction breaks scale invariance and results in nonzero bulk viscosity. The canonical example is the

Fermi liquid with nonzero albeit small bulk viscosity [49,50]. Recently, breaking of scale invariance has been extensively studied in the context of strongly interacting Fermi gas, both theoretically [51–57] and experimentally [58–60], as well as in quantum chromodynamics [61–64].

Typically, a condensed-matter electron system contains a quenched disorder. The presence of a random potential in the Hamiltonian inevitably breaks the scale invariance. Therefore, one may expect a nonzero value of the bulk viscosity even in the absence of electron-electron interactions.

In this paper, to unravel this issue, we consider a 2D noninteracting electron gas in the presence of a perpendicular static magnetic field and a random potential. Based on the Kubo formula for the bulk viscosity we demonstrate explicitly how a nonzero magnitude of the disorder-averaged bulk viscosity appears due to the presence of a random potential in the Hamiltonian. We find that the real part of the bulk viscosity as a function of frequency contains two contributions: (i) a delta-function peak with a weight which is determined by such thermodynamic quantities as pressure and isentropic compressibility and (ii) a smooth part depending on the total elastic scattering time τ_0. Within the self-consistent Born approximation (SCBA), we derive an expression for the smooth contribution to the real part of the bulk viscosity at a finite frequency. In the absence of the magnetic field, it acquires a remarkably simple form for all frequencies, ω, and temperatures, T, much smaller than the chemical potential, μ:

$$\text{Re}\zeta(\omega) = \hbar^2\nu_0/(2\tau_0), \quad \hbar|\omega|, k_B T \ll \mu. \quad (2)$$

Here ν_0 denotes the density of states at the Fermi level. We emphasize that $\text{Re}\zeta(\omega)$ is proportional to the elastic scattering rate in contrast to the shear viscosity which, as in other standard transport quantities, is proportional to the elastic scattering time. The result Eq. (2) indicates that to derive nonzero bulk viscosity within the kinetic equation approach, one needs to take into account higher order corrections due to impurity scattering (see discussion in Sec. VI). The results reported in this paper together with the results of Ref. [42]

provide the full answer for the viscosity tensor of 2D noninteracting electrons subjected to a perpendicular magnetic field.

The outline of the paper is as follows. In Sec. II, we formulate the problem and write down the Kubo-type expression for the bulk viscosity in which the delta-function peak at zero frequency is singled out. The weight of the delta-function peak is analyzed in Sec. III. In Sec. IV, the SCBA is reviewed. We present calculations of the bulk viscosity within SCBA in Sec. V. We end the paper with summary and conclusions (Sec. VI). Some details of calculations are given in the Appendix. Throughout the paper, we use the units in which $\hbar = k_B = c = 1$.

II. FORMALISM

A 2D electron gas in the presence of an external static perpendicular magnetic field B and a random potential $V(\mathbf{r})$ is described by the following Hamiltonian:

$$H = (-i\boldsymbol{\nabla} - e\mathbf{A})^2/(2m_e) + V(\mathbf{r}). \tag{3}$$

Here m_e denotes the electron mass. The vector potential $\mathbf{A}(\mathbf{r})$ corresponds to the static magnetic field B, $\boldsymbol{\nabla} \times \mathbf{A} = B\mathbf{e}_z$. We shall work in the Landau gauge: $A_y = Bx$ and $A_x = A_z = 0$. We assume the Gaussian distribution for a random potential with zero mean and characterized by the pair correlation function $\overline{V(\mathbf{r})V(\mathbf{r}')} = W(|\mathbf{r} - \mathbf{r}'|)$. The function $W(r)$ is assumed to decay at a typical length scale d_W. The magnetic field B is assumed to be strong enough to polarize the electron spins.

In the microscopic theory, the disorder-averaged viscosity tensor can be computed from the Kubo formula (see Eqs. (3.4), (3.11), and (3.14) of Ref. [30]):

$$\eta_{jk,ls}(\omega) = \frac{\delta_{jk}\delta_{ls}}{i\omega^+}(\kappa^{-1} - P) - \int \frac{d\varepsilon f_\varepsilon}{\pi\mathcal{A}\omega^+}\overline{\mathrm{Tr}[T_{jk}, J_{ls}]\mathrm{Im}G_\varepsilon^R} \\ + \int \frac{d\varepsilon d\Omega}{\pi^2\mathcal{A}}\frac{(f_\varepsilon - f_{\varepsilon+\Omega})}{i(\Omega - \omega^+)\omega^+}\overline{\mathrm{Tr}T_{jk}\mathrm{Im}G_{\varepsilon+\Omega}^R T_{ls}\mathrm{Im}G_\varepsilon^R}. \tag{4}$$

Here P stands for the internal pressure of the electron gas, κ^{-1} denotes the inverse isentropic compressibility at constant particle number, $\mathcal{A}$ is the system area, and $\omega^+ = \omega + i0$. The retarded Green's function is defined in a standard way, $G_\varepsilon^R = 1/(\varepsilon - H + i0)$; $f_\varepsilon = 1/[1 + \exp((\varepsilon - \mu)/T)]$ denotes the Fermi distribution function with the chemical potential μ and temperature T. The stress tensor operator $T_{jk} = m_e(v_j v_k + v_k v_j)/2$ is not affected by the presence of a random potential. Here $\mathbf{v} = (-i\boldsymbol{\nabla} - e\mathbf{A})/m_e$ is the velocity operator [30,42]. The strain generator operator J_{jk} is related with the stress tensor operator as $T_{jk} = -i[H, J_{jk}]$. We note that contrary to the stress tensor operator, the expression for J_{jk} is sensitive to the presence of a random potential. Disorder averaging in Eq. (4) is denoted by an overbar.

Bulk viscosity ζ can be derived from the viscosity tensor by tracing the spatial indices, $\zeta = \eta_{jj,ll}/d^2$, where $d = 2$ is the spatial dimension. Using Eq. (4), we find

$$\zeta(\omega) = \frac{\kappa^{-1} - P - X}{i\omega^+} + \int \frac{d\varepsilon d\Omega}{(\pi d)^2\mathcal{A}}\frac{(f_\varepsilon - f_{\varepsilon+\Omega})}{i(\Omega - \omega^+)\omega^+} \\ \times \overline{\mathrm{Tr}T_\Sigma\mathrm{Im}G_{\varepsilon+\Omega}^R T_\Sigma\mathrm{Im}G_\varepsilon^R}, \tag{5}$$

where $T_\Sigma = T_{jj}$ and the frequency independent quantity X is defined as

$$X = i\int \frac{d\varepsilon f_\varepsilon}{\pi d^2\mathcal{A}}\overline{\mathrm{Tr}[T_\Sigma, J_\Sigma]\mathrm{Im}G_\varepsilon^R}. \tag{6}$$

Here we introduce $J_\Sigma = J_{jj}$. Using the relation $T_\Sigma = 2(H - V)$, we can rewrite Eq. (5) as follows:

$$\zeta(\omega) = \frac{\kappa^{-1} - P - X}{i\omega^+} + 4\int \frac{d\varepsilon d\Omega}{(\pi d)^2\mathcal{A}}\frac{(f_\varepsilon - f_{\varepsilon+\Omega})}{i(\Omega - \omega^+)\omega^+} \\ \times \overline{\mathrm{Tr}V\mathrm{Im}G_{\varepsilon+\Omega}^R V\mathrm{Im}G_\varepsilon^R}. \tag{7}$$

It is worthwhile to emphasize that the last term on the right-hand side of the above expression represents the many-body two-point correlation function of a random potential. Thus, the structure of Eq. (7) resembles the structure of the Kubo formula for the interacting clean Fermi gas (see Ref. [65] and references therein). In our case, a random potential plays the role of the contact operator [66–68].

The expression Eq. (7) suggests the following sum rule for the disorder averaged bulk viscosity:

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi}\zeta(\omega) = P + X - \kappa^{-1}. \tag{8}$$

This expression is analogous to the sum rule found for the interacting clean Fermi gas [51,65].

Using Eq. (7), we obtain the following Kubo formula for the real part of the bulk viscosity:

$$\mathrm{Re}\zeta(\omega) = \frac{4}{d^2}\int \frac{d\varepsilon}{\pi\mathcal{A}}\frac{f_\varepsilon - f_{\varepsilon+\omega}}{\omega}\overline{\mathrm{Tr}V\mathrm{Im}G_{\varepsilon+\omega}^R V\mathrm{Im}G_\varepsilon^R} \\ + \pi\mathcal{D}\delta(\omega), \tag{9}$$

where the weight of the delta-function peak at $\omega = 0$ is given as

$$\mathcal{D} = P + X - \kappa^{-1} - \mathrm{Re}\frac{4}{d^2}\int \frac{d\varepsilon d\Omega}{\pi^2\mathcal{A}}\frac{f_\varepsilon - f_{\varepsilon+\Omega}}{\Omega - i0} \\ \times \overline{\mathrm{Tr}V\mathrm{Im}G_{\varepsilon+\Omega}^R V\mathrm{Im}G_\varepsilon^R}. \tag{10}$$

We emphasize that the appearance of a random potential V as vertices in Eq. (9) reflects the fact that the bulk viscosity vanishes in the clean case.

III. THE WEIGHT OF THE ZERO FREQUENCY DELTA-FUNCTION PEAK

The expression for the weight Eq. (10) involves the internal pressure which is proportional to the average value of the trace of the stress tensor, $P = \langle T_\Sigma \rangle/(d\mathcal{A})$. We note that the presence of a random potential affects the standard relation for a Fermi gas between the internal pressure and the energy,

$$P = \langle T_\Sigma \rangle/(d\mathcal{A}) = -\int \frac{d\varepsilon}{\pi d\mathcal{A}}f_\varepsilon\overline{\mathrm{Tr}T_\Sigma\mathrm{Im}G_\varepsilon^R} \\ = \frac{2}{d}\mathcal{E} + \frac{2}{d}\int \frac{d\varepsilon}{\pi\mathcal{A}}f_\varepsilon\overline{\mathrm{Tr}V\mathrm{Im}G_\varepsilon^R}, \tag{11}$$

where we used the relation $T_\Sigma = 2(H - V)$. Here $\mathcal{E} = \int d\varepsilon \nu(\varepsilon)\varepsilon f_\varepsilon$ denotes the energy density where $\nu(\varepsilon)$ stands for the disorder-averaged density of states. We mention that the relation Eq. (11) is analogous to Tan's relation for the pressure

of an interacting Fermi gas [68]. In our case, the random potential plays the role of the contact operator.

Next, using the relation $[T_\Sigma, J_\Sigma] = 2iT_\Sigma$ [69], we obtain

$$X = \frac{2}{d}P. \qquad (12)$$

Interestingly, this relation is not affected by the presence of a random potential.

Using Eq. (12), we rewrite the expression Eq. (10) for the weight as

$$\mathcal{D} = \frac{2+d}{d}P - \kappa^{-1} - \frac{4}{d^2}\sum_{a\neq b}\frac{f_{E_a} - f_{E_b}}{E_a - E_b}|\langle a|V|b\rangle|^2. \qquad (13)$$

Here E_a and $|a\rangle$ denote the exact eigenenergies and eigenstates for the Hamiltonian H, $H|a\rangle = E_a|a\rangle$. We note that the above expression for weight $\mathcal{D}$ explicitly involves a random potential. With the help of Eq. (12), the sum rule Eq. (8) can be rewritten as

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\zeta(\omega) = \frac{2+d}{d}P - \kappa^{-1}. \qquad (14)$$

We note that the right-hand side of Eq. (14) is purely real and depends on the thermodynamic quantities only .

In the absence of the magnetic field and disorder, the inverse isentropic compressibility is defined as $\kappa^{-1} = -\mathcal{A}(\partial P/\partial\mathcal{A})_{s\mathcal{A},n_e\mathcal{A}}$, where s and n_e denote the entropy and electron densities, respectively. Using the thermodynamic relation $Ts = \mathcal{E} + P - \mu n_e$, we find that a variation of the area $\delta\mathcal{A}$ under conditions $s\mathcal{A} =$ const and $n_e\mathcal{A} =$ const results in the following variation of the energy density: $\delta\mathcal{E} = -(\mathcal{E} + P)\delta\mathcal{A}/\mathcal{A}$. Also, a variation of the area leads to the variation of the electron density, $\delta n_e = -n_e\delta\mathcal{A}/\mathcal{A}$. Hence, we obtain [65]

$$\kappa^{-1} = (\mathcal{E} + P)\left(\frac{\partial P}{\partial\mathcal{E}}\right)_{n_e} + n_e\left(\frac{\partial P}{\partial n_e}\right)_{\mathcal{E}}. \qquad (15)$$

We note that κ^{-1} is related with the sound velocity, $c_s = 1/\sqrt{\kappa m_e n_e}$. In the absence of the disorder, $V = 0$, and the magnetic field, $B = 0$, the energy density and the pressure of the ideal Fermi gas are related as $P = 2\mathcal{E}/d$ [70]. This relation implies that the pressure is fixed if the energy density is fixed, i.e., $(\partial P/\partial n_e)_{\mathcal{E}} \equiv 0$. Then, from Eq. (15), we find $\kappa^{-1} = (d + 2)P/d$. As a result, we obtain that the weight of the delta-function peak is zero, $\mathcal{D} = 0$. Therefore, Eq. (7) implies that the bulk viscosity vanishes identically, $\zeta(\omega) = 0$, for the ideal Fermi gas in agreement with its scale invariance.

For clean 2D electron gas in the presence of a magnetic field, Eq. (13) simplifies to

$$\mathcal{D} = 2P - \kappa^{-1}. \qquad (16)$$

In that case, internal pressure differs from ordinary thermodynamic pressure on the contribution associated with the action of the Lorentz force on the edge current and expressed as $P = -(\partial(\mathcal{E}\mathcal{A})/\partial\mathcal{A})_{s\mathcal{A},n_e\mathcal{A},B} - \boldsymbol{m}B \equiv -(\partial(\mathcal{E}\mathcal{A})/\partial\mathcal{A})_{s\mathcal{A},n_e\mathcal{A},B\mathcal{A}}$, where $\boldsymbol{m}$ stands for the magnetization density [71]. Isentropic compressibility $\kappa^{-1} = -\mathcal{A}(\partial P/\partial\mathcal{A})_{s\mathcal{A},n_e\mathcal{A},B\mathcal{A}}$ is defined at the constant particle number and the magnetic flux.

Using the thermodynamic relation $Ts = \mathcal{E} + P + \boldsymbol{m}B - \mu n_e$, we find that a variation of the area $\delta\mathcal{A}$ under conditions $s\mathcal{A} =$ const, $n_e\mathcal{A} =$ const, and $B\mathcal{A} =$ const results in the following variation of the energy density: $\delta\mathcal{E} = -(\mathcal{E} + P)\delta\mathcal{A}/\mathcal{A}$. Also, a variation of the area yields the variations of the electron density, $\delta n_e = -n_e\delta\mathcal{A}/\mathcal{A}$, and the magnetic field, $\delta B = -B\delta\mathcal{A}/\mathcal{A}$. Hence, we obtain

$$\kappa^{-1} = (\mathcal{E} + P)\left(\frac{\partial P}{\partial\mathcal{E}}\right)_{n_e,B} + n_e\left(\frac{\partial P}{\partial n_e}\right)_{\mathcal{E},B} + B\left(\frac{\partial P}{\partial B}\right)_{\mathcal{E},n_e}. \qquad (17)$$

Again, in the absence of a random potential, the weight of the delta-function peak vanishes. It is easy to check this statement at zero temperature. Then, for N-filled Landau levels (LLs), we find $P = \mathcal{E} = m\omega_c^2 N^2/(4\pi)$ and $\kappa^{-1} = 2\mathcal{E}$. Hence Eq. (16) leads to $\mathcal{D} = 0$.

IV. SELF-CONSISTENT BORN APPROXIMATION

To take into account a random potential, we employ the SCBA [72]. This approximation is justified under the following conditions [73–75]:

$$1/k_F, d_W \ll l_B, \quad d_W \ll v_F\tau_0. \qquad (18)$$

Here $l_B = 1/\sqrt{eB}$ stands for the magnetic length and $k_F = m_e v_F$ stands for the Fermi momentum with the Fermi velocity denoted as v_F. The total elastic relaxation time, τ_0, in the absence of the magnetic field is defined by the following relation:

$$\frac{1}{\tau_n} = \nu_0\int_0^{2\pi}\frac{d\phi}{2\pi}\tilde{W}(2k_F\sin(\phi/2))\cos(n\phi), \quad n = 0, 1, 2, \ldots. \qquad (19)$$

Here $\tilde{W}(q)$ stands for the Fourier transform of $W(r)$. We note that the condition $k_F l_B \gg 1$ is equivalent to the condition $N \gg 1$, where N is the number of filled LLs.

Within the SCBA, the physical quantities of interest are usually fully expressed in terms of the disorder averaged retarded Green's function $\mathcal{G}_\varepsilon^R$. It satisfies the self-consistency equation, see Fig. 1(a),

$$\mathcal{G}_n^R = \left(\varepsilon - \epsilon_n - \Sigma_\varepsilon^R\right)^{-1}, \quad \Sigma_\varepsilon^R = \frac{\omega_c}{2\pi\tau_0}\sum_n\mathcal{G}_n^R, \qquad (20)$$

where $\epsilon_n = \omega_c(n + 1/2)$ denotes the energy of the nth LL and Σ_ε^R stands for the disorder-averaged self-energy. Here $\omega_c = eB/m_e$ is the cyclotron frequency. The self-consistency relation Eq. (20) can be solved analytically for Σ_ε^R in two limiting cases [72]. In the regime of a weak magnetic field, $\omega_c\tau_0 \ll 1$, when LLs overlap, one can perform summation over LL index n with the help of the Poisson formula and find [72]

$$\Sigma_\varepsilon^R = -\frac{i}{2\tau_0}(1 - 2\delta e^{2\pi i\varepsilon/\omega_c}), \qquad (21)$$

where $\delta = \exp(-\pi/\omega_c\tau_0) \ll 1$ is the Dingle parameter. In the opposite case of well-separated LLs, $\omega_c\tau_0 \gg 1$, one can restrict the summation over LL index n in Eq. (20) to $n = N$ only, where ϵ_N is the closest LL energy to the energy of interest: $|\varepsilon - \epsilon_N| < \omega_c/2$. Then one obtains [72]

$$\Sigma_\varepsilon^R = \frac{1}{2}(\varepsilon - \epsilon_N - i\sqrt{\Gamma^2 - (\varepsilon - \epsilon_N)^2}). \qquad (22)$$
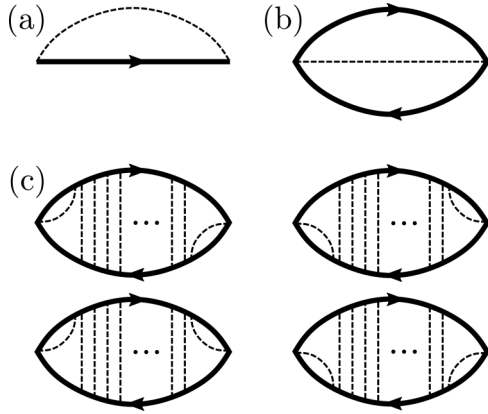
FIG. 1. Diagrams used in SCBA. (a) The self-energy diagram. (b), (c) Diagrams corresponding to the bulk viscosity within SCBA. Bold solid lines denote the disorder-averaged Green's function $\mathcal{G}_\varepsilon$, dashed lines stand for the pair correlation function $W(\mathbf{r})$.

Here the LL broadening is controlled by the energy scale $\Gamma = \sqrt{2\omega_c/(\pi\tau_0)}$. The disorder-averaged density of states can be expressed in terms of the disorder-averaged Green's function as

$$\nu_\varepsilon = -\frac{1}{2\pi^2 l_B^2}\sum_n \mathrm{Im}\,\mathcal{G}_n^R(\varepsilon) = -2\tau_0\nu_0\,\mathrm{Im}\,\Sigma_\varepsilon^R. \quad (23)$$

Using Eqs. (21) and (22), we find the disorder-averaged density of states [72]:

$$\nu_\varepsilon = \nu_0 \begin{cases} 1 - 2\delta\cos(2\pi\varepsilon/\omega_c), & \omega_c\tau_0 \ll 1 \\ \tau_0\sum_n \mathrm{Re}\sqrt{\Gamma^2 - (\varepsilon - \epsilon_n)^2}, & \omega_c\tau_0 \gg 1. \end{cases} \quad (24)$$

V. BULK VISCOSITY WITHIN SCBA

The bulk viscosity at nonzero frequency, $\omega \neq 0$, is given by the first term on the right-hand side of Eq. (9). We assume that frequency and temperature are much smaller than the chemical potential, $|\omega|, T \ll \mu$. Under this assumption, the integral over energy ε is dominated by the vicinity of the chemical potential. The unusual feature of the Kubo formula for the real part of the bulk viscosity, Eq. (9), is that vertex is a random potential. The diagrams contributing to $\mathrm{Re}\,\zeta(\omega)$ within SCBA are shown in Figs. 1(b) and 1(c).

We start from computation of the diagram of Fig. 1(b). Using Eq. (20), we can rewrite this contribution as

$$\mathrm{Re}\,\zeta^{(b)} = \int \frac{d\varepsilon}{\pi}\frac{f_\varepsilon - f_{\varepsilon+\omega}}{\omega}\frac{1}{2\pi l_B^2}\sum_n \mathrm{Im}\,\mathcal{G}_n^R(\varepsilon)\,\mathrm{Im}\,\Sigma_{\varepsilon+\omega}^R$$

$$= \int d\varepsilon \frac{f_\varepsilon - f_{\varepsilon+\omega}}{\omega}\frac{\nu_\varepsilon\nu_{\varepsilon+\omega}}{2\tau_0\nu_0}. \quad (25)$$

We note that the contribution to $\mathrm{Re}\,\zeta$ from the diagram of Fig. 1(b) can be expressed solely in terms of the density of states, ν_ε, computed within SCBA.

In addition to the diagram in Fig. 1(b), within SCBA one needs to take into account the set of diagrams shown in Fig. 1(c). They correspond to the impurity ladder insertion and describe vertex renormalization. As we shall see below, in spite of the scalar nature of the vertex (a random potential),

the diagrams of Fig. 1(c) provide a significant contribution to the real part of the bulk viscosity in the case of a strong magnetic field. Evaluation of the four diagrams in Fig. 1(c) yields (see Appendix A)

$$\mathrm{Re}\,\zeta^{(c)} = \nu_0\int d\varepsilon \frac{f_\varepsilon - f_{\varepsilon+\omega}}{\omega}\mathrm{Re}\left[\frac{(\Sigma_\varepsilon^R + \Sigma_{\varepsilon+\omega}^A)^2\Pi_0^{RA}(\omega)}{1 - \Pi_0^{RA}(\omega)/\tau_0}\right.$$

$$\left. - \frac{(\Sigma_\varepsilon^R + \Sigma_{\varepsilon+\omega}^R)^2\Pi_0^{RR}(\omega)}{1 - \Pi_0^{RR}(\omega)/\tau_0}\right]. \quad (26)$$

Here the polarization operator,

$$\Pi_0^{RA}(\omega) = \frac{\omega_c}{2\pi}\sum_n \mathcal{G}_n^R(\varepsilon+\omega)\mathcal{G}_n^A(\varepsilon) = \frac{\tau_0(\Sigma_\varepsilon^A - \Sigma_{\varepsilon+\omega}^R)}{\omega + \Sigma_\varepsilon^A - \Sigma_{\varepsilon+\omega}^R}, \quad (27)$$

provides the contribution to the bubble without the impurity ladder insertion. The expression for $\Pi_0^{RR}(\omega)$ can be obtained from Eq. (27) by changing superscript A to R. Combining contributions Eqs. (25) and (26), we find the following expression for the disorder–averaged bulk viscosity of a 2D electron gas:

$$\mathrm{Re}\,\zeta(\omega) = \int d\varepsilon \frac{f_\varepsilon - f_{\varepsilon+\omega}}{\omega}\frac{\nu_\varepsilon\nu_{\varepsilon+\omega}}{\tau_0\nu_0}\mathrm{Re}\left[\frac{1}{2} - \frac{\Sigma_{\varepsilon+\omega}^R - \Sigma_\varepsilon^R}{\omega}\right.$$

$$\left. - \frac{\nu_{\varepsilon+\omega}^2 - \nu_\varepsilon^2}{\omega\nu_\varepsilon\nu_{\varepsilon+\omega}}\left(\Sigma_{\varepsilon+\omega}^R + \Sigma_\varepsilon^R\right)\right]. \quad (28)$$

We emphasize that the above expression involves not only the density of states, ν_ε, computed within SCBA but also the real part of the SCBA self-energy. In the limit of zero frequency, the expression Eq. (28) becomes

$$\mathrm{Re}\,\zeta(\omega\to 0) = \int \frac{d\varepsilon}{2\tau_0\nu_0}(-f_\varepsilon')\nu_\varepsilon^2\left[1 - 2\frac{\partial_\varepsilon(\nu_\varepsilon^4\,\mathrm{Re}\,\Sigma_\varepsilon^R)}{\nu_\varepsilon^4}\right]. \quad (29)$$

In the absence of a magnetic field, the density of states and self-energies are independent of energy. Therefore, Eq. (28) transforms into the remarkably simple result Eq. (2). It is instructive to compare the bulk viscosity and the shear viscosity in the absence of magnetic field [31] and in the limit of zero frequency,

$$\frac{\mathrm{Re}\,\zeta(\omega\to 0)}{\eta_s} = \frac{1}{\mu^2\tau_{\mathrm{tr},2}\tau_0} \ll 1, \quad (30)$$

where $1/\tau_{\mathrm{tr},2} = 1/\tau_0 - 1/\tau_2$ denotes the inverse second transport time.

In the case of a weak magnetic field, $\omega_c\tau_0 \ll 1$, the general expression Eq. (28) can be drastically simplified. To the first order in the Dingle parameter δ, we find

$$\mathrm{Re}\,\zeta(\omega) = \frac{\nu_0}{2\tau_0}\left[1 - 4\delta\frac{\sin\Omega}{\omega}\left(1 - \frac{2\pi}{\omega_c\tau_0}\frac{\tan(\Omega/2)}{\Omega}\right)\right.$$

$$\left.\times \mathcal{F}_T\cos\frac{2\pi\mu}{\omega_c}\right]. \quad (31)$$

Here $\mathcal{F}_T = (2\pi^2 T/\omega_c)/\sinh(2\pi^2 T/\omega_c)$ and $\Omega = 2\pi\omega/\omega_c$ describe the temperature and frequency dependence of the Shubnikov–de Haas-type oscillations of the bulk viscosity, respectively. The real part of the bulk viscosity as a function of
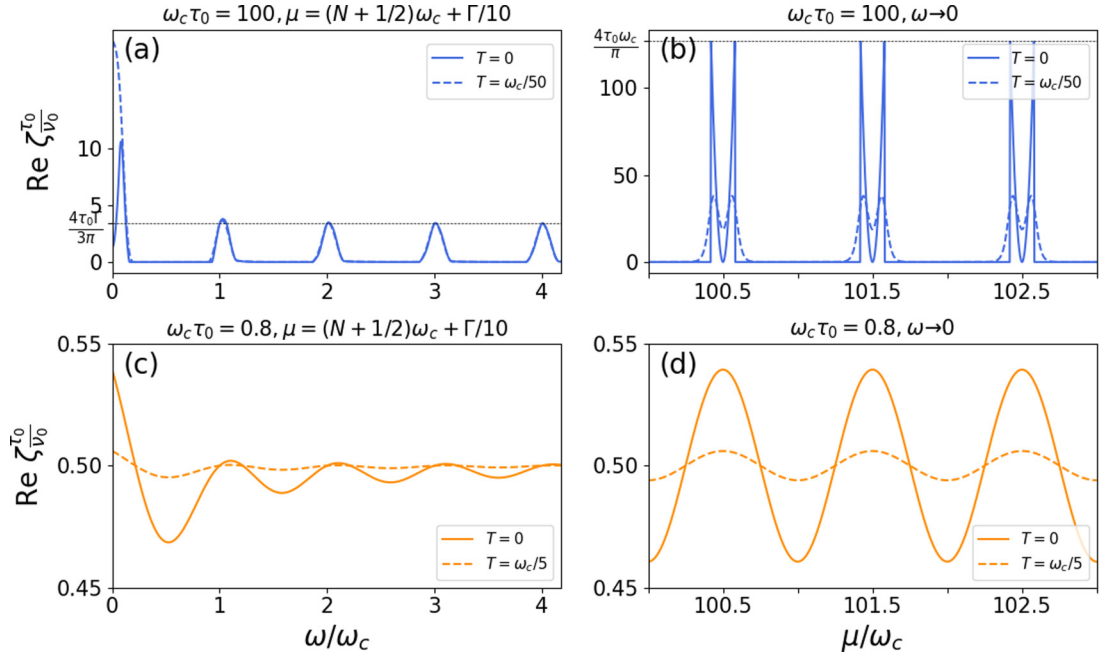
FIG. 2. The real part of the bulk viscosity in the regimes of strong [$\omega_c \tau_0 = 100$, panels (a) and (b)] and weak [$\omega_c \tau_0 = 0.8$, panels (c) and (d)] magnetic fields at zero (solid curves) and at finite (dashed curves) temperature. The dependence of $\mathrm{Re}\zeta$ on frequency is shown on panels (a) and (c). Thin black dashed line at panel (a) corresponds to the value of the bulk viscosity at the $\omega = k\omega_c$ for well-separated Landau levels, cf. Eq. (38). Panels (b) and (d) show $\mathrm{Re}\zeta(\omega \to 0)$ as a function of the chemical potential for strong and weak magnetic fields. Thin black dashed line at panel (b) corresponds to the limiting value of $\mathrm{Re}\zeta(\omega \to 0)$ at $T = 0$ in the strong magnetic field for $\mu = \epsilon_N \pm \Gamma$, cf. Eq. (32).

the frequency and the chemical potential in the case of a weak magnetic field is shown in Figs. 2(c) and 2(d). We mention that the amplitude of oscillations of $\mathrm{Re}\zeta(\omega)$ decays with the frequency as $\sim\omega^{-1}$, while at zero frequency the amplitude of oscillations of $\mathrm{Re}\zeta$ with the chemical potential is independent of μ. At zero temperature, this amplitude is enhanced by the factor $\sim 1/(\omega_c \tau_0)$ in comparison with oscillations of the density of states. Therefore, the Shubnikov–de Haas-type oscillations in the bulk viscosity are stronger than in the longitudinal conductivity [75,76] and the shear viscosity [42]. Finite temperature suppresses the amplitude of oscillations of ζ with frequency and the chemical potential, see Fig. 2.

Now we consider the case of a strong magnetic field, $\omega_c \tau_0 \gg 1$, in which LLs are well separated. Then, in the limit of zero frequency, the general result Eq. (29) can be reduced to the following expression:

$$\mathrm{Re}\zeta(\omega \to 0) = \frac{2}{\pi^2 l_B^2 \Gamma^2}\left[(\mu - \epsilon_N)^2 + \frac{\pi^2 T^2}{3}\right] \quad (32)$$

for $T, |\mu - \epsilon_N| \ll \Gamma$ and

$$\mathrm{Re}\zeta(\omega \to 0) = \frac{\Gamma}{3\pi^2 l_B^2 T}\left[1 - \frac{3\Gamma^2 + 5(\mu - \epsilon_N)^2}{20T^2}\right] \quad (33)$$

for $|\mu - \epsilon_N|, \Gamma \ll T \ll \omega_c$. For large deviations of the chemical potential from the center of the LL, we find the zero-frequency bulk viscosity as

$$\mathrm{Re}\zeta(\omega \to 0) = \frac{2e^{-(\mu - \epsilon_N)/T}}{\pi^2 l_B^2}\begin{cases}e^{\Gamma/T}, & T \ll \mu - \epsilon_N - \Gamma \\ \frac{2\Gamma}{3T}, & \Gamma \ll T \ll \mu - \epsilon_N.\end{cases} \quad (34)$$

The above results are a bit counterintuitive. At $T = 0$, the real part of the bulk viscosity vanishes when the chemical potential is at the center of the Nth LL. With deviation of μ from the center of the LL, $\mathrm{Re}\zeta(\omega \to 0)$ increases and reaches the magnitude $2/(\pi l_B)^2$ at the boundary of the disorder-broadened LL. This dependence on chemical potential is shown in Fig. 2(b). Such unusual behavior of the real part of the bulk viscosity occurs since it is proportional to the derivative of the density of states with respect to the energy. At nonzero T, a finite region of energies close to the chemical potential, $|\varepsilon - \mu| \lesssim T$, contributes to the integral over energies. Therefore, the bulk viscosity increases with rising temperature if μ lies near the center of the LL and decreases if the chemical potential is situated near the band edge.

The maximum value of $\mathrm{Re}\zeta(\omega \to 0)$ is the factor $N^2 \tau_0/\tau_{\mathrm{tr},2}$ smaller than the maximal value of the shear viscosity and the factor N^2 smaller than the maximal value of the Hall viscosity [42].

The result Eq. (28) suggests that the bulk viscosity oscillates as a function of frequency with the period ω_c. Near harmonics of the cyclotron resonance, $|\omega - k\omega_c| = |\Delta\omega| \ll \omega_c$, $k = 1, 2, \ldots$, Eq. (28) transforms into the following expression:

$$\mathrm{Re}\zeta(\omega) \approx \int d\varepsilon \frac{f_\varepsilon - f_{\varepsilon + k\omega_c + \Delta\omega}}{k\omega_c} \frac{\nu_\varepsilon \nu_{\varepsilon + \Delta\omega}}{2\tau_0 \nu_0}. \quad (35)$$

We note that at frequencies $\omega \gtrsim \omega_c$ one can neglect the terms with the self-energy on the right-hand side of Eq. (28).

At zero temperature, $T = 0$, and under the assumption that k is much smaller than the number of filled LLs, N, we obtain that the bulk viscosity near the kth harmonics of the cyclotron

resonance, $k = 1, 2, \ldots$, is given by

$$\mathrm{Re}\,\zeta(\omega) = \frac{\Gamma\Theta(2\Gamma - |\Delta\omega|)}{2\pi^2 l_B^2 \omega_c}\left[\mathcal{F}_1\left(\frac{|\Delta\omega|}{\Gamma}\right) + \frac{\mathrm{sgn}(\Delta\omega)}{k}\right.$$

$$\left.\Theta(\Gamma - |\mu - \epsilon_N|)\mathcal{F}_2\left(\frac{|\Delta\omega|}{\Gamma}, \frac{\mu - \epsilon_N}{\Gamma}\right)\right]. \quad (36)$$

Here $\Theta(x)$ stands for the Heaviside theta function and $\mathrm{sgn}(\Delta\omega)$ at $\Delta\omega = 0$ is equal to zero. The functions $\mathcal{F}_{1,2}$ are defined as $(0 \leqslant x < 2, 0 \leqslant |y| \leqslant 1)$:

$$\mathcal{F}_1(x) = \int_{-1}^{1-x} dt\,\sqrt{1-t^2}\sqrt{1-(t+x)^2},$$

$$\mathcal{F}_2(x, y) = \int_{\max\{y-x, -1\}}^{\min\{y, 1-x\}} dt\,\sqrt{1-t^2}\sqrt{1-(t+x)^2}. \quad (37)$$

We mention that this result suggests that the magnitude of the bulk viscosity at the harmonics of the cyclotron resonance is independent of the harmonics number k and the chemical potential:

$$\mathrm{Re}\,\zeta(\omega = k\omega_c) = \frac{2\Gamma}{3\pi^2 l_B^2 \omega_c}, \quad k = 1, 2, \ldots. \quad (38)$$

As one can see, the magnitude of the bulk viscosity at the harmonics of the cyclotron resonance are the factor Γ/ω_c smaller than the maximal value of the bulk viscosity at small frequencies, $|\omega| \ll \Gamma$. Dependence of $\mathrm{Re}\,\zeta$ on frequency ω is shown on Fig. 2(a). We note that the bulk viscosity decays relatively fast with detuning from the cyclotron resonance harmonics.

The effect of nonzero temperature on the bulk viscosity at finite frequency can be described as follows: Temperature enters the factor $(f_\varepsilon - f_{\varepsilon+\omega})/\omega$ in the final expression for the bulk viscosity, see Eq. (28). Adjustments of this weight function are considerable only if $\varepsilon - \mu = O(T)$ or $\varepsilon + \omega - \mu = O(T)$. Hence, at large frequencies, $\omega \gg T$, the change of the weight function due to nonzero temperature is important for a small part of the energy integration region. Therefore, at $\omega \gg T$, the temperature does not significantly affect the bulk viscosity. At low but still nonzero frequencies, the temperature effects are more significant.

VI. SUMMARY AND CONCLUSIONS

To summarize, we have developed the theory of the disorder-averaged bulk viscosity of the disordered 2D electron gas in the presence of a perpendicular magnetic field within the SCBA. We demonstrated that the real part of the bulk viscosity has two contributions: delta-function peak at zero frequency, see Eq. (16), and the smooth part, see Eq. (28). The latter is explicitly computed in the case of weak, see Eq. (31), and strong magnetic fields, see Eq. (32). Also, we analyzed the harmonics of the cyclotron resonance in the case of strong magnetic fields, see Eq. (38).

The zero-field result Eq. (2) indicates that the method of the kinetic equation is not convenient for computation of the bulk viscosity. This statement is well enough illustrated by Ref. [50] where the bulk viscosity in the clean Fermi liquid was derived from the kinetic equation. One more example

is calculations of the bulk viscosity of the clean interacting Fermi gas near the unitary limit within the kinetic equation approach [55,56,65]. However, it is worthwhile to explain for a reader how the kinetic equation can lead to the bulk viscosity which is proportional to the scattering rate, $1/\tau_0$, but not to the scattering time (as standard dissipative coefficients, e.g., the dissipative conductivity, the shear viscosity, etc.). We start from expansion of the left-hand side of the kinetic equation into formal series in $1/\tau_0$. Such an expansion can be symbolically written as $\mathcal{L}_0(n_q^{(0)} + \delta n_q) + \mathcal{L}_1 n_q^{(0)} + \cdots$. Here $n_q^{(0)}$ denotes the equilibrium distribution function and δn_q stands for the out-of-equilibrium perturbation of the distribution function induced by a bulk flow of the electron gas. The operator $\mathcal{L}_0$ coincides with the operator in the kinetic equation for the clean noninteracting electron gas [46]. As a consequence, it vanishes acting on both $n_q^{(0)}$ and δn_q. The operator $\mathcal{L}_1$ appears due to renormalization of the electron spectrum by scattering off a random potential, i.e., in other words, due to $\mathrm{Re}\,\Sigma_\varepsilon^R$. Therefore, the term $\mathcal{L}_1 n_q^{(0)}$ is proportional to $1/\tau_0$. Since the collision integral is also proportional to $\delta n_q/\tau_0$, we find that the kinetic equation yields $\delta n_q \propto (1/\tau_0)^0$. This should be contrasted with a standard situation for which $\delta n_q \propto \mathcal{L}_0 n_q^{(0)}(1/\tau_0)^{-1}$. Next, the bulk viscosity can be computed as $\zeta \propto \int d^2q\,\mathcal{C}_q \delta n_q$ [50,55,56]. However, the function $\mathcal{C}_q$ becomes nonzero only due the renormalization of the electron spectrum by scattering off a random potential, i.e., $\mathcal{C}_q \propto 1/\tau_0$ (see similar cancellation for clean interacting problem [55,56]). Again, remember that in a standard case $\mathcal{C}_q$ is independent of τ_0. Combining the estimates for δn_q and $\mathcal{C}_q$, we find that the kinetic equation results in $\zeta \propto 1/\tau_0$. We emphasize that an actual computation of $\mathcal{L}_1$ and $\mathcal{C}_q$, especially, in the presence of a magnetic field is a much more complicated task than the diagrammatic approach developed in this paper.

We mention that the viscosity tensor affects the spectrum of bulk and edge magnetoplasmons [77]. Our result for the bulk viscosity in a weak magnetic field implies that the contribution to the magnetoplasmon spectrum due to the bulk viscosity can be neglected for wave vectors $q \ll k_F$ in comparison with the contribution due to the shear viscosity.

It is instructive to estimate the magnitude of the bulk viscosity at zero magnetic field for a typical 2D electron gas in GaAs. In the absence of magnetic field, the bulk viscosity at $T = 0$ is given as $\zeta = \nu_0/2\tau_0 = e/(4\pi\mu_u) \approx 10^{-18}$ g/s, where we used the value of the mobility $\mu_u \approx 5 \times 10^4$ cm^{-2}/(V×s). For example, one may compare the above value of ζ with shear viscosity in the similar system $\eta = \hbar^2\nu_0\mu^2\tau/2 = \hbar^2 n_e^2/(4\zeta)$ [42], using the electron density $n_e \approx 10^{11}$ cm^2, one may find that $\eta \approx 10^{-15}$ g/s. Subsequently, this value may be compared with the value 10^{-12} g/s of the shear viscosity of electrons measured in graphene in the hydrodynamic regime [15]. As one can see, in our regime, the magnitude of the bulk viscosity is considerably smaller than the magnitude of the shear viscosity, which, in turn, is much smaller than typical shear viscosity in the hydrodynamic regime.

It is worthwhile to compare our result for the bulk viscosity due to a random potential with the result for the bulk viscosity in a clean weakly degenerate interacting Fermi gas. The interaction contribution to the bulk viscosity decreases with the

temperature as a power law, $\propto T^2$ (see Ref. [50] for the three-dimensional Fermi liquid). This implies that the contribution to the bulk viscosity due to disorder dominates at low enough temperatures. Therefore, we expect that our results provide the lower bound for the residual bulk viscosity in 2D interacting disordered electron system at low temperatures.

The bulk viscosity can be estimated from measurements in interacting Fermi gases [58–60]. It is an experimental challenge to extract the bulk viscosity from experiments in 2D electron systems. There are two main difficulties for possible experimental measurement of the bulk viscosity of 2D electrons that we are aware of. The first issue is that a varying in time deformation should only be applied to the electronic system while impurities should not be affected. The second one is to measure experimentally the trace of the stress tensor. The first issue may be resolved as follows. One possibility is to use a quantum well in a semiconductor heterostructure with a δ layer (in which impurities are situated). Then one can apply a time-dependent deformation only to semiconductor layers in which 2D electrons are formed. The other possibility is to use 2D electrons in van der Waals heterostructures with impurities situated in a substrate. Then again one can apply a slow time-dependent relative deformation $\delta\epsilon(t) = \epsilon \sin(2\pi f t)$ to a layer with 2D electrons only. To deal with the second problem, we propose the following. To study the bulk viscosity, one needs to measure the change in trace of the stress tensor due to an applied time-dependent deformation. Since the trace of the stress tensor is the internal pressure of the system, one can relate the change in the stress tensor with the change in the chemical potential at constant temperature, $\delta P = n_e \delta\mu$, which can be obtained from the Gibbs-Duhem relation. Then the time-dependent variation of the chemical potential, $\delta\mu(t) = (\pi f \zeta \epsilon / n_e) \cos(2\pi f t)$ can be measured, e.g., by a technique similar to one reported in Ref. [78]. Using the electron density $n_e = 10^{11}$ cm^{-2}, frequency $f = 1$ MHz, and deformation $\epsilon = 10^{-4}$, we obtain the amplitude of the change in the chemical potential of the order of 10^{-11} K.

Finally, we mention that our techniques can be extended to calculation of the bulk viscosity in a disordered graphene.

APPENDIX: LADDER CONTRIBUTION TO THE BULK VISCOSITY

In this Appendix, we present a brief derivation of Eq. (26). Diagrams corresponding to $\zeta^{(c)}$ are shown in Fig. 1(c). We find

$$\mathrm{Re}\,\zeta^{(c)} = \frac{1}{\pi\mathcal{A}} \int d\varepsilon \, \frac{f_\varepsilon - f_{\varepsilon+\omega}}{\omega} \big[X^{(1)}_{\varepsilon,\omega} + X^{(2)}_{\varepsilon,\omega} + X^{(3)}_{\varepsilon,\omega} + X^{(4)}_{\varepsilon,\omega} \big]. \tag{A1}$$

Here $X^{(1)}$ stands for the diagram in the upper-left panel in Fig. 1(c), $X^{(2)}$ for the upper-right panel, $X^{(3)}$ for the bottom-left panel, and $X^{(4)}$ for the bottom-right panel, respectively. Each of the four diagrams X_i consists of three blocks: two self-energies at the vertices and the diffuson ladder in the middle. This ladder represents an infinite sum of diagrams with the Green's functions at the top and bottom, and an arbitrary number of vertical dashed scattering lines. For computation of such diagrams, it is convenient to rewrite $\overline{\mathrm{Tr}V\,\mathrm{Im}G^R_{\varepsilon+\omega}V\,\mathrm{Im}G^R_\varepsilon}$ as $-(1/4)\overline{\mathrm{Tr}V(G^R_{\varepsilon+\omega} - G^A_{\varepsilon+\omega})V(G^R_\varepsilon - G^A_\varepsilon)}$. After such transformation, each contribution $X^{(i)}$ has four different terms with a particular combination of Green's functions. For the first contribution, we obtain

$$X^{(1)}_{\varepsilon,\omega} = -\frac{\pi\nu_0\mathcal{A}}{2}\Sigma^R_\varepsilon\Sigma^R_{\varepsilon+\omega}\Pi^{RR}_0(\omega)\sum_{n=0}^{\infty}\left(\frac{\Pi^{RR}_0(\omega)}{\tau_0}\right)^n - \frac{\pi\nu_0\mathcal{A}}{2}\Sigma^A_\varepsilon\Sigma^A_{\varepsilon+\omega}\Pi^{AA}_0(\omega)\sum_{n=0}^{\infty}\left(\frac{\Pi^{AA}_0(\omega)}{\tau_0}\right)^n$$

$$+ \frac{\pi\nu_0\mathcal{A}}{2}\Sigma^R_\varepsilon\Sigma^A_{\varepsilon+\omega}\Pi^{RA}_0(\omega)\sum_{n=0}^{\infty}\left(\frac{\Pi^{RA}_0(\omega)}{\tau_0}\right)^n + \frac{\pi\nu_0\mathcal{A}}{2}\Sigma^A_\varepsilon\Sigma^R_{\varepsilon+\omega}\Pi^{AR}_0(\omega)\sum_{n=0}^{\infty}\left(\frac{\Pi^{AR}_0(\omega)}{\tau_0}\right)^n$$

$$= -\frac{\pi\nu_0\mathcal{A}}{2}\left[\frac{\Sigma^R_\varepsilon\Sigma^R_{\varepsilon+\omega}\Pi^{RR}_0(\omega)}{1 - \Pi^{RR}_0(\omega)/\tau_0} + \frac{\Sigma^A_\varepsilon\Sigma^A_{\varepsilon+\omega}\Pi^{AA}_0(\omega)}{1 - \Pi^{AA}_0(\omega)/\tau_0} - \frac{\Sigma^R_\varepsilon\Sigma^A_{\varepsilon+\omega}\Pi^{RA}_0(\omega)}{1 - \Pi^{RA}_0(\omega)/\tau_0} - \frac{\Sigma^A_\varepsilon\Sigma^R_{\varepsilon+\omega}\Pi^{AR}_0(\omega)}{1 - \Pi^{AR}_0(\omega)/\tau_0}\right]. \tag{A2}$$

As one can check, $X^{(2)}_{\varepsilon,\omega} = X^{(1)}_{\varepsilon,\omega}$. Next, in a similar way, we find

$$X^{(3)}_{\varepsilon,\omega} = -\frac{\pi\nu_0\mathcal{A}}{2}\left[\frac{(\Sigma^R_\varepsilon)^2\Pi^{RR}_0(\omega)}{1 - \Pi^{RR}_0(\omega)/\tau_0} + \frac{(\Sigma^A_\varepsilon)^2\Pi^{AA}_0(\omega)}{1 - \Pi^{AA}_0(\omega)/\tau_0} - \frac{(\Sigma^R_\varepsilon)^2\Pi^{RA}_0(\omega)}{1 - \Pi^{RA}_0(\omega)/\tau_0} - \frac{(\Sigma^A_\varepsilon)^2\Pi^{AR}_0(\omega)}{1 - \Pi^{AR}_0(\omega)/\tau_0}\right] \tag{A3}$$

and

$$X^{(4)}_{\varepsilon,\omega} = -\frac{\pi\nu_0\mathcal{A}}{2}\left[\frac{(\Sigma^R_{\varepsilon+\omega})^2\Pi^{RR}_0(\omega)}{1 - \Pi^{RR}_0(\omega)/\tau_0} + \frac{(\Sigma^A_{\varepsilon+\omega})^2\Pi^{AA}_0(\omega)}{1 - \Pi^{AA}_0(\omega)/\tau_0} - \frac{(\Sigma^A_{\varepsilon+\omega})^2\Pi^{RA}_0(\omega)}{1 - \Pi^{RA}_0(\omega)/\tau_0} - \frac{(\Sigma^R_{\varepsilon+\omega})^2\Pi^{AR}_0(\omega)}{1 - \Pi^{AR}_0(\omega)/\tau_0}\right]. \tag{A4}$$

Combining these four contributions together, one can derive Eq. (26).

[1] R. N. Gurzhi, Hydrodynamics effects in solids at low temperature, Sov. Phys. Usp. **11**, 255 (1968).

[2] M. J. M. de Jong and L. W. Molenkamp, Hydrodynamic electron flow in high-mobility wires, Phys. Rev. B **51**, 13389 (1995).

[3] M. Müller, J. Schmalian, and L. Fritz, Graphene: A Nearly Perfect Fluid, Phys. Rev. Lett. **103**, 025301 (2009).

[4] A. V. Andreev, S. A. Kivelson, and B. Spivak, Hydrodynamic Description of Transport in Strongly Correlated Electron Systems, Phys. Rev. Lett. **106**, 256804 (2011).

[5] I. Torre, A. Tomadin, A. K. Geim, and M. Polini, Nonlocal transport and the hydrodynamic shear viscosity in graphene, Phys. Rev. B **92**, 165433 (2015).

[6] L. Levitov and G. Falkovich, Electron viscosity, current vortices and negative nonlocal resistance in graphene, Nat. Phys. **12**, 672 (2016).

[7] B. N. Narozhny, I. V. Gornyi, A. D. Mirlin, and J. Schmalian, Hydrodynamic approach to electronic transport in graphene, Ann. Phys. (Berlin) **529**, 1700043 (2017).

[8] H. Guo, E. Ilseven, G. Falkovich, and L. Levitov, Higher-than-ballistic conduction of viscous electron flows, Proc. Natl. Acad. Sci. USA **114**, 3068 (2017).

[9] O. Kashuba, B. Trauzettel, and L. W. Molenkamp, Relativistic Gurzhi effect in channels of Dirac materials, Phys. Rev. B **97**, 205129 (2018).

[10] A. Lucas and K. C. Fong, Hydrodynamics of electrons in graphene, J. Phys.: Condens. Matter **30**, 053001 (2018).

[11] E. I. Kiselev and J. Schmalian, The boundary conditions of viscous electron flow, Phys. Rev. B **99**, 035430 (2019).

[12] B. N. Narozhny, Electronic hydrodynamics in graphene, Ann. Phys. (NY) **411**, 167979 (2019).

[13] B. N. Narozhny and M. Schütt, Magnetohydrodynamics in graphene: Shear and Hall viscosities, Phys. Rev. B **100**, 035125 (2019).

[14] M. Titov, R. V. Gorbachev, B. N. Narozhny, T. Tudorovskiy, M. Schütt, P. M. Ostrovsky, I. V. Gornyi, A. D. Mirlin, M. I. Katsnelson, K. S. Novoselov, A. K. Geim, and L. A. Ponomarenko, Giant Magnetodrag in Graphene at Charge Neutrality, Phys. Rev. Lett. **111**, 166601 (2013).

[15] D. A. Bandurin, I. Torre, R. K. Kumar, M. B. Shalom, A. Tomadin, A. Principi, G. H. Auton, E. Khestanova, K. S. Novoselov, I. V. Grigorieva, L. A. Ponomarenko, A. K. Geim, and M. Polini, Negative local resistance caused by viscous electron backflow in graphene, Science **351**, 1055 (2016).

[16] J. Crossno, J. K. Shi, K. Wang, X. Liu, A. Harzheim, A. Lucas, S. Sachdev, P. Kim, T. Taniguchi, K. Watanabe, T. A. Ohki, and K. C. Fong, Observation of the Dirac fluid and the breakdown of the Wiedemann-Franz law in graphene, Science **351**, 1058 (2016).

[17] P. J. W. Moll, P. Kushwaha, N. Nandi, B. Schmidt, and A. P. Mackenzie, Evidence for hydrodynamic electron flow in PdCoO$_2$, Science **351**, 1061 (2016).

[18] R. K. Kumar, D. A. Bandurin, F. M. D. Pellegrino, Y. Cao, A. Principi, H. Guo, G. H. Auton, M. B. Shalom, L. A. Ponomarenko, G. Falkovich, K. Watanabe, T. Taniguchi, I. V. Grigorieva, L. S. Levitov, M. Polini, and A. K. Geim, Superballistic flow of viscous electron fluid through graphene constrictions, Nat. Phys. **13**, 1182 (2017).

[19] D. A. Bandurin, A. V. Shytov, G. Falkovich, R. K. Kumar, M. B. Shalom, I. V. Grigorieva, A. K. Geim, and L. S. Levitov, Fluidity onset in graphene, Nat. Commun. **9**, 4533 (2018).

[20] G. M. Gusev, A. D. Levin, E. V. Levinson, and A. K. Bakarov, Viscous transport and Hall viscosity in a two-dimensional electron system, Phys. Rev. B **98**, 161303(R) (2018).

[21] A. I. Berdyugin, S. G. Xu, F. M. D. Pellegrino, R. K. Kumar, A. Principi, I. Torre, M. B. Shalom, T. Taniguchi, K. Watanabe, I. V. Grigorieva, M. Polini, A. K. Geim, and D. A. Bandurin, Measuring Hall viscosity of graphene's electron fluid, Science **364**, 162 (2019).

[22] J. A. Sulpizio, L. Ella, A. Rozen, J. Birkbeck, D. J. Perello, D. Dutta, M. Ben-Shalom, T. Taniguchi, K. Watanabe, T. Holder, R. Queiroz, A. Stern, T. Scaffidi, A. K. Geim, and S. Ilani, Visualizing poiseuille flow of hydrodynamic electrons, Nature (London) **576**, 75 (2019).

[23] M. J. H. Ku, T. X. Zhou, Q. Li, J. K. S. Y. J. Shin, C. Burch, H. Zhang, F. Casola, T. Taniguchi, K. Watanabe, P. Kim, A. Yacoby, and R. L. Walsworth, Imaging viscous flow of the Dirac fluid in graphene using a quantum spin magnetometer, Nature (London) **583**, 537 (2020).

[24] U. Vool, A. Hamo, G. Varnavides, Y. Wang, T. X. Zhou, N. Kumar, Y. Dovzhenko, Z. Qiu, C. A. C. Garcia, A. T. Pierce, J. Gooth, P. Anikeeva, C. Felser, P. Narang, and A. Yacoby, Imaging phonon-mediated hydrodynamic flow in WTe$_2$ with cryogenic quantum magnetometry, arXiv:2009.04477.

[25] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1953).

[26] W. Marshall, *Kinetic Theory of an Ionized Gas* (Harwell, Berkshire, 1958).

[27] A. N. Kaufman, Plasma viscosity in a magnetic field, Phys. Fluids **3**, 610 (1960).

[28] W. B. Thompson, The dynamics of high temperature plasma, Rep. Prog. Phys. **24**, 363 (1961).

[29] S. I. Braginskii, in *Transport Processes in Plasma* (Gosatomizdat, Moscow, 1963), p. 183.

[30] B. Bradlyn, M. Goldstein, and N. Read, Kubo formulas for viscosity: Hall viscosity, Ward identities, and the relation with conductivity, Phys. Rev. B **86**, 245309 (2012).

[31] M. S. Steinberg, Viscosity of electron gas in metals, Phys. Rev. **109**, 1486 (1958).

[32] J. E. Avron, R. Seiler, and P. G. Zograf, Viscosity of Quantum Fluids, Phys. Rev. Lett. **75**, 697 (1995).

[33] P. Lévay, Berry phases for Landau Hamiltonians on deformed tori, J. Math. Phys. **36**, 2792 (1995).

[34] J. E. Avron, Odd viscosity, J. Stat. Phys. **92**, 543 (1998).

[35] N. Read, Non-Abelian adiabatic statistics and Hall viscosity in quantum Hall states and $p_x + ip_y$ paired superfluids, Phys. Rev. B **79**, 045308 (2009).

[36] N. Read and E. H. Rezayi, Hall viscosity, orbital spin, and geometry: Paired superfluids and quantum Hall systems, Phys. Rev. B **84**, 085316 (2011).

[37] F. D. M. Haldane, "Hall viscosity" and intrinsic metric of incompressible fractional hall fluids, arXiv:0906.1854.

[38] T. L. Hughes, R. G. Leigh, and E. Fradkin, Torsional Response and Dissipationless Viscosity in Topological Insulators, Phys. Rev. Lett. **107**, 075502 (2011).

[39] T. I. Tuegel and T. L. Hughes, Hall viscosity and momentum transport in lattice and continuum models of the integer quantum Hall effect in strong magnetic fields, Phys. Rev. B **92**, 165127 (2015).

[40] M. Sherafati, A. Principi, and G. Vignale, Hall viscosity and electromagnetic response of electrons in graphene, Phys. Rev. B **94**, 125427 (2016).

[41] M. Sherafati and G. Vignale, Hall viscosity and nonlocal conductivity of gapped graphene, Phys. Rev. B **100**, 115421 (2019).

[42] I. S. Burmistrov, M. Goldstein, M. Kot, V. D. Kurilovich, and P. D. Kurilovich, Dissipative and Hall Viscosity of a Disordered 2D Electron Gas, Phys. Rev. Lett. **123**, 026804 (2019).

[43] Y. Liao and V. Galitski, Critical viscosity of a fluctuating superconductor, Phys. Rev. B **100**, 060501(R) (2019).

[44] Y. Liao and V. Galitski, Drag viscosity of metals and its connection to Coulomb drag, Phys. Rev. B **101**, 195106 (2020).

[45] P. Rao and B. Bradlyn, Hall viscosity in Quantum Systems with Discrete Symmetry: Point Group and Lattice Anisotropy, Phys. Rev. X **10**, 021005 (2020).

[46] I. Khalatnikov, The second viscosity of monoatomic gases, Sov. Phys. JETP **2**, 169 (1955).

[47] E. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Elsevier Science, Netherlands, 1995), Vol. 10.

[48] D. T. Son, Vanishing Bulk Viscosities and Conformal Invariance of the Unitary Fermi Gas, Phys. Rev. Lett. **98**, 020604 (2007).

[49] A. A. Abrikosov and I. M. Khalatnikov, Theory of kinetic phenomena in liquid He^3, Sov. Phys. JETP **5**, 887 (1957).

[50] J. Sykes and G. A. Brooker, The transport coefficients of a Fermi liquid, Ann. Phys. (NY) **56**, 1 (1970).

[51] E. Taylor and M. Randeria, Viscosity of strongly interacting quantum fluids: Spectral functions and sum rules, Phys. Rev. A **81**, 053610 (2010).

[52] T. Enss, R. Haussmann, and W. Zwerger, Viscosity and scale invariance in the unitary Fermi gas, Ann. Phys. (NY) **326**, 770 (2011).

[53] E. Taylor and M. Randeria, Apparent Low-Energy Scale Invariance in Two-Dimensional Fermi Gases, Phys. Rev. Lett. **109**, 135301 (2012).

[54] J. Hofmann, Quantum Anomaly, Universal Relations, and Breathing Mode of a Two-Dimensional Fermi Gas, Phys. Rev. Lett. **108**, 185303 (2012).

[55] K. Dusling and T. Schäfer, Bulk Viscosity and Conformal Symmetry Breaking in the Dilute Fermi Gas Near Unitarity, Phys. Rev. Lett. **111**, 120603 (2013).

[56] C. Chafin and T. Schäfer, Scale breaking and fluid dynamics in a dilute two-dimensional Fermi gas, Phys. Rev. A **88**, 043636 (2013).

[57] T. Enss, Bulk Viscosity and Contact Correlations in Attractive Fermi Gases, Phys. Rev. Lett. **123**, 205301 (2019).

[58] M. Holten, L. Bayha, A. C. Klein, P. A. Murthy, P. M. Preiss, and S. Jochim, Anomalous Breaking of Scale Invariance in a Two-Dimensional Fermi Gas, Phys. Rev. Lett. **121**, 120401 (2018).

[59] T. Peppler, P. Dyke, M. Zamorano, I. Herrera, S. Hoinka, and C. J. Vale, Quantum Anomaly and 2D-3D Crossover in Strongly Interacting Fermi Gases, Phys. Rev. Lett. **121**, 120402 (2018).

[60] P. A. Murthy, N. Defenu, L. Bayha, M. Holten, P. Preiss, T. Enss, and J. Selim, Quantum scale anomaly and spatial coherence in a 2D Fermi superfluid, Science **365**, 268 (2019).

[61] F. Karsch, D. Kharzeev, and K. Tuchin, Universal properties of bulk viscosity near the QCD phase transition, Phys. Lett. B **663**, 217 (2008).

[62] G. D. Moore and O. Saremi, Bulk viscosity and spectral functions in QCD, J. High Energy Phys. 09 (2008) 015.

[63] P. Romatschke and D. T. Son, Spectral sum rules for the quark-gluon plasma, Phys. Rev. D **80**, 065021 (2009).

[64] Y. Akamatsu, A. Mazeliauskas, and D. Teaney, Bulk viscosity from hydrodynamic fluctuations with relativistic hydrokinetic theory, Phys. Rev. C **97**, 024902 (2018).

[65] K. Fujii and Y. Nishida, Bulk viscosity of resonating fermions revisited: Kubo formula, sum rule, and the dimer and high-temperature limits, Phys. Rev. A **102**, 023310 (2020).

[66] S. Tan, Energetics of a strongly correlated Fermi gas, Ann. Phys. (NY) **323**, 2952 (2008).

[67] S. Tan, Large momentum part of a strongly correlated Fermi gas, Ann. Phys. (NY) **323**, 2971 (2008).

[68] S. Tan, Generalized virial theorem and pressure relation for a strongly correlated Fermi gas, Ann. Phys. (NY) **323**, 2987 (2008).

[69] This result can be derived from the expansion of the relation $e^{-2\lambda}(H-V)+V = e^{-i\lambda J_\Sigma}He^{i\lambda J_\Sigma}$ to the second order in λ. See Ref. [30] for more details.

[70] L. Landau and E. Lifshitz, *Statistical Physics* (Elsevier Science, Netherlands, 2013), Vol. 5.

[71] L. Landau, E. Lifshitz, and L. Pitaevskii, *Electrodynamics of Continuous Media* (Elsevier Science, Netherlands, 1995), Vol. 8.

[72] T. Ando, A. B. Fowler, and F. Stern, Electronic properties of 2D systems, Rev. Mod. Phys. **54**, 437 (1982).

[73] M. E. Raikh and T. V. Shahbazyan, High Landau levels in a smooth random potential for two-dimensional electrons, Phys. Rev. B **47**, 1522 (1993).

[74] B. Laikhtman and E. L. Altshuler, Quasiclassical theory of Shubnikov-de Haas effect in 2D electron gas, Ann. Phys. (NY) **232**, 332 (1994).

[75] I. A. Dmitriev, A. D. Mirlin, and D. G. Polyakov, Cyclotron-Resonance Harmonics in the AC Response of a 2D Electron Gas with Smooth Disorder, Phys. Rev. Lett. **91**, 226802 (2003).

[76] I. A. Dmitriev, A. D. Mirlin, D. G. Polyakov, and M. A. Zudov, Nonequilibrium phenomena in high Landau levels, Rev. Mod. Phys. **84**, 1709 (2012).

[77] R. Cohen and M. Goldstein, Hall and dissipative viscosity effects on edge magnetoplasmons, Phys. Rev. B **98**, 235103 (2018).

[78] F. Yang, A. A. Zibrov, R. Bai, T. Taniguchi, K. Watanabe, M. P. Zaletel, and A. F. Young, Experimental Determination of the Energy Per Particle in Partially Filled Landau Levels, Phys. Rev. Lett. **126**, 156802 (2021).