Comment on "Super-Universality in Anderson Localization"

In a recent Letter, Horváth and Markoš [1] investigated the quantity

$$\mathcal{N}_* = \sum_j \min\{L^d|\psi(\boldsymbol{r}_j)|^2, 1\}, \tag{1}$$

termed the "minimal effective amount" or "minimal counting scheme," at the Anderson transitions in orthogonal (AI), unitary (A), symplectic (AII), and chiral unitary (AIII) classes in three spatial dimensions, $d = 3$. Here $\psi(\boldsymbol{r}_j)$ is a wave function on a site with coordinate $\boldsymbol{r}_j$, and L is the system size. The authors of Ref. [1] presented numerical evidence for a "super-universal" (intact for all four symmetry classes studied) power-law scaling of the quantity [Eq. (1)] averaged over disorder realizations,

$$\langle\mathcal{N}_*\rangle \sim L^{d_{\mathrm{IR}}} \quad \text{with} \quad d_{\mathrm{IR}} \approx 8/3. \tag{2}$$

In this Comment, for the standard Wigner-Dyson (WD) symmetry classes, we shall demonstrate that the result [Eq. (2)] is incorrect. The quantity can be computed explicitly in the limit of large L as

$$\langle\mathcal{N}_*\rangle = 4\left[|f''(\alpha_0)|/(2\pi\ln L)\right]^{1/2}L^{f(d)}. \tag{3}$$

Here, $f(\alpha)$ is the singularity spectrum characterizing multifractality at Anderson transitions (see Ref. [2] and references therein) and α_0 is the point of its maximum, $f'(\alpha_0) = 0$. Equation (3) shows that (i) d_{IR} is nothing but $f(d)$ and (ii) $\langle\mathcal{N}_*\rangle$ does not demonstrate "super-universality" in a strict sense: its behavior does depend on the symmetry class via $f(d)$ and $f''(\alpha_0)$ (see Table I).

The derivation of Eq. (3) is based on the *exact* relation $\langle\mathcal{N}_*\rangle = c_0\sqrt{\ln L}\int d\alpha L^{f(\alpha)}\min\{L^{d-\alpha}, 1\}$, where the normalization constant can be estimated as $c_0 \simeq \sqrt{|f''(\alpha_0)|/(2\pi)}$. Then, using properties of $f(\alpha)$ and the symmetry relation $f(2d-\alpha) = f(\alpha) + d - \alpha$ [2], one obtains Eq. (3), provided $\ln L \gg |f''(d)|$.

It is worthwhile to emphasize that, according to Eq. (3), the scaling of $\langle\mathcal{N}_*\rangle$ with the system size is not purely power-law-like, in contrast to the assumption of Ref. [1]. The presence of $\sqrt{\ln L}$ in the denominator affects significantly the analysis of the L dependence at not too large L; see Fig. 1. This is the reason why the exponent d_{IR} found in Ref. [1] by extrapolating the results for $L \leq 128$ to $L \to \infty$ is smaller than $f(d)$.

At the same time, we note a striking numerical closeness of the values of $f(d = 3)$, which might indeed suggest a kind of universality, as hypothesized by the authors of Ref. [1]. Moreover, the whole singularity spectrum functions are very close (albeit certainly distinct) in $d = 3$ for classes A, AI, and AII [3]. However, this fact is specific for Anderson transitions in $d = 3$. Indeed, in $d = 2 + \epsilon$ dimensions, one finds $f(d) \simeq d - b\epsilon/16$, where $b = 4$ and 1 for the classes AI and A, respectively (see, e.g., Ref. [2]). Therefore, in $d = 2 + \epsilon$ dimensions $\langle\mathcal{N}_*\rangle$ clearly demonstrates no "super-universality." For the class AII, the situation is even more interesting, since the Anderson transition occurs already in $d = 2$ dimensions and $f(d = 2) \simeq 2 - 0.04$ [6]. Thus, there is no reason to expect exact "super-universality" in $d = 3$, either.

TABLE I. Numerical results for $f(\alpha = d = 3)$ and c_0 at Anderson transitions in the WD classes in three dimensions. In order to estimate c_0 we used data from Ref. [3].

	Class AI	Class A	Class AII
$f(3)$	$2.730 \div 2.736$ [3]	$2.719 \div 2.721$ [3]	$2.712 \div 2.715$ [3]
	$2.7307 \div 2.7328$ [4]	$2.7187 \div 2.7195$ [5]	
c_0	0.291	0.282	0.278

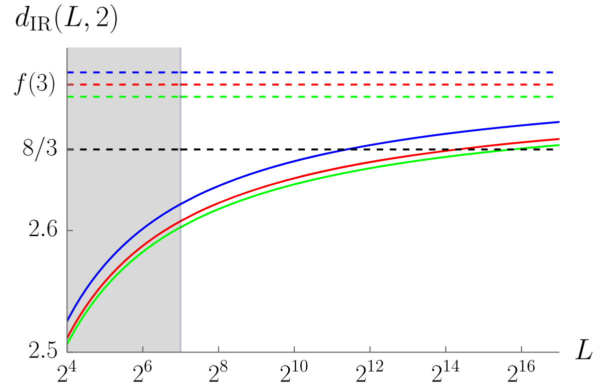

FIG. 1. Plot of $d_{\mathrm{IR}}(L, s) = \{\ln[\langle\mathcal{N}_*(L)\rangle/\langle\mathcal{N}_*(L/s)\rangle]\}/\ln s$ introduced in Ref. [1], as a function of L on the logarithmic scale for $s = 2$. Blue, red, and green curves correspond to Eq. (3) for the symmetry classes AI, A, and AII, respectively. The parameters of the curves are taken from Table I. The limiting value 8/3 for d_{IR} proposed in Ref. [1] is shown by the black dashed line. The blue (AI), red (A), and green (AII) dashed lines indicate the asymptotic expression $f(d = 3)$ for $d_{\mathrm{IR}}(L, 2)$ in the limit $L \to \infty$. The shaded area denotes the region of system sizes for which numerical simulations in Ref. [1] were performed.

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