Letter

Bulk-edge correspondence at the spin-to-integer quantum Hall effect crossover in topological superconductors

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The spin and integer quantum Hall effects are two cousins of topological phase transitions in two-dimensional electronic systems. Their close relationship makes it possible to convert spin to integer quantum Hall effect by continuous increase in a symmetry breaking Zeeman magnetic field. We study microscopic and mesoscopic peculiarities of bulk-edge correspondence and a fate of massless edge and bulk topological (instantons) excitations at such a crossover in topological superconductors. We propose possible experimental verification of our predictions.

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Introduction. Topological phase transitions (TPT) are a constant focus of physics research. The discovery of topological insulators and superconductors [1–4] gave a boost to research on TPT in disordered electronic systems [5–20]. Perhaps the most famous example of a TPT is the integer quantum Hall effect (iqHe) in which different topological phases are labeled by $\mathbb{Z}$ (the set of the integer numbers). The iqHe reflects the existence of the $\mathbb{Z}$-valued topological charge in a two-dimensional (2D) realization of class A in the Altland-Zirnbauer symmetry classification of disordered Hamiltonians [21–26]. The iqHe has two close cousins in 2D topological superconductors, whose distinct topological phases are labeled by integers: the spin (class C) [27–29] and thermal (class D) [30] quantum Hall effects.

The iqHe has been investigated extensively in experiments [31–40] as opposed to the spin quantum Hall effect (sqHe). However, the latter has an advantage since its criticality is analytically tractable [41]. In particular, the position of the critical point [42,43], the correlation length exponent, and the infinite subset of generalized multifractal exponents are known exactly through the mapping to percolation [44]. Class C can be thought as a parent class for the classes A and D due to the following crossovers: C $\to$A with breaking of the SU(2) spin rotation symmetry down to U(1) [45] and C $\to$D which corresponds to the complete breaking of SU(2) symmetry while preserving superconductivity [28,29,44,46–48].

Although the crossover phenomena in the context of phase transitions are thoroughly studied [49], the crossovers between topologically nontrivial phases are much less investigated. An immediate difficulty can be readily appreciated from the observation that the topological phases of the sqHe are enumerated by even integers while the topological phases of the iqHe are labeled by all integers. Thus the transformation $2\mathbb{Z} \to \mathbb{Z}$ should occur during the C $\to$A crossover. The understanding of crossovers between topological phases is complicated by the presence of topological excitations (instantons) in the bulk and massless edge excitations, which are related by the bulk-boundary correspondence. From practical point of view, interest in the crossovers lies in their potential experimental applications. For instance, does the iqHe realized in a topological superconductor due to the C $\to$A crossover differ from the ordinary iqHe experimentally?

The goal of this Letter is to study the sqHe-to-iqHe crossover and answer the following *physical* questions: (i) Is it possible to describe the crossover in terms of the edge theory only? (ii) How do physical observables depend on a bare spin Hall conductance after the crossover? (iii) What is the structure of the emergent iqHe staircase?

Edge modes for sqHe. Both sqHe and iqHe possess nondissipative gapless edge modes. First we discuss their transformation across the crossover. We begin with a reminder of the edge theory for the sqHe [29]. We consider chiral fermion quasiparticles at the edge of a (2D) disordered $d_{x^2-y^2}+id_{xy}$ superconductor. To be able to average over quenched disorder we will use the replica trick. The imaginary time replica action for the sqHe edge can be written in terms of N_r copies of spin 1/2 chiral fermions [29]:

$$S_{\mathrm{e}} = \int_0^\beta d\tau \int dy[\bar\psi(iv\partial_y - \partial_\tau - \eta_3)\psi + \eta_-\bar\psi\Sigma_+\bar\psi^{\mathrm{T}} + \eta_+\psi^{\mathrm{T}}\Sigma_-\psi]. \tag{1}$$

Here $\bar\psi = \{\bar\psi_{\uparrow,1},\dots,\bar\psi_{\downarrow,N_r}\}$ and $\psi = \{\psi_{\uparrow,1},\dots,\psi_{\downarrow,N_r}\}^T$ are Grassmann variables corresponding to fermion creation and annihilation operators, $\Sigma_\pm = \sigma_\pm \otimes 1_r$ with 1_r being the identity matrix in the replica space and $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$, where σ_j are standard Pauli matrices acting in the spin space. A quasiparticle edge velocity is denoted as v, and β stands for the inverse temperature. The random Gaussian fields $\eta_\pm = \eta_1 \pm i\eta_2$ and η_3 mimic fluctuations of a superconducting order parameter and scattering off impurities, respectively. They have the zero mean and are delta correlated in space: $\langle \eta_j(y)\eta_k(y')\rangle = \varkappa\delta_{jk}\delta(y-y')$. The action (1) does not conserve the number of ψ fermions but has SU(2) symmetry corresponding to spin conservation.

To elucidate symmetries of action (1) inherent in class C, we introduce new fields: $\bar{\chi}_{\uparrow,\alpha}=\bar{\psi}_{\uparrow,\alpha}$, $\chi_{\uparrow,\alpha} = \psi_{\uparrow,\alpha}$, $\bar{\chi}_{\downarrow,\alpha} = \psi_{\downarrow,\alpha}$, and $\chi_{\downarrow,\alpha} = \bar{\psi}_{\downarrow,\alpha}$, where $\alpha = 1, \dots, N_r$ [29]. In this representation the edge action (1) becomes

$$S_{\rm e} = \int_0^\beta d\tau \int dy\, \bar{\chi}(-\partial_\tau - H\otimes 1_r)\chi, \quad H = -iv\partial_y + \boldsymbol{\eta}\boldsymbol{\sigma}. \tag{2}$$

The above action conserves the number of χ fermions, which coincides with the z projection of the spin of ψ fermions. Thus, the χ (ψ) fermions serve as spin (charge) carriers, respectively. Hamiltonian (2) manifests antiunitary Bogoliubov–de Gennes (BdG) symmetry, $H = -\sigma_2 H^T \sigma_2$, as expected for class C. The action (2) describes two spin-degenerate hybridized electron-hole edge modes that propagate in the same direction and transfer a quantum of the transverse spin conductivity each [29]. Therefore, in the case of a clean system (for which $\eta_j \equiv 0$ and $N_r = 1$), applying a generalized Thouless-Kohmoto-Nightingale-Nijs (TKNN) formula [50], we obtain that the spin Hall conductance is quantized in units of $G_0^{(s)} = \hbar/8\pi$ [27,29],

$$g_{\rm H} = 2k\, G_0^{(s)}. \tag{3}$$

Here k is the number of edge modes [$k = 1$ for Eq. (2)].

Edge theory for sqHe. As expected, the $2\mathbb{Z}$ quantization of $g_{\rm H}$, Eq. (3), holds in the presence of the disorder. Averaging action (1) over disorder and employing the non-Abelian bosonization [51–57], we derive the noninear sigma model (NLσM) action for the soft diffusive edge modes (see Supplemental Material [58] and also references [59–62] therein):

$$\mathcal{S}_{\rm e} = \frac{k}{2}{\rm Tr}\Lambda T\partial_y T^{-1} + \pi k\nu_{\rm e}{\rm Tr}\hat{\epsilon}Q. \tag{4}$$

Here $\nu_{\rm e} = 1/(2\pi v)$ is the density of edge states. $Q = T^{-1}\Lambda T$ is Hermitian traceless matrix acting in $N_r\times N_r$ replica space, $2N_m\times 2N_m$ Matsubara frequency space, and 2×2 Nambu space. The matrix Q satisfies the following relations:

$$Q^2 = 1, \quad Q = Q^\dagger = -L_0 \mathsf{s}_2 Q^T \mathsf{s}_2 L_0. \tag{5}$$

Here and below, $\mathsf{s}_{0,1,2,3}$ stand for the standard Pauli matrices in the Nambu space, $(L_0)_{nm}^{\alpha_1\alpha_2} = \delta_{\varepsilon_n,-\varepsilon_m}\delta^{\alpha_1\alpha_2}\mathsf{s}_0$, $\Lambda_{nm}^{\alpha_1\alpha_2} = {\rm sgn}(\varepsilon_n)\delta_{nm}\delta^{\alpha_1\alpha_2}\mathsf{s}_0$, $\hat{\epsilon}_{nm}^{\alpha_1\alpha_2} = \varepsilon_n\delta_{nm}\delta^{\alpha_1\alpha_2}\mathsf{s}_0$, where $\varepsilon_n = \pi(2n+1)/\beta$ denotes the fermionic Matsubara frequency. Symbol "Tr" includes spatial integration as well as the trace over replica, Matsubara, and Nambu spaces. As the consequence of SU(2) symmetry, the *spin space* is *not present* in action (4). The relations (5) determine the NLσM target manifold of class C, $Q \in {\rm Sp}(2N)/{\rm U}(N)$, where $N = 2N_r N_m$, whereas $T \in {\rm Sp}(2N)$.

The information about the quantization of $g_{\rm H}$ is encoded in the first term of the NLσM (4), which is nothing but the edge form of Pruisken's θ term [63]. The factor $k/2$ is responsible for exactly the same result for $g_{\rm H}$ as in the clean case, Eq. (3). It is expected since the gauge transformation $\tilde{\chi}(y) = \mathcal{T}_y \exp[i\int^y dy'\boldsymbol{\eta}(y')\boldsymbol{\sigma}/v_{\rm e}]\chi(y)$ ($\mathcal{T}_y$ is spatial ordering) [29] excludes disorder from Eq. (2).

The sqHe-to-iqHe crossover at the edge. In order to remove the spin degeneracy of the chiral edge states, we introduce the Zeeman magnetic field B_z by adding the term $\mu_{\rm B}B_z\int d\tau\, dy\, \bar{\psi}(\sigma_3\otimes 1_r)\psi$ to the action (1). Here $\mu_{\rm B}$ is the Bohr magneton. In Eq. (2), it works as the shift $H \to H + \mu_{\rm B}B_z\sigma_0$, which explicitly breaks the BdG symmetry. Thus the resulting Hamiltonian becomes just a Hermitian operator belonging to class A.

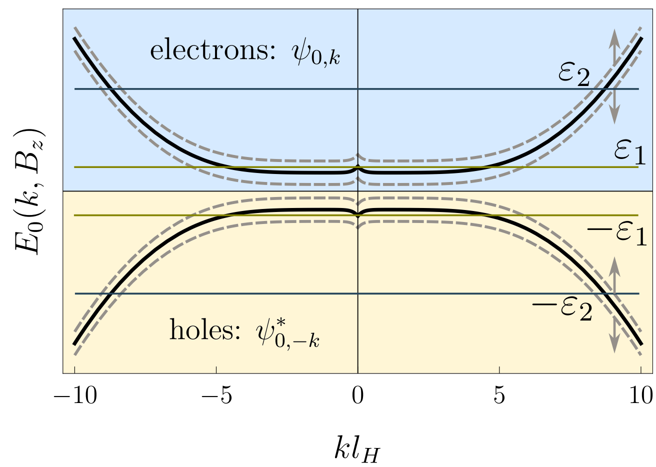

FIG. 1. The quasiparticle spectrum in the toy model: 2D fermions on a stripe in the presence of a perpendicular magnetic field, constant superconducting pairing amplitude, and Zeeman splitting (see SM [58] and Refs. [64–66] for details).

In the clean case, the spectrum of ψ fermions remains linear in momentum p_y, but modes with different spin projections are split by the momentum difference $\Delta p_y = \mu_{\rm B}B_z/v$ (see Fig. 1 for the energy level ε_2). Since in the presence of orbital magnetic field the real space coordinate is proportional to the quasimomentum in the perpendicular direction, the Zeeman field results in splitting of the chiral edge modes with different spin projections in a real space. However Δp_y can be absorbed into the phase of ψ fermions, thus the magnitude of $g_{\rm H}$ remains insensitive to the presence of B_z; see Eq. (3). Therefore, in the case of the Zeeman field acting at the edge only, the $2\mathbb{Z}$ quantization of $g_{\rm H}$ survives.

The edge modes have curvature due to merging with the bulk states. Then the spectrum of spin-$\uparrow$ (spin-$\downarrow$) ψ fermion floats up (down) in energy with increasing B_z. Hence there are energy levels (e.g., energy ε_1 in Fig. 1) for which only a single edge mode remains. Thus the spin Hall conductance becomes $g_{\rm H} = (2k-1)G_0^{(s)}$, in agreement with the $\mathbb{Z}$ quantization for the iqHe.

Now let us turn on disorder at the edge again. The Zeeman splitting emerges in the NLσM action as [58]

$$\mathcal{S}_{\rm e}^{(Z)} = i\pi\mu_{\rm B}B_z\nu_{\rm e}{\rm Tr}\mathsf{s}_3 Q. \tag{6}$$

We emphasize that the *physical* magnetic field B_z enters the NLσM as the Zeeman splitting acting in the *Nambu* space. Though $\mathcal{S}_{\rm e}^{(Z)}$ is consistent with the symmetry (5), it breaks rotation symmetry in the Nambu space from SU(2) down to U(1). The term (6) acts as the mass term for an otherwise massless theory (4). At long distances, $|y| \gg 1/\Delta p_y$, only the rotations T commuting with the matrix s_3 survive, enforcing the diagonal form of the matrix Q in the Nambu space. Substituting $Q = {\rm diag}\{Q_{\mathsf{u}}, Q_{\mathsf{d}}\}$ into Eq. (4) and using the relation $Q_{\mathsf{d}} = -L_0 Q_{\mathsf{u}}^T L_0$, we find that $\mathcal{S}_{\rm e}$ is given by Eq. (4) with T,

Q, and k substituted by T_{U}, Q_{U}, and $2k$ respectively, with traces now over replica and Matsubara spaces alone. Since the Hermitian matrix Q_{U} has no additional constraints except the nonlinear one, $Q_{\mathsf{U}}^2 = 1$, at long distances the NLσM edge action in the presence of Zeeman splitting, Eqs. (4) and (6), becomes the iqHe edge action with $Q_{\mathsf{U}} \in \mathrm{U}(N)/[\mathrm{U}(N/2)\times \mathrm{U}(N/2)]$. That action describes $2k$ chiral edge channels and leads to Eq. (3) for g_{H}. As in the clean case, we see that within the edge theory only the Zeeman field *does not change* the $2\mathbb{Z}$ quantization of g_{H}. Thus, to get the $2\mathbb{Z} \to \mathbb{Z}$ transformation of g_{H}'s quantization, we have to study the bulk theory.

Bulk theory for sqHe. Now we recall the NLσM description of the 2D bulk of a system with class C symmetry [67–70]:

$$\mathcal{S}_{\mathrm{b}} = -\frac{\bar{g}}{16}\mathrm{Tr}(\nabla Q)^2 + i\pi \bar{g}_{\mathrm{H}}\mathcal{C} + \pi\bar{\nu}\mathrm{Tr}[\hat{\epsilon} + i\mu_{\mathrm{B}}B_z \mathsf{s}_3]Q. \quad (7)$$

Here $\bar{\nu}$ denotes the bare bulk density of states and $\bar{g}$ and $\bar{g}_{\mathrm{H}}$ stand for the bare dimensional spin longitudinal and Hall conductances (in units $G_0^{(s)}$). The topology of the class C is encoded in the $\mathbb{Z}$ quantized topological charge

$$\mathcal{C}[Q] = \mathrm{Tr}(\varepsilon_{jk}Q\nabla_j Q\nabla_k Q)/(16\pi i), \quad (8)$$

where ε_{jk} is the Levi-Civita symbol with $\varepsilon_{xy} = -\varepsilon_{yx} = 1$. For $\bar{g}_{\mathrm{H}} = 2k$ the term proportional to $\mathcal{C}[Q]$ in Eq. (7) coincides with the first term in the edge theory (4). The term in Eq. (7) proportional to B_z describes breaking the SU(2) symmetry in the Nambu space. As expected, its form is the same as for the edge theory, Eq. (6) [58].

The crossover in the bulk. The NLσM action (7) is renormalized such that the parameters g, g_{H}, and ν become length-scale (L) dependent. Their renormalization group (RG) equations are well known [68–73]. The class C $\to$A crossover can be seen already at the level of the NLσM action. At long distances, $L \gg L_B = \sqrt{g(L_B)/[\nu(L_B)\mu_{\mathrm{B}}B_z]}$, the Zeeman term in Eq. (7) forces Q to become a diagonal matrix in the Nambu space. As a result, the NLσM action of class A arises. It is given by Eq. (7) with Q, $\bar{g}$, $\bar{g}_{\mathrm{H}}$, $\bar{\nu}$ substituted by Q_{U}, $2\bar{\bar{g}} = 2g(L_B)$, $2\bar{\bar{g}}_{\mathrm{H}} = 2g_{\mathrm{H}}(L_B)$, $2\bar{\bar{\nu}} = 2\nu(L_B)$, respectively, and with $B_z = 0$. Thus, the sqHe-to-iqHe crossover can be considered roughly as follows. At $\ell \leqslant L \leqslant L_B$ the system is described by the RG equations for class C with initial conditions $g(\ell) = \bar{g}$ and $g_{\mathrm{H}}(\ell) = \bar{g}_{\mathrm{H}}$. At $L = L_B$ the conductivities reach magnitudes $\bar{\bar{g}}$ and $\bar{\bar{g}}_{\mathrm{H}}$, respectively. Then at $L > L_B$ the system is governed by the RG equations for class A with initial conditions $g(L_B) = \bar{\bar{g}}$ and $g_{\mathrm{H}}(L_B) = \bar{\bar{g}}_{\mathrm{H}}$. Consequently, a physical observable $\mathcal{O}$ at $L > L_B$ depends on $\bar{\bar{g}}$ and $\bar{\bar{g}}_{\mathrm{H}}$.

The crossover scenario characterized by the length scale L_B is universal for a relevant symmetry-breaking parameter in the renormalization-group sense and is therefore applicable to both topologically trivial and nontrivial systems. In our case, the corresponding relevant parameter is the Zeeman splitting, which drives the crossover between the class C and class A [44,47]. It is straightforward to verify that, at the perturbative level, the presence of B_z generates a mass for the diffusive modes of the NLσM (7) that do not belong to class A [58]. This is exactly the mechanism that converts the perturbative part of the RG equations for class C to the ones for class A. However, the topological nontrivial systems have topological excitations (instanton configurations Q_W in our case) with integer quantized value of the topological charge $\mathcal{C}[Q_W] = W$. It is these topological excitations that are responsible for the nonperturbative part of the RG equations and for the periodicity of the physical observables with the bare Hall conductance in the cases of iqHe [74–78] and sqHe [70]. The $2\mathbb{Z}$ [70] ($\mathbb{Z}$ [31]) quantization in the case of sqHe (iqHe) implies the periodicity of each physical observable $\mathcal{O}$ with respect to $\bar{g}_{\mathrm{H}}$ ($\bar{\bar{g}}_{\mathrm{H}}$) with period 2 (1). Consequently, we suggest that at the sqHe-to-iqHe crossover the following transformation occurs:

$$\mathcal{O} = \sum_{W\in\mathbb{Z}} \underbrace{\mathcal{O}_W^{(\mathrm{C})} e^{i\pi\bar{g}_{\mathrm{H}}W}}_{L\ll L_B} \longrightarrow \mathcal{O} = \sum_{W\in\mathbb{Z}} \underbrace{\mathcal{O}_W^{(\mathrm{A})} e^{i2\pi\bar{\bar{g}}_{\mathrm{H}}W}}_{L\gg L_B}, \quad (9)$$

where $\mathcal{O}_W^{(\mathrm{C})} \propto \exp(-\pi\bar{g}|W|)$ and $\mathcal{O}_W^{(\mathrm{A})} \propto \exp(-2\pi\bar{\bar{g}}|W|)$. The only consistent possibility to realize Eq. (9) is the following picture of the crossover in the nonperturbative contributions to the RG equations. At $L \gg L_B$ the nonperturbative class C contributions with odd W have to be suppressed [58], while contributions with even W transform smoothly into the class A contributions.

Topological excitations at the crossover. To argue for the above scenario, we consider the class C instantons with $W = 1, 2$. For simplicity, we present expressions for instanton solutions for $N_r = N_m = 1$. The $W = 1$ instanton is given as $Q_1 = T^{-1}\Lambda_1(\boldsymbol{x})T$, where T represents a spatially uniform rotational matrix that defines orientation of the instanton within the NLσM manifold and [70]

$$\Lambda_1 = \begin{pmatrix} \mathsf{s}_0\cos^2\theta - \mathsf{s}_1\sin^2\theta & \frac{(i\mathsf{s}_2-\mathsf{s}_3)}{2}e^{i\phi}\sin 2\theta \\ -\frac{(i\mathsf{s}_2+\mathsf{s}_3)}{2}e^{-i\phi}\sin 2\theta & -\mathsf{s}_0\cos^2\theta - \mathsf{s}_1\sin^2\theta \end{pmatrix}. \quad (10)$$

Here $\theta = \arctan(\lambda/|z - z_0|)$, $\phi = \arg(z - z_0)$, $z = x + iy$ is the complex coordinate, λ is the instanton scale size, and z_0 is the position of instanton. In the absence of the last term in Eq. (7), we find $\mathcal{S}_{\mathrm{b}}[Q_1] = -\pi\bar{g} + i\pi\bar{g}_{\mathrm{H}}$, such that the parameters λ, z_0, and the generators of the T rotations constitute the zero mode manifold of the $W = 1$ instanton. We note that there is no analog of such an instanton solution in the case of the iqHe. This can be understood from the Nambu structure of Eq. (10): the presence of nonzero off-diagonal (superconducting) elements prevents the reduction described above. Due to the Zeeman term, rotational zero modes with $[T, \mathsf{s}_3] \neq 0$ acquire a mass $\propto B_z\ln(L/\lambda)$, i.e., modification of the zero mode manifold from $T \in \mathrm{Sp}(2N)$ (class C) to $T \in \mathrm{U}(N)/[\mathrm{U}(N/2)\times\mathrm{U}(N/2)]$ (class A) occurs. However, λ and z_0 remain zero modes, i.e., the $W = 1$ instanton is not fully suppressed by B_z at the classical level [79].

Accounting for fluctuations around the $W = 1$ instanton leads to logarithmically divergent renormalizations in physical observables. These divergences can be resummed within the RG framework. Without the Zeeman splitting, the resummation process continues until the RG flow reaches a scale where the instanton size becomes comparable to the dynamically generated localization length in class C, $\lambda \sim \xi^{(C)} \simeq \ell\exp(\pi\bar{g})$ [70]. In the presence of a nonzero B_z, the RG procedure for $W = 1$ instantons halts at $\lambda \sim L_B$, because all instantons with sizes $\lambda > L_B$ fail to contribute logarithmic corrections to physical observables [58]. It leads to suppression of contributions from the $W = 1$ instantons to the RG equations beyond the length scale L_B.

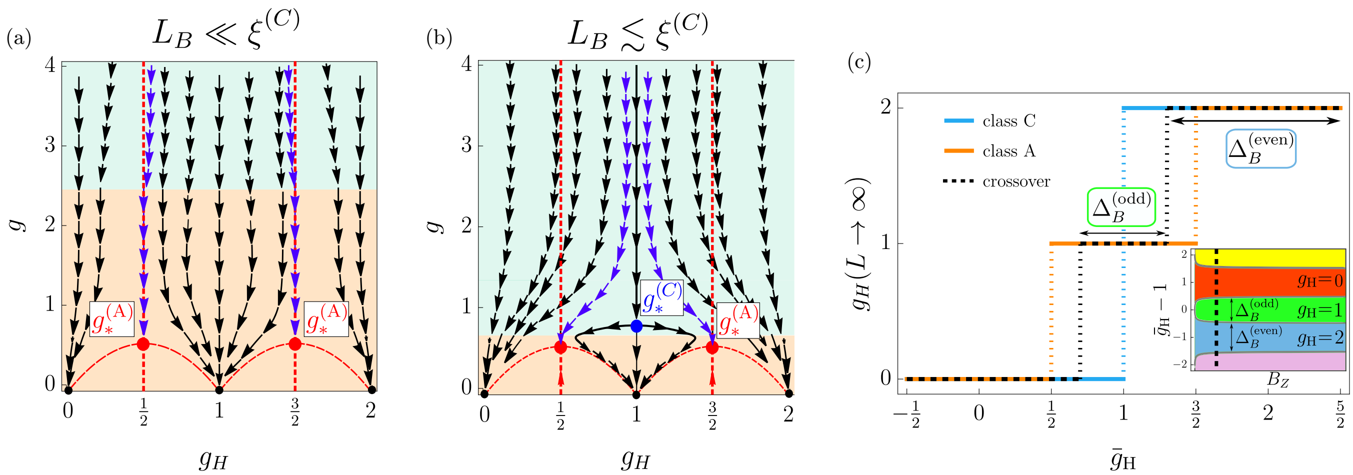

FIG. 2. Sketch of the crossover RG flow for strong (a) and weak (b) Zeeman splitting. For bare values $\bar{g} \gg 1$, the RG flows in blue (orange) regions are governed by the RG equations for class C (A). Blue lines correspond to separatrixes that define the width of the iqHe plateaus. (c) The quantization of the spin Hall conductance at $L \to \infty$ as a function of its bare value $\bar{g}_{\text{H}}$ for the sqHe (blue solid line), the ordinary iqHe (orange solid line), and a finite B_z (dashed black line). The inset shows the phase diagram in the $\{\bar{g}_{\text{H}}, B_z\}$ plane. The black dashed line indicates the Zeeman splitting for which the main panel is plotted.

The $W = 2$ instanton solution can be expressed as $Q_2 = \tilde{T}^{-1}\Lambda_2(\boldsymbol{x})\tilde{T}$, where the matrix $\tilde{T}$ contains rotational zero modes and (for $N_r = N_m = 1$ as above)

$$\Lambda_2(\boldsymbol{x}) = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix} \mathcal{K}^{-1} \begin{pmatrix} \cos 2\hat{\theta} & \sin 2\hat{\theta} \\ -\sin 2\hat{\theta} & \cos 2\hat{\theta} \end{pmatrix} \mathcal{K}. \tag{11}$$

Here $\mathcal{K} = \text{diag}\{\mathcal{U}, \mathcal{U}^*\}$ with an arbitrary U(2) matrix $\mathcal{U}$ and a matrix $\hat{\theta} = \text{diag}\{\theta_1, \theta_2\}$. The instanton angles θ_j are defined similarly to those of the $W = 1$ instanton and involve two sets of zero modes $z_0^{(j)}$ and λ_j. The resulting classical bulk action for this solution can be divided into two parts: the classical action for the $W = 2$ instanton, $\mathcal{S}_{\text{b}}^{(\text{cl})} = -2\pi\bar{g} + 2\pi i\bar{g}_{\text{H}}$, and the Zeeman contribution, $\mathcal{S}_{\text{b}}^{(\text{Z})} \propto B_z \int d^2\boldsymbol{x}[\cos 2\theta_1(\boldsymbol{x}) - \cos 2\theta_2(\boldsymbol{x})]$, with a coefficient depending on the matrices $\tilde{T}$ and $\mathcal{U}$ (see [58] for details). The Zeeman term enforces synchronization of the instanton scale sizes, $\lambda_1 = \lambda_2$, and positions, $z_0^{(1)} = z_0^{(2)}$. As a result, the $W = 2$ instanton becomes a diagonal matrix in Nambu space, which can be interpreted as two identical class A $W = 1$ instantons with pinned centers and equal sizes. Thus the Zeeman splitting forces the class C $W = 2$ instanton to transform into the class A $W = 1$ instanton already at the level of the classical action (for details, see [58]).

Physical predictions. The above picture of the sqHe-to-iqHe crossover has implications for the length-scale dependence of physical observables. For example, the dependence of g and g_{H} on L can be visualized as a two-parameter scaling diagram shown in Fig. 2. We assume that RG flow starts from a weak coupling region, $\bar{g} \gg 1$. At strong Zeeman splitting, $L_B \ll \xi^{(C)}$ [80], the crossover occurs in weak coupling region, $\bar{\bar{g}} \gg 1$; see Fig. 2(a). In contrast, at weak B_z such that $L_B \lesssim \xi^{(C)}$, the crossover occurs in the strong coupling regime close to the class C unstable fixed point $g_*^{(C)} = \sqrt{3}/2$ [42,43] (for class A, $g_*^{(A)} \simeq 0.5$–0.6 [81–83]); see Fig. 2(b). In cases of both strong and weak B_z, the flow lines starting at $|\bar{g}_{\text{H}} - 1| \leqslant \Delta_B^{(\text{odd})}/2$ approach the stable fixed point at $g = 0$ and $g_{\text{H}} = 1$ as $L \to \infty$. Thus for $B_z \neq 0$ the RG flow in Fig. 2 shows the $\mathbb{Z}$ quantization of g_{H} as $L \to \infty$. The RG flow in Fig. 2 looks similar to the crossover RG flow due to breaking of spin degeneracy in an ordinary iqHe [84] and mixing of valleys for the iqHe in graphene [85]. However, those crossovers occur within the same class A.

For $|\bar{g}_{\text{H}} - 1| \leqslant \Delta_B^{(\text{odd})}/2$ the dependence of g_{H} on L is nonmonotonic, with the extremum at $L \sim L_B$. Plateaus at odd integer values in dependence of g_{H} on $\bar{g}_{\text{H}}$ start to develop as L grows beyond L_B. In the limit $L \to \infty$ the dependence of g_{H} on $\bar{g}_{\text{H}}$ becomes steplike with plateaus at $\mathbb{Z}$; see Fig. 2(c). However, the widths of the odd, $\Delta_B^{(\text{odd})}$, and even, $\Delta_B^{(\text{even})} = 2 - \Delta_B^{(\text{odd})}$, plateaus are different. This fact reflects periodic dependence of physical observables on $\bar{g}_{\text{H}}$ with period $2G_0^{(s)}$, as follows from Eq. (9).

At small values of B_z, corresponding to $L_B \sim \xi^{(C)}$, the width of the odd plateaus can be estimated as $\Delta_B^{(\text{odd})} \sim |B_z|^{3/7}$ [44]. At strong Zeeman splitting, $L_B \ll \xi^{(C)}$, the odd-plateau width approaches 1 as [58]

$$1 - \Delta_B^{(\text{odd})} \sim [(L_B - \ell)/\xi^{(C)}] \ln^3(\xi^{(C)}/L_B). \tag{12}$$

We note that the staircase with $\Delta_B^{(\text{odd})} \neq \Delta_B^{(\text{even})}$ distinguishes the iqHe obtained in a result of crossover from the sqHe in a topological superconductor, and from the ordinary iqHe.

Summary. We developed a coherent physical picture of the spin-to-integer quantum Hall effect crossover in the bulk and at the edge of topological superconductors. We demonstrated that it is not possible to describe the crossover in terms of the edge theory only. The correct description of the crossover involves the bulk theory, in particular, topological excitations (instantons). We found that although the spin Hall conductance becomes quantized in units $G_0^{(s)}$ as a result of the crossover, the periodic dependence of the physical observables on the bare spin Hall conductance has the period $2G_0^{(s)}$ as for the sqHe. We found that after the crossover the widths of the odd and even iqHe plateaus are different, in contrast to the conventional iqHe staircase. Although we study the sqHe-to-iqHe crossover in the absence of electron-electron interaction, we expect that it does not alter the developed physical picture.

Finally, we mention that twisted $Bi_2Sr_2CaCu_2O_{8+x}$ bilayers have been recently shown to spontaneously break time-reversal symmetry [86], in agreement with theoretical predictions for an emergent $d_{x^2-y^2}+id_{xy}$ topological superconducting state [87,88]. Results of our work suggest that a magnetic field parallel to bilayers is an efficient tool to control and manipulate the edge spin-current-carrying states in such topological superconductors in a way similar to manipulation of edge current channels in the conventional iqHe [89–92].

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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