


Instanton analysis for the spin quantum Hall symmetry class: Nonperturbative corrections to physical observables and generalized multifractal spectrum

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Recently, there has been renewed interest in studies of criticality in the spin quantum Hall effect, realized in the Altland-Zirnbauer symmetry class C of disordered, noninteracting fermions in two spatial dimensions. In our study, we develop a nonperturbative analysis of the replica two-dimensional nonlinear sigma model in class C. We explicitly construct the instanton solution with a unit topological charge. By treating fluctuations around the instanton at the Gaussian level, we calculate the instanton correction to the disorder-averaged logarithm of the partition function. We compute nonperturbative corrections to the anomalous dimensions of pure power-law scaling local operators, which determine the spectrum of generalized multifractality. We also calculate instanton corrections to the renormalized longitudinal and Hall spin conductivities and determine the topology of the phase diagram for class C. Our results demonstrate that the spin quantum Hall effect is indeed a close cousin of the integer quantum Hall effect.

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I. INTRODUCTION

The prototypical example of a quantum phase transition in disordered noninteracting fermionic systems is the Anderson localization/delocalization transition [1]. After more than 60 years of active research on Anderson localization, its physics and related phenomena are well understood (see Refs. [2,3] for a review). Perhaps the most intriguing discovery in the realm of Anderson localization was the discovery of the integer quantum Hall effect (iqHe) [4,5] that realizes Anderson transitions between topologically nontrivial localized phases. The understanding of the synergy between topology and quantum interference in the iqHe has stimulated search of other topological Anderson transitions. It was found [6–11] that there exist ten different (Altland-Zirnbauer) symmetry classes of disordered noninteracting Hamiltonians. In addition, each spatial dimension admits five out of ten symmetry classes with nontrivial topology [12–14]. Physics in each of ten symmetry classes is described by the corresponding effective long-wave field theory—nonlinear sigma model (NL σ M) (see Ref. [2] for a review). The nontrivial topology is reflected by the presence of the topological term (either ϑ term or Wess-Zumino-Witten-Novikov term) in the NL σ M action.

In two spatial dimensions the corresponding Anderson transitions between different topological phases occur at strong coupling typically. Thus, criticality is not accessible within effective long-wave description in terms of NL σ M. Nevertheless, the nonperturbative (instanton) analysis of NL σ M in a weak coupling regime allows one to understand the structure of the phase diagram and explain quantization of the proper physical observable. A well-known example of such a situation is the integer quantum Hall effect (iqHe). The iqHe phase diagram and the quantization of the Hall

conductance has been understood on the basis of existence of instantons—topologically nontrivial solutions of classical equations of motion for the NL σ M action [15–22].

Recently, the interest in the iqHe has been renewed in the context of a critical theory for the Anderson transition between different topological phases (plateau-plateau transition). For a long time it has been suggested [23–28] that the Wess-Zumino-Novikov-Witten models are an ultimate conformal critical theory for the Anderson transition in the iqHe. However, the critical theory predicts not only the localization length exponent but also anomalous dimensions of various local operators. Examples of such operators are the disorder averaged moments of the local density of states (LDoS), $\nu(\mathbf{x})$, which demonstrate pure power-law scaling with the system size, $\langle \nu^q(\mathbf{x}) \rangle \sim L^{-\chi(q)}$, at criticality [29–31]. In addition to the moments of LDoS there are many more pure scaling observables [32]. The corresponding operators can be expressed in terms of the disorder averages of specific combinations of wave functions [33–36]. The corresponding set of generalized multifractal exponents x_λ is unique for each symmetry class and dimensionality (see Refs. [37,38] for a review). Provided the local conformal invariance and Abelian fusion rules for the pure scaling operators hold, the generalized multifractal exponents x_λ were proven to have a parabolic form with a single free parameter only [39–41]. The validity of this prediction has been debated in numerical simulations [2,40,42]. One more interesting aspect of recent discussions of the iqHe criticality is the numerical evidence that the magnitude of the localization length exponent in the iqHe transition varies with a change of geometry of a random potential [43–47].

The iqHe has a close cousin, the spin quantum Hall effect (sqHe) that occurs in superconducting class C [48–50]. The

analog of the integer quantized Hall conductivity is the spin Hall conductivity that describes the response of the spin current to the gradient of the external magnetic field¹. The sqHe has very similar phenomenology as the iqHe, in particular, it has even integer quantized spin Hall conductivity, Anderson transitions between different topological phases, description in terms of the NL σ M (see Ref. [2] for a review). Additionally, the sqHe has an advantage: an infinite set of anomalous dimensions of local operators has been computed analytically at criticality by mapping to the classical percolation problem [34,53–57]. Recently, the sqHe criticality has been intensively tested against description by the conformal field theory. It was shown [34–36,40,55,58,59] that although the numerical data for the generalized multifractal exponents x_λ reproduce exact analytical results obtained from mapping to percolation, there is a clear evidence for a violation of parabolicity. These results prove a lack of the local conformal invariance at the sqHe transition in $d = 2$. There is an alternative point of view on breaking of the generalized parabolicity and the presence of local conformal invariance. As it has recently been proposed [60], it can be explained by a nonperturbative reconstruction of nonlinear σ -model manifold, when renormalization group (RG) flow reaches strong coupling limit.

The absence of plausible candidates for the critical theory of the sqHe transition makes nonperturbative weak coupling analysis of the corresponding NL σ M to be of interest. Surprisingly, the nonperturbative analysis of the NL σ M for class C is absent so far in the literature. In our paper we fill this gap by developing the instanton analysis of the replica NL σ M in the class C. In particular,

(i) we construct the instanton solution for the replica NL σ M in the class C, cf. Eqs. (15) and (107);

(ii) employing the Pauli-Villars regularization scheme [61], we calculate the instanton correction to the partition function, cf. Eq. (64);

(iii) we apply the developed methodology to computation of nonperturbative corrections to the average LDoS, cf. Eq. (86), and to the anomalous dimensions of the local derivativeless eigenoperators (with respect to the renormalization group), cf. Eq. (96), that determine the spectrum of generalized multifractality;

(iv) we calculate instanton corrections to the renormalized longitudinal and Hall spin conductivities and extract the two-parameter nonperturbative renormalization group equations, cf. Eq. (106).

Our results support a general idea that the sqHe is in many ways very similar to the iqHe.

The outline of the paper is as follows. We start from the formalism of the replica Pruisken's NL σ M for class C (Sec. II). In Sec. III we present construction of instantons with topological charge ± 1 for class C and analysis of Gaussian fluctuations around it. The instanton contribution to the partition function is computed in Sec. IV. Next we present computation and analysis of instanton corrections to the anomalous dimensions of pure scaling eigenoperators

(Sec. V). In Sec. VI we compute instanton corrections to the longitudinal and Hall spin conductivities and rewrite them in terms of two-parameter renormalization group equations. We end the paper with discussions and conclusions (Sec. VII). Details of lengthy calculations are given in Appendixes.

II. PRUISKEN'S NL σ M FOR CLASS C

A. NL σ M action

We use formalism of Finkel'stein's NL σ M for the symmetry of class C (see Refs. [62–66] for details) adapted for noninteracting electrons. We exclude interaction term, reduce the space of positive and negative Matsubara frequencies to the retarded-advanced (RA) space, leaving two frequencies only. In this way we obtain the following form of the Pruisken's NL σ M:

$$S_\sigma = -\frac{g}{16} \int_{\mathbf{x}} \text{Tr}(\nabla \mathcal{Q})^2 + \frac{g_H}{16} \int_{\mathbf{x}} \text{Tr}[\varepsilon_{jk} \mathcal{Q} \nabla_j \mathcal{Q} \nabla_k \mathcal{Q}], \quad (1)$$

where $\int_{\mathbf{x}} = \int d^2\mathbf{x}$ and ε_{jk} denotes Levi-Civita symbol with $\varepsilon_{xy} = -\varepsilon_{yx} = 1$. The field \mathcal{Q} is a traceless Hermitian matrix, defined on $N_r \times N_r$ replica, 2×2 retarded-advanced and 2×2 spin spaces. It satisfies a nonlinear constraint and Bogoliubov-de Gennes (BdG) symmetry constraint:

$$\begin{aligned} \mathcal{Q}^2 = 1, \quad \mathcal{Q} = -\bar{\mathcal{Q}}, \quad \bar{\mathcal{Q}} = \mathbf{s}_2 L_0 \mathcal{Q}^T L_0 \mathbf{s}_2, \\ L_0 = \begin{pmatrix} 0 & \mathbf{s}_0 \otimes \hat{\Gamma}_r \\ \mathbf{s}_0 \otimes \hat{\Gamma}_r & 0 \end{pmatrix}_{\text{RA}}. \end{aligned} \quad (2)$$

Here subscript RA implies matrix acting in the RA space. The parameter $g(g_H)$ denotes bare dimensionless longitudinal (transverse) spin conductance, $\hat{\Gamma}_r$ stands for the unit matrix in replica space, and \mathbf{s}_j are standard Pauli matrices. We note that the last (topological) term in the right-hand side (r.h.s.) of Eq. (1) has exactly the same form as the one in class A [17].

Non-linear constraint on \mathcal{Q} matrix and BdG symmetry relation define σ -model target manifold: $\mathcal{Q} \in \text{Sp}(4N_r)/\text{U}(2N_r)$. In the end of all calculation we should take the replica limit: $N_r \rightarrow 0$. In order to resolve nonlinear constraint we can rewrite \mathcal{Q} matrix in terms of nonuniform matrix rotations:

$$\mathcal{Q} = \tilde{\mathcal{T}}^{-1} \Lambda \tilde{\mathcal{T}}, \quad \Lambda = \sigma_3 \otimes \mathbf{s}_0 \otimes \hat{\Gamma}_r, \quad (3)$$

where $\tilde{\mathcal{T}} \in \text{Sp}(4N_r)$, σ_3 is the corresponding Pauli matrix in the RA space. The matrix Λ is so-called metallic saddle-point. It is convenient to realize the rotation matrices $\tilde{\mathcal{T}}$ belonging to $\text{Sp}(4N_r)$ as the $4N_r \times 4N_r$ matrices satisfying the following conditions:

$$\tilde{\mathcal{T}}^{-1} = \tilde{\mathcal{T}}^\dagger, \quad (\tilde{\mathcal{T}}^{-1})^T L_0 \mathbf{s}_2 = \mathbf{s}_2 L_0 \tilde{\mathcal{T}}. \quad (4)$$

The first relation in Eq. (4) restricts the number of independent real variables of the matrix $\tilde{\mathcal{T}}$ to be equal to $(4N_r)^2$ as given for $\text{U}(4N_r)$ group. The second condition in Eq. (4) reduces the number of independent real variables down to $2(2N_r)^2 + (2N_r)$ as it should be for $\text{Sp}(4N_r)$.

B. Nonunitary matrix rotation

The model defined in Sec. II A reduces to the 4×4 matrix theory in single-replica limit $N_r = 1$. Even with such size of

¹We note that similar relation between the spin current and the gradient of the magnetic field is realized in thin films of the superfluid ³He-A [51,52].

matrix \mathcal{Q} calculations might become too tedious. In order to avoid this difficulty, we perform nonunitary rotation of the matrix basis, introducing new matrix \mathcal{Q} as

$$\mathcal{Q} = U^{-1} \mathcal{Q} U, \quad U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}_{\text{RA,S}} \otimes \frac{\hat{1}_r}{\sqrt{2}}. \quad (5)$$

Here the subscript RA, S implies matrix acting in the combined RA and spin spaces. We use peculiarities of symplectic group to change the antisymmetric matrix, which defines BdG symmetry relation in such a way that it acts in the spin space only:

$$\mathcal{Q} = -\bar{\mathcal{Q}}, \quad \bar{\mathcal{Q}} = \mathbf{s}_2 \mathcal{Q}^T \mathbf{s}_2. \quad (6)$$

Also we note that the rotation (5) changes the definition of the metallic saddle point:

$$\Lambda \rightarrow \underline{\Lambda} = \hat{1}_{\text{RA}} \otimes \mathbf{s}_3 \otimes \hat{1}_r. \quad (7)$$

It is easy to check, that the transformation (5) does not change the form of the action (1). After the transformation (5) we cannot distinguish RA and replica spaces, therefore, it is convenient to introduce new notation: $n = 2N_r$ for dimension of the combined RA/replica space. After the above transformation, the NL σ M reduces to the 2×2 matrix theory in single-replica limit, $n = 1$.

III. INSTANTONS WITH TOPOLOGICAL CHARGES $\mathcal{C} = \pm 1$

A. Construction of the instanton solution

In order to obtain the saddle-point solutions with nontrivial topology we use Bogomolny inequality:

$$\text{Tr}(\nabla_x \mathcal{Q} \pm i \mathcal{Q} \nabla_y \mathcal{Q})^2 \geq 0. \quad (8)$$

It can be rewritten equivalently as

$$\frac{1}{16} \int_x \text{Tr}(\nabla \mathcal{Q})^2 \geq \pi |\mathcal{C}[\mathcal{Q}]|. \quad (9)$$

Here we introduce the topological charge

$$\mathcal{C}[\mathcal{Q}] = \frac{1}{16\pi i} \int_x \text{Tr} \varepsilon_{jk} \mathcal{Q} \nabla_j \mathcal{Q} \nabla_k \mathcal{Q}. \quad (10)$$

Therefore, stable matrix field configurations, which minimize the action should satisfy the self-duality equation:

$$\nabla_x \mathcal{Q} \pm i \mathcal{Q} \nabla_y \mathcal{Q} = 0. \quad (11)$$

We note that Eq. (11) is invariant under transformation (5). Therefore, we construct solution of this equation in the rotated basis as follows. At first, we set the number of replicas in the rotated basis equal to unity, $n = 1$. Then, similar to class A [19], we use Belavin-Polyakov instanton to solve Eq. (11) by the matrix

$$\Lambda_{\text{inst}}^{(n=1)} = \begin{pmatrix} |e_1|^2 - e_0^2 & 2e_0 e_1 \\ 2e_0 e_1^* & -(|e_1|^2 - e_0^2) \end{pmatrix}, \quad (12)$$

where

$$e_0 = \frac{\lambda}{\sqrt{|z - z_0|^2 + \lambda^2}}, \quad e_1 = \frac{z - z_0}{\sqrt{|z - z_0|^2 + \lambda^2}}. \quad (13)$$

Here $z = x + iy$ stands for the complex coordinate, the complex z_0 denotes a position of the instanton's center and λ stands for its scale size. Generalization of solution (12) to the case of $n > 1$ is constructed in the form that explicitly violates the replica symmetry:

$$\Lambda_{\text{inst}}^{(n>1)} = \begin{pmatrix} \Lambda_{\text{inst}}^{(n=1)} & 0 & \dots & 0 \\ 0 & \mathbf{s}_3 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbf{s}_3 \end{pmatrix}_r. \quad (14)$$

Here the lower index r denotes that the matrix is written in the replica space. Topological charge (10) for this solution is equal to one: $\mathcal{C}[\Lambda_{\text{inst}}^{(n>1)}] = 1$. Solution with negative topological charge can be obtained by complex conjugation of solution (14). Therefore, full instanton manifold can be written in terms of the unitary rotations \mathcal{T}_0 and \tilde{R} about the metallic saddle point $\underline{\Lambda}$:

$$\mathcal{Q}_{\text{inst}} = \mathcal{T}_0^{-1} \Lambda_{\text{inst}}^{(n>1)} \mathcal{T}_0 = \mathcal{T}_0^{-1} \tilde{R}^{-1} \underline{\Lambda} \tilde{R} \mathcal{T}_0, \quad (15)$$

$$\tilde{R} = \begin{pmatrix} R_0 & 0 \\ 0 & \hat{1}_r \end{pmatrix}_r = \begin{pmatrix} e_1^* & 0 & e_0 & 0 \\ 0 & \hat{1}_r & 0 & 0 \\ -e_0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & \hat{1}_r \end{pmatrix}_{\text{S,r}}.$$

Here $\mathcal{T}_0 \in \text{Sp}(2n)$ stands for an arbitrary global unitary rotation, which describes the orientation of the instanton in the coset space $\text{Sp}(2n)/\text{U}(n)$. One can check that $\mathcal{Q}_{\text{inst}}$ satisfies BdG symmetry relation (6). The classical action for the instanton solution (15) is finite:

$$S_{\text{cl}} = -\pi g + i\pi g_H. \quad (16)$$

We note that the classical action S_{cl} is independent of z_0 , λ , and \mathcal{T}_0 , i.e., they can be identified as zero modes. In contrast to the case of class A, it is convenient to split g_H on even integer part and fractional part:

$$g_H = 2k + \vartheta/\pi, \quad k \in \mathbb{Z}, \quad -\pi < \vartheta \leq \pi. \quad (17)$$

We note that a change of ϑ from $-\pi$ to π corresponds to change of g_H on 2 rather than on 1 as in the class A.

B. Fluctuations near the instanton solution

In order to construct perturbation theory around the instanton solution we use exponential parametrization of the \mathcal{Q} matrix:

$$\mathcal{Q} = \tilde{R}^{-1} \mathcal{V} \tilde{R}, \quad \mathcal{V} = e^{-W/2} \underline{\Lambda} e^{W/2}. \quad (18)$$

The matrix W has to satisfy the following constraints: $\{W, \Lambda\} = 0$, $W^\dagger = -W$, and $W = -\bar{W}$. Consequently, the matrix W can be parametrized by $n \times n$ complex symmetric matrix \hat{w} ,

$$W = \begin{pmatrix} 0 & \hat{w} \\ -\hat{w}^* & 0 \end{pmatrix}_{\text{sp}}, \quad \hat{w}^T = \hat{w}. \quad (19)$$

The presence of instanton can be interpreted as appearance of the non-Abelian vector potential, $A_j = \tilde{R} \nabla_j \tilde{R}^{-1}$, in the

NL σ M action, e.g.,

$$-\frac{g}{16} \int_x \text{Tr}(\nabla \mathcal{Q})^2 = -\frac{g}{16} \int_x \text{Tr}(\nabla_j \mathcal{V} - [\mathcal{V}, A_j])^2. \quad (20)$$

Using symmetries of the fields \hat{w} and \hat{w}^* we can expand action (20) to the second order in \hat{w} , \hat{w}^* :

$$\begin{aligned} \delta S_\sigma = & -\frac{g}{8} \int dx \mu^2(r) \left[w^{11} O^{(2)} w^{*11} + \sum_{\alpha=2}^n w^{\alpha\alpha} O^{(0)} w^{*\alpha\alpha} \right. \\ & \left. + 2 \sum_{1 < \alpha < \beta \leq n} w^{\alpha\beta} O^{(0)} w^{*\alpha\beta} + 2 \sum_{\alpha=2}^n w^{1\alpha} O^{(1)} w^{*1\alpha} \right], \end{aligned} \quad (21)$$

where $r^2 = x^2 + y^2$ and the Greek indices denote matrix structure in the replica space. Here we define the operators

$$O^{(a)} = -\frac{(r^2 + \lambda^2)^2}{4\lambda^2} \left[\nabla_j + \frac{ia}{r^2 + \lambda^2} \varepsilon_{jkl} x_k \right]^2 - \frac{a}{2}, \quad (22)$$

and the measure

$$\mu(r) = \frac{2\lambda}{r^2 + \lambda^2}. \quad (23)$$

We note that the set of operators $O^{(a)}$ and the measure $\mu(r)$ are exactly the same as the ones that arise in analysis of fluctuations around the instanton in class A [19].

The natural appearance of the measure $\mu^2(r)$ indicates that it is convenient to employ inverse stereographic projection from the flat space onto the sphere with a radius λ . Therefore, we should introduce new coordinates, the spherical angles:

$$\cos \phi = \frac{r^2 - \lambda^2}{r^2 + \lambda^2} = \eta, \quad \theta = \arctan\left(\frac{y}{x}\right). \quad (24)$$

In terms of the spherical coordinates, the quantities e_1 and e_0 can be written as

$$e_0 = \sqrt{\frac{1-\eta}{2}}, \quad e_1 = \sqrt{\frac{1+\eta}{2}} e^{i\theta}, \quad (25)$$

while the operators $O^{(a)}$ become

$$\begin{aligned} O^{(a)} = & -\frac{\partial}{\partial \eta} \left[(1-\eta^2) \frac{\partial}{\partial \eta} \right] - \frac{1}{1-\eta^2} \frac{\partial^2}{\partial \theta^2} + \frac{ia}{1-\eta} \frac{\partial}{\partial \theta} \\ & + \frac{a^2}{4} \frac{1+\eta}{1-\eta} - \frac{a}{2}. \end{aligned} \quad (26)$$

Eigensystem for operators (26) can be found in a standard way as the solution of Schrödinger-type equation

$$O^{(a)} \Phi^{(a)}(\eta, \theta) = E^{(a)} \Phi^{(a)}(\eta, \theta). \quad (27)$$

Here the eigenfunctions are normalized with respect to the measure $d\eta d\theta$. The eigenfunctions are expressed in terms of Jacobi polynomials:

$$P_n^{\alpha, \beta}(\eta) = \frac{(-1)^n (1-\eta)^{-\alpha}}{2^n n!} \frac{d^n}{d\eta^n} \frac{(1-\eta)^{n+\alpha}}{(1+\eta)^{-n-\beta}}. \quad (28)$$

as [19]

$$\Phi_{J,M}^{(a)} = C_{J,M}^{(a)} e^{-iM\theta} (1-\eta)^{\frac{a}{2}} (1-\eta^2)^{\frac{M}{2}} P_{J-M-s_a}^{M+a, M}(\eta), \quad (29)$$

TABLE I. Number of fields for operators $O^{(a)}$.

Operator	Number of fields $w^{\alpha\beta}$	Degeneracy
$O^{(0)}$	$(n^2 - n)/2$	1
$O^{(1)}$	$(n - 1)$	2
$O^{(2)}$	1	3

where $s_a = 0, 1, 1$ for $a = 0, 1, 2$, respectively. The eigenstates are enumerated by the angular momentum $J = 0, 1, 2, \dots$ for $a = 0$ and $J = 1, 2, 3, \dots$ for $a = 1, 2$. The corresponding momentum projections satisfy $-J - a(a - 1)/2 \leq M \leq J - s_a$. The normalization constants read [67]

$$\begin{aligned} C_{J,M}^{(a)} = & \frac{\sqrt{\Gamma(J+M+1+a(a-1)/2)\Gamma(J-M+1-s_a)}}{2^{M+1+a(a-1)/2} \sqrt{\pi} \Gamma(J)} \\ & \times \begin{cases} \frac{\sqrt{2J+1}}{J+1}, & a = 0, \\ 1, & a = 1, \\ \frac{\sqrt{2J+1}}{\sqrt{J(J+1)}}, & a = 2. \end{cases} \end{aligned} \quad (30)$$

The eigenenergies are given as

$$E_J^{(a)} = (J - s_a)(J + 1 - s_a + a). \quad (31)$$

We emphasize that the eigenenergy vanishes for the smallest allowed angular momentum, $E_{J=s_a}^{(a)} = 0$.

C. Analysis of the zero modes

There are several zero-energy modes for the operators $O^{(a)}$. From Table I we can compute the number of the zero modes to be equal to $n^2 + 3n + 2$.

We also note, that one can rewrite eigenfunctions corresponding to the modes with zero eigenenergies in terms of e_0 and e_1 :

$$\begin{aligned} \Phi_{0,0}^{(0)} = \frac{1}{2\sqrt{\pi}}, \quad \Phi_{1,-1}^{(1)} = \frac{1}{\sqrt{2\pi}} e_1, \quad \Phi_{1,0}^{(1)} = \frac{1}{\sqrt{2\pi}} e_0, \\ \Phi_{1,-2}^{(2)} = \sqrt{\frac{3}{4\pi}} e_1^2, \quad \Phi_{1,-1}^{(2)} = \sqrt{\frac{3}{2\pi}} e_0 e_1, \quad \Phi_{1,0}^{(2)} = \sqrt{\frac{3}{4\pi}} e_0^2. \end{aligned} \quad (32)$$

Now we show that each zero mode is related with corresponding instanton degree of freedom (collective coordinate). For this purpose we introduce small deviations of instanton degrees of freedom: $\xi_j = \{\lambda, x_0, y_0\}$ and generators t , which can be defined as an expansion of global rotation matrices \mathcal{T}_0 near the identity matrix:

$$\mathcal{T}_0 = 1 + it. \quad (33)$$

In this way we find

$$\begin{aligned} Q(\xi_j + \delta\xi_j) = \tilde{R}^{-1}(\xi_j)(\Lambda + [\Lambda, B])\tilde{R}(\xi_j), \\ B = i\tilde{R}(\xi_j)t\tilde{R}^{-1}(\xi_j) - \delta\xi_j\tilde{R}(\xi_j)(\partial_{\xi_j}\tilde{R}^{-1}(\xi_j)). \end{aligned} \quad (34)$$

Comparing the above equations with Eq. (18), we relate fluctuation matrix \hat{w} and the zero modes as

$$w = 2(i\tilde{R}t\tilde{R}^{-1} - \delta\xi_j\tilde{R}(\partial_{\xi_j}\tilde{R}^{-1}))_{12}. \quad (35)$$

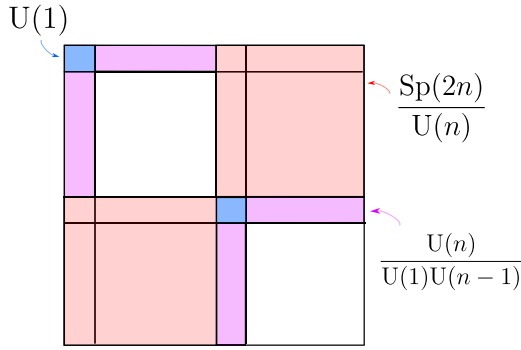


FIG. 1. The symmetry-breaking pattern in terms of generators (33) for an arbitrary matrix $T_0 \in \text{Sp}(2n)$. The off-diagonal elements corresponding to $\mathcal{T}' \in \text{Sp}(2n)/\text{U}(n)$ are shown by pink color. The diagonal elements corresponding to $\mathbf{T} \in \text{U}(1) \cup \text{U}(n)/[\text{U}(1) \times \text{U}(n-1)]$ are shown in blue and violet colors.

Here the subscripts 12 correspond to the spin space structure in the rotated basis. Explicit expressions for matrices and symmetry relations for \hat{w} allows us to obtain the following results:

$$\begin{aligned} w^{11} &= -4e_0 e_1^* \left(it_{11}^{11} - \frac{\delta\lambda}{2\lambda} \right) + 2e_0^2 \left(\frac{\delta z^*}{\lambda} - it_{12}^{*11} \right) + 2ie_1^{*2} t_{12}^{11}, \\ w^{1\alpha} &= w^{\alpha 1} = 2i(e_1^* t_{12}^{1\alpha} - e_0 t_{11}^{*1\alpha}), \\ w^{\alpha\beta} &= 2it_{12}^{\alpha\beta}, \quad \beta \geq \alpha > 1. \end{aligned} \quad (36)$$

If one considers a similar task for the trivial topological sector ($\mathcal{C}[\mathcal{Q}] = 0$), i.e., one makes an expansion around the metallic saddle point $\underline{\Delta}$, one obtains

$$w^{\alpha\beta} = 2it_{12}^{\alpha\beta}. \quad (37)$$

These are trivial zero modes corresponding to $\mathcal{N}_0 = n^2 + n$ real parameters. The number of zero modes for the instanton solution can be split as follows:

$$\begin{aligned} n^2 + 3n + 2 &= \underbrace{n^2 + n}_{\text{trivial}} + \underbrace{2n + 2}_{\text{instanton}} = \mathcal{N}_0 \\ &+ \underbrace{2n - 2 + 1}_{\text{instanton rotations}} + \underbrace{3}_{\xi_i}. \end{aligned} \quad (38)$$

In the last line of the above equation we separate the number of zero modes corresponding to the instanton parameters $\xi_i = \{z_0, \lambda\}$ from the number of zero modes corresponding to generators of instanton rotations t . The above zero mode structure corresponds to the symmetry-breaking pattern shown in Fig. 1. The instanton breaks $U(n)$ down to $U(n-1)$ explicitly. Therefore, we can present the zero modes corresponding to rotations \mathcal{T}_0 around the instanton as a product $\mathcal{T}_0 = \mathbf{T}\mathcal{T}'$, where $\mathcal{T}' \in \text{Sp}(2n)/\text{U}(n)$ and $\mathbf{T} \in \text{U}(1) \cup \text{U}(n)/[\text{U}(1) \times \text{U}(n-1)]$ describes the additional rotational zero modes.

IV. CALCULATION OF THE PARTITION FUNCTION

In order to calculate the partition function we use the fact, that total partition function can be written as sum over

contributions from all topological sectors:

$$\mathcal{Z} = \sum_{\mathcal{C}=-\infty}^{\infty} \mathcal{Z}_{\mathcal{C}}, \quad (39)$$

where \mathcal{C} is the integer-valued topological charge, see Eq. (10). For a reason to be explained shortly, it is more convenient to work with the quantity $\mathcal{F} = [1/(2n)] \ln \mathcal{Z}$. Analogously to the statistical mechanics, \mathcal{F} can be referred as the free energy. In the replica limit, $n \rightarrow 0$, \mathcal{F} determines the disorder-averaged logarithm of the partition function. Due to exponential smallness of configurations with $|\mathcal{C}| > 1$, we can expand the logarithm of the partition function in a series:

$$\ln \mathcal{Z} \approx \ln \mathcal{Z}_0 + \frac{\mathcal{Z}_{+1}}{\mathcal{Z}_0} + \frac{\mathcal{Z}_{-1}}{\mathcal{Z}_0} + \dots \quad (40)$$

Therefore, our main goal for this section is computation of the instanton corrections to $\ln \mathcal{Z}$, which we call the normalized partition function in all calculations below.

Calculation of the partition function can be separated into two parts. The first one is calculation of the contribution of massive modes in Gaussian approximation. The second part is integration over the zero modes, that should be carried out exactly due to divergence of the corresponding determinants in the Gaussian approximation.

Also, we notice that we are interested in calculation of the normalized partition function, which is defined as

$$\frac{\mathcal{Z}_{\text{inst}}}{\mathcal{Z}_0} = \frac{\int \mathcal{D}[w, w^*] \exp(S_{\text{cl}} + \delta S_{\sigma})}{\int \mathcal{D}[w, w^*] \exp(\delta S_0)}, \quad (41)$$

where $\mathcal{Z}_{\text{inst}} = \mathcal{Z}_{+1} = \mathcal{Z}_{-1}^*$ and δS_0 has the same form as δS_{σ} (21) with all operators $O^{(a)}$ replaced by $O^{(0)}$.

A. Determinants for the massive modes

In this section we exclude all calculations related to the zero modes, concentrating on obtaining the determinant due to integration over the massive modes. Therefore, schematically we rewrite Eq. (41) as:

$$\frac{\mathcal{Z}_{\text{inst}}}{\mathcal{Z}_0} = \int_{\text{zm}} A_{\text{zm}}[\lambda, z_0, g, \vartheta] e^{S_{\text{cl}} + \mathcal{D}}. \quad (42)$$

Here \int_{zm} denotes the integral over the manifold of the zero modes, A_{zm} is a functional associated with the contribution of the zero modes, which will be defined below, and \mathcal{D} is the contribution of the massive modes. In this section our aim is to compute \mathcal{D} in the Gaussian approximation.

It is convenient to introduce Green's functions for the operators $O^{(a)}$ in a standard way:

$$\mathcal{G}_a(\eta, \theta; \eta', \theta'; \omega) = \sum_{JM} \frac{|JM\rangle_{(a)(a)} \langle JM|}{E_J^{(a)} + \omega}. \quad (43)$$

Then we can calculate integral over w fields explicitly, using expansion in terms of the eigenfunctions of operators $O^{(a)}$:

$$w^{\alpha\beta}(\eta, \theta) = \frac{2\sqrt{2}}{\sqrt{g}} \sum_{J,M} u_{JM}^{\alpha\beta} \Phi_{J,M}^{(a)*}. \quad (44)$$

Performing Gaussian integration over complex coefficients $u_{J,M}^{\alpha\beta}$ we obtain

$$\begin{aligned} \mathcal{D} &= (n-1) \left(\text{Tr} \ln \frac{1}{2} \mathcal{G}_1(0) - \text{Tr} \ln \frac{1}{2} \mathcal{G}_0(0) \right) + \text{Tr} \ln \mathcal{G}_2(0) \\ &\quad - \text{Tr} \ln \mathcal{G}_0(0) = -(n-1)D^{(1)} - D^{(2)}. \end{aligned} \quad (45)$$

Appearance of factors $1/2$ under $\text{Tr} \ln$ after integration over $u_{J,M}^{\alpha\beta}$ is due to last two terms in Eq. (21). It reflects symmetry relations between matrix elements of fluctuation matrices W , cf. Eq. (18).

The traces in Eq. (45) can be readily written in terms of the eigenvalues of the operators $O^{(a)}$. However, these sums are divergent. This fact reflects the infrared divergencies which are well known to appear in the course of the perturbative background field renormalization of the NL σ M action. Usually, such divergences are treated by means of the dimensional regularization scheme. However, since the instanton solution exists strictly in two spatial dimensions we cannot use dimensional regularization scheme. Fortunately, there exists other regularization scheme, Pauli-Villars method [61], which can be employed in two dimensions. We note that it was used for computation of nonperturbative corrections in Yang-Mills theory due to Belavin-Polyakov-Schwarz-Tyupkin instanton [68]. Later this methodology has been adapted to the studies of instanton effects in the iqHe [19,22,67]. Below we will sketch the derivation of the results for the regularized determinants. More details can be found in Ref. [67].

The idea of the Pauli-Villars method is to introduce $K+1$ copies of the quantum theory δS_σ . In each copy the operators $O^{(a)}$ are supplemented by the mass term \mathcal{M}_f^2 , $f=0, \dots, K$, i.e. $O^{(a)} \rightarrow O^{(a)} + \mathcal{M}_f^2$. It is assumed that $\mathcal{M}_0 = 0$ while

$\mathcal{M}_f \gg 1$ for $f=1, \dots, K$. The additional K copies of the theory are served to cancel all the divergencies except the logarithmic one. In order to extract the logarithmically divergent contribution some of the additional copies are assumed to contribute to the logarithm of the determinant, \mathcal{D} , as if they result from the integration over Grassmanian variables, i.e., contributing with the opposite sign in front of $\text{Tr} \ln$ in Eq. (45). In other words, one has to use the following substitution in Eq. (45),

$$\text{Tr} \ln r_b^{-1} \mathcal{G}_a(0) \rightarrow \sum_{f=0}^K \varepsilon_f \text{Tr} \ln r_b^{-1} \mathcal{G}_a(\mathcal{M}_f^2). \quad (46)$$

Here $r_b = 1, 2$ [see Eq. (45)], $\varepsilon_0 = 1$, and $\varepsilon_f = \pm 1$ for $f=1, \dots, K$. The particular sign of each ε_f is chosen to be able to cancel all divergencies except the logarithmic one. Then the regularized versions of the functions $D^{(1,2)}$ become

$$D_{\text{reg}}^{(a)} = \lim_{\Lambda \rightarrow \infty} \left[\Phi_\Lambda \left(\frac{1+a}{2}, r_a \right) - \Phi_\Lambda \left(\frac{1}{2}, r_a \right) \right], \quad (47)$$

where $r_1 = 2$ and $r_2 = 1$, and

$$\begin{aligned} \Phi_\Lambda(p, r) &= \sum_{J=p+1}^{\Lambda} 2J \ln[r(J^2 - p^2)] \\ &\quad + \sum_{f=1}^K \varepsilon_f \sum_{J=p}^{\Lambda} 2J \ln[r(J^2 - p^2 + \mathcal{M}_f^2)]. \end{aligned} \quad (48)$$

Evaluating the function $\Phi_\Lambda(p, r)$ under assumptions that $\Lambda \gg \mathcal{M}_f \gg 1$, we find (see Ref. [67] for details)

$$\begin{aligned} \Phi_\Lambda(p, r) &= -2p \ln r + 4 \sum_{J=1}^{\Lambda} J \ln J + 2p^2 - 2 \sum_{J=1}^{2p} (J-p) \ln J - 2p^2 \ln \Lambda + 2 \ln r \sum_{J=p+1}^{\Lambda} J + \sum_{f=1}^K \varepsilon_f \left[2\Lambda(\Lambda+1) \ln \Lambda - \Lambda^2 \right. \\ &\quad \left. + \frac{\ln e\Lambda}{3} + 2 \ln r \sum_{J=p+1}^{\Lambda} J - \frac{1-6p}{3} \ln \mathcal{M}_f - 2\mathcal{M}_f^2 \ln \mathcal{M}_f + 2\mathcal{M}_f^2 \ln \Lambda - 2p^2 \ln \Lambda \right]. \end{aligned} \quad (49)$$

Now we choose $K=5$ and set $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = -1$. The masses satisfy the following two equations:

$$\sum_{f=1}^K \varepsilon_f \mathcal{M}_f^2 = 0, \quad \sum_{f=1}^K \varepsilon_f \mathcal{M}_f^2 \ln \mathcal{M}_f = 0. \quad (50)$$

Then, the above expression for the function $\Phi_\Lambda(p, r)$ simplifies drastically,

$$\begin{aligned} \Phi_\Lambda(p, r) &= -2p \ln r + \frac{1-6p}{3} \ln \mathcal{M} + 2p^2 - 2 \sum_{J=1}^{2p} (J-p) \ln J \\ &\quad + 4 \sum_{J=1}^{\Lambda} J \ln J - 2\Lambda(\Lambda+1) \ln \Lambda + \Lambda^2 - \frac{\ln e\Lambda}{3}, \end{aligned} \quad (51)$$

where we introduced the so-called Pauli-Villars mass

$$\ln \mathcal{M} = - \sum_{f=1}^K \varepsilon_f \ln \mathcal{M}_f. \quad (52)$$

Next, using the result (51), we can compute the regularized determinants $D_{\text{reg}}^{(a)}$. Interestingly, they can be expressed in terms of regularized determinants for the similar problem in class A, see Ref. [67],

$$\begin{aligned} D_{\text{reg}}^{(1)} &= D_{\text{reg}, A}^{(1)} - \ln 2, \quad D_{\text{reg}}^{(2)} = D_{\text{reg}, A}^{(2)}, \\ D_{\text{reg}, A}^{(1)} &= -\ln \mathcal{M} + \frac{3}{2} - 2 \ln 2, \\ D_{\text{reg}}^{(2)} &= -2 \ln \mathcal{M} + 4 - \ln 2 - 3 \ln 3. \end{aligned} \quad (53)$$

Hence we obtain the final result for the regularized determinant, coming from the integration over the massive

modes

$$\mathcal{D} = (n+1) \ln \mathcal{M} e^{\gamma-1/2} - \frac{3(n-1)}{2} + (3n-2) \ln 2 - 4 + 3 \ln 3 - (n+1)(\gamma-1/2). \quad (54)$$

Here for the purposes that will be clear further, we added and subtracted a constant $(n+1)(\gamma-1/2)$, where $\gamma \approx 0.577$ stands for Euler constant. We emphasize that the Pauli-Villars mass \mathcal{M} controls divergencies in the fluctuation determinant \mathcal{D} . As we discussed above such divergences arise similar to divergencies in the course of the perturbative renormalization of the action due to elimination of the fast fluctuations on the top of slow background field configuration.

B. Jacobian for the zero modes

Detailed calculation of the Jacobian for collective modes can be found in Ref. [19]. In this section we present resulting expression and explain peculiarities arising in the case of class C. In order to derive the zero-mode Jacobian we use explicit expressions for coefficients $u_{J,M}^{\alpha\beta}$ in terms of the instanton degrees of freedom (36):

$$\begin{aligned} u_{1,-1}^{11} &= -\sqrt{\frac{4\pi g}{3}} \left(i t_{11}^{11} - \frac{\delta\lambda}{2\lambda} \right), & u_{1,-2}^{11} &= \sqrt{\frac{2\pi g}{3}} i t_{12}^{11}, \\ u_{1,0}^{11} &= \sqrt{\frac{2\pi g}{3}} \left(\frac{\delta z^*}{\lambda} - i t_{12}^{*11} \right), & u_{1,-1}^{1\alpha} &= \sqrt{\pi g} i t_{12}^{1\alpha}, \\ u_{1,0}^{1\alpha} &= -\sqrt{\pi g} i t_{11}^{*1\alpha}, & u_{0,0}^{\alpha\beta} &= \sqrt{2\pi g} i t_{12}^{\alpha\beta}, \quad \beta \geq \alpha > 1. \end{aligned} \quad (55)$$

The above expressions can be rewritten in terms of Jacobi matrices with block structure in replica space:

$$(\text{Re } u_{1,-1}^{11}, \text{Im } u_{1,-1}^{11}) = \begin{pmatrix} \sqrt{\frac{\pi g}{3\lambda^2}} & 0 \\ 0 & -\sqrt{\frac{4\pi g}{3}} \end{pmatrix} \begin{pmatrix} \delta\lambda \\ t_{11}^{11} \end{pmatrix}, \quad (56)$$

$$\begin{aligned} &(\text{Re } u_{1,-2}^{11}, \text{Im } u_{1,-2}^{11}, \text{Re } u_{1,0}^{11}, \text{Im } u_{1,0}^{11}) \\ &= \begin{pmatrix} 0 & -\sqrt{\frac{2\pi g}{3}} & 0 & 0 \\ \sqrt{\frac{2\pi g}{3}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{2\pi g}{3}} & \sqrt{\frac{2\pi g}{3\lambda^2}} & 0 \\ -\sqrt{\frac{2\pi g}{3}} & 0 & 0 & -\sqrt{\frac{2\pi g}{3\lambda^2}} \end{pmatrix} \begin{pmatrix} \text{Re } t_{12}^{11} \\ \text{Im } t_{12}^{11} \\ x_0 \\ y_0 \end{pmatrix}, \end{aligned} \quad (57)$$

$$\begin{aligned} &(\text{Re } u_{1,-1}^{1\alpha}, \text{Im } u_{1,-1}^{1\alpha}, \text{Re } u_{1,0}^{1\alpha}, \text{Im } u_{1,0}^{1\alpha}) \\ &= \begin{pmatrix} 0 & -\sqrt{\pi g} & 0 & 0 \\ \sqrt{\pi g} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\pi g} \\ 0 & 0 & -\sqrt{\pi g} & 0 \end{pmatrix} \begin{pmatrix} \text{Re } t_{12}^{1\alpha} \\ \text{Im } t_{12}^{1\alpha} \\ \text{Re } t_{11}^{1\alpha} \\ \text{Im } t_{11}^{1\alpha} \end{pmatrix}, \end{aligned} \quad (58)$$

$$(\text{Re } u_{0,0}^{\alpha\beta}, \text{Im } u_{0,0}^{\alpha\beta}) = \begin{pmatrix} 0 & -\sqrt{2\pi g} \\ \sqrt{2\pi g} & 0 \end{pmatrix} \begin{pmatrix} \text{Re } t_{12}^{\alpha\beta} \\ \text{Im } t_{12}^{\alpha\beta} \end{pmatrix}, \quad (59)$$

with $\beta \geq \alpha > 1$. Using these explicit expressions for the blocks of the full Jacobi matrix, we derive the following Jacobian for collective modes:

$$|J_{\text{inst}}| = \frac{2g}{3\lambda} \cdot \frac{4g}{9\lambda^2} \cdot g^{2n-2} \cdot (2g)^{n(n-1)/2}. \quad (60)$$

Here we present the result as a product of four factors, which correspond to block matrices (56)–(59). Jacobian for the trivial topological sector can be derived in similar way, using Eq. (37):

$$|J_0| = (2g)^{n(n+1)/2}. \quad (61)$$

Therefore, resulting answer for the contribution from the zero modes acquires the following form:

$$\int_{\text{zm}} A_{\text{zm}} = \frac{8g^{n+1}}{27 \cdot 2^n} \int \frac{d\mathbf{r}_0 d\lambda}{\lambda^3} \frac{\int \mathcal{D}[\mathcal{T}'] \int \mathcal{D}[\mathcal{T}]}{\int \mathcal{D}[\mathcal{T}']}, \quad (62)$$

where global rotations $\mathcal{T}' \in \text{Sp}(2n)/\text{U}(n)$ corresponding to the zero modes in the absence of the instanton, while $\mathcal{T} \in \text{U}(1) \cup \text{U}(n)/[\text{U}(1) \times \text{U}(n-1)]$ are additional rotational zero mode induced by the presence of the instanton (see Fig. 1).

C. Partition function

Using Eq. (42), we calculate the instanton correction to the partition function. Since \mathcal{T}' and \mathcal{T} describe the zero modes, integration over them is reduced to the volume of the corresponding manifolds, which were computed, for example, in Refs. [69,70]:

$$\int \mathcal{D}[\mathcal{T}] = \text{vol}[\text{U}(1)] \text{vol} \left[\frac{\text{U}(n)}{\text{U}(1) \times \text{U}(n-1)} \right] = \frac{2\pi^n}{\Gamma(n)}. \quad (63)$$

After all substitutions we find the following one-instanton contribution to the partition function

$$\begin{aligned} \frac{\mathcal{Z}_{\text{inst}}}{\mathcal{Z}_0} &= \frac{n(n+1)}{2} G_n \int \frac{d\mathbf{r}_0 d\lambda}{\lambda^3} (\pi g)^{n+1} e^{-\pi g(\mathcal{M}) + i\vartheta}, \\ G_n &= \frac{2^{2n+3} e^{-n-(n+1)\gamma}}{\pi e^2 \Gamma(n+2)}. \end{aligned} \quad (64)$$

Here we introduce one-loop renormalized spin conductance in Pauli-Villars regularization scheme (see Appendix A),

$$g(\mathcal{M}) = g - \beta_0 \ln \mathcal{M} e^{\gamma-1/2}, \quad \beta_0 = \frac{n+1}{\pi}. \quad (65)$$

In the replica limit, $n \rightarrow 0$, the correction $\mathcal{Z}_{\text{inst}}/\mathcal{Z}_0$ is proportional to n .

The result (64) is derived within the Gaussian theory for the fluctuations around the instanton solutions. Therefore, g in the prefactor of the exponent under the integral sign in Eq. (64) remains unrenormalized. It is natural to expect [71–74] that treatment of fluctuations beyond the Gaussian approximation will result in substitution of g by $g(\mathcal{M})$ in the preexponential factor also. However, such a calculation is beyond the scope of the present paper.

In order the result (64) for $\mathcal{Z}_{\text{inst}}/\mathcal{Z}_0$ becomes operative, one has to relate the Pauli-Villars mass \mathcal{M} with the instanton size λ . Fortunately, the form of Eq. (65) suggests the form of such a relation. A key observation is that the correction to the conductance in the flat space due to elimination of fast fluctuations with spatial scales between the ultraviolet length scale ℓ and the running scale λ has exactly the same form as Eq. (65) but with λ/ℓ under the logarithm instead of \mathcal{M} . Therefore, one concludes that $\mathcal{M} = \zeta\lambda/\ell$ where ζ is some constant. Within a particular scheme, this constant can be fixed to the magnitude $\zeta = e/2$ (see Appendix E). Therefore, the correction $\mathcal{Z}_{\text{inst}}/\mathcal{Z}_0$ depends on a particular value of ζ , i.e., on a method of transformation from the theory of fluctuations on the sphere and in the flat space. This known to occur in the class A also [67].

V. INSTANTON CORRECTIONS FOR PURE SCALING EIGENOPERATORS

As was outlined in Sec. I, there is an infinite set of local operators demonstrating pure scaling at sqHe criticality. In this section, we will compute instanton corrections to the anomalous dimensions of gradientless local operators $\mathcal{K}_\lambda[\mathcal{Q}]$. Each such operator is an eigenoperator with respect to the renormalization group and, consequently, the corresponding physical observable demonstrates a pure scaling behavior at criticality characterized by a critical exponent x_λ , $\mathcal{K}_\lambda[\mathcal{Q}] \sim L^{-x_\lambda}$ where L is the system size. The operators $\mathcal{K}_\lambda[\mathcal{Q}]$ can be enumerated by a tuple $\lambda = (\lambda_1, \dots, \lambda_s)$ of integer numbers, $\lambda_1 \geq \lambda_2 \geq \dots \lambda_s > 0$, which are the highest weight of corresponding irreducible representation of $\text{Sp}(2n)/\text{U}(n)$ [32]. Each operator $\mathcal{K}_\lambda[\mathcal{Q}]$ involves $|\lambda| = \lambda_1 + \dots + \lambda_s$ operators \mathcal{Q} . The simplest example of such pure scaling observables are the disorder-averaged moments of the LDoS, $\langle \nu^\lambda \rangle$. They correspond to the operators $\mathcal{K}_\lambda[\mathcal{Q}]$ with $\lambda = (\mathbf{q})$ (see for details Ref. [33]).

An operator O averaged over the NL σ M can be written as a sum over topological sectors,

$$\langle O \rangle \approx \langle O \rangle_0 \left(1 - \frac{\mathcal{Z}_{\text{inst}} + \mathcal{Z}_{\text{inst}}^*}{\mathcal{Z}_0} \right) + \langle O \rangle_{+1} + \langle O \rangle_{-1} + \dots \quad (66)$$

For computation of averages at nontrivial topological sectors, we employ the saddle-point approximation near the instanton solution, taking into account Gaussian fluctuations in the action only. As above, we restrict our considerations by the contribution from the topological sector with $\mathcal{C} = \pm 1$. We note that one has to take into account that in fact there are many instanton solutions with a given topological charge parametrized by the zero-mode manifold of λ , z_0 , and \mathbf{T} . Therefore, one has to sum contributions to $\langle O \rangle_{\pm 1}$ from all such instanton solutions (for more detailed discussion see Ref. [67]). The weight of each contribution is fixed by the expression (64) for $\mathcal{Z}_{\text{inst}}/\mathcal{Z}_0$. Then, we find

$$\langle O \rangle_{\pm 1} \simeq \frac{n(n+1)}{2} G_n \int \frac{d\mathbf{r}_0 d\lambda}{\lambda^3} \langle O \rangle_{\mathbf{T}} (\pi g)^{n+1} e^{-\pi g(\mathcal{M}) \pm i\vartheta}, \quad (67)$$

where we introduce

$$\langle O \rangle_{\mathbf{T}} = \frac{\int \mathcal{D}[\mathbf{T}] O[\mathcal{Q}_{\text{inst}}]}{\text{vol } \mathbf{T}}, \quad (68)$$

We remind that here $\mathbf{T} \in \text{U}(1) \cup \text{U}(n)/[\text{U}(1) \times \text{U}(n-1)]$. As it was shown in Ref. [67], for computation of anomalous dimensions for pure-scaling operators without derivatives, it is enough to restrict instanton zero-modes manifold to only such rotational zero modes, which commute with $\underline{\lambda}$, i.e., to the diagonal blocks in Fig. 1. They are precisely \mathbf{T} rotations. We took explicitly into account that the operator O evaluated on the instanton solution might depend on the unitary rotations \mathbf{T} . We note that below we will work with operators expressed in terms of the original \mathcal{Q} matrices.

A. LDoS

The disorder-averaged LDoS $\langle \nu \rangle$ corresponds to the pure scaling operator $\mathcal{K}_{(1)}[\mathcal{Q}]$, which involves a single \mathcal{Q} matrix. It can be written explicitly as [66]

$$\mathcal{K}_{(1)}[\mathcal{Q}] = \frac{1}{4} [\text{tr} \mathcal{Q}_{RR}^{\alpha\alpha} - \text{tr} \mathcal{Q}_{AA}^{\alpha\alpha}], \quad (69)$$

where tr is trace over spin space and α is a fixed replica index. We note that the average LDoS is determined as $\langle \nu \rangle = \nu_0 \langle \mathcal{K}_{(1)}[\mathcal{Q}] \rangle$ where ν_0 is the bare value of the LDoS. As well-known [2] the disorder-averaged LDoS depends on the energy as a power-law at sqHe criticality. To simplify calculations we will study the scaling of $\langle \nu \rangle$ at zero energy with the system size L . Then, lowest order perturbative treatment results in the following expression:

$$\langle \nu \rangle = \nu_{\text{pert}}(L) = \nu_0 \left(1 + \frac{\gamma_{(1)}^{(0)}}{g} \ln \frac{L}{\ell} \right), \quad \gamma_{(1)}^{(0)} = -\frac{1+n}{\pi}. \quad (70)$$

It is convenient to use parametrization of instanton solution in terms of deviation from a metallic saddle point Λ :

$$\begin{aligned} \mathcal{Q}_{\text{inst}} &= \Lambda + U^{-1} \mathbf{T}^{-1} \rho \mathbf{T} U, \\ \mathbf{T} &\in \frac{\text{U}(n)}{\text{U}(1) \times \text{U}(n-1)} \cup \text{U}(1). \end{aligned} \quad (71)$$

Here, we remind, the rotation matrix U is defined in Eq. (5). The matrix ρ has only four nonzero matrix elements:

$$\rho_{00}^{11} = -\rho_{-1-1}^{11} = -2e_0^2, \quad \rho_{0-1}^{11} = (\rho_{-10}^{11})^* = 2e_0 e_1. \quad (72)$$

We write down expansion for the operator $\mathcal{K}_{(1)}$ computed on the instanton solution (71) and averaged over \mathbf{T} rotations:

$$\begin{aligned} \langle \mathcal{K}_{(1)}[\mathcal{Q}_{\text{inst}}] \rangle_{\mathbf{T}} &= \frac{1}{4} \text{tr} [\Lambda_{RR}^{\alpha\alpha} - \Lambda_{AA}^{\alpha\alpha}] \\ &+ \frac{1}{4} \sum_{p_1=\pm} p_1 \langle \text{tr} [U^{-1} \mathbf{T}^{-1} \rho \mathbf{T} U]_{p_1 p_1}^{\alpha\alpha} \rangle_{\mathbf{T}}. \end{aligned} \quad (73)$$

We note that the matrix \mathbf{T} acts as a $2n \times 2n$ block-diagonal matrix in a spin space, see Fig. 1. Therefore, we can write $\mathbf{T} = \mathbf{T}_+(1 + \mathbf{s}_3)/2 + \mathbf{T}_-(1 - \mathbf{s}_3)/2$, where \mathbf{T}_+ is $n \times n$ matrix belonging to $\mathbf{T} \in \text{U}(1) \cup \text{U}(n)/[\text{U}(1) \times \text{U}(n-1)]$, while $\mathbf{T}_- = (\mathbf{T}_+^{-1})^T$.

For averaging in Eq. (73) over T rotations we use the following relations (see Refs. [75] and [22]):

$$\langle (\mathbb{T}^{-1})_+^{\alpha 1} \mathbb{T}_+^{1b} \rangle_{\mathbb{T}} = \frac{\delta^{\alpha\beta}}{n}, \quad (74)$$

$$\langle (\mathbb{T}^{-1})_+^{\alpha 1} \mathbb{T}_+^{1\beta} (\mathbb{T}^{-1})_+^{\gamma 1} \mathbb{T}_+^{1\delta} \rangle_{\mathbb{T}} = \frac{[\delta^{\beta\gamma} \delta^{\alpha\delta} + \delta^{\alpha\beta} \delta^{\gamma\delta}]}{n + n^2}. \quad (75)$$

After averaging, we should contract the similar contribution from the trivial topological sector proportional to normalized partition function. Then we obtain

$$\begin{aligned} \langle \mathcal{K}_{(1)}[\mathcal{Q}_{\text{inst}}] \rangle_{\mathbb{T}} &= -\frac{(n+1)}{2} G_n \int \frac{d\lambda}{\lambda} (\pi g)^{n+1} \\ &\times e^{-\pi g(\mathcal{M}) + i\vartheta} \int d\mathbf{r}_0 \frac{\mu(r_0)}{\lambda}. \end{aligned} \quad (76)$$

Here $\mu(r_0)$ is the measure, induced by instanton, see Eq. (23). The term in the second line of Eq. (76) has an ultraviolet divergence due to integration over the instanton position \mathbf{r}_0 . To treat this divergence we are forced to take into account Gaussian fluctuations around the instanton in the preexponential factor. After performing straightforward calculations (see Appendix D), we find the following expression for the instanton correction to the disorder-averaged LDoS,

$$\begin{aligned} \delta v_{\text{inst}} &= v_0 (\langle \mathcal{K}_{(1)}[\mathcal{Q}_{\text{inst}}] \rangle_{\mathbb{T}} + \langle \mathcal{K}_{(1)}[\mathcal{Q}_{\text{inst}}^*] \rangle_{\mathbb{T}}) \simeq \pi \gamma_{(1)}^{(0)} G_n \\ &\times \int \frac{d\lambda}{\lambda} (\pi g)^{n+1} e^{-\pi g(\mathcal{M})} \cos \vartheta \int d\mathbf{r}_0 \frac{\mu(r_0)}{\lambda} \\ &\times v_0 \left(1 + \frac{\gamma_{(1)}^{(0)}}{g} \ln \mathcal{M} \right). \end{aligned} \quad (77)$$

We note that the last line of Eq. (77) coincides with the perturbative renormalization of the LDoS in the flat space, cf. Eq. (70). The Pauli-Villars masses in Eq. (77) can be translated into the expression for the flat space by a trick with a spatial varying mass method [21]. Physical idea behind this method is that the instanton solution centered at the spatial point r_0 acts for the Gaussian fluctuations of $\mathcal{Q}(\mathbf{r}=0)$ as a slow-varying background field with a spatial scale $1/\mu(r_0)$. Taking this into account, we write

$$\begin{aligned} \delta v_{\text{inst}} &= v_0 (\langle \mathcal{K}_{(1)}[\mathcal{Q}_{\text{inst}}] \rangle_{\mathbb{T}} + \langle \mathcal{K}_{(1)}[\mathcal{Q}_{\text{inst}}^*] \rangle_{\mathbb{T}}) \simeq \pi \gamma_{(1)}^{(0)} G_n \\ &\times \int \frac{d\lambda}{\lambda} (\pi g)^{n+1} e^{-\pi g(\zeta\lambda)} \cos \vartheta \int d\mathbf{r}_0 \frac{\mu(r_0)}{\lambda} v_{\text{pert}} \\ &\times [1/\mu(r_0)]. \end{aligned} \quad (78)$$

Here we also substituted the Pauli-Villars mass \mathcal{M} by $\zeta\lambda/\ell$ in the argument of g in the exponent. To be precise, we define [cf. Eq. (65)]

$$g(\lambda) = g - \beta_0 \ln(e^{\gamma-1/2}\lambda/\ell). \quad (79)$$

The decay of $v_{\text{pert}}(1/\mu)$ at small μ (at long length scales) makes the integral over \mathbf{r}_0 in the last line of Eq. (78) convergent (see Fig. 2). Performing integration over \mathbf{r}_0 in Eq. (78)

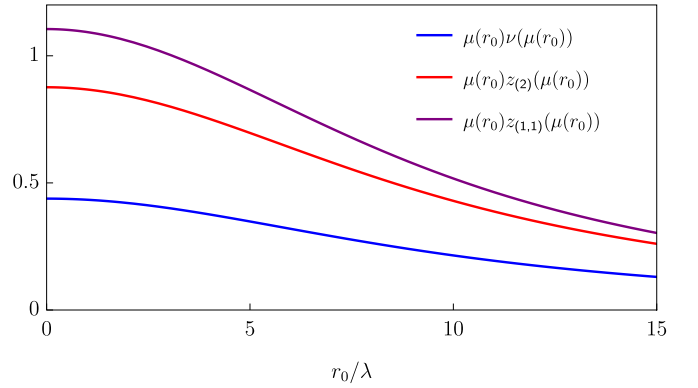


FIG. 2. Schematic plot of r_0 integrands in Eqs. (78) and (91).

(see Ref. [67] and Appendix E), we find

$$\delta v_{\text{inst}} = G_n \int \frac{d\lambda}{\lambda} (\pi g)^{n+1} v(\zeta\lambda) \mathcal{H}_{(1)}[g(\zeta\lambda)] e^{-\pi g(\zeta\lambda)} \cos \vartheta, \quad (80)$$

where $\zeta = e/2$, and we introduced

$$v(\lambda) = v_0 \left[1 + \frac{\gamma_{(1)}^{(0)}}{g} \ln(e^{\gamma-1/2}\lambda/\ell) \right]. \quad (81)$$

Also we defined the function

$$\mathcal{H}_{(1)}(g) = \frac{2\pi^2 g \gamma_{(1)}^{(0)}}{\beta_0 - \gamma_{(1)}^{(0)}} \equiv \pi^2 g. \quad (82)$$

It is worthwhile to mention that the function $\mathcal{H}_{(1)}$ is proportional to g . This fact can be understood in a following way. The integral over r_0 in Eq. (76) diverges logarithmically at large r_0 . However, the one-loop RG Eq. (79) implies a dynamically generated localization length $\sim \exp(g/\beta_0)$. This length serves as a natural infrared cut off for the otherwise logarithmically divergent integral in Eq. (76) thus resulting in a finite contribution proportional to g .

Combining together the perturbative contribution (81) and instanton correction (80), we find

$$\frac{\langle v \rangle}{v_0} = 1 + \int_{\zeta\ell}^{\zeta L} \frac{d\lambda}{\lambda} \gamma_{(1)}[g(\lambda), \vartheta]. \quad (83)$$

Here the function

$$\gamma_{(1)}(g, \vartheta) = \frac{\gamma_{(1)}^{(0)}}{g} + G_n (\pi g)^{n+1} \mathcal{H}_{(1)}(g) e^{-\pi g} \cos \vartheta. \quad (84)$$

can be interpreted as the anomalous dimension of the disorder-averaged LDoS, which determine its scaling with the system size L ,

$$\frac{d \ln \langle v \rangle}{d \ln L} = \gamma_{(1)}(g, \vartheta). \quad (85)$$

Taking into account explicit expressions for $\gamma_{(1)}^{(0)}$ and β_0 , see Eqs. (70) and (79), we find in the replica limit $n \rightarrow 0$,

$$\gamma_{(1)}(g, \vartheta) = -\frac{1}{\pi g} - \pi G_0 (\pi g)^2 e^{-\pi g} \cos \vartheta. \quad (86)$$

B. Operators with two \mathcal{Q} matrices

There exist only two eigenoperators for $|\lambda| = 2$. They involve two \mathcal{Q} matrices and correspond to two tuples: $\lambda = (2)$ and $\lambda = (1, 1)$. While the former corresponds to the second moment of the LDoS, the later describes more involved correlations of four wave functions of a random Hamiltonian of class C symmetry [40]. These two eigenoperators can be constructed explicitly as [66]

$$\mathcal{K}_\lambda[\mathcal{Q}] = \frac{1}{16} \sum_{p_1, p_2=R/A} (\sigma_3)_{p_1 p_1} (\sigma_3)_{p_2 p_2} \mathcal{P}_\lambda^{\alpha_1 \alpha_2; p_1 p_2}. \quad (87)$$

Here the correlation function $\mathcal{P}_\lambda^{\alpha_1 \alpha_2; p_1 p_2}$ is defined as

$$\begin{aligned} \mathcal{P}_\lambda^{\alpha_1 \alpha_2; p_1 p_2} &= \text{tr} \mathcal{Q}_{p_1 p_1}^{\alpha_1 \alpha_1}(\mathbf{x}) \text{tr} \mathcal{Q}_{p_2 p_2}^{\alpha_2 \alpha_2}(\mathbf{x}) \\ &+ \mu_\lambda \text{tr} [\mathcal{Q}_{p_1 p_2}^{\alpha_1 \alpha_2}(\mathbf{x}) \mathcal{Q}_{p_2 p_1}^{\alpha_2 \alpha_1}(\mathbf{x})]. \end{aligned} \quad (88)$$

The coefficients μ_λ can be fixed by the condition that the operator $\mathcal{P}_\lambda^{\alpha_1 \alpha_2; p_1 p_2}$ is the eigenoperator with respect to renormalization group transformation [66]. In particular, they are given as $\mu_{(2)} = -1$ and $\mu_{(1,1)} = 2$.

In order to study perturbative and nonperturbative renormalization of the pure scaling operators $\mathcal{K}_\lambda[\mathcal{Q}]$ it is convenient to introduce the physical observable $z_\lambda = z_\lambda^{(0)} \langle \mathcal{K}_\lambda[\mathcal{Q}] \rangle$ with the $z_\lambda^{(0)} = 1$. The lowest-order perturbative renormalization of the operators yield

$$z_\lambda(L) = z_\lambda^{(0)} \left(1 + \frac{\gamma_\lambda^{(0)}}{g} \ln \frac{L}{\ell} \right), \quad (89)$$

where the one-loop coefficients are given as [40]

$$\gamma_{(2)}^{(0)} = -\frac{1+2n}{\pi}, \quad \gamma_{(1,1)}^{(0)} = -\frac{4+2n}{\pi}. \quad (90)$$

We repeat exactly the same steps as for computation of the instanton corrections to LDoS in the previous section. After substitution of parametrization (71) into Eq. (88) and with the help of Eq. (75), we obtain

$$\begin{aligned} \delta z_\lambda^{(\text{inst})} &= z_\lambda^{(0)} ((K_\lambda[\mathcal{Q}_{\text{inst}}])_{\text{T}} + (K_\lambda[\mathcal{Q}_{\text{inst}}^*])_{\text{T}}) \simeq \pi \gamma_\lambda^{(0)} G_n \\ &\times \int \frac{d\lambda}{\lambda} (\pi g)^{n+1} e^{-\pi g(\zeta\lambda)} \cos \vartheta \int d\mathbf{r}_0 \frac{\mu(r_0)}{\lambda} \\ &\times z_\lambda(1/\mu(r_0)). \end{aligned} \quad (91)$$

We note that under the integral over \mathbf{r}_0 in Eq. (91) we omitted terms, which do not diverge in the ultraviolet, i.e., at $r_0 \rightarrow \infty$. Surprisingly, the structure of Eq. (91) repeats exactly the structure of instanton correction to the DOS. The specifics of the eigenoperator is hidden into the one-loop coefficient $\gamma_\lambda^{(0)}$, which appears in two places: as an overall factor and in the expression for $z_\lambda(1/\mu(r_0))$.

After integration over \mathbf{r}_0 , we find

$$\delta z_\lambda^{(\text{inst})} = G_n \int \frac{d\lambda}{\lambda} (\pi g)^{n+1} z_\lambda(\zeta\lambda) \mathcal{H}_\lambda[g(\zeta\lambda)] e^{-\pi g(\zeta\lambda)} \cos \vartheta, \quad (92)$$

where the function

$$\mathcal{H}_\lambda(g) = \frac{2\pi^2 g \gamma_\lambda^{(0)}}{\beta_0 + |\gamma_\lambda^{(0)}|}. \quad (93)$$

We note that in the derivation of expression (93) (see Appendix E) the negative sign of $\gamma_\lambda^{(0)}$ has been important. It is this negative sign results to decay of the integrand at $r_0 \rightarrow \infty$, see Fig. 2. However, one can extend the expression to the positive $\gamma_\lambda^{(0)}$ as well [67]. The expression (93) holds for both cases.

Repeating the same steps as in the previous section for the LDoS, using Eqs. (92) and (89), we find the instanton correction to the anomalous dimension of the operators $K_{(2)}$ and $K_{(1,1)}$ as follows:

$$\gamma_\lambda = \frac{d \ln z_\lambda}{d \ln L} = \frac{\gamma_\lambda^{(0)}}{g} + \frac{2\pi \gamma_\lambda^{(0)}}{\beta_0 + |\gamma_\lambda^{(0)}|} G_n (\pi g)^{n+2} e^{-\pi g} \cos \vartheta. \quad (94)$$

C. Operators with an arbitrary number of \mathcal{Q} -matrices

As one can check, Eq. (91) is valid for an arbitrary pure scaling operator corresponding to the tuple λ (see Appendix B). Therefore, the result (94) holds also for an arbitrary eigenoperator. In the replica limit, $n \rightarrow 0$, the perturbative coefficient $\gamma_\lambda^{(0)}$ for $\lambda = (\lambda_1, \dots, \lambda_s)$ is given as [40]

$$\gamma_\lambda^{(0)} = \frac{1}{2\pi} \sum_{j=1}^s \lambda_j (\lambda_j + c_j), \quad c_j = 1 - 4j. \quad (95)$$

Then, we find the following result for the anomalous dimension of the pure scaling operator in the replica limit, $n \rightarrow 0$,

$$\gamma_\lambda = \frac{\lambda(\lambda + \mathbf{c})}{2\pi g} + \frac{\lambda(\lambda + \mathbf{c})}{2 + |\lambda(\lambda + \mathbf{c})|} \mathcal{C}_C (\pi g)^2 e^{-\pi g} \cos \vartheta, \quad (96)$$

where $\mathcal{C}_C = 16e^{-2-\gamma}$ and we introduce the vector $\mathbf{c} = (c_1, \dots, c_s)$. We note that the result (94) holds also for the eigenoperator $\mathcal{K}_{(1)}$ corresponding to the LDoS. We emphasize that the instanton correction is expressed via the quadratic Casimir operator $\lambda(\lambda + \mathbf{c})$ similar to the one-loop perturbative correction. Therefore, the result (96) remains invariant under symmetry transformations, which are consequence of Weyl-group invariance.

We note that the instanton contribution to the anomalous dimensions γ_λ at $\vartheta = \pi$ is of opposite sign with respect to the one-loop perturbative correction. It implies that the instanton effects reduce the multifractal behavior at $\vartheta = \pi$.

VI. CORRECTIONS TO THE SPIN CONDUCTIVITIES

Our next aim is to compute instanton corrections to conductivities with the help of NL σ M formalism. The Kubo-type expressions for longitudinal spin conductivity can be written as follows (see Appendix C)

$$\begin{aligned} g' &= g + \frac{g}{8n(n+1)} \left\{ \frac{1}{2} \text{Tr}[\Lambda, \mathcal{Q}]^2 + (\text{Tr} \Lambda \mathcal{Q})^2 - (\text{Tr} 1)^2 \right\} \\ &- \frac{g^2}{64n(n+1)} \int d\mathbf{x}' \langle \text{Tr}[\Lambda, \mathcal{J}(\mathbf{x})][\Lambda, \mathcal{J}(\mathbf{x}')] \rangle. \end{aligned} \quad (97)$$

Here we introduce the matrix current $\mathcal{J} = \mathcal{Q} \nabla \mathcal{Q}$. We note that Eq. (97) produces correct perturbative renormalization for g .

Similar expression can be derived for the transverse spin conductivity (see Appendix C),

$$g'_H = g_H + \frac{g^2}{8n(n+1)} \int_{x'} \varepsilon_{\mu\nu} \langle \text{Tr}[\Lambda_- \mathcal{J}_\mu(\mathbf{x}) \Lambda_+ \mathcal{J}_\nu(\mathbf{x}')] \rangle, \quad (98)$$

where $\Lambda_\pm = (1 \pm \Lambda)/2$ stands for the projector on the retarded and advanced blocks.

A. Longitudinal spin conductivity

We start from calculation of the instanton corrections to g' . Here we will proceed in a similar way as in Sec. V. We use the approximation (66) in our calculations below. An important remark is in order here. The longitudinal spin conductivity involves two operators, cf. Eq. (97). These operators individually are not the eigenoperators of the renormalization group. This situation is similar to the eigenoperator $\mathcal{K}_{(2)}$, cf. Eq. (88), which is composed from two operators, each of which is not eigenoperator under the action of the renormalization group. Nevertheless, the full operator of spin conductivity is the RG eigenoperator. In other words, within the background field renormalization method, the renormalized conductivity can be written as

$$\delta g[\mathcal{Q}] \rightarrow Z_g \delta g[\mathcal{Q}_0] = Z_g (g_{dm}[\mathcal{Q}_0] + g_{j-j}[\mathcal{Q}_0]), \quad (99)$$

where $\mathcal{Q}_0 = \mathcal{T}_0^{-1} \Lambda \mathcal{T}_0$ is a slow field and Z_g is a renormalization factor for the conductivity. Here g_{dm} corresponds to the operator in the first line of Eq. (97) (diamagnetic contribution) while g_{j-j} is the operator in the second line of the same equation (current-current contribution). Therefore, our task is simplified and it is enough to compute the instanton contribution to Z_g from the one of the operators. For reasons to be explained shortly, we compute the instanton contribution to the current-current part of the spin conductivity, g_{j-j} . It reads

$$\begin{aligned} & -\frac{g^2}{64n(n+1)} \int dx' \langle \text{Tr}[\Lambda, \mathcal{J}(x)][\Lambda, \mathcal{J}(x')] \rangle \\ & \rightarrow -G_n \int \frac{d\lambda}{\lambda} (\pi g)^{n+3} e^{-\pi g(\mathcal{M})} \cos \vartheta. \end{aligned} \quad (100)$$

Hence we obtain the final expression to the single instanton correction to the dissipative spin conductance

$$\delta g_{\text{inst}} = -G_n \int \frac{d\lambda}{\lambda} (\pi g)^{n+3} e^{-\pi g(\zeta\lambda)} \cos \vartheta, \quad (101)$$

where, we remind, $g(\zeta\lambda)$ is the spin conductance renormalized within the perturbation theory, cf. Eq. (79). As in the previous section, one can interpret the above correction in terms of the nonperturbative contribution to the β function (for the finite number of replica)

$$\beta_g(g, \vartheta) = -\frac{dg}{d \ln L} = \frac{1+n}{\pi} + G_n (\pi g)^{n+3} e^{-\pi g} \cos \vartheta. \quad (102)$$

B. Transverse spin conductivity

Our next step is to compute the renormalization of the transverse spin conductivity. We note that there are no perturbative corrections to g_H . Using Eq. (98) and the

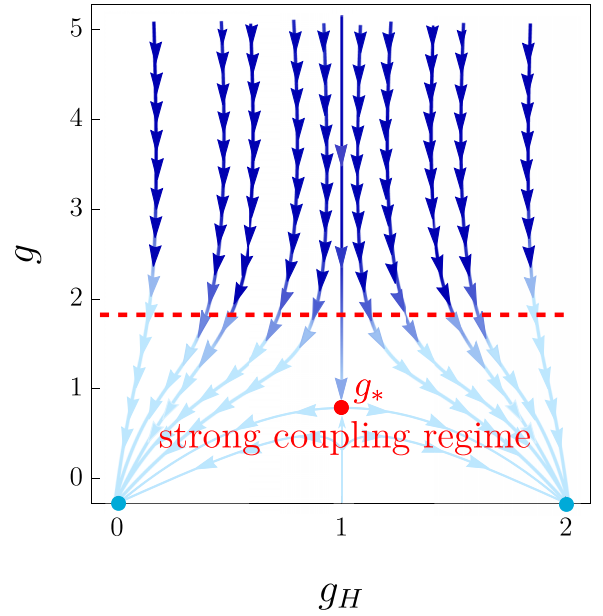


FIG. 3. Sketch of the RG-flow diagram for spin conductivities in class C, see Eqs. (106).

approximation (66), we find

$$\begin{aligned} & \frac{g^2}{8n(n+1)} \left\langle \int_{x'} \varepsilon_{\mu\nu} \text{Tr}[\Lambda_- \mathcal{J}_\mu(\mathbf{x}) \Lambda_+ \mathcal{J}_\nu(\mathbf{x}')] \right\rangle \\ & \rightarrow -G_n \int \frac{d\lambda}{\lambda} (\pi g)^{n+3} e^{-\pi g(\mathcal{M})} \sin \vartheta. \end{aligned} \quad (103)$$

With the help of Eq. (17), we write the instanton correction to the θ angle, cf. Eq. (17), as

$$\delta \vartheta_{\text{inst}} = -\pi G_n \int \frac{d\lambda}{\lambda} (\pi g)^{n+3} e^{-\pi g(\zeta\lambda)} \sin \vartheta. \quad (104)$$

The corresponding β function takes the following form:

$$\beta_\vartheta(g, \vartheta) = -\frac{d(\vartheta/2\pi)}{d \ln L} = \frac{G_n}{2} (\pi g)^{n+3} e^{-\pi g} \sin \vartheta. \quad (105)$$

C. Renormalization group equations for spin conductivities

In the replica limit, $n \rightarrow 0$, using Eqs. (102) and (105), the RG equations for spin conductivities acquire the following form:

$$\begin{aligned} \frac{dg}{d \ln L} &= -\frac{1}{\pi} - \frac{2}{\pi^2 g} - \mathcal{D}_C (\pi g)^3 e^{-\pi g} \cos \vartheta, \\ \frac{d\vartheta}{d \ln L} &= -\pi \mathcal{D}_C (\pi g)^3 e^{-\pi g} \sin \vartheta, \end{aligned} \quad (106)$$

where $\mathcal{D}_C = 8e^{-2-\gamma}/\pi \approx 0.2$. Here we added the two-loop perturbative correction to the longitudinal spin conductivity [76]. Equations (106) describe the flow of longitudinal conductivity g and the fractional part of the transverse spin conductivity ϑ , cf. Eq. (17), with increase of the system size L .

Although the RG Eqs. (106) have been derived in the weak coupling regime, $g \gg 1$, they help to understand the overall phase diagram of the class C and quantization of the transverse spin conductance (see Fig. 3). Similar to the

case of class A, the instantons provide the mechanism for the scale dependence of the theta-angle ϑ , which is a fractional part of g_H . RG Eqs. (106) predict that $\vartheta = 0$ ($g_H = 2k$) is the stable fixed line of the RG flow, while $\vartheta = \pi$ ($g_H = 2k + 1$) is the unstable fixed line. It is the latter that corresponds to the transition between spin quantum Hall phases with $g_H = 2k$ where $k \in \mathbb{Z}$. Although the instanton correction to the renormalization of longitudinal spin conductivity is of antilocalization character at $\vartheta = \pi$, its magnitude is not sufficient to compensate the perturbative localization corrections. This situation is not surprising since, as we have already mentioned above, RG Eqs. (106) are applicable at weak coupling, $g \gg 1$. Based on numerical [77] and analytical [78] studies of the critical spin conductance at the spin quantum Hall effect we know that there exists an unstable fixed point at $\vartheta = \pi$ and $g = \sqrt{3}/2$. This fixed point describes the transition between different topological phases in class C (see Fig. 3).

VII. DISCUSSIONS AND CONCLUSIONS

Before closing the paper, we will discuss some important aspects of the obtained results.

A. Instanton solution for unrotated \mathcal{Q} matrix

Although instanton configuration has been discussed for the \mathcal{Q} matrix, it is interesting to see how it looks like before the rotation. Performing the inverse transformation, cf. Eq. (5), we obtain the instanton solution $\mathcal{Q}_{\text{inst}}$ in the following form (we set $N_r = 1$ and $\mathbf{r}_0 = 0$)

$$\mathcal{Q}_{\text{inst}} = \begin{pmatrix} \frac{r^2 s_0 - \lambda^2 s_1}{r^2 + \lambda^2} & \frac{r \lambda e^{i\vartheta} (i s_2 - s_3)}{r^2 + \lambda^2} \\ -\frac{r \lambda e^{-i\vartheta} (i s_2 + s_3)}{r^2 + \lambda^2} & -\frac{r^2 s_0 + \lambda^2 s_1}{r^2 + \lambda^2} \end{pmatrix}_{\text{RA}}. \quad (107)$$

It is crucial that the solution (107) has nontrivial structure in the spin space. In other words rotations in the spin and RA spaces are entangled. The naive construction of the instanton solution for class C would be to place the class A instanton in the upper diagonal block of matrix $\mathcal{Q}_{\text{inst}}$ and constructing the rest in a way consistent with the BdG symmetry. However, such a procedure results in the instanton solution with the topological charge 2. Therefore, the structure (107) is not a trivial generalization of the class A instanton.

B. Manifestation of Weyl symmetry

The anomalous dimensions γ_λ , cf. Eq. (96), determine the flow of the physical observables z_λ (corresponding to the eigenoperators \mathcal{K}_λ) with the system size L . At criticality, the dependence becomes a power law,

$$z_\lambda \sim L^{-x_\lambda}, \quad x_\lambda = |\lambda| x_{(1)} + \Delta_\lambda, \quad (108)$$

where the scaling exponents x_λ are given by magnitudes of $-\gamma_\lambda$ at the critical point. We note that anomalous dimension Δ_λ describes the scaling of the pure scaling observables normalized to the proper power of the disorder-averaged LDoS, $z_\lambda / \langle \nu \rangle^{|\lambda|}$. Important outcome of our result (96) for the instanton contributions to the anomalous dimensions of the eigenoperators under the action of RG is that they preserve the Weyl-group symmetry relations. This symmetry relates the

anomalous dimensions of operators, which can be obtained from each other by the following symmetry operations acting on the tuple $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$: reflection $\lambda_j \rightarrow -c_j - \lambda_j$ and permutation of some pair: $\lambda_{j/i} \rightarrow \lambda_{i/j} + (c_{i/j} - c_{j/i})/2$. Since it is known [33] that the existence of the Weyl symmetry is not limited to the criticality, the Weyl symmetry of our nonperturbative result (96) provides a strong consistency check.

We note also that the instanton correction, Eq. (96), breaks the generalized parabolicity [anomalous dimension is not a linear function of the combination $\boldsymbol{\lambda}(\boldsymbol{\lambda} + \mathbf{c})$]. Thus, instanton analysis signals breaking of the generalized parabolicity already in weak coupling regime. This fact is consistent with numerical data and analytical results from mapping to percolation at criticality that demonstrate clear evidence of a violation of generalized parabolicity for the multifractal exponents x_λ [34–36, 40, 55, 58, 59, 79].

C. Comparison with the integer quantum Hall effect

The instanton effects in the sqHe discussed in this paper is a counterpart of the similar nonperturbative effects in the iqHe which has been studied with the help of the same technique, NL σ M [67]. Below, we will highlight the main distinctions between these two cases.

First, we note the different target manifolds of NL σ M. In the case of the iqHe, the \mathcal{Q} matrices lie in the coset $U(2n)/U(n) \times U(n)$, while, as mentioned above, the target manifold for class C has the form $\text{Sp}(2n)/U(n)$. The latter is a consequence of the presence of the additional BdG symmetry, cf. Eq. (2). The difference in the target manifolds affects the mechanism of symmetry breaking due to the instanton (see Fig. 1). Due to the presence of an additional constraint on the diagonal blocks of the \mathcal{Q} matrix in class C, the volume of the zero-mode manifold scales as $\sim n$ at $n \rightarrow 0$. Hence the instanton correction to the partition function turns out to be linear in the replica number, which guarantees the presence of a nonzero correction to the average logarithm of the partition function and to the density of states. In contrast, in class A, the volume of the zero-mode manifold scales $\sim n^2$ in the replica limit, $n \rightarrow 0$ such that all corrections to the average logarithm of the partition function and to the average density of states (both perturbative and nonperturbative) vanish.

The second difference is the structure of the RG equations (106). The nonperturbative corrections to the β functions for the spin conductivities, (102) and (105), have a stronger power-law dependence on g in the pre-exponential factor than it is in class A [19, 67] (g^2 versus g^3). This suggests that in a weak coupling regime, $g \gg 1$, the rate of change of the ϑ angle in the class C is larger than that in the class A. Interestingly, the situation remains similar at criticality. There the rate of RG flow of the ϑ angle is controlled by inverse of the localization length exponent ν . As known [49, 53, 80], the magnitude of ν for the iqHe is almost two times large than in the case of the sqHe. Also we mention that the relative magnitude of the instanton correction with respect to the perturbative one is the same for both classes. The g^3 prefactor of the instanton correction in the case of class C is compensated by the large perturbative contribution, $\sim g^0$ (in contrast with class A, where it is $\sim 1/g$ only).

D. Future directions

Our results pave the way for a future research in the sqHe. First, it would be interesting to understand the instanton corrections in the NL σ M approach through the lens of the percolation mapping.

Second, it is known [2,42,57,81,82] that scaling with the system size of wave functions at the boundary of a system undergoing bulk Anderson transition is different from the corresponding scaling in the bulk. Recently, it was shown to be true for generalized multifractality exponents [79]. Thus it would be interesting to understand how to treat instanton configurations in the presence of a boundary and to compute instanton corrections to the anomalous dimensions of pure scaling local operators at the boundary.

Thirdly, as known in the theory of iqHe [22,67], the fluctuations of the topological term at the boundary corresponds to the edge theory of chiral spinless fermions. It would be interesting to study the edge theory that follows from the fluctuations of the topological term at the boundary for the sqHe. Also, it would be challenging to relate the thus derived edge theory with microscopic theory of the sqHe.

Fourth, in the context of the iqHe the instanton analysis has been extended to the Finkel'stein NL σ M that takes into account the electron-electron interaction [22]. Recently, the generalized multifractality for the sqHe has been extended to the interacting case [66]. It would be interesting to adapt the instanton analysis for the class C presented in this paper in such a way to be able include the electron-electron interaction.

Fifth, as known, by breaking the SU(2) spin-rotational symmetry, the class C transforms into the class D, which hosts the thermal quantum Hall effect in two dimensions [83]. The corresponding NL σ M looks similar to the NL σ M for class C and involves the topological θ term. It would be interesting to develop the instanton analysis for the case of class D, in particular, since in that class there exists other topological objects, domain walls, describing jumps of the \mathcal{Q} matrix between two disconnected pieces of the NL σ M target manifold [84–87].

Sixth, class A can be obtained from the class C by breaking SU(2) symmetry down to U(1). It would be interesting to implement such a symmetry breaking into NL σ M and to study transformation of the instanton solution (107) into the instanton of class A.

E. Summary

In conclusion, we summarize all results, which are discussed above. We developed the nonperturbative analysis of topological Anderson transition in the sqHe. Using NL σ M for the class C we found instanton solution with nontrivial topological charge equal to ± 1 , cf. Eqs. (15) and (107). We identified all collective coordinates (zero modes) of the instanton and integrated over them exactly. We integrated within Gaussian approximation over fluctuations around the instanton. Thus we derived the instanton correction to the logarithm of the partition function, cf. Eq. (64). Remarkably, due to the structure of the NL σ M manifold in class C, Sp(2N)/U(N), this correction survives in the replica limit (in contrast to the vanishing correction in class A). In addition, applying the same methodology, we computed the instanton corrections to the anomalous dimensions of all pure scaling local operators,

which determine the generalized multifractal spectrum in the sqHe, cf. Eq. (96). Remarkably, instanton corrections do not spoil the Weyl-group symmetry relations between anomalous dimensions of different eigenoperators. We computed also instanton corrections to the longitudinal and Hall spin conductivities. Interpreting the derived results as corrections to the two parameter renormalization group equations, cf. Eq. (106), we constructed the phase diagram for the sqHe. Finally, we listed several new directions, which our work opens.

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APPENDIX A: EVALUATION OF LDOs AND THE SPIN CONDUCTIVITY IN PAULI-VILLARS REGULARIZATION

In this Appendix we present the calculation of LDoS and the spin conductivity in the trivial topological sector with the help of the Pauli-Villars regularization scheme for class C. We benefit from similar calculations for class A [22,67]. We find

$$\begin{aligned} \nu_{\text{pert}}(\mathcal{M}) &= \frac{\nu}{\mathcal{K}_1(\Lambda)} \int \mathcal{D}\mathcal{Q} \text{Tr} \underline{\Lambda} \mathcal{Q} e^{-S[\mathcal{Q}]} \\ &= \nu - \frac{\nu}{4n} \int \mathcal{D}W \text{Tr} W^2 e^{-S_0[W]} \\ &= \nu - \frac{\nu}{2n} \langle w^{\alpha\beta} w^{*\alpha\beta} \rangle_0 \\ &= \nu \left(1 - \frac{2(1+n)}{g} \mathcal{G}_0(\eta\theta; \eta\theta) \right). \end{aligned} \quad (\text{A1})$$

We can rewrite the Green's function in coinciding points in terms of new function:

$$\mathcal{G}_0(\eta\theta; \eta\theta) = \frac{1}{4\pi} Y^{(0)}, \quad Y^{(s)} = \sum_{J=J_s}^{\infty} \frac{2J + (1-s)^2}{E_J^{(s)}}, \quad (\text{A2})$$

where $J_s = 2 - (s-2)(s-1)/2$. Next we introduce the function

$$Y^{(\Lambda)}(p) = \sum_{J=p}^{\Lambda} \frac{2J}{J^2 - p^2}. \quad (\text{A3})$$

Similar to (49), the regularized function $Y_{\text{reg}}^{(\Lambda)}(p)$ is given as follows:

$$Y_{\text{reg}}^{(\Lambda)}(p) = \sum_{f=1}^K e_f \sum_{J=p}^{\Lambda} \frac{2J}{J^2 - p^2 + \mathcal{M}_f^2} + \sum_{J=p+1}^{\Lambda} \frac{2J}{J^2 - p^2}. \quad (\text{A4})$$

We note that

$$Y_{\text{reg}}^{(s)} = \lim_{\Lambda \rightarrow \infty} Y_{\text{reg}}^{(\Lambda)} \left(\frac{1+s}{2} \right). \quad (\text{A5})$$

As in the main text, we assume existence of the cutoff $\Lambda \gg \mathcal{M}_f$. Applying the Euler-Maclaurin formula,

$$\sum_{J=p+1}^{\Lambda} g(J) = \int_{p+1}^{\Lambda} g(x) dx + \frac{g(\Lambda) + g(p+1)}{2} + \frac{g'(x)}{12} \Big|_{p+1}^{\Lambda}, \quad (\text{A6})$$

and after some straightforward calculations we obtain

$$\lim_{\Lambda \rightarrow \infty} Y_{\text{reg}}^{(\Lambda)}(p) = 2 \ln \mathcal{M} + \gamma - \psi(1+2p), \quad (\text{A7})$$

where $\psi(z)$ stands for the digamma function. Therefore, the regularized expressions for $Y^{(s)}$ are given as

$$Y_{\text{reg}}^{(0)} = 2 \ln \mathcal{M} + 2\gamma - 1, \quad Y_{\text{reg}}^{(1)} = 2 \ln \mathcal{M} + 2\gamma - \frac{3}{2}. \quad (\text{A8})$$

Using the above results, we find the perturbative correction to the disorder-averaged LDoS:

$$\nu_{\text{pert}}(\mathcal{M}) \approx \nu \left(1 + \frac{\gamma_1^{(0)}}{g} \ln \mathcal{M} e^{\gamma-1/2} \right). \quad (\text{A9})$$

Next, we consider the one-loop renormalization of the spin conductivity in the Pauli-Villars regularization scheme. We start from considering the diamagnetic part of Eq. (97). Using exponential parametrization for fluctuations, Eq. (18), we write

$$\delta g_{dm} = \frac{g}{8} (V_{1,1} + V_2) \langle \text{Tr} W^2 (2 + \text{Tr} 1) \rangle_{S_0}. \quad (\text{A10})$$

As one can see this expression is analogous to the expression obtained for the LDoS, cf. Eq. (A9). Therefore we obtain immediately,

$$\delta g_{dm} = -\frac{(1+n)}{\pi} \ln \mathcal{M} e^{\gamma-1/2}. \quad (\text{A11})$$

At the next step, we take into account the current-current part (j - j) of the correlation function for the spin conductivity. There are only three non-zero contributions, which can be written as

$$\begin{aligned} & \langle \text{Tr} \nabla_{\mu} W(\mathbf{x}) \nabla_{\mu'} W(\mathbf{x}') \rangle, \\ & - 2 \langle \text{Tr} [\nabla_{\mu} W(\mathbf{x}) W(\mathbf{x}') [\nabla_{\mu'} W(\mathbf{x}') W(\mathbf{x}')] \rangle, \\ & \frac{2}{3} \langle \text{Tr} (\nabla_{\mu} W(\mathbf{x}) \nabla_{\mu'} W^3(\mathbf{x}') \rangle. \end{aligned} \quad (\text{A12})$$

All of these expressions can be obtained from expansion of the following term:

$$\begin{aligned} \int d\mathbf{x} \text{Tr} [E \nabla Q(\mathbf{x})] &= \int_{\partial\Omega} ds \mathbf{n} \nabla \cdot \text{Tr} [E Q(\mathbf{x})], \\ E &= \int d\mathbf{x}' Q(\mathbf{x}') \nabla Q(\mathbf{x}') \Lambda. \end{aligned} \quad (\text{A13})$$

Using the fact that the Q matrix at the boundary is the constant, $Q(\mathbf{x})|_{\partial\Omega} = Q_b = \text{const.}$, to be consistent with the quantization of the topological charge, we obtain

$$\int_{\partial\Omega} ds \mathbf{n} \nabla \cdot \text{Tr} (E Q_b) = 0, \quad \div \text{Tr} (E Q_b) = 0. \quad (\text{A14})$$

Therefore, there are no full-derivative contributions to the longitudinal spin conductivity in Pauli-Villars regularization. Finally, we obtain that only one relevant one-loop correction to longitudinal spin conductivity g comes from diamagnetic part, cf. Eq. (A11). It reads

$$g(\mathcal{M}) = g - \frac{(1+n)}{\pi} \ln \mathcal{M} e^{\gamma-1/2}. \quad (\text{A15})$$

APPENDIX B: RENORMALIZATION OF RG EIGENOPERATORS WITH THREE Q MATRICES

In this Appendix we confirm the fulfillment of Eq. (94) for the eigenoperators with three Q matrices. Following [66], we define RG eigenoperators in terms of correlation functions $\mathcal{P}_{\lambda}^{\alpha_1 \alpha_2 \alpha_3; p_1 p_2 p_3}$:

$$\mathcal{K}_{\lambda}[Q] = \frac{1}{64} \sum_{p_j=R/A} (\sigma_3)_{p_1 p_1} (\sigma_3)_{p_2 p_2} (\sigma_3)_{p_3 p_3} \mathcal{P}_{\lambda}^{\alpha_1 \alpha_2 \alpha_3; p_1 p_2 p_3}, \quad (\text{B1})$$

where $|\lambda| = 3$ and $\mathcal{P}_{\lambda}^{\alpha_1 \alpha_2 \alpha_3; p_1 p_2 p_3}$ can be written in terms of the Q matrices as follows:

$$\begin{aligned} \mathcal{P}_{\lambda}^{\alpha \beta \mu; p_1 p_2 p_3} &= \text{tr} Q_{p_1 p_1}^{\alpha\alpha} \text{tr} Q_{p_2 p_2}^{\beta\beta} \text{tr} Q_{p_3 p_3}^{\mu\mu} \\ &+ \mu_{2,1}^{(\lambda)} \text{tr} Q_{p_1 p_1}^{\alpha\alpha} \text{tr} Q_{p_2 p_3}^{\beta\mu} Q_{p_3 p_2}^{\mu\beta} \\ &+ \mu_3^{(\lambda)} \text{tr} Q_{p_1 p_2}^{\alpha\beta} Q_{p_2 p_3}^{\beta\mu} Q_{p_3 p_1}^{\mu\alpha}. \end{aligned} \quad (\text{B2})$$

Here α, β, μ denote fixed different replica indices. In order to $\mathcal{K}_{\lambda}[Q]$ be the RG eigenoperator, the constants $\mu_{2,1}^{(\lambda)}$ and $\mu_3^{(\lambda)}$ should take only some specific values:

$$\begin{aligned} \lambda = (3): & \quad \mu_{2,1}^{(3)} = -3, \quad \mu_3^{(3)} = 2, \\ \lambda = (2,1): & \quad \mu_{2,1}^{(2,1)} = 1, \quad \mu_3^{(2,1)} = -2, \\ \lambda = (1,1,1): & \quad \mu_{2,1}^{(1,1,1)} = 6, \quad \mu_3^{(1,1,1)} = 8. \end{aligned} \quad (\text{B3})$$

Our aim is to compute nonperturbative renormalization of $\mathcal{K}_{\lambda}[Q]$. It can be performed with the help of the saddle-point approximation (66). Using the parametrization of Q in terms of a deviation from the trivial saddle point Λ (71) and taking

into account fluctuations in the action only, we obtain

$$\langle \mathcal{K}_\lambda \rangle_{\pm 1} = \left\langle \frac{\mu_{2,1}^{(\lambda)} |\rho_{12}|^2}{2n(n+1)} + \frac{3\rho_{11}}{n} - \frac{2\mu_{2,1}^{(\lambda)} + 3\mu_3^{(\lambda)} |\rho_{12}|^2 \rho_{11}}{8(n+2)} + \frac{\rho_{11}^2 (\mu_{2,1}^{(\lambda)} + 6)}{2n(n+1)} + \frac{\rho_{11}^3 (2\mu_{2,1}^{(\lambda)} + \mu_3^{(\lambda)} + 4)}{4n(n+1)(n+2)} \right\rangle_{\pm 1}. \quad (\text{B4})$$

We note that we averaged over T rotations in Eq. (B4). Technically, it can be performed with the help of the generalization of the expression (75) to the case with six unitary matrices, which can be found, for example, in Ref. [75]. In Eq. (B4) we use some specific notation for the functions averaged over the instanton manifold

$$\langle f \rangle_{\pm 1} = G_n \int \frac{d\lambda}{\lambda^3} (\pi g)^{n+1} e^{-\pi g(\mathcal{M}) + i\vartheta} \int d\mathbf{r}_0 f(\mathbf{r}_0). \quad (\text{B5})$$

Next step is to omit contributions, which are finite in the ultraviolet. Surprisingly, it leads to Eq. (91), where we should replace only expressions for the one-loop coefficients $\gamma_\lambda^{(0)}$:

$$\gamma_{(3)}^{(0)}(g) = -\frac{3n}{\pi}, \quad \gamma_{(2,1)}^{(0)}(g) = -\frac{4+3n}{\pi}, \quad (\text{B6})$$

$$\gamma_{(1,1,1)}^{(0)}(g) = -\frac{9+3n}{\pi}. \quad (\text{B7})$$

Repeating the spatial-varying mass procedure we obtain instanton correction to the anomalous dimensions of the eigenoperators \mathcal{K}_λ with $|\lambda| = 3$ in the form of Eq. (94).

APPENDIX C: DERIVATION OF KUBO FORMULA FOR THE SPIN CONDUCTIVITY

Our aim is to derive Kubo-type formulas for spin conductivities, cf. Eqs. (97) and (98). It can be done in several ways. Here we present derivation based on Matsubara-Kubo formula, which was obtained in Ref. [66]. First, we introduce the generalization of NL σ M to the case of Matsubara frequency space (Finkel'stein NL σ M). The field \hat{Q} becomes a traceless Hermitian matrix, defined on $N_r \times N_r$ replica, $2N_m \times 2N_m$ Matsubara and 2×2 spin spaces. It satisfies the same nonlinear constraint, $\hat{Q}^2 = 1$, and BdG symmetry relation:

$$\hat{Q} = -\bar{\hat{Q}}, \quad \bar{\hat{Q}} = \mathbf{s}_2 \hat{L}_0 \hat{Q}^T \hat{L}_0 \mathbf{s}_2, \quad (\hat{L}_0)_{nm}^{\alpha\beta} = \delta_{\varepsilon_n, -\varepsilon_m} \delta^{\alpha\beta} \mathbf{s}_0. \quad (\text{C1})$$

Here we define fermionic Matsubara frequencies in a standard way: $\varepsilon_n = \pi T(2n+1)$. Therefore, the trivial saddle point of NL σ M, taking into account Matsubara frequency space, has the following form [66]:

$$\hat{\Lambda}_{nm}^{\alpha\beta} = \text{sgn } \varepsilon_n \delta_{nm} \delta^{\alpha\beta} \mathbf{s}_0. \quad (\text{C2})$$

In such extended representation, Kubo formula for longitudinal spin conductivity can be written in terms of two operators:

$$\sigma_s(i\omega_k) = -\frac{g}{8kL^2} \int_x \langle \text{Tr} [I_k^\alpha \mathbf{s}_3, \hat{Q}] [I_{-k}^\alpha \mathbf{s}_3, \hat{Q}] \rangle + \frac{g^2}{32kL^2} \int_x \int_{x'} \langle \langle \text{Tr} I_k^\alpha \mathbf{s}_3 \hat{Q}(x) \nabla \hat{Q}(x) \text{Tr} I_{-k}^\alpha \mathbf{s}_3 \hat{Q}(x') \nabla \hat{Q}(x') \rangle \rangle, \quad (\text{C3})$$

where $(I_k^\gamma)_{nm}^{\alpha\beta} = \delta_{n-m,k} \delta^{\alpha\beta} \delta^{\alpha\gamma} \mathbf{s}_0$. Next step is to average these two operators over unitary rotations of \hat{Q} , which commute with $\hat{\Lambda}$. It can be performed with the help of the expressions similar to Eq. (75), where we should replace all indices on vector index $a \rightarrow \{a_r, a_S, a_M\}$ in replica, spin, and Matsubara spaces. The averaged longitudinal spin conductivity operator consists of three different averages only: $\langle \text{Tr} I_k^\alpha \mathbf{s}_3 \mathcal{U}^{-1} A \mathcal{U} \rangle_{\mathcal{U}} = 0$,

$$\begin{aligned} \langle \text{Tr} I_k^\alpha \mathbf{s}_3 \mathcal{U}^{-1} A \mathcal{U} I_{-k}^\alpha \mathbf{s}_3 \mathcal{U}^{-1} B \mathcal{U} \rangle_{\mathcal{U}} &= V_{1,1} [(N_m - k) (\text{Tr} A \text{Tr} B + \text{Tr} \hat{\Lambda} A \text{Tr} \hat{\Lambda} B + \text{Tr} [A\bar{B} - \hat{\Lambda} A \hat{\Lambda} \bar{B}]) \\ &\quad + 2k (\text{Tr} \hat{\Lambda}_- A \text{Tr} \hat{\Lambda}_+ B + \text{Tr} \hat{\Lambda}_- A \hat{\Lambda}_- \bar{B})] + V_2 [2(N_m - k) \text{Tr} \hat{\Lambda} A \hat{\Lambda} B \\ &\quad + 2k (\text{Tr} \hat{\Lambda}_- A \text{Tr} \hat{\Lambda}_+ B + \text{Tr} \hat{\Lambda}_- A \hat{\Lambda}_- \bar{B})], \end{aligned} \quad (\text{C4})$$

and

$$\begin{aligned} \langle \text{Tr} I_k^\alpha \mathbf{s}_3 \mathcal{U}^{-1} A \mathcal{U} \text{Tr} I_{-k}^\alpha \mathbf{s}_3 \mathcal{U}^{-1} B \mathcal{U} \rangle_{\mathcal{U}} &= V_{1,1} [(N_m - k) \text{Tr} [A(B - \bar{B}) + \hat{\Lambda} A \hat{\Lambda} (B - \bar{B})] + 2k \text{Tr} \hat{\Lambda}_- A \hat{\Lambda}_+ (B - \bar{B})] \\ &\quad + V_2 [2(N_m - k) \text{Tr} \hat{\Lambda} A \text{Tr} \hat{\Lambda} B + 2k \text{Tr} \hat{\Lambda}_- A \hat{\Lambda}_+ (B - \bar{B})]. \end{aligned} \quad (\text{C5})$$

where the coefficients $V_{a,b}$ are given as follows:

$$V_1 = \frac{1}{n}, \quad V_{1,1} = \frac{1}{n^2 - 1}, \quad V_2 = -\frac{1}{n(n^2 - 1)}. \quad (\text{C6})$$

Here $n = 2N_r N_m$ and $\hat{\Lambda}_\pm = (1 \pm \hat{\Lambda})/2$ denote projectors on the positive and negative Matsubara frequencies. After that, choosing matrices A and B to be equal to \hat{Q} in Eq. (C4) and $\hat{Q} \nabla \hat{Q}$ in Eq. (C5), we are able to write down the full expressions for the spin conductivity. For convenience, we divide the derived expression into two parts: diamagnetic (without spatial derivatives)

and current-current correlation function (with derivatives):

$$g_{dm}(i\omega_k) = -\frac{g}{4k} \left\langle V_{1,1} \left[(N_m - k)(\text{Tr}(\hat{\Lambda} \hat{Q})^2 - \text{Tr} 1 + (\text{Tr} \hat{\Lambda} \hat{Q})^2) - \frac{k}{2}(\text{Tr} 1 + \text{Tr}(\hat{\Lambda} \hat{Q})^2 + (\text{Tr} \hat{\Lambda} \hat{Q})^2) \right] \right. \\ \left. + V_2 [2(N_m - k) \text{Tr}(\hat{\Lambda} \hat{Q})^2 - \frac{k}{2}(\text{Tr} 1 + \text{Tr}(\hat{\Lambda} \hat{Q})^2 + (\text{Tr} \hat{\Lambda} \hat{Q})^2)] - 4(2N_m - k) \right\rangle \quad (\text{C7})$$

and

$$g_{j-j}(i\omega_k) = \frac{g^2}{32k} \int_{\mathbf{x}'} \langle V_{1,1} [2(N_m - k)(\text{Tr} \mathcal{J}(\mathbf{x}) \cdot \mathcal{J}(\mathbf{x}') + \text{Tr} \hat{\Lambda} \mathcal{J}(\mathbf{x}) \cdot \hat{\Lambda} \mathcal{J}(\mathbf{x}') + k(\text{Tr} \mathcal{J}(\mathbf{x}) \cdot \mathcal{J}(\mathbf{x}') - \text{Tr} \hat{\Lambda} \mathcal{J}(\mathbf{x}) \cdot \hat{\Lambda} \mathcal{J}(\mathbf{x}'))] \rangle \\ + V_2 [2(N_m - k)(\text{Tr} \hat{\Lambda} \mathcal{J}(\mathbf{x}) \cdot \hat{\Lambda} \mathcal{J}(\mathbf{x}') + k(\text{Tr} \mathcal{J}(\mathbf{x}) \cdot \mathcal{J}(\mathbf{x}') - \text{Tr} \hat{\Lambda} \mathcal{J}(\mathbf{x}) \cdot \hat{\Lambda} \mathcal{J}(\mathbf{x}'))] \rangle, \quad (\text{C8})$$

respectively. Here we introduce the matrix current $\mathcal{J}(\mathbf{x}) = \hat{Q}(\mathbf{x}) \nabla \hat{Q}(\mathbf{x})$. After performing the averaging, we reduce the Matsubara frequency space to a single frequency and, thus, set $N_m = 1$. Our consideration restricts only RA space, therefore, we can set $k = 1$, in other words we consider processes, which change energy only at the first bosonic Matsubara frequency. A strong consistency check of this reasoning is that the entire part of Eqs. (C7) and (C8) proportional to $N_m - k$ vanishes during the background field renormalization procedure. Therefore, for $N_m = k = 1$, Eqs. (C7) and (C8) reduce to Eq. (97) after some straightforward algebra.

Kubo formula for the transverse spin conductivity has the form:

$$\sigma_s'(i\omega_k) = \frac{g^2}{32kL^2} \int_{\mathbf{x}'} \int_{\mathbf{x}} \langle \langle \varepsilon_{\mu\nu} \text{Tr} I_k^\alpha s_3 \mathcal{Q}(\mathbf{x}) \nabla_\mu \mathcal{Q}(\mathbf{x}) \rangle \rangle \\ \times \text{Tr} I_{-k}^\alpha s_3 \mathcal{Q}(\mathbf{x}') \nabla_\nu \mathcal{Q}(\mathbf{x}') \rangle. \quad (\text{C9})$$

In order to average the above expression, it is enough to use Eq. (C5) only. After that we set $N_m = k = 1$ and obtain Eq. (98).

APPENDIX D: ONE-LOOP CORRECTIONS TO LDoS ON THE INSTANTON BACKGROUND

In this Appendix we justify the appearance of the renormalized observables in preexponential factors for instanton corrections to the RG eigenoperators. We consider into account quantum fluctuations near instanton saddle point in the

preexponential factor. Using Eq. (66), we write the following expression for the renormalized LDoS:

$$\nu'(\mathcal{M}) = \frac{\nu}{2n} \langle \text{Tr} \Lambda \mathcal{Q} \rangle_0 \left(1 - \frac{\mathcal{Z}_{\text{inst}} + \mathcal{Z}_{\text{inst}}^*}{\mathcal{Z}_0} \right) \\ + \frac{\nu}{2n} \langle \text{Tr} \Lambda \mathcal{Q} \rangle_{+1} + \frac{\nu}{2n} \langle \text{Tr} \Lambda \mathcal{Q} \rangle_{-1}. \quad (\text{D1})$$

The first term in the right-hand side of the above equation was calculated previously (see Appendix A). Here we focus on calculation of the last two terms. Using exponential parametrization for \mathcal{Q} , we find

$$\langle \text{Tr} \Lambda \mathcal{Q} \rangle_{+1} = \langle \text{Tr} \underline{\Lambda} \mathcal{Q} \rangle_{+1} \approx \langle \text{Tr} \tilde{R} \underline{\Lambda} \tilde{R}^{-1} \underline{\Lambda} \rangle_{+1} \\ + \frac{1}{2} \langle \text{Tr} \tilde{R} \underline{\Lambda} \tilde{R}^{-1} \underline{\Lambda} W^2 \rangle_{+1}. \quad (\text{D2})$$

We start from the term without fluctuation matrix field W :

$$\langle \text{Tr} \tilde{R} \underline{\Lambda} \tilde{R}^{-1} \underline{\Lambda} \rangle_{+1} = 2n \frac{\mathcal{Z}_{\text{inst}}}{\mathcal{Z}_0} - n(n+1)G_n \\ \times \int \frac{d\lambda}{\lambda} (\pi g)^{n+1} e^{-\pi g(\mathcal{M})+i\theta} \int d\mathbf{r}_0 \frac{\mu(r_0)}{\lambda}. \quad (\text{D3})$$

We note that the first term, proportional to $\mathcal{Z}_{\text{inst}}$ can be rewritten as $\langle \text{Tr} \underline{\Lambda} \underline{\Lambda} \rangle_0 \mathcal{Z}_{\text{inst}} / \mathcal{Z}_0$ and it cancels out with the second term in brackets in Eq. (D1). The second contribution in Eq. (D2), which depends on quantum fluctuations, can be calculated as follows:

$$\frac{1}{2} \langle \text{Tr} \tilde{R} \underline{\Lambda} \tilde{R}^{-1} \underline{\Lambda} W^2 \rangle_{+1} = - \left(\langle (|e_1|^2 - e_0^2) w^{11} w^{*11} \rangle_{+1} + \sum_{\alpha=2}^n \langle (|e_1|^2 - e_0^2 + 1) w^{1\alpha} w^{*1\alpha} \rangle_{+1} \right) \\ + \sum_{\alpha=2}^n \langle w^{\alpha\alpha} w^{*\alpha\alpha} \rangle_{+1} + 2 \sum_{1 < \alpha < \beta \leq n} \langle w^{\alpha\beta} w^{*\alpha\beta} \rangle_{+1}. \quad (\text{D4})$$

The above averages can be rewritten in terms of the Green's functions at coinciding points:

$$\frac{1}{2} \langle \text{Tr} \tilde{R} \underline{\Lambda} \tilde{R}^{-1} \underline{\Lambda} W^2 \rangle_{+1} = - \left((|e_1|^2 - e_0^2) \mathcal{G}_2 \right)_{+1} + \frac{n-1}{2} (|e_1|^2 - e_0^2 + 1) \mathcal{G}_1 \Big|_{+1} + (n-1) \mathcal{G}_0 \Big|_{+1} + \frac{(n-1)(n-2)}{2} \langle \mathcal{G}_0 \rangle_{+1}. \quad (\text{D5})$$

We should subtract from this equation the contribution of the trivial topological sector $\frac{1}{2}\langle\text{Tr}\ \underline{\Lambda}\underline{\Lambda}W^2\rangle_0\mathcal{Z}_{\text{inst}}/\mathcal{Z}_0$ [the second term in brackets in Eq. (D1)], after that we obtain:

$$-\left(\left(|e_1|^2 - e_0^2\right)\mathcal{G}_2\right)_{+1} - \frac{\mathcal{Z}_{\text{inst}}}{\mathcal{Z}_0}\mathcal{G}_0 + \frac{n-1}{2}\left(\left(|e_1|^2 - e_0^2 + 1\right)\mathcal{G}_1\right)_{+1} - (n-1)\frac{\mathcal{Z}_{\text{inst}}}{\mathcal{Z}_0}\mathcal{G}_0. \quad (\text{D6})$$

Using the Pauli-Villars regularization scheme and representation of regularized Green's functions in terms of Y function, cf. Eq. (A5), we obtain with logarithmic accuracy, i.e., with neglect of all terms without large Pauli-Villars mass \mathcal{M} :

$$(\delta v'(\mathcal{M}))_{+1} = \frac{(1+n)^2 \ln \mathcal{M}}{2\pi g} G_n \int \frac{d\lambda}{\lambda} (\pi g)^{n+1} e^{-\pi g(\mathcal{M})+i\vartheta} \int d\mathbf{r}_0 \frac{\mu(r_0)}{\lambda}. \quad (\text{D7})$$

Taking into account a similar contribution with the negative topological charge, we obtain expression (78) for renormalized LDoS.

APPENDIX E: CURING THE ULTRAVIOLET DIVERGENCES BY MEANS OF THE SPATIALLY VARYING MASS METHOD

In this Appendix, we discuss the transformation of Eq. (77) into Eq. (78). In order to cure the ultraviolet divergences, we employ the scheme of spatially varying mass adapted from Ref. [67]. First, we transfer from the curved space in which the quantum correction to the LDoS is controlled by the Pauli-Villars mass \mathcal{M} to the flat space in which a relevant length scale, which limits the quantum fluctuations is $1/\mu(r_0)$,

$$\begin{aligned} v(\mathcal{M}) &\rightarrow v(\mathcal{M}) \left(1 - \frac{\gamma_{(1)}^{(0)}}{g} \ln[\mu(r_0)\ell\mathcal{M}]\right) \\ &= v \left(1 + \frac{\gamma_{(1)}^{(0)}}{g} \ln \frac{e^{\gamma-1/2}}{\mu(r_0)\ell}\right) \equiv v[1/\mu(r_0)]. \end{aligned} \quad (\text{E1})$$

This expression we substitute into integral over instanton position r_0 in Eq. (78). Next we convert the perturbative correction with spatially dependent mass into the perturbative correction at the length scale $\zeta\lambda$ where $\zeta = e/2$,

$$\begin{aligned} v(\zeta\lambda) &= \frac{1}{4\pi} \int d\mathbf{r}_0 \mu^2(r_0) v[\mu(r_0)] \\ &= v_0 \left(1 + \frac{\gamma_{(1)}^{(0)}}{g} \ln \lambda \mu_0 \zeta e^{\gamma-1/2}\right), \quad \zeta = \frac{e}{2}. \end{aligned} \quad (\text{E2})$$

The above expression suggests the following correspondence between Pauli-Villars mass and the instanton size λ :

$$\mathcal{M} \rightarrow \zeta\lambda\mu_0. \quad (\text{E3})$$

Next we rewrite Eq. (78) in a more transparent form

$$\delta v_{\text{inst}} = G_n \int \frac{d\lambda}{\lambda} (\pi g)^{n+1} e^{-\pi g(\zeta\lambda)} \mathcal{A}_{(1)} \cos \vartheta. \quad (\text{E4})$$

Here we introduce the amplitude \mathcal{A}_λ

$$\begin{aligned} \mathcal{A}_\lambda &= \pi \gamma_\lambda^{(0)} \int_0^L d\mathbf{r}_0 \frac{\mu(r_0)}{\lambda} z_\lambda[1/\mu(r_0)] = -2\pi^2 \gamma_\lambda^{(0)} z_\lambda[1/\mu(0)] \\ &\quad \times \int_{\mu(0)}^{\mu(L)} d[\ln \mu(r_0)] \frac{z_\lambda[1/\mu(r_0)]}{z_\lambda[1/\mu(0)]}, \end{aligned} \quad (\text{E5})$$

where the integral over r_0 runs until the system size L . We assume that \mathcal{A}_λ corresponds to the operator with negative anomalous dimension, $\gamma_\lambda < 0$. In particular, this is the case of LDoS. Using definition of the anomalous dimension γ_λ , we can rewrite the integrand as follows:

$$\frac{z_\lambda[\mu(r_0)]}{z_\lambda[\mu(0)]} = \exp\left(-\int_{\ln \mu(0)}^{\ln \mu(r_0)} d[\ln \mu] \gamma_\lambda\right). \quad (\text{E6})$$

We note that the condition $\gamma_\lambda < 0$ guarantees the convergence of integral over $\mu(r_0)$ as $\mu(L) \rightarrow 0$.

Now it is convenient to change integral variable from $\ln \mu$ to g with the help of perturbative RG equation for g , see Eq. (102). Then we obtain $\mathcal{A}_\lambda = z_\lambda[1/\mu(0)] \mathcal{H}_\lambda\{g[1/\mu(0)]\}$, where we introduce the function

$$\mathcal{H}_\lambda = -2\pi^2 \gamma_\lambda^{(0)} \int_{g[1/\mu(0)]}^{g[1/\mu(L)]} \frac{dg}{\beta_g(g)} \exp\left(-\int_{g[1/\mu(0)]}^{g[1/\mu(r_0)]} \frac{dg'}{\beta_g(g')} \gamma_\lambda(g')\right). \quad (\text{E7})$$

Using the one-loop results for γ_λ and β_g and integrating over g' and g , we obtain the expression (93). The result in the case of positive anomalous dimensions γ_λ can be obtained with the help of a kind of analytic continuation (see Ref. [67] for details).

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