Boundary multifractality in the spin quantum Hall symmetry class with interaction

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Generalized multifractality characterizes system size dependence of pure scaling local observables at Anderson transitions in all 10 symmetry classes of disordered systems. Recently, the concept of generalized multifractality has been extended to boundaries of critical disordered noninteracting systems. Here we study the generalized boundary multifractality in the presence of electron-electron interaction, focusing on the spin quantum Hall symmetry class (class C). Employing the two-loop renormalization group analysis within the Finkel'stein nonlinear sigma model, we compute the anomalous dimensions of the pure scaling operators located at the boundary of the system. We find that generalized boundary multifractal exponents are twice larger than their bulk counterparts. Exact symmetry relations between generalized boundary multifractal exponents in the case of noninteracting systems are explicitly broken by the interaction.

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I. INTRODUCTION

A fascinating example of a quantum phase transition in a free fermion system is the Anderson transition [1]. This transition is controlled by disorder and separates metallic and insulating phases. An additional boost to studies of the Anderson transition is provided by the fact that some Anderson transitions occur between distinct topological (insulating) phases, e.g., integer quantum Hall plateau-plateau transitions. An intriguing feature of the Anderson transition is the strong mesoscopic fluctuations of electron wave functions at criticality [2,3]. Consequently, the disorder-averaged moments of the local density of states (LDOS) demonstrate *pure* power-law scaling with the system size, $\langle \rho^q \rangle \sim L^{-x_{(q)}}$. Here, values of the *multifractal* exponents $x_{(q)}$ depend on a symmetry class of the considered random Hamiltonian (see Refs. [4,5] for a review).

There are many more pure scaling observables in addition to the moments of LDOS [6]. They can be expressed in terms of disorder averages of specific combinations of wave functions [7–10]. The corresponding set of multifractal exponents x_λ, termed as *generalized multifractality*, is a unique characteristic of the Anderson transition in each symmetry class. Exponents x_λ are related by symmetry relations specific for each symmetry class [7,11,12].

Recently, it has been established that the statistics of wave functions at the surface (s) of a system undergoing a bulk Anderson transition is different from the statistics in the bulk [13–17]. In particular, the scaling of the LDOS moments at the boundary is given as $\langle \rho^q(\boldsymbol{r}\in s)\rangle \sim L^{-x^{(s)}_{(q)}}$, with $x^{(s)}_{(q)} \neq x_{(q)}$. In Ref. [18], the theory of generalized multifractality has been extended to boundaries of critical systems.

The picture of generalized multifractality at Anderson transitions has recently been fully supported by numerics in the symmetry classes A, C, AIII, AII, D, and DIII [8–10,19,20]. However, multifractality is of relevance for experiments as well. Light waves spreading in an array of dielectric nanoneedles demonstrated multifractal behavior in experiments reported in Ref. [21]. Multifractal behavior of ultrasound waves was observed while they propagated through a system of randomly packed Al beads [22]. In the experiment [23], the electron LDOS was measured by scanning tunneling microscopy on a surface of diluted magnetic semiconductor $Ga_{1-x}Mn_xAs$. While it was tuned through a bulk Anderson transition, multifractal signatures in LDOS have been measured. In the experiment on $Ga_{1-x}Mn_xAs$, the surface multifractality was presumably observed. The multifractal behavior of the LDOS amplitude has recently been measured in a weakly disordered superconducting state in the stripped incommensurate phase of monolayer Pb/Si(111) [24].

Multifractality is responsible for many nontrivial physical effects. It was shown [25–32] that multifractal correlations effectively increase electron-electron attraction and, thus, lead to strong enhancement of the superconducting transition temperature and the superconducting gap at zero temperature. Moreover, it was found that multifractality is responsible for the log-normal distribution of the superconducting order parameter [26,33,34] and LDOS [30,31,35] in dirty superconducting films. Multifractal correlations result in instabilities of surface states in topological superconductors [36,37]. The multifractal behavior of LDOS causes strong mesoscopic fluctuations of the Kondo temperature [38–40]. Multifractality affects electron-phonon coupling, making the cooling of electrons more efficient [41]. The Anderson orthogonality catastrophe is also affected by the multifractal properties of wave functions [42]. Multifractality in LDOS enhances the depairing effect of magnetic impurities on the superconducting state in dirty films [43] and the superconducting LDOS around Yu-Shiba-Rusinov states [44].

Recently, it has been suggested that multifractality can serve as a sensitive instrument to test critical theories proposed to describe Anderson transitions. Although, for each of the 10 Altland-Zirnbauer symmetry classes, an effective, long-wave description in terms of a nonlinear sigma model (NLσM) is known (see Ref. [5] for a review), the Anderson transition typically occurs in strong coupling. Thus, Anderson transition criticality is beyond the standard treatment of NLσM. A prime example of such a situation is the integer quantum Hall plateau transition for which the Wess-Zumino-Novikov-Witten models were conjectured to be an ultimate conformal critical theory [45–50]. It turns out that assumptions of the local conformal invariance and Abelian fusion rules result in the parabolic form[1] of the generalized multifractal exponents x_λ with a single free parameter only [19,52]. However, available numerical computations of the multifractal spectrum for the integer quantum Hall plateau transitions demonstrate significant deviations from the exact parabolicity [16,17,19]. This makes the theoretical suggestions of the Wess-Zumino-Novikov-Witten models as critical theories for the integer quantum Hall transition to be highly questionable.

An even more dramatic situation is in the superconducting cousin of the integer quantum Hall effect—the spin quantum Hall effect (class C) [53–55]. An advantage of the spin quantum Hall transition in $d = 2$ is that an infinite subset of generalized multifractal exponents is known analytically from exact mapping to the percolation problem [8,15,56–59]. The rigorous analytical results serve as a benchmark against numerical computations. Although numerical data for the generalized multifractal spectrum reproduce exact analytical results, they demonstrate clear evidence for a violation of parabolicity [8–10,19,20,58,60]. Similarly, parabolicity is expected to hold for the surface generalized multifractal exponents in the presence of the local conformal invariance and the Abelian fusion. Again, for the class C, the numerics does not support parabolicity of the boundary multifractal exponents, but coincides, simultaneously, with the exact analytical values of the exponents [18]. These results prove a lack of the local conformal invariance at the spin quantum Hall transition in $d = 2$.

Electron-electron interaction, typically being a relevant perturbation, modifies the scaling properties of the observables at Anderson (or, in that case, the so-called Mott-Anderson) transitions (see Refs. [61,62] for a review). Surprisingly, the generalized multifractality exists even in the presence of interaction, i.e., at Mott-Anderson criticality [63–66]. In this case, the pure scaling operators can be formulated as proper correlations of single-particle Green's functions. In particular, the moments of LDOS remain pure scaling operators. Although interaction does not change the form of the pure scaling operators (except straightforward generalization to incorporate a set of Matsubara frequencies), it affects the generalized multifractal exponents. In particular, it breaks the symmetry relations between different multifractal exponents.

In this paper, we develop the theory of the generalized boundary multifractality for the spin quantum Hall symmetry class in the presence of electron-electron interaction. Using the Finkel'stein NLσM for class C, we compute the anomalous dimensions of the pure scaling derivativeless local operators situated near the boundary in the two-loop renormalization group (RG) approximation. Surprisingly, within the two-loop approximation, we find that the anomalous dimensions of pure scaling operators at the boundary and in the bulk differ by a factor of 2. Also, the interaction breaks the symmetry relations between the generalized surface multifractal exponents in the same way as for the bulk ones.

Throughout the paper, we use terms "surface" and "boundary" interchangeably, as they both have been used in the previous literature on multifractality. Also we note that in d dimensions, the surface is understood as a $(d-1)$-dimensional boundary.

The outline of the paper is as follows. In Sec. II, we remind the reader of the formalism of the Finkel'stein NLσM for class C. We summarize the results for generalized bulk multifractality in the presence of interaction (Sec. III). The original results for generalized surface multifractality in the presence of interaction are presented in Sec. IV. We end the paper with discussions and conclusions in Sec. V. The details of the computations are given in the Appendix.

II. FINKEL'STEIN NLσM FOR CLASS C

A. NLσM action

We start from a brief reminder of the Finkel'stein NLσM for the class C (see Refs. [66–70] for details). We use notations from Ref. [66]. The grand canonical partition function is given as

$$Z = \int D[Q] \exp S, \quad S = S_0 + S_{\rm int}, \tag{1}$$

where S_0 and $S_{\rm int}$ are free and interacting parts of the action. We note that the action also involves the topological term similar to the class A. However, we omit the topological term in this paper since we focus on the perturbative treatment of the model. S_0 and $S_{\rm int}$ are as follows:

$$S_0 = -\frac{g}{16}\int_{\boldsymbol{r}} \operatorname{Tr}(\nabla Q)^2 + Z_\omega \int_{\boldsymbol{r}} \operatorname{Tr}\hat{\varepsilon} Q, \tag{2a}$$

$$S_{\rm int} = -\frac{\pi T \Gamma_t}{4}\sum_{\alpha,n}\int_{\boldsymbol{r}} \operatorname{Tr}\big(I_n^\alpha \mathsf{s} Q\big)\operatorname{Tr}\big(I_{-n}^\alpha \mathsf{s} Q\big), \tag{2b}$$

where $\int_{\boldsymbol{r}} \equiv \int d^d\boldsymbol{r}$ and T stands for temperature. The matrices I_n^α, $\hat{\varepsilon}$, and s are defined below; cf. Eqs. (5) and (6). The field variable Q is a Hermitian matrix, $Q^\dagger = Q$, acting in the 2×2 spin space, in the $N_r \times N_r$ replica space, and in the $2N_m \times 2N_m$ space of the Matsubara fermionic energies, $\varepsilon_n = \pi T(2n+1)$. The matrix Q satisfies the nonlinear local constraint

$$Q^2(\boldsymbol{r}) = 1, \tag{3}$$

and obeys the Bogoliubov–de Gennes symmetry relation

$$Q = -\bar{Q}, \quad \bar{Q} = \mathsf{s}_2 L_0 Q^{\mathsf{T}} L_0 \mathsf{s}_2. \tag{4}$$

Here, superscript T denotes the matrix transposition operation. Several of the matrices introduced above are given as

[1]We note that the parabolicity of x_λ arises in any dimensionality $d \geqslant 2$ in the case of conformal invariance [51].

follows:

$$(I_n^\alpha)_{km^{\beta\gamma}} = \delta_{k-m,n}\delta^{\beta\alpha}\delta^{\gamma\alpha}\mathsf{s}_0, \quad \hat{\varepsilon}_{nm^{\alpha\beta}} = \varepsilon_n\,\delta_{nm}\delta^{\alpha\beta}\mathsf{s}_0,$$
$$(L_0)_{nm^{\alpha\beta}} = \delta_{\varepsilon n,-\varepsilon m}\delta^{\alpha\beta}\mathsf{s}_0, \tag{5}$$

where s_0 is the 2×2 identity matrix in the spin space. The Latin indices represent Matsubara energies, whereas the Greek indices correspond to replica space. The vector $\mathsf{s} = \{\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3\}$ is the vector of three nontrivial Pauli matrices,

$$\mathsf{s}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{s}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathsf{s}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

Nonlinear constraint (3) can be resolved by

$$Q = \mathtt{T}^{-1}\Lambda\mathtt{T}, \quad \Lambda_{nm}^{\alpha\beta} = \operatorname{sgn}\varepsilon_n\,\delta_{nm}\delta^{\alpha\beta}\mathsf{s}_0. \tag{7}$$

Here the rotation $\mathtt{T}$ is a unitary matrix satisfying

$$\mathtt{T}^{-1} = \mathtt{T}^\dagger, \quad (\mathtt{T}^{-1})^{\mathsf{T}}L_0\mathsf{s}_2 = \mathsf{s}_2L_0\mathtt{T}. \tag{8}$$

Parametrization (7) and condition (8) fix the target space of the NLσM as $Q\in\mathrm{Sp}(2N)/\mathrm{U}(N)$, where $N = 2N_rN_m$. We note that one needs to take the limits $N_m \to \infty$ and $N_r \to 0$.

The NLσM action (1) involves a bare dimensionless spin conductance g, a bare exchange interaction Γ_t, and a parameter Z_ω, which is responsible for frequency renormalization.

As we shall see below, in order to extract singular infrared behavior within the NLσM action, it is convenient to add the following regulator into the action (1):

$$S_{\mathrm{h}} = \frac{gh^2}{8}\int_{\boldsymbol{r}}\mathrm{Tr}\Lambda Q. \tag{9}$$

We note that the NLσM action (2a), (2b), and (9) can be reduced to the NLσM for the class A by breaking spin rotation symmetry from SU(2) down to U(1) such that the Q matrix in the spin space acquires the diagonal form,

$$Q = \begin{pmatrix} Q_\uparrow & 0 \\ 0 & Q_\downarrow \end{pmatrix}, \quad Q_\downarrow = -L_0Q_\uparrow^{\mathsf{T}}L_0. \tag{10}$$

B. Perturbation theory

In order to proceed, we need to develop perturbation theory in $1/g \ll 1$. Since in this work we are interested in boundary multifractality, we consider a two-dimensional (2D) sample with the boundary at $x = 0$ (see Fig. 1). In what follows, we will employ the dimensional regularization method such that we will work in $d = 2+\epsilon$ dimensions. We parametrize a d-dimensional coordinate vector as $\boldsymbol{r}=\{x, y_1, \dots, y_{d-1}\}$.

Also we will use the square-root parametrization of the Q matrix,[2]

$$Q = W + \Lambda\sqrt{1-W^2}, \quad W = \begin{pmatrix} 0 & w \\ w^\dagger & 0 \end{pmatrix}, \tag{11}$$

where we adopt the following notations: $W_{n_1n_2}=w_{n_1n_2}$ and $W_{n_2n_1}=w^\dagger_{n_2n_1}$ with $\varepsilon_{n_1} > 0$ and $\varepsilon_{n_2}<0$. Making expansion

[2]This transformation has the Jacobian different from the unity [71]. However, the additional contribution to the action due to the Jacobian vanishes in the dimensional regularization scheme which we employ in this work [72].

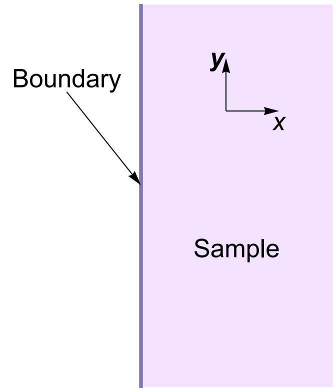

FIG. 1. Sketch of the system with a boundary perpendicular to the x axis and situated at $x = 0$.

$w=\sum_{\mathrm{j}=0}^3 w_{\mathrm{j}}\mathsf{s}_{\mathrm{j}}$, we find that the elements of w_{j} satisfy the symmetry relations [cf. Eq. (4)]

$$(w_{\mathrm{j}})_{n_1n_2}^{\alpha\beta} = \mathsf{v}_{\mathrm{j}}(w_{\mathrm{j}})_{-n_2,-n_1}^{\beta\alpha}, \tag{12}$$

where $\mathsf{v}_{\mathrm{j}}= -\operatorname{tr}(\mathsf{s}_{\mathrm{j}}\mathsf{s}_2\mathsf{s}_{\mathrm{j}}^{\mathsf{T}}\mathsf{s}_2)/2=\{-1, 1, 1, 1\}$.

From the second-order expansion of Eq. (1) in W, we find the propagators of Gaussian theory,

$$\begin{aligned}&\left\langle (w_{\mathrm{j}})_{n_1n_2}^{\alpha\beta}(\boldsymbol{r})(w_{\mathrm{j}}^\dagger)_{n_4n_3}^{\mu\nu}(\boldsymbol{r}')\right\rangle \\ &\quad = \frac{2}{g}\Big[\big(\delta^{\alpha\nu}\delta^{\beta\mu}\delta_{n_1n_3}\delta_{n_2n_4} + \mathsf{v}_{\mathrm{j}}\delta^{\alpha\mu}\delta^{\beta\nu}\delta_{n_1,-n_4}\delta_{n_2,-n_3}\big) \\ &\quad\quad \times \hat{\mathcal{D}}\big(i\omega_{n_{12}};\boldsymbol{r},\boldsymbol{r}'\big) - \frac{4\pi T\gamma}{D}(1-\delta_{\mathrm{j}0})\delta^{\alpha\nu}\delta^{\beta\mu}\delta^{\alpha\beta}\delta_{n_{12},n_{34}} \\ &\quad\quad \times \widehat{\mathcal{D}\mathcal{D}^t}\big(i\omega_{n_{12}};\boldsymbol{r},\boldsymbol{r}'\big)\Big],\end{aligned} \tag{13}$$

where we denote $\omega_{n_{12}}=\varepsilon_{n_1}-\varepsilon_{n_2}$ and $n_{12}=n_1-n_2$. Next, $D=g/(4Z_\omega)$ and $\gamma=\Gamma_t/Z_\omega$ are a bare diffusion coefficient and a dimensionless interaction strength, respectively. Diffuson and diffuson dressed by interaction via ladder resummation are given as

$$\hat{\mathcal{D}}\big(i\omega_{n_{12}};\boldsymbol{r},\boldsymbol{r}'\big) = \sum_{s=\pm}\mathcal{D}\big(i\omega_{n_{12}};x-sx',\boldsymbol{y}-\boldsymbol{y}'\big), \tag{14a}$$

$$\hat{\mathcal{D}}^t\big(i\omega_{n_{12}};\boldsymbol{r},\boldsymbol{r}'\big) = \sum_{s=\pm}\mathcal{D}^t\big(i\omega_{n_{12}};x-sx',\boldsymbol{y}-\boldsymbol{y}'\big), \tag{14b}$$

where $\boldsymbol{y} = \{y_1,\dots,y_{d-1}\}$. Here, $\mathcal{D}(i\omega_n;x,\boldsymbol{y})$ and $\mathcal{D}^t(i\omega_n;x,\boldsymbol{y})$ correspond to the diffusons in an infinite sample $[\int_{\boldsymbol{q}} \equiv \int d^d\boldsymbol{q}/(2\pi)^d]$,

$$\mathcal{D}^{/t}(i\omega_n;x,\boldsymbol{y}) = \int_{\boldsymbol{q}}\mathcal{D}_q^{/t}(i\omega_n)e^{iq_xx+i\boldsymbol{q}_\parallel\boldsymbol{y}}, \tag{15}$$

with the standard momentum representation

$$\mathcal{D}_q(i\omega_n) = [q^2+h^2+\omega_n/D]^{-1}, \tag{16a}$$

$$\mathcal{D}_q^t(i\omega_n) = [q^2+h^2+(1+\gamma)\omega_n/D]^{-1}. \tag{16b}$$

We note that the diffusons (14a) and (14b) are the Green's functions of the corresponding diffusive equations with the Neumann boundary condition. The latter guarantees the absence of current flowing out of the system (see Appendix A).

Also we introduced the following notation:

$$\widehat{\mathcal{D}\mathcal{D}^t}(i\omega;\boldsymbol{r},\boldsymbol{r_1}) = \int' dx_2 \int d^{d-1}\boldsymbol{y_2}\hat{\mathcal{D}}(i\omega;\boldsymbol{r},\boldsymbol{r_2})\hat{\mathcal{D}}^t(i\omega;\boldsymbol{r_2},\boldsymbol{r_1}) = \int_q \mathcal{D}\mathcal{D}^t_q(i\omega)\sum_{s=\pm} e^{iq_x(x-sx_1)+i\boldsymbol{q}_\parallel(\boldsymbol{y}-\boldsymbol{y_1})}. \quad (17)$$

Here the "prime" sign on the integral indicates that we integrate over $x_2>0$. Also we introduced the short-hand notation $\mathcal{D}\mathcal{D}^t_q(i\omega_n)\equiv\mathcal{D}_q(i\omega_n)\mathcal{D}^t_q(i\omega_n)$.

The NLσM action [see Eqs. (2a), (2b), and (9)] is subjected to renormalization. Within one-loop order [the lowest order in $t=1/(\pi g)$], the renormalized parameters (denotes by prime signs) are given as (for a system without the boundary)

$$h'^2 = \frac{gh^2Z}{g'} = h^2\left[1-\frac{bth^\epsilon}{\epsilon}\right], \quad g' = g\left[1+\frac{a_1th^\epsilon}{\epsilon}\right],$$
$$\frac{Z'_\omega}{Z_\omega} = \frac{\Gamma'_t}{\Gamma_t} = 1+(1-3\gamma)\frac{th^\epsilon}{\epsilon}, \quad t=\frac{1}{\pi g},$$
$$a_1 = \mathsf{v}/2+6f(\gamma), \quad b = 3\ln(1+\gamma)+6f(\gamma). \quad (18)$$

Here we introduced $\mathsf{v}=\sum_{j=0}^3 \mathsf{v_j}\equiv 2$. The above results can be rewritten in the form of the one-loop renormalization group equations (with usage of the minimal subtraction scheme [73]),

$$\frac{dt}{d\ell} = -\epsilon t + [\mathsf{v}/2+6f(\gamma)]t^2 + O(t^3), \quad (19a)$$
$$\frac{d\gamma}{d\ell} = 0 + O(t^2), \quad (19b)$$
$$\frac{d\ln Z_\omega}{d\ell} = -(\mathsf{v}/2-3\gamma)t + O(t^2), \quad (19c)$$
$$\frac{d\ln Z}{d\ell} = -[\mathsf{v}/2-3\ln(1+\gamma)]t + O(t^2). \quad (19d)$$

Here, $\ell=\ln 1/h'$ stands for the logarithm of the infrared length scale, which is just a system size at $T=0$. At finite temperature, the infrared scale is set by the temperature length $\sim\sqrt{D/T}$. We note that Eqs. (19) have been derived in Refs. [66–70] by various techniques.

III. GENERALIZED BULK MULTIFRACTALITY

We start with a reminder of the generalized multifractality in the bulk for class C reported in Ref. [66]. An operator without derivatives which involves the number q of Q fields can be constructed as follows [65,66]. We introduce the quantity

$$\mathcal{K}_q(E_1,\dots,E_q) = \frac{1}{4^q}\sum_{p_1,\dots,p_q=\pm}\left(\prod_{j=1}^q p_j\right) \times \mathcal{P}_q^{\alpha_1,\dots,\alpha_q;p_1,\dots,p_q}(E_1,\dots,E_q), \quad (20)$$

depending on the set $\{E_1,\dots,E_q\}$ of real energies. The correlation function $\mathcal{P}_q^{\alpha_1,\dots,\alpha_q;p_1,\dots,p_q}(E_1,\dots,E_q)$ can be obtained from its Matsubara counterpart $P_q^{\alpha_1,\dots,\alpha_q}(i\varepsilon_{n_1},\dots,i\varepsilon_{n_q})$ by the analytic continuation to the real frequencies: $\varepsilon_{n_j}\to E_j+ip_j0^+$. The corresponding Matsubara correlation function is given as

$$P_q^{\alpha_1,\dots,\alpha_q}(i\varepsilon_{n_1},\dots,i\varepsilon_{n_q}) = \sum_{\{k_1,\dots,k_q\}}\mu_{k_1,\dots,k_s}\langle R_{k_1,\dots,k_s}\rangle,$$
$$R_{k_1,\dots,k_s} = \prod_{r=k_1}^{k_s}\mathrm{tr}Q^{\alpha_{j_1}\alpha_{j_2}}_{n_{j_1}n_{j_2}}Q^{\alpha_{j_2}\alpha_{j_3}}_{n_{j_2}n_{j_3}}\dots Q^{\alpha_{j_r}\alpha_{j_1}}_{n_{j_r}n_{j_1}}. \quad (21)$$

The summation on the right-hand side of Eq. (21) is performed over all partitions[3] of the integer number q. We note that all replica indices in Eq. (20) are different: $\alpha_j\neq\alpha_k$ if $j\neq k$ for $j,k=1,\dots,q$. One coefficient among the set $\{\mu_{k_1,\dots,k_s}\}$ can be chosen to be arbitrary. We adopt the following convention: $\mu_{1,1,\dots,1}=1$.

The energy dependence of the operator $\mathcal{K}_q$, given by Eq. (20), complicates its renormalization. The energies E_j provide infrared regularization of otherwise divergent terms in the perturbative renormalization scheme for the operator $\mathcal{K}_q$ (see details in Ref. [65]). In order to avoid such a complication, we introduced the infrared regulator h^2; cf. Eq. (9). It allows us to compute the renormalization of the operator $\mathcal{K}_q$, setting all $E_j=0$. However, in the Matsubara counterpart of the operator $\mathcal{K}_q$, we cannot set Matsubara energies ε_{n_j} to zero from the very beginning. In order to be able to make the proper analytic continuation to the real frequencies, $\varepsilon_{n_j}\to E_j+ip_j0^+$, we have to keep track of the signs of ε_{n_j} since $p_j\equiv\mathrm{sgn}\varepsilon_{n_j}$. Therefore, once signs of ε_{n_j} are treated properly, it is convenient to set all ε_{n_j} to zero. In this way, the simplified operator [66]

$$K_q = \frac{1}{4^q}\sum_{p_k=\pm}\left(\prod_{j=1}^q p_j\lim_{\varepsilon_{n_j}\to 0}\right)P_q^{\alpha_1,\dots,\alpha_q}(ip_1|\varepsilon_{n_1}|,\dots,ip_q|\varepsilon_{n_q}|) \quad (22)$$

can be used to study the renormalization of the operator $\mathcal{K}_q(E_1,\dots,E_q)$. Therefore, in what follows, we will work with K_q instead of $\mathcal{K}_q$.

In the absence of interaction, $\gamma=0$, the NLσM action reduces to Eq. (2a). Then one can project the Q matrix to the 2×2 subspace of a given single pair of positive and negative Matsubara frequencies.[4] The projection corresponds to reduction of Sp($2N$) to Sp($4N_r$). The effective action becomes invariant under rotations $Q\to\mathtt{T}^{-1}Q\mathtt{T}$ with $\mathtt{T}\in$U($2N_r$). This allows one to average operators $\mathcal{K}_q$ over U($2N_r$) rotations. The resulting rotationally invariant operators can be classified with respect to the irreducible representations of Sp($2N$). Each irreducible representation contains a single rotationally invariant pure scaling operator [6,7,19]. The corresponding eigenoperators can uniquely be characterized by the Young tableau $\lambda=(\mathsf{k_1},\dots,\mathsf{k_s})$ (with $|\lambda|=\sum_{\mathsf{j=1}}^{\mathsf{s}}\mathsf{k_j}=q$).

[3]The partitions are a set of positive integer numbers $\{k_1,\dots,k_s\}$ which satisfy the following conditions: $k_1+k_2+\dots k_s=q$ and $k_1\geqslant k_2\geqslant\dots\geqslant k_s>0$.

[4]It is possible since the Matsubara indices of the Q matrix are not mixed in the absence of interaction (the energy of diffusive modes conserves).

TABLE I. The coefficients $\mu_{k_1,\ldots,k_s}$ for eigenoperators with $q = 2, 3, 4$.

q = 2	$\mu_{1,1}$	μ_2			
(2)	1	−1			
(1,1)	1	2			
q = 3	$\mu_{1,1,1}$	$\mu_{2,1}$	μ_3		
(3)	1	−3	2		
(2,1)	1	1	−2		
(1,1,1)	1	6	8		
q = 4	$\mu_{1,1,1,1}$	$\mu_{2,1,1}$	$\mu_{3,1}$	$\mu_{2,2}$	μ_4
(4)	1	−6	8	3	−6
(3,1)	1	−1	−2	−2	4
(2,2)	1	2	−8	7	−2
(2,1,1)	1	5	4	−2	−8
(1,1,1,1)	1	12	32	12	48

Although interaction breaks the beautiful mathematical structure of the NLσM manifold, surprisingly, it does not spoil the structure of non-U($2N_rN_m$)-invariant eigenoperators $\mathcal{K}_\lambda$ [63–66]. The coefficients $\mu_{k_1,\ldots,k_s}$ for $|\lambda| = 2, 3, 4$ are listed in Table I. Not surprisingly, the anomalous dimension of $\mathcal{K}_\lambda$ is changed by the interaction. The renormalized eigenoperator can be written as

$$\mathcal{K}_\lambda = Z^q M_\lambda \mathcal{K}_\lambda[\Lambda], \tag{23}$$

where the factor Z describing renormalization of the local density of states is governed by the following RG equation:

$$\eta_{(1)} = -\frac{d\ln Z}{d\ell} = [1 - 3\ln(1+\gamma)]t + O(t^2). \tag{24}$$

We note that in the presence of interaction, the expression for $\eta_{(1)}$ is known up to the one-loop approximation only. The quantity M_λ determines the anomalous dimension,

$$\eta_\lambda = -\frac{d\ln M_\lambda}{d\ell} = \mu_{2,1,\ldots,1} t[1 + 3c(\gamma)t] + O(t^3), \tag{25}$$

where $\mu_{2,1,\ldots,1}$ is a coefficient in the expansion of the eigenoperator in series in the basis operators $R_{k_1,\ldots,k_s}$; see Eq. (21). For the eigenoperator characterized by the Young tableau $\lambda = (\mathsf{k}_1, \ldots, \mathsf{k}_\mathsf{s})$, this coefficient is given as [19] (see Table I for $|\lambda| = 2, 3, 4$)

$$\mu_{2,1,\ldots,1} = -\frac{1}{2}\sum_{\mathsf{j}=1}^{\mathsf{s}} \mathsf{k}_\mathsf{j}(\mathsf{c}_\mathsf{j} + 2 + \mathsf{k}_\mathsf{j}), \quad \mathsf{c}_\mathsf{j} = 1 - 4\mathsf{j}. \tag{26}$$

The function $c(\gamma)$ contains information about the interaction and is given as [63–66]

$$c(\gamma) = 2 + \frac{1+\gamma}{2\gamma}\ln^2(1+\gamma) + \frac{2+\gamma}{\gamma}\mathrm{li}_2(-\gamma). \tag{27}$$

The anomalous dimensions η_λ determine the scaling with the system size L of the eigenoperators at the fixed point,

$$\mathcal{K}_\lambda \sim L^{-x_\lambda}, \quad x_\lambda = |\lambda| x_{(1)} + \Delta_\lambda. \tag{28}$$

Here the exponent $x_{(1)}$ coincides with the magnitude of $\eta_{(1)}$ at the fixed point, $x_{(1)} = \eta_{(1)}^*$. Similarly, the exponent Δ_λ is equal to the anomalous dimension of M_λ at the fixed point, $\Delta_\lambda = \eta_\lambda$.

Next we discuss how Eqs. (24) and (25) are modified for the local eigenoperators situated near the boundary.

IV. GENERALIZED SURFACE MULTIFRACTALITY

In this section, we compute anomalous dimensions of the renormalization group eigenoperators without derivatives near the boundary.

A. Operator with a single Q matrix

We start the analysis from the local eigenoperator with a single Q matrix,

$$P_1^\alpha(i\varepsilon_n) = \mathrm{tr}\langle Q_{nn}^{\alpha\alpha}\rangle. \tag{29}$$

Physically, it corresponds to the average local density of states near the boundary. Substituting the expansion $Q \simeq \Lambda + W - \Lambda W^2/2$, we find that

$$P_1^\alpha(i\varepsilon_n) = 2Z^{(s)}(i\varepsilon_n)\mathrm{sgn}\varepsilon_n, \tag{30}$$

where

$$\begin{aligned} Z^{(s)}(i\varepsilon_n) = 1 &- \frac{\mathsf{v}}{g}\hat{\mathcal{D}}(2i|\varepsilon_n|;\boldsymbol{r},\boldsymbol{r}) \\ &+ \frac{12\pi T\gamma}{gD}\sum_{\omega_m>|\varepsilon_n|}\widehat{\mathcal{D}\mathcal{D}^t}(i\omega_m;\boldsymbol{r},\boldsymbol{r}). \end{aligned} \tag{31}$$

Assuming that the point $\boldsymbol{r}$ is close to the boundary at $x = 0$, we find that

$$\hat{\mathcal{D}}(i\omega_n;\boldsymbol{r},\boldsymbol{r}) \simeq 2\int_q \mathcal{D}_q(i\omega_n), \tag{32a}$$

$$\widehat{\mathcal{D}\mathcal{D}^t}(i\omega_n;\boldsymbol{r},\boldsymbol{r}) \simeq 2\int_q \mathcal{D}\mathcal{D}_q^t(i\omega_n). \tag{32b}$$

We note that the factors 2 in the above equations reflect the well-known physical result of increase of the return probability near the reflecting boundary. Therefore, we find

$$\begin{aligned} \mathcal{K}_{(1)} &= Z^{(s)}\mathcal{K}_{(1)}[\Lambda], \\ Z^{(s)} &= 1 + \left[\frac{\mathsf{v}}{2} - 3\ln(1+\gamma)\right]\frac{2th^\epsilon}{\epsilon} \\ &= 1 + [\mathsf{v}/2 - 3\ln(1+\gamma)]2th'^\epsilon/\epsilon. \end{aligned} \tag{33}$$

Applying the minimal subtraction scheme, we deduce the anomalous dimension of the operator $\mathcal{K}_{(1)}$,

$$\eta_{(1)}^{(s)} = -\frac{d\ln Z^{(s)}}{d\ell} = 2t[1 - 3\ln(1+\gamma)] + O(t^2). \tag{34}$$

We note that similar to the bulk anomalous dimension $\eta_{(1)}$, the interaction affects the anomalous dimension of $Z^{(s)}$ already in the one-loop approximation. The effect of the boundary is a factor 2 in front of t on the right-hand side of Eq. (34); cf. Eq. (24). This factor 2 comes from the factor 2 in Eqs. (32a) and (32b).

Since the two-loop expression for the bulk anomalous dimension $\eta_{(1)}$ is not known at the moment, in this work we restrict our computation of the surface anomalous dimension $\eta_{(1)}^{(s)}$ to the one-loop order only. As we shall see below (cf. Sec. IV B 3), one-loop renormalization of $Z^{(s)}$ will be enough in order to determine the surface anomalous dimensions of eigenoperators with $q \geqslant 2$ within a two-loop approximation.

B. Local eigenoperators with two Q matrices

1. One-loop renormalization

As known, there are two local eigenoperators with two Q matrices denoted as $\mathcal{K}_{(2)}$ and $\mathcal{K}_{(1,1)}$. It will be convenient to consider the irreducible part of the corresponding correlation function,

$$P_2^{\alpha\beta;(\mathrm{irr})}(i\varepsilon_n, i\varepsilon_m) = \langle\langle \mathrm{tr} Q_{nn}^{\alpha\alpha} \mathrm{tr} Q_{mm}^{\beta\beta}\rangle\rangle + \mu_2 \langle \mathrm{tr} Q_{nm}^{\alpha\beta} Q_{mn}^{\beta\alpha}\rangle. \quad (35)$$

Here, $\mu_2 = -1$ and 2 corresponds to the operator $\mathcal{K}_{(2)}$ and $\mathcal{K}_{(1,1)}$, respectively. We note that the full correlation function can be restored as follows:

$$P_2^{\alpha\beta}(i\varepsilon_n, i\varepsilon_m) = (2Z^{(s)})^2 \mathrm{sgn}\varepsilon_n \mathrm{sgn}\varepsilon_m + P_2^{\alpha\beta;(\mathrm{irr})}(i\varepsilon_n, i\varepsilon_m). \quad (36)$$

After expansion of Q to the first order in W, the one-loop contribution becomes

$$\begin{aligned} P_{2,1}^{\alpha\beta;(\mathrm{irr})}&(i\varepsilon_n, i\varepsilon_m) \\ &= \mu_2 \langle \mathrm{tr} W_{nm}^{\alpha\beta} W_{mn}^{\beta\alpha}\rangle \\ &= \frac{16\mu_2}{g} \frac{1-\mathrm{sgn}\varepsilon_n \mathrm{sgn}\varepsilon_m}{2} \hat{\mathcal{D}}(i|\varepsilon_n| + i|\varepsilon_m|; \boldsymbol{r}, \boldsymbol{r}). \end{aligned} \quad (37)$$

Neglecting the energy dependence in the diffusive propagators and using Eq. (32a), we find the following one-loop result for the irreducible part of the operator K_2:

$$K_{2,1}^{(\mathrm{irr})} = 2t\mu_2 h^{\epsilon}/\epsilon. \quad (38)$$

We note the same additional factor 2 as in the one-loop expression for $Z^{(s)}$.

2. Two-loop renormalization

Next expanding Q to the second order in W, we obtain the two-loop contribution as

$$\begin{aligned} P_{2,2}^{\alpha\beta;(\mathrm{irr})} = {}& \frac{1}{4} \mathrm{sgn}\varepsilon_n \mathrm{sgn}\varepsilon_m \langle\langle \mathrm{tr}(W^2)_{nn}^{\alpha\alpha} \mathrm{tr}(W^2)_{mm}^{\beta\beta}\rangle\rangle + \mu_2 \frac{1+\mathrm{sgn}\varepsilon_n \mathrm{sgn}\varepsilon_m}{8} \langle \mathrm{tr}(W^2)_{nm}^{\alpha\beta}(W^2)_{mn}^{\beta\alpha}\rangle \\ & + \mu_2 \frac{1-\mathrm{sgn}\varepsilon_n \mathrm{sgn}\varepsilon_m}{2} \left\langle\!\!\left\langle \mathrm{tr} W_{nm}^{\alpha\beta} W_{mn}^{\beta\alpha} \begin{bmatrix} S_0^{(4)} + S_{\mathrm{h}}^{(4)} + S_{\mathrm{int}}^{(4)} \\ +\frac{1}{2}\left(S_{\mathrm{int}}^{(3)}\right)^2 \end{bmatrix} \right\rangle\!\!\right\rangle. \end{aligned} \quad (39)$$

In order to compute (39), we need to calculate several contractions of the W matrices. At first, using Eq. (13), we find

$$\langle\langle \mathrm{tr}(W^2)_{nn}^{\alpha\alpha} \mathrm{tr}(W^2)_{mm}^{\beta\beta}\rangle\rangle = \frac{64}{g^2}[\hat{\mathcal{D}}(i|\varepsilon_n| + i|\varepsilon_m|; \boldsymbol{r}, \boldsymbol{r})]^2 \simeq \frac{64}{g^2}\left[2\int_q \mathcal{D}_q(i|\varepsilon_n| + i|\varepsilon_m|)\right]^2 \to 16\frac{(2t)^2 h^{2\epsilon}}{\epsilon^2}. \quad (40)$$

In the last line, we use Eq. (32a) and neglect the energy dependence in the propagators.

Next, we proceed as follows:

$$\begin{aligned} \langle \mathrm{tr}(W^2)_{nm}^{\alpha\beta}(W^2)_{mn}^{\beta\alpha}\rangle = {}& -3\frac{2^7\pi T\gamma}{g^2 D} \sum_{\varepsilon_k>0} \hat{\mathcal{D}}(i|\varepsilon_m| + i\varepsilon_k; \boldsymbol{r}, \boldsymbol{r}) \widehat{\mathcal{D}\mathcal{D}^t}(i|\varepsilon_n| + i\varepsilon_k; \boldsymbol{r}, \boldsymbol{r}) + \frac{32\mathrm{v}}{g^2} \hat{\mathcal{D}}(2i|\varepsilon_n|; \boldsymbol{r}, \boldsymbol{r}) \hat{\mathcal{D}}(i|\varepsilon_n| + i|\varepsilon_m|; \boldsymbol{r}, \boldsymbol{r}) + (n \leftrightarrow m) \\ \simeq {}& -3\frac{2^9\pi T\gamma}{g^2 D} \sum_{\varepsilon_k>0} \int_{qp} \mathcal{D}_q(i|\varepsilon_m| + i\varepsilon_k) \mathcal{D}\mathcal{D}_p^t(i|\varepsilon_n| + i\varepsilon_k) + \frac{32\mathrm{v}}{g^2} 4 \int_{qp} \mathcal{D}_q(2i|\varepsilon_n|) \mathcal{D}_p(i|\varepsilon_n| + i|\varepsilon_m|) + (n \leftrightarrow m) \\ \to {}& 16\mathrm{v}\frac{(2t)^2 h^{2\epsilon}}{\epsilon^2} - 48\frac{(2t)^2 h^{2\epsilon}}{\epsilon^2}\left[\ln(1+\gamma) - \frac{\epsilon}{4}\ln^2(1+\gamma)\right]. \end{aligned} \quad (41)$$

Here we use Eqs. (32a) and (32b). We refer the reader to Ref. [64] for details on the computation of integrals over momenta and frequency involved in Eq. (41).

Next we have to introduce the following non-Gaussian terms stemming from the expansion of the Q matrix in powers of W of the NLσM action,

$$S_0^{(4)} + S_{\mathrm{h}}^{(4)} = -\frac{g}{64}\int_{\boldsymbol{r}} \sum_{\alpha_i, n_i} \left(\nabla_{12}\nabla_{34} + \nabla_{14}\nabla_{23} + \frac{\omega_{n_{12}+n_{34}}}{D} + 2h^2\right) \mathrm{tr}\left\{[w(\boldsymbol{r}_1)]_{n_1 n_2}^{\alpha_1\alpha_2} [w^\dagger(\boldsymbol{r}_2)]_{n_2 n_3}^{\alpha_2\alpha_3} [w(\boldsymbol{r}_3)]_{n_3 n_4}^{\alpha_3\alpha_4} [w^\dagger(\boldsymbol{r}_4)]_{n_4 n_1}^{\alpha_4\alpha_1}\right\}\Big|_{\boldsymbol{r}_i=\boldsymbol{r}} \quad (42)$$

(here we use a shorthand notation $\nabla_{12}\equiv\nabla_1+\nabla_2$) and

$$S_{\rm int}^{(3)}=\frac{\pi T\Gamma_t}{4}\sum_{\alpha,n}\int_{\boldsymbol r}\mathrm{tr}I_n^\alpha \mathsf{s}W\,\mathrm{Tr}I_{-n}^\alpha\mathsf{s}\Lambda W^2, \tag{43a}$$

$$S_{\rm int}^{(4)}=-\frac{\pi T\Gamma_t}{16}\sum_{\alpha,n}\int_{\boldsymbol r}\mathrm{Tr}I_n^\alpha \mathsf{s}\Lambda W^2\,\mathrm{Tr}I_{-n}^\alpha\mathsf{s}\Lambda W^2. \tag{43b}$$

Performing averaging with the help of Wick theorem and Eq. (13), we obtain

$$\begin{aligned}\langle\langle\mathrm{Tr}\big[W_{nm}^{\alpha\beta}W_{mn}^{\beta\alpha}\big]\big[S_0^{(4)}+S_{\rm h}^{(4)}\big]\rangle\rangle&=-\frac{8\mathsf{v}}{g^2}\int_{\boldsymbol r'}\left[\nabla_{12}\nabla_{34}+\nabla_{14}\nabla_{32}+\frac{2|\varepsilon_n|+|\omega_{nm}|}{D}+2h^2\right]\hat{\mathcal D}(i2|\varepsilon_n|;\boldsymbol r_1,\boldsymbol r_2)\hat{\mathcal D}(i|\omega_{nm}|;\boldsymbol r_3,\boldsymbol r)\\
&\quad\times\hat{\mathcal D}(i|\omega_{nm}|;\boldsymbol r,\boldsymbol r_4)|_{\boldsymbol r_i=\boldsymbol r'}+\frac{96\pi T\gamma}{g^2D}\sum_{\omega_k>|\varepsilon_n|}\left[\nabla_{12}\nabla_{34}+\nabla_{14}\nabla_{32}+\frac{|\omega_k|+|\omega_{nm}|}{D}+2h^2\right]\\
&\quad\times\widehat{\mathcal{D}\mathcal{D}^t}(i|\omega_k|;\boldsymbol r_1,\boldsymbol r_2)\hat{\mathcal D}(i|\omega_{nm}|;\boldsymbol r_3,\boldsymbol r)\hat{\mathcal D}(i|\omega_{nm}|;\boldsymbol r,\boldsymbol r_4)|_{\boldsymbol r_i=\boldsymbol r'}+(n\leftrightarrow m)\\
&=-\frac{16\mathsf{v}}{g^2}\int_{qp}\left(p^2+q^2+\frac{2|\varepsilon_n|+|\omega_{nm}|}{D}+2h^2\right)\mathcal D_p(i2|\varepsilon_n|)\mathcal D_q^2(i|\omega_{nm}|)\\
&\quad-\frac{16\mathsf{v}}{g^2}\int_{qp}\left(4p_x^2+2p_xq_x+p^2+q^2+\frac{2|\varepsilon_n|+|\omega_{nm}|}{D}+2h^2\right)\mathcal D_p(i2|\varepsilon_n|)\mathcal D_q(i|\omega_{nm}|)\\
&\quad\times\mathcal D_{q_x+2p_x,\boldsymbol q_\parallel}(i|\omega_{nm}|)+2\frac{96\pi T\gamma}{g^2D}\sum_{\omega_k>|\varepsilon_n|}\int_{qp}\left(4p_x^2+2p_xq_x+p^2+q^2+\frac{|\omega_k|+|\omega_{nm}|}{D}+2h^2\right)\\
&\quad\times\mathcal{D}\mathcal{D}^t_p(i\omega_k)\mathcal D_q(i|\omega_{nm}|)\mathcal D_{q_x+2p_x,\boldsymbol q_\parallel}(i|\omega_{nm}|)+\frac{192\pi T\gamma}{g^2D}\\
&\quad\times\sum_{\omega_k>|\varepsilon_n|}\int_{qp}\left(p^2+q^2+\frac{|\omega_k|+|\omega_{nm}|}{D}+2h^2\right)\mathcal{D}\mathcal{D}^t_p(i\omega_k)\mathcal D_{\boldsymbol q}^2(i|\omega_{nm}|)+(n\leftrightarrow m)\\
&\to-5\mathsf{v}\frac{(2t)^2h^{2\epsilon}}{\epsilon^2}+6\frac{(2t)^2h^{2\epsilon}}{\epsilon^2}\left[5\ln(1+\gamma)+\frac{\epsilon\gamma}{1+\gamma}\right]+\frac{192\gamma}{g^2}I_{110}^0(1+\gamma).\end{aligned} \tag{44}$$

Here we introduced the following notation:

$$\big[\mathcal D_{q_x,\boldsymbol q_\parallel}(i\omega)\big]^{-1}=q_x^2+\boldsymbol q_\parallel^2+\omega/D+h^2. \tag{45}$$

We emphasize that the appearance of such diffuson as defined in Eq. (45) is specific for the problem of boundary multifractality. The corresponding integrals are evaluated in Appendix B. The definition of the integral I_{110}^0 is given in Appendix B. Instead of computing the integral I_{110}^0 separately, it is convenient to calculate it in combination with two other similar integrals; see below.

The last contribution in Eq. (39) can be evaluated using the following simplification, which is possible due to different replica indices, $\alpha\neq\beta$:

$$S_{\rm int}^{(4)}+\frac12\big(S_{\rm int}^{(3)}\big)^2\to-\sum_{\nu n}\int_{\boldsymbol r,\boldsymbol r'}\left[\delta(\boldsymbol r-\boldsymbol r')-\frac{\gamma|\omega_n|}{D}\widehat{\mathcal D^t}(i|\omega_n|;\boldsymbol r,\boldsymbol r')\right]\frac{\pi T\Gamma_t}{4}\sum_{\mathrm j=1}^{3}\mathrm{Tr}I_n^\nu\mathsf{s}_{\mathrm j}\Lambda W^2(\boldsymbol r)\mathrm{Tr}I_{-n}^\nu\mathsf{s}_{\mathrm j}\Lambda W^2(\boldsymbol r'). \tag{46}$$

After tedious but straightforward calculations, we obtain

$$\begin{aligned}&\left\langle\!\!\left\langle\mathrm{tr}W_{nm}^{\alpha\beta}W_{mn}^{\beta\alpha}\left[S_{\rm int}^{(4)}+\frac{\big(S_{\rm int}^{(3)}\big)^2}{2}\right]\right\rangle\!\!\right\rangle\\
&\quad=\frac{96\pi T\gamma}{g^2D}\int_{\boldsymbol r',\boldsymbol r''}\left(\sum_{|\varepsilon_n|>\omega_k}+\sum_{|\varepsilon_m|>\omega_k}\right)\left[\frac{\gamma|\omega_k|}{D}\widehat{\mathcal D^t}(i|\omega_k|;\boldsymbol r',\boldsymbol r'')-\delta(\boldsymbol r'-\boldsymbol r'')\right]\hat{\mathcal D}(i|\varepsilon_n|+i|\varepsilon_m|;\boldsymbol r,\boldsymbol r')\\
&\quad\times\hat{\mathcal D}(i|\varepsilon_n|+i|\varepsilon_m|;\boldsymbol r,\boldsymbol r'')\hat{\mathcal D}(i|\varepsilon_n|+i|\varepsilon_m|-i\omega_k;\boldsymbol r',\boldsymbol r'')\end{aligned}$$

$$\simeq \frac{192\pi T\gamma}{g^2 D}\int_{qp}\left(\sum_{|\varepsilon_n|>\omega_k}+\sum_{|\varepsilon_m|>\omega_k}\right)\left[\frac{\gamma|\omega_k|}{D}\mathcal{D}^t_{\boldsymbol{p}+\boldsymbol{q}}(i|\omega_k|)-1\right]\mathcal{D}_q(i|\varepsilon_n|+i|\varepsilon_m|)[\mathcal{D}_q(i|\varepsilon_n|+i|\varepsilon_m|)+\mathcal{D}_{q_x+2p_x,\boldsymbol{q}_\parallel}(i|\varepsilon_n|+i|\varepsilon_m|)]$$
$$\times\ \mathcal{D}_p(i|\varepsilon_n|+i|\varepsilon_m|-i\omega_k)$$
$$=6\frac{(2t)^2h^{2\epsilon}}{\epsilon^2}\left\{\frac{2\gamma-(2+\gamma)\ln(1+\gamma)}{\gamma}-\epsilon\frac{(2+\gamma)\ln(1+\gamma)}{\gamma}-\frac{\epsilon\gamma}{1+\gamma}-\epsilon\frac{2+\gamma}{\gamma}\left[\mathrm{li}_2(-\gamma)-\frac{1}{4}\ln^2(1+\gamma)\right]\right\}$$
$$-\frac{192\gamma}{g^2}\left[I^0_{110}(1)-\gamma I^1_{111}(1+\gamma)\right]. \tag{47}$$

Here, $\mathrm{li}_2(z)=\sum_{k=1}^{\infty}z^k/k^2$ denotes the polylogarithm. Again we emphasize the emergence of boundary diffusons (45) in the expression (47). Combining the above results, given by Eqs. (40)–(44) and (47), we find

$$K^{(\mathrm{irr})}_{2,2}=\left(\mu_2[\mathsf{v}-6\ln(1+\gamma)]+1+\frac{\mu_2\mathsf{v}}{8}-\frac{3\mu_2}{2}f(\gamma)+\frac{3\epsilon\mu_2}{4}\left\{\frac{2+3\gamma}{4\gamma}\ln^2(1+\gamma)+\frac{2+\gamma}{\gamma}[\mathrm{li}_2(-\gamma)+\ln(1+\gamma)]\right\}\right)\frac{(2t)^2h^{2\epsilon}}{\epsilon^2}$$
$$-\frac{24\gamma\mu_2}{g^2}\left[I^0_{110}(1+\gamma)-I^0_{110}(1)+\gamma I^1_{111}(1+\gamma)\right]. \tag{48}$$

Using the result for the combination of I integrals from Eq. (B11) in Appendix B, we obtain

$$K^{(\mathrm{irr})}_{2,2}=\left\{\mu_2[\mathsf{v}-6\ln(1+\gamma)]+\left(b_2^{(2)}+\epsilon\mu_2 b_3\right)\right\}\frac{(2t)^2h^{2\epsilon}}{\epsilon^2}, \tag{49}$$

where

$$b_2^{(2)}=1+\frac{\mu_2\mathsf{v}}{8}-\frac{3\mu_2}{2}f(\gamma),$$
$$b_3=\frac{3}{4}\left\{\frac{2+3\gamma}{4\gamma}\ln^2(1+\gamma)+\frac{2+\gamma}{\gamma}[\mathrm{li}_2(-\gamma)+\ln(1+\gamma)]-\frac{\gamma}{4}\Phi(\gamma)\right\}. \tag{50}$$

Here we introduced the function $\Phi(\gamma)=\ln^2(1+\gamma)/\gamma$ [see Eqs. (B12) and (B13)]. We note that $\Phi(\gamma)$ appears from the combination of I integrals.

3. Anomalous dimension

Employing the one-loop [see Eq. (38)] and two-loop [see Eq. (49)] results, we write the operator K_2 in the following form:

$$K_2=(Z^{(s)})^2M_2^{(s)}K_2[\Lambda]. \tag{51}$$

Here, $K_2[\Lambda]=1$ is the classical value of K_2 and

$$M_2^{(s)}=1+Z^{-2}\left(K^{(\mathrm{irr})}_{2,1}+K^{(\mathrm{irr})}_{2,2}\right)$$
$$=1+\mu_2\frac{2th^\epsilon}{\epsilon}+(b_2^{(2)}+\epsilon\mu_2b_3)\frac{(2t)^2h^{2\epsilon}}{\epsilon^2}$$
$$=1+\mu_2\frac{2th'^\epsilon}{\epsilon}+\left(b_2^{(2)}+\epsilon\mu_2\tilde{b}_3\right)\frac{(2t)^2h'^{2\epsilon}}{\epsilon^2}, \tag{52}$$

where $\tilde{b}_3=b_3+b/4$, with b given by Eq. (18). Next we apply the minimal subtraction scheme to Eq. (52). We note that the following relation holds (for $\mu_2=-1$ and 2):

$$2\mu_2(2\mu_2-a_1)=8b_2^{(2)}, \tag{53}$$

which guarantees the finiteness of the anomalous dimension at $\epsilon\to 0$. Hence, we obtain the anomalous dimensions for two eigenoperators $\mathcal{K}_{(2)}$ and $\mathcal{K}_{(1,1)}$ at the boundary

$$\mu_2=-1,\quad \eta^{(s)}_{(2)}=-2t[1+3c(\gamma)t]+O(t^3),$$
$$\mu_2=2,\quad \eta^{(s)}_{(1,1)}=4t[1+3c(\gamma)t]+O(t^3). \tag{54}$$

C. Local eigenoperators with arbitrary number of Q matrices

The above results for the local eigenoperators with two Q matrices can be extended to the case of an arbitrary number of Q matrices in the same way as has been done for the bulk generalized multifractality (see Ref. [66]). The eigenoperator with the number q of the Q matrices involved characterized by the Young tableau $\lambda=(\mathsf{k}_1,\dots,\mathsf{k}_\mathsf{s})$ (with $\sum_{\mathsf{j}=1}^{\mathsf{s}}\mathsf{k}_\mathsf{j}=|\lambda|$) becomes

$$\mathcal{K}_\lambda=(Z^{(s)})^{|\lambda|}M_\lambda^{(s)}\mathcal{K}_\lambda[\Lambda]. \tag{55}$$

The quantity $M_\lambda^{(s)}$ determines the anomalous dimension,

$$\eta_\lambda^{(s)}=-\frac{d\ln M_\lambda^{(s)}}{d\ell}=2\mu_{2,1,\dots,1}t[1+3c(\gamma)t]+O(t^3), \tag{56}$$

where $\mu_{2,1,\dots,1}$ is given by Eq. (26). Equation (56) is the main result of our work.

The anomalous dimensions $\eta_\lambda^{(s)}$ determine the scaling with the system size L of the eigenoperators near the boundary at criticality,

$$\mathcal{K}_\lambda\sim L^{-x_\lambda^{(s)}},\quad x_\lambda^{(s)}=|\lambda|x_{(1)}^{(s)}+\Delta_\lambda^{(s)}. \tag{57}$$

Here the exponent $x_{(1)}^{(s)}$ coincides with the magnitude of $\eta_{(1)}^{(s)}$, given by Eq. (34), at the fixed point, $x_{(1)}^{(s)}=\eta_{(1)}^{(s)*}$. Similarly, the exponent $\Delta_\lambda^{(s)}$ is equal to the anomalous dimension of $M_\lambda^{(s)}$ at the fixed point, $\Delta_\lambda^{(s)}=\eta_\lambda^{(s)}$.

V. DISCUSSIONS AND CONCLUSIONS

A. Generalization to higher orders in t

In this paper, we determine the anomalous dimensions of the local eigenoperators situated near the boundary for the symmetry class C in the presence of interaction. We apply perturbative renormalization group expansion for the anomalous dimensions up to the second order in t. It was known that bulk and surface anomalous dimensions within the first order in t are related by the factor 2. We find that the same factor 2 appears within the second order. Interestingly, it happens in spite of the fact that the two-loop contribution to anomalous dimension is a nontrivial function of interaction strength γ.

A naïve idea could be that the bulk and surface exponents are related by the factor 2 in all orders of expansion in t. However, it is definitely not the case for the spin quantum Hall transition in $d = 2$ dimensions. The set of bulk and surface exponents which are known exactly from mapping to percolation [8,15,56–59] are not related by a factor of 2, e.g., $x_{(2)} = 1/4$ while $x^{(s)}_{(2)} = 1/3$. Additionally, the numerical computation of x_λ and $x^{(s)}_\lambda$ indicates that the ratio between them is not a universal factor equal to 2 [18].

Having in mind the above discussion, it would be interesting to develop a scenario where a factor of 2 in weak coupling transforms into nontrivial factors that are different for different operators. Is the factor 2 a feature of the perturbative expansion to all orders in t while nonperturbative instanton effects are responsible for transformation to nontrivial factors? Or is the factor 2 limited to the lowest-order terms of the series in t only?

B. Relations to other symmetry classes

The results reported in this paper for the surface anomalous dimensions can be directly translated to the standard Wigner-Dyson classes (classes A, AI, and AII) where the bulk generalized multifractality in the presence of interaction has been recently developed [63–65]. Similarly, within a two-loop approximation, the boundary multifractal exponents are twice larger than the bulk ones. Moreover, our results can be extended to the other two superconducting classes, CI and DIII, that allow interaction within the Finkel'stein NLσM. We will provide details for the above-mentioned results elsewhere.

C. The role of topology

Similar to the class A, the NLσM for class C allows the presence of the topological θ term. The topological term does not change the classification of the local pure scaling operators but, certainly, contributes to their anomalous dimensions. At weak disorder, $t \ll 1$, where the instanton effects can be treated in a controlled manner, the question of how instantons affect the anomalous dimension of an arbitrary local operator is still not well understood. The only exception are the anomalous dimensions of bilinear in Q eigenoperators for class A in the absence of interaction [74]. Instantons are expected to affect both the bulk and boundary anomalous dimensions.

D. Breakdown of the Weyl symmetry

In the absence of interaction, the Weyl-group invariance [7] forces not only the bulk generalized multifractal dimensions x_λ, but also surface generalized multifractal dimensions $x^{(s)}_\lambda$ to obey the symmetry relations [18]. These symmetry relations make the exponents $x^{(s)}_\lambda$ the same for the eigenoperators related by the following symmetry operations: reflection, $\mathsf{k_j} \to -\mathsf{c_j} - \mathsf{k_j}$, and permutation of some pair, $\mathsf{k_{j/i}} \to \mathsf{k_{i/j}} + (\mathsf{c_{i/j}} - \mathsf{c_{j/i}})/2$. Our one-loop results for the boundary anomalous dimensions are consistent with the Weyl-group invariance symmetry in the absence of interaction. The presence of interaction is known to break the symmetry relations between exponents characterizing bulk generalized multifractality [66]. A similar situation—interaction-induced breaking of Weyl symmetry relations—occurs with the surface exponents within the two-loop approximation considered in this paper. To illustrate how it occurs, let us consider the Mott-Anderson transition in $d = 2 + \epsilon$ dimensions. Then, as follows from Eq. (19), there is a line of fixed points at $t_* = \epsilon/[1 + 6f(\gamma)]$ with arbitrary γ. The surface generalized multifractal exponents become (to the order ϵ)

$$x^{(s)}_\lambda = \frac{\epsilon}{[1 + 6f(\gamma)]} \sum_{\mathsf{j}=1}^{\mathsf{s}} \mathsf{k_j}[-\mathsf{c_j} - 3\ln(1+\gamma) - \mathsf{k_j}]. \tag{58}$$

The above expression is inconsistent with Weyl symmetry in the presence of interaction, $\gamma \neq 0$. It occurs due to the appearance of γ dependence in $x^{(s)}_{(1)}$. Such a situation also suggests breaking Weyl symmetry for $\gamma \neq 0$ at the spin quantum Hall transition in $d = 2$. Unfortunately, the present numerical computing power [75–78] is not enough to access generalized multifractal exponents and to check our predictions, in particular, to test violation of symmetry relations in the presence of interaction.

E. Summary

To summarize, we developed the theory of the generalized boundary multifractality in class C in the presence of electron-electron interaction. Employing the two-loop renormalization group approximation controlled by inverse spin conductance t, we computed the anomalous dimensions of the pure scaling operators at the boundary of the sample. At the one-loop approximation, we found the expected result that the boundary anomalous dimensions are two times larger than the bulk ones. Surprisingly, we found that the same relation (a factor 2 difference) holds within the two-loop approximation in spite of the nontrivial dependence of bulk and surface anomalous dimensions on interaction parameter γ. Consequently, we showed that the presence of interaction invalidates the exact symmetry relations between generalized surface multifractal exponents, which are a consequence of Weyl symmetry in the noninteracting case. We discussed future developments and applications of our theory.

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APPENDIX A: DIFFUSION IN THE PRESENCE OF A REFLECTING BOUNDARY

The diffuson propagator satisfies the following diffusion equation:

$$[-\nabla^2 + h_n^2]\hat{\mathcal{D}}(i\omega_n; \boldsymbol{r}, \boldsymbol{r}') = \delta(x - x')\delta(\boldsymbol{y} - \boldsymbol{y}'), \tag{A1}$$

where $h_n^2 = h^2 + \omega_n/D$. Following Ref. [79], we supplement Eq. (A1) by the Neumann boundary condition

$$\partial_x \hat{\mathcal{D}}(i\omega_n;\boldsymbol{r},\boldsymbol{r}')|_{x=0} = 0. \tag{A2}$$

Solving, by standard means, Eqs. (A1) and (A2), we find, for $x, x' \geqslant 0$,

$$\hat{\mathcal{D}}(i\omega_n;\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2}\int_{\boldsymbol{q}_\parallel} \frac{e^{i\boldsymbol{q}_\parallel(\boldsymbol{y}-\boldsymbol{y}')}}{\sqrt{h_n^2+\boldsymbol{q}_\parallel^2}}\left[e^{-|x-x'|\sqrt{h_n^2+\boldsymbol{q}_\parallel^2}} + e^{-(x+x')\sqrt{h_n^2+\boldsymbol{q}_\parallel^2}}\right]. \tag{A3}$$

We note that $\hat{\mathcal{D}}(i\omega_n;\boldsymbol{r},\boldsymbol{r}')$ is symmetric under the interchange of spatial coordinates $\boldsymbol{r}\leftrightarrow\boldsymbol{r}'$. The result (A3) coincides with Eq. (14a).

In order to clarify the physical meaning of the Neumann boundary condition (A2), we compute the matrix current $\mathcal{J}_x = Q\nabla_x Q$ flowing perpendicular to the boundary in the lowest order of perturbation theory. Then we find

$$\mathcal{J}_{nm}^{\alpha\beta}(x=0,\boldsymbol{y}) \simeq \frac{2}{g}\delta_{nm}\delta^{\alpha\beta}\mathsf{s}_0\partial_x\left[\mathsf{v}\hat{\mathcal{D}}(2i|\varepsilon_n|;\boldsymbol{r},\boldsymbol{r}') - \frac{12\pi T\gamma}{g}\sum_{\omega>|\varepsilon_n|}\widehat{\mathcal{D}\mathcal{D}^t}(i\omega;\boldsymbol{r},\boldsymbol{r}')\right]\Bigg|_{\substack{x=0,x'=0^+,\\ \boldsymbol{y}'=\boldsymbol{y}}} \equiv 0, \tag{A4}$$

as it should occur for a fully reflecting boundary.

APPENDIX B: EVALUATION OF CONTRACTIONS

1. Equation (44)

We start by rewriting the integrals over momenta in Eq. (44) as follows:

$$\begin{aligned}\langle\langle \mathrm{tr} W_{nm}^{\alpha\beta}W_{mn}^{\beta\alpha}\big[S_0^{(4)}+S_{\mathrm{h}}^{(4)}\big]\rangle\rangle \to& -\frac{32\mathsf{v}}{g^2}\int_{qp}\big[\mathcal{D}_p(0)\mathcal{D}_q(0)+\mathcal{D}_q^2(0)\big] + \frac{384\pi T\gamma}{g^2 D}\int_{qp}\sum_{\omega>0}\big[\mathcal{D}\mathcal{D}_p^t(i\omega)\mathcal{D}_q(0)+\mathcal{D}_p^t(i\omega)\mathcal{D}_q^2(0)\big]\\ &-\frac{32\mathsf{v}}{g^2}\int_{qp}\big[\mathcal{D}_{p_x-q_x,\boldsymbol{q}_\parallel}(0)\mathcal{D}_{p_x+q_x,\boldsymbol{q}_\parallel}(0)+\mathcal{D}_p(0)\mathcal{D}_{p_x+q_x,\boldsymbol{q}_\parallel}(0)+2p_x^2\mathcal{D}_p(0)\mathcal{D}_{p_x+q_x,\boldsymbol{q}_\parallel}(0)\mathcal{D}_{p_x-q_x,\boldsymbol{q}_\parallel}(0)\big]\\ &+\frac{384\pi T\gamma}{g^2 D}\int_{qp}\sum_{\omega>0}\big[\mathcal{D}_p^t(i\omega)\mathcal{D}_q(0)+\mathcal{D}\mathcal{D}_p^t(i\omega)+2p_x(q_x+2p_x)\mathcal{D}\mathcal{D}_p^t(i\omega)\mathcal{D}_q(0)\big]\mathcal{D}_{q_x+2p_x,\boldsymbol{q}_\parallel}(0).\end{aligned} \tag{B1}$$

Next we find

$$\begin{aligned}\langle\langle \mathrm{tr} W_{nm}^{\alpha\beta}W_{mn}^{\beta\alpha}\big[S_0^{(4)}+S_{\mathrm{h}}^{(4)}\big]\rangle\rangle \to& -4\mathsf{v}\frac{(2t)^2h^{2\epsilon}}{\epsilon^2} - \frac{64\mathsf{v}}{g^2}I_1 + \frac{192\gamma}{g^2}\big[2J_{110}^0(1+\gamma)+J_{020}^0(1+\gamma)+I_{110}^0(1+\gamma)+2\ln(1+\gamma)I_1\big]\\ \to& -4\mathsf{v}\left(1+\frac{1}{4}\right)\frac{(2t)^2h^{2\epsilon}}{\epsilon^2}+24\frac{(2t)^2h^{2\epsilon}}{\epsilon^2}\left[\left(1+\frac{1}{4}\right)\ln(1+\gamma)+\frac{\epsilon\gamma}{4(1+\gamma)}\right]+\frac{192\gamma}{g^2}I_{110}^0(1+\gamma).\end{aligned} \tag{B2}$$

Here we introduce the following notations for integral over momenta and frequency:

$$J_{\nu\mu\eta}^\delta(a) = \int_{qp}\int_0^\infty ds\, s^\delta\frac{1}{(p^2+h^2+s)^\nu}\frac{1}{(p^2+h^2+as)}\frac{1}{(q^2+h^2)^\mu}\frac{1}{[(\boldsymbol{p}+\boldsymbol{q})^2+h^2+s]^\eta}. \tag{B3}$$

Also we used the following relations:

$$\begin{aligned}&\int_q\mathcal{D}_q(0) = -\frac{2\Omega_d h^\epsilon\Gamma(1-\epsilon/2)\Gamma(1+\epsilon/2)}{\epsilon}, \quad \int_{qp}\mathcal{D}_q^2(0) = \int_{qp}\mathcal{D}_{p_x-q_x,\boldsymbol{q}_\parallel}(0)\mathcal{D}_{p_x+q_x,\boldsymbol{q}_\parallel}(0) = 0,\\ &\frac{2\pi T}{D}\sum_{\omega>0}\mathcal{D}\mathcal{D}_p^t(i\omega) = \frac{\ln(1+\gamma)}{\gamma}\mathcal{D}_p(0),\\ &\frac{2\pi T\gamma}{D}\int_{qp}\sum_{\omega>0}2p_x(q_x+2p_x)\mathcal{D}\mathcal{D}_p^t(i\omega)\mathcal{D}_q(0)\mathcal{D}_{q_x+2p_x,\boldsymbol{q}_\parallel}(0)\\ &\quad=\frac{2\pi T\gamma}{D}\int_{qp}\sum_{\omega>0}2p_x(q_x+p_x)\mathcal{D}\mathcal{D}_p^t(i\omega)\mathcal{D}_{q_x-p_x,\boldsymbol{q}_\parallel}(0)\mathcal{D}_{q_x+p_x,\boldsymbol{q}_\parallel}(0)\\ &\quad=2\ln(1+\gamma)\int_{qp}p_x^2\mathcal{D}_p(0)\mathcal{D}_{q_x-p_x,\boldsymbol{q}_\parallel}(0)\mathcal{D}_{q_x+p_x,\boldsymbol{q}_\parallel}(0)=2\ln(1+\gamma)I_1,\end{aligned} \tag{B4}$$

where $\Omega_d = S_d/[2(2\pi)^d]$ and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the d-dimensional sphere. We use $t = 4\Omega_d/g$ at arbitrary dimensionality such that $t = 1/(\pi g)$ at $d = 2$. The integral I_1 is evaluated as follows:

$$\begin{aligned}
I_1 &= \int_{qp} p_x^2 \mathcal{D}_p(0)\mathcal{D}_{p_x+q_x,\boldsymbol{q}_\parallel}(0)\mathcal{D}_{p_x-q_x,\boldsymbol{q}_\parallel}(0) = \int_0^1 dz \int_{qp} \frac{p_x^2}{(p^2+h^2)\{\boldsymbol{q}_\parallel^2 + [q_x + p_x(1-2z)]^2 + 4p_x^2 z(1-z) + h^2\}^2} \\
&= \int_0^1 dz \int_{Qp} \frac{p_x^2}{(p^2+h^2)\big[4p_x^2 z(1-z) + h^2\big]^{2-d/2}(Q^2+1)^2} \\
&= \Omega_d \frac{\Gamma(d/2)\Gamma(2-d/2)}{\Gamma(2)} \frac{\Gamma(3-d/2)}{\Gamma(2-d/2)} \int_0^1 dz \int_0^1 du \int_p \frac{p_x^2 u^{1-d/2}}{\big\{\boldsymbol{p}_\parallel^2(1-u) + p_x^2[1-u(1-2z)^2] + h^2\big\}^{3-d/2}} \\
&= h^{2\epsilon}\Omega_d^2 \frac{\Gamma(d/2)\Gamma(2-d/2)}{\Gamma(2)} \frac{\Gamma(3-d/2)}{\Gamma(2-d/2)} \frac{\Gamma(d/2+1)\Gamma(2-d)}{d\Gamma(3-d/2)} \int_0^1 du \int_0^1 dv u^{1-d/2}(1-u)^{-(d-1)/2}(1-uv^2)^{-3/2} \\
&= -\frac{h^{2\epsilon}\Omega_d^2}{2\epsilon}\Gamma^2(d/2)\Gamma(3-d) \int_0^1 du\, u^{1-d/2}(1-u)^{-d/2} \\
&= -\frac{h^{2\epsilon}\Omega_d^2}{2\epsilon}\Gamma^2(d/2)\Gamma(3-d)\frac{\Gamma(2-d/2)\Gamma(1-d/2)}{\Gamma(3-d)} = \frac{A_\epsilon h^{2\epsilon}}{\epsilon^2},
\end{aligned} \tag{B5}$$

where $A_\epsilon = \Omega_d^2\Gamma^2(1-\epsilon/2)\Gamma^2(1+\epsilon/2)$. The evaluation of integrals $J^\delta_{\nu\mu\eta}(a)$ is described in Ref. [64]. Also we introduced the following new integrals:

$$I^\delta_{\nu\mu\eta}(a) = \int_{qp}\int_0^\infty ds\, s^\delta \frac{1}{(q^2+h^2)^\nu} \frac{1}{(p^2+h^2+as)} \frac{1}{[(q_x+2p_x)^2 + \boldsymbol{q}_\parallel^2 + h^2]^\mu} \frac{1}{[(\boldsymbol{p}+\boldsymbol{q})^2+h^2+s]^\eta}. \tag{B6}$$

2. Equation (47)

$$\begin{aligned}
\bigg\langle\!\!\bigg\langle \mathrm{tr} W^{\alpha\beta}_{nm} W^{\beta\alpha}_{mn}\bigg[S^{(4)}_{\mathrm{int}} + \frac{1}{2}\big(S^{(3)}_{\mathrm{int}}\big)^2\bigg]\bigg\rangle\!\!\bigg\rangle &\to -\frac{384\pi T\gamma}{g^2 D}\int_{qp}\sum_{\omega>0}\bigg[1 - \frac{\gamma\omega}{D}\mathcal{D}^t_{\boldsymbol{p}+\boldsymbol{q}}(i\omega)\bigg]\mathcal{D}_p(i\omega)[\mathcal{D}_q(0) + \mathcal{D}_{q_x+2p_x,\boldsymbol{q}_\parallel}(0)] \\
&\quad - \frac{192\gamma}{g^2}\big[J^0_{020}(1) - \gamma J^1_{021}(1+\gamma) + I^0_{110}(1) - \gamma I^1_{111}(1+\gamma)\big] \\
&= -6\frac{(2t)^2 h^{2\epsilon}}{\epsilon^2}\bigg\{-\frac{2\gamma - (2+\gamma)\ln(1+\gamma)}{\gamma} + \frac{\epsilon\gamma}{1+\gamma} + \epsilon\frac{(2+\gamma)\ln(1+\gamma)}{\gamma} \\
&\quad + \epsilon\frac{2+\gamma}{\gamma}\bigg[\mathrm{li}_2(-\gamma) + \frac{1}{4}\ln^2(1+\gamma)\bigg]\bigg\} - \frac{192\gamma}{g^2}\big[I^0_{110}(1) - \gamma I^1_{111}(1+\gamma)\big].
\end{aligned} \tag{B7}$$

Here we used the known results for the integrals $J^\delta_{\nu\mu\eta}(a)$ from Ref. [64]. Instead of the computation of integrals I^0_{110}, I^0_{110}, and I^1_{111} separately, it is more convenient to evaluate the combination as they appear together:

$$\begin{aligned}
&I^0_{110}(1+\gamma) - I^0_{110}(1) + \gamma I^1_{111}(1+\gamma) \\
&= \gamma\int_{qp}\int_0^\infty ds\, s\bigg[\frac{1}{[(\boldsymbol{p}+\boldsymbol{q})^2 + (1+\gamma)s + h^2]} - \frac{1}{[p^2 + (1+\gamma)s + h^2]}\bigg]\frac{1}{(p^2+s+h^2)}\frac{1}{(q^2+h^2)[(q_x+2p_x)^2 + \boldsymbol{q}_\parallel^2 + h^2]} \\
&= -\gamma\int_{qp}\int_0^\infty ds \frac{(q^2 + 2\boldsymbol{pq})s}{[(\boldsymbol{p}+\boldsymbol{q})^2 + (1+\gamma)s + h^2][p^2 + (1+\gamma)s + h^2](p^2+s+h^2)}\frac{1}{(q^2+h^2)\big[(q_x+2p_x)^2 + \boldsymbol{q}_\parallel^2 + h^2\big]} \\
&= \int_{qp}\int_0^\infty ds\bigg[\frac{1}{[p^2 + (1+\gamma)s + h^2]} - \frac{1}{(p^2+s+h^2)}\bigg]\frac{(q^2+2\boldsymbol{pq})}{(q^2+h^2)[(q_x+2p_x)^2 + \boldsymbol{q}_\parallel^2 + h^2]}\frac{1}{[(\boldsymbol{p}+\boldsymbol{q})^2 + (1+\gamma)s + h^2]} \\
&= -\int_{qp}\int_0^\infty ds \frac{(q^2+2\boldsymbol{pq})}{(p^2+s+h^2)[(\boldsymbol{p}+\boldsymbol{q})^2 + (1+\gamma)s + h^2](q^2+h^2)\big[(q_x+2p_x)^2 + \boldsymbol{q}_\parallel^2 + h^2\big]}.
\end{aligned} \tag{B8}$$

Here, in the last line, we employed the following transformation: $\boldsymbol{p} \to \boldsymbol{P} + \boldsymbol{Q}$ and $\boldsymbol{q} \to -\boldsymbol{Q}$, that makes $q^2 + 2\boldsymbol{pq} \to -Q^2 - 2\boldsymbol{PQ}$. Now we employ the Feynman trick and find

$$
\begin{aligned}
&I^0_{110}(1+\gamma) - I^0_{110}(1) + \gamma I^1_{111}(1+\gamma)\\
&\quad = -\Gamma(4)\int_0^1 dz \int dx_1 dx_2 dx_3 \delta(1-x_1-x_2-x_3)\int_{qp}\int_0^\infty ds x_1 (q^2 + 2\boldsymbol{pq})\Big[x_1(\boldsymbol{p}_\parallel + z\boldsymbol{q}_\parallel)^2 + [x_2+x_3+z(1-z)x_1]\boldsymbol{q}_\parallel^2\\
&\qquad + (x_1+4x_3)\left(p_x + \frac{zx_1+2x_3}{x_1+4x_3}q_x\right)^2 + \frac{\{x_1[x_2+x_3+z(1-z)x_1]+4x_2x_3\}}{x_1+4x_3}q_x^2 + (1+\gamma z)sx_1 + h^2\Big]^{-4}\\
&\quad = -\Gamma(4)\int_0^1 dz \int dx_1 dx_2 dx_3 \delta(1-x_1-x_2-x_3)\frac{h^{8-d}}{x_{1^{(d-1)/2}}\sqrt{x_1+4x_3}}\int_{pq}\int_0^\infty ds\left(q^2 - 2q_x^2\frac{zx_1+2x_3}{x_1+4x_3} - 2z\mathbf{q}_{||}^2\right)\\
&\qquad \times \frac{1}{[p^2+h^2]^4}\left\{[x_2+x_3+z(1-z)x_1]\mathbf{q}_{||^2} + \frac{\{x_1[x_2+x_3+z(1-z)x_1]}{+}4x_2x_3\}x_1 + 4x_3 q_{x^2} + (1+\gamma z)s + h^2\right\}^{d/2-4}.
\end{aligned}
\tag{B9}
$$

Performing integration over momenta and frequency and using the parametrization $x_1 = s/(s+1)$, $x_2 = u/(s+1)$, $x_3 = (1-u)/(s+1)$, where $0 \leqslant s < \infty$ and $0 \leqslant u \leqslant 1$ [with the Jacobian $1/(s+1)^3$], we obtain

$$
\begin{aligned}
&I^0_{110}(1+\gamma) - I^0_{110}(1) + \gamma I^1_{111}(1+\gamma)\\
&\quad = \frac{h^{2\epsilon}\Omega_d^2}{2\epsilon}\Gamma^2(d/2)\Gamma(3-d)\int_0^1 dz\frac{(1-2z)}{1+\gamma z}\int_0^\infty ds\int_0^1 du(s+1)^{d-2}s^{(1-d)/2}[1+z(1-z)s]^{(1-d)/2}\\
&\qquad \times \{s[1+z(1-z)s] + 4u(1-u)\}^{-1/2}\left[\frac{d-1}{1+z(1-z)s} + \frac{s}{s[1+z(1-z)s]+4u(1-u)}\right].
\end{aligned}
\tag{B10}
$$

The integrals over z, s, and u are convergent in $d = 2$; therefore, we can set $d = 2$. Then, we find

$$
\begin{aligned}
&I^0_{110}(1+\gamma) - I^0_{110}(1) + \gamma I^1_{111}(1+\gamma)\\
&\quad = \frac{h^{2\epsilon}A_\epsilon}{2\epsilon}\int_0^1 dz\frac{(1-2z)}{1+\gamma z}\int_0^\infty ds\frac{1}{\sqrt{s[1+z(1-z)s]^3}}\left[\arctan\frac{1}{\sqrt{s[1+z(1-z)s]}} + \frac{\sqrt{s[1+z(1-z)s]}}{\{1+s[1+z(1-z)s]\}}\right]\\
&\quad = \frac{h^{2\epsilon}A_\epsilon}{2\epsilon}\int_0^1 dz\frac{(1-2z)}{1+\gamma z}\int_0^\infty dy\left[\frac{1}{\sqrt{z(1-z)}}\frac{1}{\sqrt{y(1+y)^3}}\arctan\frac{\sqrt{z(1-z)}}{\sqrt{y(1+y)}} + \frac{1}{(y+1)[z(1-z)+y(1+y)]}\right]\\
&\quad = \frac{h^{2\epsilon}A_\epsilon}{2\epsilon}\int_0^1 \frac{dz}{1+\gamma z}\left[\frac{(1-2z)}{\sqrt{z(1-z)}}\int_0^\infty \frac{dv}{\cosh^2(v/2)}\arctan\frac{2\sqrt{z(1-z)}}{\sinh v} + \frac{\ln(1-z)}{z} - \frac{\ln z}{1-z}\right]\\
&\quad = \frac{h^{2\epsilon}A_\epsilon}{2\epsilon}\Phi(\gamma).
\end{aligned}
\tag{B11}
$$

Here we introduced $y = z(1-z)s$ and $v = 2\operatorname{arcsinh}\sqrt{y}$. The function $\Phi(\gamma)$ is given as follows:

$$
\Phi(\gamma) = \int_0^1 dz\frac{F(z)}{1+\gamma z},\quad F(z) = -(1-2z)\left(\frac{\ln z}{1-z} + \frac{\ln(1-z)}{z}\right) + \frac{\ln(1-z)}{z} - \frac{\ln z}{1-z} = 2\ln(1-z) - 2\ln z.
\tag{B12}
$$

Finally, integrating over z exactly, we find

$$
\Phi(\gamma) = \frac{\ln^2(1+\gamma)}{\gamma}.
\tag{B13}
$$

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