Editors' Suggestion

Class *C* quantum network model with random tunneling and its nonlinear sigma model representation

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The spin quantum Hall effect is a relative of the integer quantum Hall effect, characterized by integer quantized spin Hall conductance. In this work, we formulate and investigate a quantum network model consisting of N channels per chiral link, preserving the fundamental symmetries of the spin quantum Hall effect. We demonstrate that, in the general case, the triplet sector of the theory remains coupled to the singlet sector. In the large-N limit, we systematically derive the effective long-distance, low-energy field theory, identified as a nonlinear sigma model. Our analysis reveals that while triplet modes are typically massive and do not influence the large-N nonlinear sigma model, specific conditions exist where these modes become soft, thereby increasing the diffusive length scale, which is the ultraviolet cutoff length of the effective theory. Furthermore, by calculating the bare longitudinal and spin Hall conductances, we show that the standard saddle-point approximation fails in regimes with significant tunneling asymmetry between even and odd links. Finally, we establish that the introduction of a Zeeman field not only breaks the SU(2) symmetry of the nonlinear sigma model action but also generates a term that explicitly violates inversion symmetry.

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I. INTRODUCTION

Anderson localization [1] in a system of noninteracting fermions in a random potential has been attracting researchers for more than 65 years (see Refs. [2,3] for reviews). As is known [4–9], there exist ten different (Altland-Zirnbauer) symmetry classes of disordered noninteracting Hamiltonians. In each spatial dimension, there are five of these ten symmetry classes with nontrivial topology that affects physics [10–12]. A well-known example of such a situation is the integer quantum Hall effect (iqHe) [13,14]. Localization effects in each of the ten symmetry classes can be studied using the effective low-energy field theory—the nonlinear sigma model (NLσM) (see Ref. [15] for a review). The corresponding NLσM may include the topological term (either the theta term or the Wess-Zumino-Witten-Novikov term). These terms only arise in symmetry classes with nontrivial topology and are essential for understanding deviations from the conventional picture of Anderson localization.

In two spatial dimensions (2D), a topological Anderson transition is typically a strong coupling phenomenon that may not be accessible within the NLσM. For example, in the case of the iqHe, a critical theory for the plateau-to-plateau transitions is not known and remains a subject of debate in the literature [16–24]. On the other hand, assuming the local conformal invariance and Abelian fusion rules for pure scaling field theory operators, their scaling exponents at the iqHe criticality have been proven to have a parabolic form with only one free parameter [21,25,26]. However, numerical simulations do not support this prediction [15,25,27]. Recent studies have found that the variation of the localization length exponent in the iqHe transition depends on the geometry of the random potential [28–33]. Despite these controversies regarding critical theory, the nonperturbative analysis of NLσM with the topological theta term [34,35] allows us to understand the structure of the phase diagram [36,37], explain the integer-valued quantization of the Hall conductance [38–42], and develop a coherent picture of iqHe consistent with experimental findings [43–52].

In some ways, the opposite situation exists with the spin quantum Hall effect (sqHe), which is a close relative of the iqHe. The sqHe occurs in the unitary superconducting class *C* [53–55]. In the absence of Zeeman splitting, systems in this class possess a global SU(2) spin-rotation symmetry. This symmetry allows one to split the theory into spin-singlet and spin-triplet sectors. The response of the spin current to the gradient of the external magnetic field is characterized by the integer quantized spin Hall conductivity [56]. In contrast to the iqHe, the strong coupling regime of the sqHe is partially accessible via a mapping to the classical percolation problem. In particular, the critical value of the spin conductance, the critical exponent of the localization length, and an infinite set of anomalous dimensions of local operators were computed analytically and the obtained results are in agreement with numerical data [57–68]. These analytical findings indicate a lack of local conformal symmetry at the sqHe transition, which complicates the search for a critical theory describing the sqHe transition. The lack of a critical theory for the sqHe transition (similar to the iqHe transition) makes nonperturbative weak-coupling analysis of the corresponding NLσM relevant. In contrast to the iqHe, such nonperturbative analysis of the replica NLσM has only recently been started [69,70].

From the experimental point of view, the realization of the sqHe remains an open problem. Originally, the observation of the sqHe was suggested in the $d_{x^2-y^2} + id_{xy}$ topological superconducting state [53–55]. Although direct experimental evidence for a superconducting film with this order parameter symmetry is lacking, recent experiments on twisted $Bi_2Sr_2CaCu_2O_{8+x}$ bilayers near a 45° twist angle have reported indications of spontaneous time-reversal symmetry breaking [71,72]. These observations are consistent with theoretical predictions for an emergent $d_{x^2-y^2} + id_{xy}$ topological superconducting state in twisted superconducting flakes [73,74]. This provides a possible physical setting in which class C quasiparticle transport and, consequently, sqHe-type network descriptions may be relevant. Another proposal to realize the sqHe is based on the observation that edge states at the interface between a two-dimensional electron gas in a perpendicular magnetic field and an s-wave superconductor can be interpreted as the sqHe edge states [75]. This suggests that quantum network models of the type studied below may also be relevant for a two-dimensional electron gas coupled to superconducting islands in a perpendicular magnetic field (see, e.g., Ref. [76]).

Our current understanding of the critical exponents that characterize the iqHe transitions is based on the Chalker-Coddington network model [77] (see Refs. [15,78] for a review). It has been shown [79] that this model can be mapped to the supersymmetric NLσM. Recently, the SU(2) extension of the Chalker-Coddington network model which belongs to the symmetry class C has been mapped to a supersymmetric NLσM for the class C [80]. The Chalker-Coddington network model has nonrandom amplitudes for the tunneling between the links. For the symmetry class A, the quantum network model with a random tunneling between the links has been proposed and mapped to NLσM in the continuum limit [81]. However, there exists no similar mapping to NLσM for the quantum network model with the symmetry class C.

In this work, we generalize the quantum network model, which was proposed for the sqHe in Ref. [55], by extending each link to have N channels and by considering the most general random tunneling consistent with symmetry class C. The microscopic starting point of our analysis is the replicated imaginary-time action, cf. Eq. (1), or, equivalently, its Bogoliubov–de Gennes representation, cf. Eq. (9). We demonstrate that the continuum limit of this model at large N is mapped to the NLσM action, cf. Eq. (43), in a weak coupling metallic regime. We find that, in general, the triplet sector of the theory is not decoupled from the singlet sector. Typically, the modes in the triplet sector are massive and, thus, do not affect the NLσM action at large N. That is, the picture of Anderson localization/delocalization transitions in the infrared limit remains unchanged. However, there exist special cases where the triplet modes become soft and, as a result, increase the diffusive length scale for the NLσM. By deriving the expressions for the bare spin longitudinal and Hall conductance, we find that in cases with relatively strong asymmetry in tunneling between even and odd links, the standard saddle point approximation breaks down. We also find that the presence of a Zeeman field not only breaks the SU(2) symmetry of the NLσM action but also leads to a term that breaks inversion symmetry.

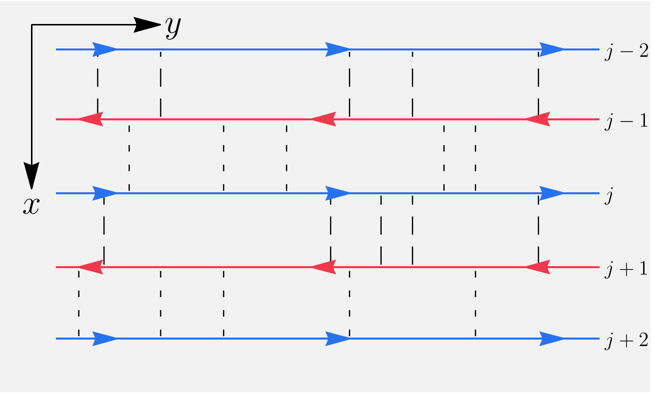

FIG. 1. Sketch of the quantum network model. Red and blue lines indicates one-dimensional chiral links, dashed lines corresponds to random tunneling between links.

The outline of the paper is as follows. In Sec. II, we formulate the quantum network model with N channels in each link and introduce the microscopic action. The averaging over random tunneling is discussed in Sec. III. The derivation of NLσM is given in Sec. IV. In Sec. V, we analyze the effect of the triplet modes on the NLσM action. We end the paper with discussions and conclusions in Sec. VI. Some technical details are presented in Appendixes.

II. MODEL

A. Formulation

We consider fermions on a two-dimensional quantum network that belongs to symmetry class C (see Fig. 1). At the microscopic level, the system is composed of one-dimensional channels (links) labeled by an integer index $j = 1, \dots, M$. The fermions' velocity alternates in sign on even and odd links, reflecting the chiral motion induced by a perpendicular magnetic field. We encode this by defining the velocity on the jth link as $v_j = (-1)^j v$. Each link hosts N species of spin-1/2 fermions. To distinguish them, we introduce a flavor index $\mathsf{f} = 1, \dots, N$.

Fermions are subject to random scattering. To address this, we employ the replica method and introduce N_r copies of the system with the same realization of the random disorder. The fermions in each copy are designated by a replica index $\alpha = 1, \dots, N_r$. The action in imaginary time ($\beta = 1/T$, where T is the temperature) reads

$$\mathcal{S} = \int_0^\beta d\tau \int dy \Bigg[\sum_{\sigma=\uparrow,\downarrow} \bar{\psi}_\sigma (i\hat{V}\partial_y - \partial_\tau - \hat{\Sigma}_0)\psi_\sigma - \bar{\psi}_\uparrow \hat{\Sigma}_+ \bar{\psi}_\downarrow^T - \psi_\downarrow^T \hat{\Sigma}_- \psi_\uparrow - \bar{\psi}_\uparrow \hat{\Sigma}_3^{(Z)} \psi_\uparrow + \bar{\psi}_\downarrow \hat{\Sigma}_3^{(Z)} \psi_\downarrow \Bigg]. \tag{1}$$

Here $\bar{\psi}_\sigma(y) = \{\bar{\psi}_{\sigma,\alpha,j,\mathsf{f}}(y)\}$ and $\psi_\sigma(y) = \{\psi_{\sigma,\alpha,j,\mathsf{f}}(y)\}^T$ are Grassmann fields corresponding to the creation and annihilation fermionic field operators. The velocity matrix is given by

$\hat{V} = 1_{N_r} \otimes \hat{v} \otimes 1_N$, where $\hat{v}_{jj'}=v_j\delta_{jj'}$. Four matrices $\hat{\Sigma}_{0,\pm}(y)$ and $\hat{\Sigma}_3^{(Z)}(y)$ depend on the y coordinate and act in the replica, link, and flavor spaces and are defined as follows:

$$\hat{\Sigma}_0 = 1_{N_r} \otimes (\hat{\eta}^{(3)} + i\hat{\eta}^{(0)}), \quad \hat{\Sigma}_\pm = 1_{N_r} \otimes (\hat{\eta}^{(1)} \mp i\hat{\eta}^{(2)}),$$
$$\hat{\Sigma}_3^{(Z)} = 1_{N_r} \otimes \hat{b} \otimes 1_N. \tag{2}$$

Here the real symmetric matrices

$$\eta^{(c)}(y) = \eta^{(c)T}(y) = \eta^{(c)\dagger}(y), \quad c = 1, 2, 3 \tag{3}$$

and the real skew-symmetric matrix

$$\eta^{(0)}(y) = -\eta^{(0)T}(y) = -\eta^{(0)\dagger}(y) \tag{4}$$

operate in the combined link and flavor spaces. They describe random scattering of fermions. The matrix $\hat{b}_{jj'}(y) = \mu_B B_j(y)\delta_{jj'}$ describes the Zeeman term with a spatially dependent magnetic field $B_j(y)$.

B. Symmetries

The Hamiltonian (1) contains terms with two annihilation or creation fermionic operators. Thus, the $U(1)$ charge symmetry is broken in the Hamiltonian (1). As is well known in the context of superconductivity, it is convenient to double the variables. Also, for convenience, we perform a transformation of fermionic fields from the imaginary time to the fermionic Matsubara frequencies, $\varepsilon_n = \pi T(2n + 1)$. It implies that fermionic fields $\bar{\psi}$ and ψ acquire additional index ε_n. Thus, we introduce

$$\Xi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_\uparrow \\ L_0\bar{\psi}_\downarrow^T \\ -i\psi_\downarrow \\ iL_0\bar{\psi}_\uparrow^T \end{pmatrix}, \quad \overline{\Xi} = \frac{1}{\sqrt{2}}(\bar{\psi}_\uparrow, \psi_\downarrow^T L_0, i\bar{\psi}_\downarrow, -i\psi_\uparrow^T L_0), \tag{5}$$

where the matrix

$$L_0 = \ell_0 \otimes 1_{N_r} \otimes 1_M \otimes 1_N, \quad (\ell_0)_{\varepsilon_n,\varepsilon_m} = \delta_{\varepsilon_n,-\varepsilon_m}. \tag{6}$$

We note that the fields $\overline{\Xi}$ and Ξ are not independent. They are related by the linear transformation

$$\overline{\Xi} = \Xi^T\hat{C} \tag{7}$$

with the charge conjugation matrix

$$\hat{C} = L_0 \otimes (is_2) \otimes \sigma_2 = \begin{pmatrix} 0 & L_0 \otimes \sigma_2 \\ -L_0 \otimes \sigma_2 & 0 \end{pmatrix},$$
$$\hat{C}^2 = -1, \quad \hat{C}^T = \hat{C}. \tag{8}$$

Here we introduce two sets of standard Pauli matrices σ_c and s_c with $c = 0, 1, 2, 3$. Following standard convention, we consider that the matrices σ_c act on the spin space while the matrices s_c act on the particle-hole (Nambu) space.

Next we rewrite the action (1) as follows:

$$\mathcal{S} = \int dy\, \bar{\Xi}\, \mathcal{H}\, \Xi,$$
$$\mathcal{H} = i\check{V}\partial_y + i\check{\varepsilon} - i\check{\Upsilon}_0 - \sum_{c=1}^{3}\check{\Upsilon}_c - \check{\Upsilon}^{(Z)}, \tag{9}$$

where $\check{V} = 1_{2N_m} \otimes \hat{V} \otimes s_0 \otimes \sigma_0$. We also introduce the following matrices of size $2N_m \times N_r \times M \times N \times 4$ ($2N_m$ is the number of Matsubara frequencies involved) [82]:

$$\check{\Upsilon}_c = 1_{2N_m} \otimes 1_{N_r} \otimes \hat{\eta}^{(c)} \otimes s_0 \otimes \sigma_c,$$
$$\check{\Upsilon}^{(Z)} = 1_{2N_m} \otimes 1_{N_r} \otimes \hat{b} \otimes s_3 \otimes \sigma_0, \tag{10}$$
$$\check{\varepsilon} = \hat{\varepsilon} \otimes 1_{N_r} \otimes 1_M \otimes 1_N \otimes s_0 \otimes \sigma_0.$$

Here $\hat{\varepsilon}_{\varepsilon_n,\varepsilon_m} = \varepsilon_n\delta_{\varepsilon_n,\varepsilon_m}$ is the Matsubara frequency matrix. We note that although $\check{\Upsilon}^{(Z)}$ is proportional to σ_0, the corresponding Zeeman term in the action (9) nevertheless correctly preserves the structure of the spin-dependent term in Eq. (1). The transformation in Eq. (5) involves a simultaneous rotation in both spin and Nambu spaces. Under this change of basis, the original spin matrix σ_3 is mapped onto the Nambu matrix s_3.

We emphasize that the relation (7) is nothing but a realization of the antiunitary Bogoliubov–de Gennes (charge-conjugation) symmetry [2]. Indeed, the Hamiltonian in Eq. (9) satisfies the symmetry relation $\mathcal{H}^T = \hat{C}\mathcal{H}\hat{C}$.

In the absence of the Zeeman splitting, $\check{\Upsilon}^{(Z)} = 0$, the action (9) involves only the matrix s_0. It indicates that the action is invariant under the global SU(2) rotation in the Nambu space: $\Xi \to \hat{U}\,\Xi$ and $\bar{\Xi} \to \bar{\Xi}\hat{U}^\dagger$, where $\hat{U} = 1_{2N_m} \otimes 1_{N_r} \otimes 1_M \otimes 1_N \otimes \hat{u} \otimes \sigma_0$ with $\hat{u} \in$ SU(2) [83]. The Zeeman term explicitly breaks the SU(2) spin rotational symmetry down to $U(1)$, i.e., the action (9) is invariant under the global rotation with $\hat{u} = \exp(i\theta s_3)$.

C. Random tunneling

To formulate a specific microscopic model, we assume that the random tunneling of fermions occurs only between neighboring links. Additionally, during the tunneling process, the flavor index of the particles is randomly changed. Therefore, we introduce the following parametrization for the matrices $\hat{\eta}^{(c)}$ and $\hat{\eta}^{(0)}$:

$$\hat{\eta}^{(c)}_{j\mathsf{f};j'\mathsf{f}'} = t_j^{(c)}\nu^{(c)}_{\mathsf{f}\mathsf{f}'}\delta_{j',j+1} + t_{j'}^{(c)}\nu^{(c)}_{\mathsf{f}'\mathsf{f}}\delta_{j,j'+1} + \mu_j^{(c)}\phi^{(c)}_{\mathsf{f}\mathsf{f}'}\delta_{jj'},$$
$$\hat{\eta}^{(0)}_{j\mathsf{f};j'\mathsf{f}'} = t_j^{(0)}\nu^{(0)}_{\mathsf{f}\mathsf{f}'}\delta_{j',j+1} - t_{j'}^{(0)}\nu^{(0)}_{\mathsf{f}'\mathsf{f}}\delta_{j,j'+1}. \tag{11}$$

Here $t_j^{(0)}(y)$, $t_j^{(c)}(y)$, and $\mu_j^{(c)}(y)$ are independent uncorrelated real random Gaussian variables with zero mean and the following variances:

$$\langle t_j^{(0)}(y)t_{j'}^{(0)}(y')\rangle = \delta_{jj'}\Delta_j\delta(y - y'),$$
$$\langle t_j^{(c)}(y)t_{j'}^{(c')}(y')\rangle = \delta_{cc'}\delta_{jj'}\gamma_j\delta(y - y'), \tag{12}$$
$$\langle \mu_j^{(c)}(y)\mu_{j'}^{(c')}(y')\rangle = \delta_{cc'}\delta_{jj'}m_j\delta(y - y').$$

For reasons to be explained shortly, in what follows we assume that $\Delta_j, \gamma_j, m_j > 0$. The parametrization (11) is chosen to satisfy the class C constraints (3) and (4). The matrices $\nu^{(a)}$, $a = 0, 1, 2, 3$, and $\phi^{(c)}$, $c = 1, 2, 3$ are the independent blocks entering the random tunneling matrices. The matrices $\nu^{(a)}$ are arbitrary real matrices in flavor space and describe tunneling between neighboring links; the reverse-link blocks are then fixed by the constraints (3) and (4). The matrices $\phi^{(c)}$ describe the diagonal blocks of $\eta^{(c)}$ and are therefore real symmetric, $\phi^{(c)} = \phi^{(c)T}$. In this work, we use the factorized large-N ensemble, in which the link dependence is carried by $t_j^{(a)}$ and $\mu_j^{(c)}$, whereas the flavor structure is carried by $\nu^{(a)}$ and $\phi^{(c)}$. We take the random variables in (11) to be independent Gaussian variables with zero mean and finite variances:

$$\begin{aligned}
&\big\langle \phi^{(c)}_{\mathsf{ff}'}\phi^{(c')}_{\mathsf{f}_1\mathsf{f}_1'}\big\rangle = \tfrac{1}{2}\delta_{cc'}\big(\delta_{\mathsf{ff}_1}\delta_{\mathsf{f}'\mathsf{f}_1'}+\delta_{\mathsf{f}'\mathsf{f}_1}\delta_{\mathsf{ff}_1'}\big),\\
&\big\langle \nu^{(c)}_{\mathsf{ff}'}\nu^{(c')}_{\mathsf{f}_1\mathsf{f}_1'}\big\rangle = \delta_{cc'}\delta_{\mathsf{ff}_1}\delta_{\mathsf{f}'\mathsf{f}_1'},\\
&\big\langle \nu^{(0)}_{\mathsf{ff}'}\nu^{(0)}_{\mathsf{f}_1\mathsf{f}_1'}\big\rangle = \delta_{\mathsf{ff}_1}\delta_{\mathsf{f}'\mathsf{f}_1'}.
\end{aligned} \tag{13}$$

We note that our model for $N = 1$ is equivalent to the most general form of a network for the class C introduced in Ref. [55]. In particular, we generalize it by introducing an arbitrary number of fermion flavors N.

The factorized form of the ensemble is chosen so, after disorder averaging, the flavor index can be traced out and N appears only as an overall prefactor of the tr ln term. This renders the saddle-point approximation controlled by $1/N$. Allowing more general link-dependent flavor-space matrices $\nu_j^{(a)}$ and $\phi_j^{(a)}$ would not alter the class C constraints (3) and (4) but would obscure this controlled large-N construction.

To elucidate the meaning of the variances Δ_j and γ_j, it is instructive to consider the 2×2 matrix in spin space:

$$\mathcal{T}_j = it_j^{(0)}\sigma_0 + \sum_{c=1}^{3} t_j^{(c)}\sigma_c. \tag{14}$$

This matrix has the following eigenvalues:

$$t_j^{(\pm)} = it_j^{(0)} \pm \sqrt{t_j^{(1)2}+t_j^{(2)2}+t_j^{(3)2}}. \tag{15}$$

Importantly, $|t_j^{(+)}| = |t_j^{(-)}|$, which reflects the presence of the spin symmetry. We note that

$$\begin{aligned}
&\langle |t_j^{(\pm)}|^2\rangle = \lambda_j = \Delta_j + 3\gamma_j,\\
&\langle \sin^2(\arg t_j^{(\pm)})\rangle = \frac{\Delta_j}{(\sqrt{\Delta_j}+\sqrt{\gamma_j})^2}.
\end{aligned} \tag{16}$$

At $\Delta_j = \gamma_j$ we find $\langle \sin^2(\arg t_\pm)\rangle = 1/4$ that corresponds to the uniform distribution of the unit vector in the 4-dimensional space.

III. DISORDER AVERAGING

The derivation of the NLσM requires performing an average over disorder realizations. Using the replica trick, we can average the partition function over disorder. The resulting action then contains terms of fourth order in fermionic operators.

As the first step toward the derivation of the NLσM, we split the action (9) into two parts:

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{\text{dis}}, \quad \mathcal{S}_0 = \int dy\, \bar{\Xi}(i\check{V}\partial_y + i\check{\varepsilon} - \check{\Upsilon}^{(Z)})\Xi \tag{17}$$

and

$$\mathcal{S}_{\text{dis}} = -\int dy \sum_j [\overline{\Xi}_j\Phi_j\Xi_j + 2\overline{\Xi}_j\mathcal{N}_j\Xi_{j+1}]. \tag{18}$$

Here we introduce two random matrices:

$$\begin{aligned}
\Phi_j &= \sum_{c=1}^{3} 1_{2N_m}\otimes 1_{N_r}\otimes\phi^{(c)}\otimes s_0\otimes\big(\mu_j^{(c)}\sigma_c\big),\\
\mathcal{N}_j &= it_j^{(0)}1_{2N_m}\otimes 1_{N_r}\otimes\nu^{(0)}\otimes s_0\otimes\sigma_0\\
&\quad+\sum_{c=1}^{3} t_j^{(c)}1_{2N_m}\otimes 1_{N_r}\otimes\nu^{(c)}\otimes s_0\otimes\sigma_c.
\end{aligned} \tag{19}$$

Next we perform averaging of the partition function over the random Gaussian matrices Φ_j and $\mathcal{N}_j$. The averaging results in the substitution of $\mathcal{S}_{\text{dis}}$ by [84]

$$\mathcal{S}_{\text{av}} = \frac{1}{2}\langle \mathcal{S}_{\text{dis}}^2\rangle = -\frac{1}{2}\int dy \sum_{j,j'}\sum_{c=0}^{3}\beta_{jj'}^{(c)}\text{tr}D_j^{(c)}D_{j'}^{(c)}, \tag{20}$$

where

$$\begin{aligned}
\beta_{jj'}^{(0)} &= \tfrac{3}{2}m_j\delta_{jj'} + \lambda_j\delta_{j',j+1} + \lambda_{j'}\delta_{j,j'+1},\\
\beta_{jj'}^{(c)} &= \tfrac{1}{2}m_j\delta_{jj'} + (\gamma_j - \Delta_j)\delta_{j',j+1} + (\gamma_{j'} - \Delta_{j'})\delta_{j,j'+1}.
\end{aligned} \tag{21}$$

In addition, we introduce the matrices $D_j^{(c)}$ operating on the Matsubara, replica, and Nambu spaces. They are formally defined as ($c = 0, 1, 2, 3$)

$$\big[D_j^{(c)}\big]_{\varepsilon_n,\alpha,s;\varepsilon_m,\beta,s'} = \text{tr}_{N\times 2}\,\sigma_c\,\Xi_{\varepsilon_n,\alpha,j,s}\overline{\Xi}_{\varepsilon_m,\beta,j,s'}, \tag{22}$$

where the trace is taken over the spin and flavor spaces. We note that the matrices $D_j^{(c)}$ satisfy the following symmetry relations:

$$\begin{aligned}
&D_j^{(0)} = -CD_j^{(0)T}C, \quad D_j^{(c)} = CD_j^{(c)T}C, \quad c = 1,2,3\\
&C = \ell_0\otimes 1_{N_r}\otimes s_2, \quad C^2 = 1, \quad C^T = -C.
\end{aligned} \tag{23}$$

The action after averaging has an additional symmetry related to the global SU(2) rotation in the spin space. We mention that this global symmetry can be extended to a local one, $\Xi_j \to u_j\Xi_j$ and $\bar{\Xi}_j \to \bar{\Xi}_j u_j^\dagger$, in the special case $\Delta_j = \gamma_j$. It is this case that was studied in Refs. [55,85]. Also, we note that for $\Delta_j = \gamma_j$ and $N = 1$, the triplet matrices $D_j^{(c)}$, where $c = 1, 2, 3$, can be excluded from the action (20). In the following, we consider the general case where this local rotational symmetry is absent.

Equation (20) will be used below as the effective disorder-averaged action at the scale where the diffusive description sets in. However, before the diffusive regime is reached, the four-fermion term (20) undergoes renormalization (the so-called ballistic renormalization). As shown in Appendix A, integrating out fermionic modes at ballistic scales renormalizes the matrix $\beta^{(c)}$ and, in general, generates matrix elements of $\beta^{(c)}$ corresponding to tunneling between non-nearest-neighbor links. Therefore, in the following sections, except in the simplified limit considered explicitly in Sec. IV D, the matrices $\beta^{(c)}$ are not assumed to retain the tridiagonal form of Eq. (21).

IV. DERIVATION OF THE NLσM

In this section, we perform the derivation of the NLσM action. We do this in several steps. First, we perform the Hubbard-Stratonovich transformation and integrate over fermions. Next, we analyze the saddle point of the theory and the structure of the massive and massless modes. Finally, integrating out the massive modes allows us to obtain the low-energy effective field theory.

A. Hubbard-Stratonovich transformation

We start by decoupling the quadratic terms in Eq. (20) by means of the Hubbard-Stratonovich matrix fields $\tilde{Q}_j^{(c)}$, where $c = 0, 1, 2, 3$, conjugated to $D_j^{(c)}$. Naturally, the matrix field $\tilde{Q}_j^{(c)}$ acts on the same spaces as $D_j^{(c)}$ and obeys the same symmetry, cf. Eq. (23):

$$\tilde{Q}_j^{(0)} = -C\tilde{Q}_j^{(0)T}C, \quad \tilde{Q}_j^{(c)} = C\tilde{Q}_j^{(c)T}C, \quad c = 1, 2, 3. \tag{24}$$

We note the difference in symmetry relations for the singlet, $\tilde{Q}_j^{(0)}$, and triplet, $\tilde{Q}_j^{(1,2,3)}$, matrix modes. After integrating out the fermionic fields Ξ and $\bar{\Xi}$, the action reduces to

$$\begin{aligned}\mathcal{S} = {}& \frac{N}{2}\int dy \sum_j \mathrm{tr} \\ & \times \ln\left[(i\tilde{v}_j\partial_y + i\tilde{\varepsilon} - \tilde{b}_j)\sigma_0 + i\sum_{j'}\sum_{c=0}^{3}\beta_{jj'}^{(c)}\tilde{Q}_{j'}^{(c)}\sigma_c\right] \\ & - \frac{1}{2}\int dy \sum_{jj'}\sum_{c=0}^{3}\beta_{jj'}^{(c)}\mathrm{tr}\tilde{Q}_j^{(c)}\tilde{Q}_{j'}^{(c)}.\end{aligned} \tag{25}$$

Here we introduce the following matrices acting in the Matsubara frequency, replica, and Nambu spaces:

$$\begin{aligned}\tilde{v}_j &= v_j\, 1_{2N_m} \otimes 1_{N_r} \otimes s_0, \\ \tilde{\varepsilon} &= \hat{\varepsilon} \otimes 1_{N_r} \otimes s_0, \\ \tilde{b}_j &= \mu_B B_j\, 1_{2N_m} \otimes 1_{N_r} \otimes s_3.\end{aligned} \tag{26}$$

As we discussed above, the action (20) has a global SU(2) rotational symmetry in spin space, which can be extended to a local symmetry in the special case with $\Delta_j = \gamma_j$. In this case, $\beta_{jj'}^{(c)} = (m_j/2)\delta_{jj'}$, for $c = 1, 2, 3$, so using Fierz identities in the spin space (see [84]) the action (25) can be written solely in terms of the invariant combinations of the matrix $\hat{Q}_j = \sum_{c=0}^{3}\tilde{Q}_j^{(c)}\sigma_c$, e.g., $\mathrm{tr}_\sigma\hat{Q}_j$, $\mathrm{tr}_\sigma\hat{Q}_j^2$, and $(\mathrm{tr}_\sigma\hat{Q}_j)^2$, where tr_σ is the trace over spin space only. Therefore, the action (25) becomes invariant under local SU(2) rotations in the spin space for the case $\Delta_j = \gamma_j$.

B. The saddle point

The first line of the action (25) is proportional to the number of flavors N. Therefore, at $N \gg 1$ we can treat the action (25) in the saddle-point approximation. By varying the action (25) (with $\tilde{b}_j = 0$ and $\tilde{\varepsilon} = 0$) over the matrix field $\tilde{Q}_j^{(c)}$, we obtain the saddle-point equations

$$\sum_{j'}\beta_{jj'}^{(c)}\tilde{Q}_{j'}^{(c)}(y) = i\frac{N}{2}\sum_{j'}\beta_{jj'}^{(c)}\mathrm{tr}_\sigma\sigma_c G_{j'}(y, y), \tag{27}$$

where G_j stands for the Green's function corresponding to the inverse operator under the trace-logarithm:

$$G_j^{-1} = i\tilde{v}_j\sigma_0\partial_y + i\sum_{j'}\sum_{c=0}^{3}\beta_{jj'}^{(c)}\tilde{Q}_{j'}^{(c)}\sigma_c. \tag{28}$$

As one can check, the saddle-point equation has the standard solution

$$\underline{\tilde{Q}}_j^{(c)} = \frac{N}{2v}\Lambda\delta_{c0}, \quad \Lambda = \hat{\eta} \otimes 1_{N_r} \otimes s_0, \tag{29}$$

where $\hat{\eta}_{\varepsilon_n,\varepsilon_m} = \mathrm{sgn}\varepsilon_n\delta_{\varepsilon_n,\varepsilon_m}$. We note that although no explicit retarded-advanced index is introduced, positive and negative Matsubara frequencies play the role of the two sectors which, after the standard analytic continuation $i\varepsilon_n \to E \pm i0$, become the retarded and advanced sectors. Accordingly, the soft diffusive modes mix Matsubara frequencies of opposite signs, anticommuting with Λ, while fluctuations within the same sector, corresponding to RR and AA modes, are massive. This is the standard way in which the retarded-advanced structure appears in the Matsubara replica NLσM. For an explicit discussion of the analytic continuation, see Ref. [86].

The corresponding saddle-point Green's function is diagonal in the spin space:

$$\underline{G}_j^{-1} = \left(i\tilde{v}_j\partial_y + \frac{i}{2\tau_j}\Lambda\right)\sigma_0, \quad \frac{1}{\tau_j} = \frac{N}{v}\sum_{j'}\beta_{jj'}^{(0)}. \tag{30}$$

We note that there are more solutions to Eq. (27). We can freely rotate the solution (29) in the Matsubara frequency, replica and Nambu spaces with a matrix t independent of the link index j and the spatial coordinate y. Then the saddle-point manifold is given by

$$\underline{\tilde{Q}}_j^{(c)} = \frac{N}{2v}t^{-1}\Lambda t\,\delta_{c0}, \quad Ct^{-1} = t^T C. \tag{31}$$

Since the matrix $\tilde{Q}_j^{(c)}$ is conjugated (in the sense of the Hubbard-Stratonovich transformation) to the matrix $D_j^{(c)}$, see Eq. (22), it is natural to impose the Hermiticity condition on $\underline{\tilde{Q}}_j^{(0)}$ to ensure convergence, i.e., to assume that $\underline{\tilde{Q}}_j^{(0)} = \underline{\tilde{Q}}_j^{(0)\dagger}$. Then together with the symmetry relation (24), the saddle-point manifold corresponds to the NLσM target space of class C, $\underline{\tilde{Q}}_j^{(0)} \in G/K$ with $G = \mathrm{Sp}(4N_rN_m)$ and $K = \mathrm{U}(2N_rN_m)$.

C. Massive modes

We start with the triplet spin sector, i.e., the matrix fields $\tilde{Q}_j^{(c)}$ (with $c = 1, 2, 3$). By expanding the tr ln in Eq. (25) to the second order in the matrix fields $\tilde{Q}_j^{(c)}$ and neglecting spatial derivatives, we obtain the following Gaussian action:

$$\begin{aligned}\mathcal{S}_{\mathrm{t}}^{(2)} \simeq {}& -\frac{1}{8}\int dy \sum_{c=1}^{3}\sum_{jj'}\beta_{jj'}^{(c)}\mathrm{tr}\{\tilde{Q}_j^{(c)}, \Lambda\}\{\tilde{Q}_{j'}^{(c)}, \Lambda\}, \\ & + \frac{1}{8}\int dy \sum_{c=1}^{3}\sum_{jj'}\tilde{\beta}_{jj'}^{(c)}\mathrm{tr}[\tilde{Q}_j^{(c)}, \Lambda][\tilde{Q}_{j'}^{(c)}, \Lambda],\end{aligned} \tag{32}$$

where

$$\tilde{\beta}^{(c)}_{jj'} = \beta^{(c)}_{jj'} - \frac{N}{v}\sum_{j''}\beta^{(c)}_{jj''}\tau_{j''}\beta^{(c)}_{j''j'}. \tag{33}$$

We note that the above expression for $\tilde{\beta}^{(c)}$ is similar to the ballistic renormalization, cf. Eq. (A10), computed at the momentum scale corresponding to the inverse mean free path $q_\Lambda \sim 1/(v\tau_j)$. Provided the matrices $\beta^{(c)}$ and $\tilde{\beta}^{(c)}$ have no zero eigenvalues, the matrix fields $\tilde{Q}^{(c)}_j$ (with $c = 1, 2, 3$) are massive and, thus, can be ignored in the course of the derivation of the NLσM action. We will discuss their effect on the NLσM action in Sec. V below.

For the singlet sector, a naive expansion in the deviation of the $\tilde{Q}^{(0)}_j$ from its saddle-point value yields exactly the same expressions as in Eq. (32), where $\tilde{Q}^{(c)}_j$ is substituted by $\tilde{Q}^{(0)}_j - \underline{\tilde{Q}}^{(0)}_j$. The first line of Eq. (32) then corresponds to the modes that commute with Λ. They are massive if the matrix $\beta^{(0)}$ does not have any zero eigenvalues (see Appendix B) [87]. The second line of Eq. (32) describes modes that anticommute with Λ. They can be massless, since $\det\tilde{\beta}^{(0)} = 0$ by construction. These modes are the ones we are interested in below.

D. The case of strong diagonal disorder $m \gg \gamma_\pm, \Delta_\pm$

Although we are interested in deriving the NLσM action for the most general form of the matrices $\beta^{(c)}$, we start with the simplest case. Namely, we start from the tridiagonal matrices $\beta^{(c)}$, cf. Eq. (21), with the assumption that there is an even/odd-link asymmetry such that $\gamma_{2j} = \gamma_+$, $\gamma_{2j+1} = \gamma_-$, $\Delta_{2j} = \Delta_+$, and $\Delta_{2j+1} = \Delta_-$. Correspondingly, $\lambda_{2j} = \lambda_+$ and $\lambda_{2j+1} = \lambda_-$. In this case, from Eq. (30) we find that the scattering time is independent of the link index: $1/\tau_j = 1/\tau = N(3m + 2\lambda_+ + 2\lambda_-)/(2v)$. We also assume that $m_j = m \gg \gamma_\pm, \Delta_\pm$. In this limit, the diagonal part of $\beta^{(c)}$ is dominant. The ballistic renormalization discussed in Appendix A can still generate non-nearest-neighbor matrix elements, but these elements are of higher order in the small off-diagonal tunneling amplitudes. Therefore, to leading order in $\gamma_\pm/m$ and $\Delta_\pm/m$, we may use the tridiagonal form of $\beta^{(c)}$ given in Eq. (21).

We parametrize the singlet matrix field as

$$\tilde{Q}^{(0)}_j(y) = \frac{N}{2v}t^{-1}_j(y)\Lambda t_j(y), \tag{34}$$

where $t_j(y)$ has the same symmetries as the matrix t in Eq. (31). Then the effective action (25) becomes

$$\mathcal{S} = N\int dy\sum_j \operatorname{tr}\ln\left[i\tilde{v}_j\partial_y + i\tilde{\varepsilon} - \tilde{b}_j + \frac{iN}{2v}\sum_{j'}\beta^{(0)}_{jj'}t^{-1}_{j'}\Lambda t_{j'}\right] - \frac{N^2}{8v^2}\int dy\sum_{jj'}\beta^{(0)}_{jj'}\operatorname{tr}t^{-1}_j\Lambda t_j t^{-1}_{j'}\Lambda t_{j'}. \tag{35}$$

Here we neglected all massive modes as discussed above. As usual, we assume that the matrix t_j varies slowly along the link (y coordinate) and changes slowly from link to link (below we denote this direction as x). Then, we employ the following expansion up to second order in spatial gradients:

$$t^{-1}_{j\pm1}\Lambda t_{j\pm1} \simeq t^{-1}_j\left\{\Lambda \pm a\left[L^{(x)}_j, \Lambda\right] + \frac{a^2}{2}\left[\partial_x L^{(x)}_j, \Lambda\right] + \frac{a^2}{2}\left[L^{(x)}_j, \left[L^{(x)}_j, \Lambda\right]\right]\right\}t_j. \tag{36}$$

Here $L^{(x,y)}_j = t_j\partial_{x,y}t^{-1}_j$ and a is the spatial period of the network in real space. By substituting expression (36) into Eq. (35) and expanding tr ln up to the second order in the spatial gradients, we obtain

$$\begin{aligned}\mathcal{S} \simeq -\sum_j\int dy\Bigg\{&\sum_{r=x,y}\pi^{(0)}_r\operatorname{tr}\left[L^{(r)}_j, \Lambda\right]^2 - \frac{N}{2}(-1)^j\operatorname{tr}\Lambda L^{(y)}_j\\ &+\frac{N^2 a\tau(\lambda_+ - \lambda_-)}{2v}\operatorname{tr}\Lambda\left[L^{(x)}_j, L^{(y)}_j\right] - \frac{N}{2v}\operatorname{tr}(\tilde{\varepsilon} + i\tilde{b}_j)t^{-1}_j\Lambda t_j\\ &-\frac{N\tau}{8v}\operatorname{tr}\left[t_j\tilde{b}_j t^{-1}_j, \Lambda\right]^2 + \frac{iN\tau}{4v}v_j\operatorname{tr}\left[t_j\tilde{b}_j t^{-1}_j, \Lambda\right]\left[L^y_j, \Lambda\right]\Bigg\},\end{aligned} \tag{37}$$

where

$$\pi^{(0)}_x = \frac{N^2a^2(\lambda_+ + \lambda_-)}{16v^2}\left[1 - \frac{2N\tau}{v}\frac{(\lambda_+ - \lambda_-)^2}{(\lambda_+ + \lambda_-)}\right],\quad \pi^{(0)}_y = \frac{Nv\tau}{8}. \tag{38}$$

To proceed further with the action (37), we have to resolve three issues. First, we have to express all contributions in terms of the matrix field $Q_j = t^{-1}_j\Lambda t_j$ to restore the symmetry of the saddle-point manifold. Second, we have to take the continuum limit, $a \to 0$, explicitly. Third, we have to restore the spatial isotropy, i.e., the symmetry between the x and y directions.

To express the effective action (37) in terms of the matrix field $Q_j = t^{-1}_j\Lambda t_j$, we use the following identities:

$$\operatorname{tr}\left[L^{(x)}_j, \Lambda\right]^2 = \operatorname{tr}(\partial_x Q_j)^2,\ \operatorname{tr}[L^{(y)}_j, \Lambda]^2 = \operatorname{tr}(\partial_y Q_j)^2,$$
$$\operatorname{tr}\Lambda\left[L^{(x)}_j, L^{(y)}_j\right] = -\tfrac{1}{2}\operatorname{tr}Q_j\partial_x Q_j\partial_y Q_j. \tag{39}$$

Next, we define the continuum limit, $a \to 0$, by the following transformation from the sum to the integral, $a\sum_j \to \int dx$. Also, we rescale the coordinates as

$$x \to x\zeta\sqrt{\pi_x{}^{(0)}},\quad y \to y\zeta\sqrt{\pi_y{}^{(0)}}. \tag{40}$$

Here, the parameter ζ does not appear in the terms involving spatial derivatives. We fix ζ by requiring that the term proportional to the Matsubara energy has the standard coefficient equal to the density of states (see below). Finally, we note that in the continuum limit, the term proportional to $(-1)^j\operatorname{tr}\Lambda L^{(y)}_j$ becomes the boundary term and can then be rewritten as a bulk term:

$$\int dy\sum_j(-1)^j\operatorname{tr}\Lambda L^{(y)}_j = \oint ds\operatorname{tr}\Lambda t\partial_s t^{-1} = \frac{1}{8}\int dxdy\epsilon_{\mu\nu}\operatorname{tr}Q\partial_\mu Q\partial_\nu Q. \tag{41}$$

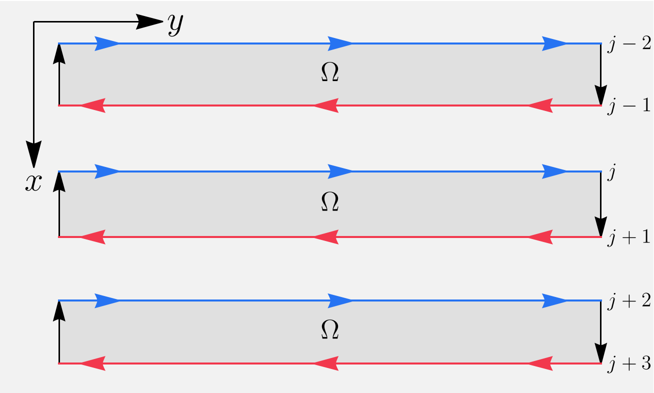

FIG. 2. Illustration for the calculation in Eq. (41). The alternating sum $\int dy \sum_j (-1)^j \mathrm{tr}\Lambda L_j^{(y)}$ is interpreted as the integral over the oriented boundary of the shaded region Ω, formed by every odd strip between neighboring links. Since Ω covers half of the network area, Stokes' theorem converts the boundary integral into half the integral of the topological density over the full sample.

Here $\epsilon_{\mu\nu}$ stands for the antisymmetric tensor with nonzero elements $\epsilon_{xy} = -\epsilon_{yx} = 1$. The factor $(-1)^j$ assigns opposite orientations to neighboring chiral links. To interpret the alternating sum as an integral over a closed contour, we impose the boundary condition in the longitudinal direction [88],

$$t_j(y) \to 1, \quad Q_j(y) \to \Lambda, \quad y \to \pm\infty, \tag{42}$$

for all links j. Equivalently, one may require that all Q_j approach the same constant matrix at the two ends of the sample. This boundary condition fixes the topological sector and is needed for the topological term to be well defined. It also allows us to close the contours in Fig. 2 by the auxiliary black segments at $y \to \pm\infty$. Since t_j is constant on these closing segments, their contribution to the boundary integral $\oint ds\, \mathrm{tr}\Lambda t \partial_s t^{-1}$ vanishes. Thus, the black segments merely complete the oriented contour and do not modify the value of the alternating sum.

After the auxiliary closing segments are introduced, the alternating sum in Eq. (41) is naturally interpreted as the integral over the oriented boundary $\partial\Omega$ of the shaded region in Fig. 2. The region Ω consists of every odd strip between neighboring links and therefore covers one-half of the total area. Stokes' theorem then yields the bulk topological density integrated over Ω, or, equivalently, one-half of the integral over the full area in the continuum limit.

The choice of $\partial\Omega$ is fixed by the physical embedding of the network. We take Ω to be the strip/puddle whose two horizontal edges are neighboring counterpropagating chiral links. The interior side of a chiral link is determined by the direction of the quantizing perpendicular magnetic field, which fixes the chirality pattern of the edge modes. Choosing the complementary region (the area between links $j-1$ and j and between $j+1$ and $j+2$ in Fig. 2) would reverse the sign of the contribution in Eq. (41) ($-1/8$ instead of $+1/8$). However, the first link at the bottom ($j+3$ in Fig. 2) and the last link at the top ($j-2$ in Fig. 2) form an additional contour with opposite circulation enclosing the whole system. This contour contributes $+1/4$, yielding a total $-1/8 + 1/4 = +1/8$, the same result as for the choice of Ω shown in Fig. 2.

In summary, the final form of the NLσM action can be written as

$$\begin{aligned}
\mathcal{S} = &-\frac{\bar{g}}{16}\int d^2\boldsymbol{r}\mathrm{tr}(\nabla Q)^2 + \frac{\bar{g}_H}{16}\int d^2\boldsymbol{r}\epsilon_{\mu\nu}\mathrm{tr}Q\partial_\mu Q\partial_\nu Q \\
&+ \pi\bar{\nu}\int d^2\boldsymbol{r}\,\mathrm{tr}\Big(\hat{\varepsilon}Q + i\mu_B\overline{B}s_3Q + \frac{\tau\mu_B^2}{4}\overline{B^2}[s_3, Q]^2\Big) \\
&- \frac{iN\tau\mu_B}{4}\int d^2\boldsymbol{r}\,\overline{B}\,\epsilon_{\mu\nu}\mathrm{tr}s_3\partial_\mu Q\partial_\nu Q \\
&- \frac{i\mu_B\sqrt{\pi\tau\bar{\nu}\bar{g}}}{2v}\int d^2\boldsymbol{r}\,\overline{B\mathrm{v}}\mathrm{tr}s_3 Q\partial_y Q.
\end{aligned} \tag{43}$$

The matrix Q satisfies the following constraints:

$$\begin{aligned}
&Q^\dagger = Q, \quad Q = -CQ^TC, \quad Q^2 = 1, \\
&Q \in \mathrm{Sp}(4N_rN_m)/\mathrm{U}(2N_rN_m).
\end{aligned} \tag{44}$$

Here the matrix C is defined in Eq. (23). The bare dimensionless spin longitudinal conductance (in units $G_0^s = \hbar/(8\pi)$) is given by

$$\begin{aligned}
\bar{g} &= N\left[\frac{2N\tau}{v}(\lambda_+ + \lambda_-)\left(1 - \frac{2N\tau}{v}\frac{(\lambda_+ - \lambda_-)^2}{(\lambda_+ + \lambda_-)}\right)\right]^{1/2} \\
&\simeq 2N\left(\frac{\lambda_+ + \lambda_-}{3m}\right)^{1/2}.
\end{aligned} \tag{45}$$

The bare dimensional spin Hall conductance becomes

$$\bar{g}_H = N\left[1 + \frac{2N\tau}{v}(\lambda_+ - \lambda_-)\right] \simeq N\left[1 + \frac{4(\lambda_+ - \lambda_-)}{3m}\right]. \tag{46}$$

We also introduce the bare density of states as

$$\bar{\nu} = N/(\pi^2 av) \equiv N\zeta^2\sqrt{\pi_x^{(0)}\pi_y^{(0)}}/(2va\pi). \tag{47}$$

The latter equality fixes the rescaling parameter ζ as

$$\zeta^2 = 2/\left(\pi\sqrt{\pi_x^{(0)}\pi_y^{(0)}}\right). \tag{48}$$

We note that for $\lambda_+ = \lambda_-$, the bare Hall conductance is equal to the integer, $\bar{g}_H = N$. Therefore, for an odd number of flavors the NLσM for the symmetric case $\lambda_+ = \lambda_-$ corresponds to the unstable critical line separating phases with the quantized spin Hall conductance in even integers. Interestingly, the bare value of the spin longitudinal conductance is almost insensitive to deviations from the line $\lambda_+ = \lambda_-$ by virtue of the assumption $\lambda_\pm \ll m$.

The quantities $\overline{B}(y)$, $\overline{B^2}(y)$, and $\overline{B\mathrm{v}}(y)$ represent the link average of the Zeeman magnetic field, its square, and its product with velocity, respectively. In particular, we define

$$\overline{B^m}(y) = \frac{1}{M}\sum_j B_j^m(y), \quad \overline{B\mathrm{v}}(y) = \frac{1}{M}\sum_j B_j(y)v_j. \tag{49}$$

All four terms in Eq. (43) that involve the Zeeman magnetic field explicitly break the SU(2) symmetry of the NLσM action. In the case of alternating signs of the Zeeman field at even/odd links, $B_j(y) = (-1)^j B$, the average is zero, $\overline{B} = 0$. In such a situation, the term in Eq. (43), proportional to $\overline{B^2}$, is responsible for the breaking of the SU(2) symmetry.

In addition, the last term in Eq. (43) breaks the inversion symmetry if the Zeeman splitting is synchronized with the velocities v_j, i.e., $\overline{B\mathrm{v}} \neq 0$. This happens, for example, in the case $B_j(y) = (-1)^j B$.

E. The symmetric case

In this section, we derive the NLσM for the symmetric case, where the scattering nodes on the even and odd links are identical. For the microscopic model in Eq. (21), the symmetric case corresponds to the choice $\lambda_+ = \lambda_-$. However, as shown in Appendix A, the ballistic renormalization does not preserve the nearest-neighbor form of the tunneling matrix $\beta^{(0)}$: non-nearest-neighbor matrix elements are generated. For this reason, in the symmetric case we consider a general symmetric Toeplitz-type matrix $\beta^{(0)}_{jj'}$, which depends only on the distance $|j-j'|$. As before, we ignore all massive modes in this derivation.

In the following, we focus on the infinite number of links, $M \to \infty$. Therefore, it is convenient to consider the matrix $\beta^{(0)}_{jj'}$ to be of infinite size and define its Fourier transform as

$$\beta^{(0)}_{jj'} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \underline{\beta}(k) e^{ik(j-j')}. \tag{50}$$

Introducing new matrix variable $\tilde{Q}_j = \sum_{j'} \beta^{(0)}_{jj'} \tilde{Q}^{(0)}_{j'}$, we rewrite the action (25) as

$$\mathcal{S} = N \int dy \sum_j \operatorname{tr} \ln[i\tilde{v}_j \partial_y + i\tilde{\varepsilon} - \tilde{b}_j + i\tilde{Q}_j] - \frac{1}{2} \int dy \sum_{jj'} [\beta^{(0)}]^{-1}_{jj'} \operatorname{tr} \tilde{Q}_j \tilde{Q}_{j'}. \tag{51}$$

We assume that the function $\underline{\beta}(k)$ has no zeros, such that the matrix $\beta^{(0)}_{jj'}$ is invertible. The saddle-point solution for the matrix $\tilde{Q}_j$ becomes

$$\underline{\tilde{Q}}_j = \frac{1}{2\tau} t^{-1} \Lambda t, \quad \frac{1}{\tau} = \frac{N}{v} \sum_{j'} \beta^{(0)}_{jj'} = \frac{N}{v} \underline{\beta}(0), \tag{52}$$

where, similar to Eq. (31), the matrix t is independent of y. Making the rotation t spatially and link dependent, we perform the gradient expansion of the action as before. Hence, we obtain

$$\mathcal{S} \simeq -\sum_j \int dy \bigg\{ \pi^{(s)}_y \operatorname{tr}\big[L^{(y)}_j, \Lambda\big]^2 + \pi^{(s)}_x \operatorname{tr}(\partial_x Q_j)^2 - \frac{N}{2}(-1)^j \operatorname{tr} \Lambda L^{(y)}_j - \frac{N}{2v} \operatorname{tr}(\tilde{\varepsilon} + i\tilde{b}_j) Q_j - \frac{N\tau}{8v} \operatorname{tr}\big[t_j \tilde{b}_j t_j^{-1}, \Lambda\big]^2 \bigg\}, \tag{53}$$

where $Q_j = t_j^{-1} \Lambda t_j$ and

$$\pi^{(s)}_x = -\frac{a^2 \underline{\beta}''(0)}{16\tau^2 \underline{\beta}^2(0)}, \quad \pi^{(s)}_y = \frac{N v \tau}{8}. \tag{54}$$

Proceeding as in Sec. IV D, with the replacement $\pi^{(0)}_{x,y} \to \pi^{(s)}_{x,y}$ in Eqs. (40), (47), and (48), we obtain the effective action in the form of Eq. (43) but with

$$\bar{g} = N \left(\frac{2|\underline{\beta}''(0)|}{\underline{\beta}(0)} \right)^{1/2}, \quad \bar{g}_H = N. \tag{55}$$

We note that in the case $\underline{\beta}(k) = 3m/2 + 2\lambda \cos k$, where $\lambda = \lambda_+ = \lambda_-$ with $\lambda \ll m$, the results (55) reproduce the expressions (45) and (46).

F. The asymmetric case

In this section, we allow for both long-range tunneling and an asymmetry between the nodes on the odd and even links. In this case, $\beta^{(0)}_{jj'}$ depends not only on the distance $|j-j'|$ but also on the parity of the jth link. We can write the link index j as $j = 2n + s$, where n enumerates links on two sublattices which are labeled by $s = 0, 1$ for the even/odd link, respectively. Then, for $j = 2n + s$ and $j' = 2n' + s'$, we can write $\beta^{(0)}_{jj'} \equiv \beta^{(0)}_{ss'}(n - n')$. Due to the symmetry of the tunneling matrix, $\beta^{(0)}_{jj'} = \beta^{(0)}_{j'j}$, we have $\beta^{(0)}_{ss'}(n-n') = \beta^{(0)}_{s's}(n'-n)$. To make the scattering time τ independent of the sublattice index s, we also assume that $\beta^{(0)}_{00}(n) = \beta^{(0)}_{11}(n)$ for all n. With these assumptions, introducing the matrix field $\tilde{Q}_j = \sum_{j'} \beta^{(0)}_{jj'} \tilde{Q}^{(0)}_{j'}$ brings us back to the action (51). The saddle-point solution remains unchanged. For an arbitrary integer displacement r we perform a gradient expansion, assuming that the matrix field varies slowly in the x direction:

$$t^{-1}_{j+r} \Lambda t_{j+r} \simeq t_j^{-1} \bigg[\Lambda + ra\big[L^x_j, \Lambda\big] + \frac{(ra)^2}{2} \Big(\big[\partial_x L^x_j, \Lambda\big] + \big[L^x_j, \big[L^x_j, \Lambda\big]\big] \Big) + \cdots \bigg] t_j. \tag{56}$$

Then, using sublattice representation for the matrix $\beta^{(0)}_{jj'}$, we find

$$\tilde{Q}_j = \sum_{s=0,1} \sum_n \beta^{(0)}_{j,j+2n+s} \underline{\tilde{Q}}^{(0)}_{j+2n+s} \simeq \frac{N}{2v} t_j^{-1} \bigg[A_0 \Lambda + a(-1)^j \times A_1 \big[L^x_j, \Lambda\big] + a^2 \frac{A_2}{2} \Big(\big[\partial_x L^x_j, \Lambda\big] + \big[L^x_j, \big[L^x_j, \Lambda\big]\big] \Big) \bigg] t_j. \tag{57}$$

Here we introduce the following parameters ($m = 0, 1, 2$):

$$A_m = \sum_n \big[(2n)^m \beta^{(0)}_{00}(-n) + (2n+1)^m \beta^{(0)}_{01}(-n) \big]. \tag{58}$$

The coefficients A_m can be written in a more compact form in terms of the function

$$\underline{\beta}(k) = \sum_{j'} \beta^{(0)}_{2j,2j+j'} e^{ikj'} = \sum_n \beta^{(0)}_{00}(-n) e^{ik(2n)} + \sum_n \beta^{(0)}_{01}(-n) e^{ik(2n+1)}. \tag{59}$$

We note that $\underline{\beta}(k)$ coincides with the Fourier transform introduced in the Toeplitz case, cf. Eq. (50). Then, we obtain

$$A_0 = \underline{\beta}(0), \quad A_1 = -i\underline{\beta}'(0), \quad A_2 = -\underline{\beta}''(0). \tag{60}$$

For the expansion in ra in Eq. (56) to be controlled, we require the gradients to be sufficiently small. We assume that $\beta^{(0)}_{ss'}(n)$ decays exponentially on a scale n_0. Therefore, the matrix field t must change on length scales greater than the scale an_0. After carrying out the gradient expansion in a standard manner, we arrive at an effective action:

$$\begin{aligned}\mathcal{S} \simeq &-\sum_j \int dy \Big\{ \pi_y \mathrm{tr}\big[L_j^{(y)}, \Lambda\big]^2 + \pi_x \mathrm{tr}\big[L_j^{(x)}, \Lambda\big]^2 \\ &+ \pi_H \mathrm{tr}\,\Lambda\big[L_j^{(x)}, L_j^{(y)}\big] - \frac{N}{2v}\mathrm{tr}(\tilde{\varepsilon} + i\tilde{b}_j)Q_j \\ &- \frac{N\tau}{8v}\mathrm{tr}\big[t_j\tilde{b}_j t_j^{-1}, \Lambda\big]^2 - (-1)^j \frac{N}{2}\mathrm{tr}\Lambda L_j^{(y)} \\ &+ \frac{iN\tau}{4v} v_j \mathrm{tr}\big[t_j\tilde{b}_j t_j^{-1}, \Lambda\big]\big[L_j^{y}, \Lambda\big]\Big\}, \end{aligned} \tag{61}$$

where coefficients are given by

$$\begin{aligned}\pi_x &= -\frac{a^2\underline{\beta}''(0)}{16\tau^2\underline{\beta}^2(0)}\left(1 - \frac{2[\underline{\beta}'(0)]^2}{\underline{\beta}''(0)\underline{\beta}(0)}\right), \quad \pi_y = \frac{Nv\tau}{8}, \\ \pi_H &= i\frac{N^2\tau a}{2v}\underline{\beta}'(0). \end{aligned} \tag{62}$$

Repeating the same steps of derivation as before with the replacement $\pi^{(0)}_{x,y} \to \pi_{x,y}$ in Eqs. (40), (47), and (48), we again obtain the NLσM action (43) with scattering time τ given by Eq. (52), as well as with the following longitudinal and Hall conductivities:

$$\begin{aligned}\bar{g} &= N\left[\frac{2|\underline{\beta}''(0)|}{\underline{\beta}(0)}\left(1 - \frac{2[\underline{\beta}'(0)]^2}{\underline{\beta}''(0)\underline{\beta}(0)}\right)\right]^{1/2}, \\ \bar{g}_H &= N\left(1 + \frac{2i\underline{\beta}'(0)}{\underline{\beta}(0)}\right). \end{aligned} \tag{63}$$

We note that in the symmetric case $\underline{\beta}'(0) \equiv 0$ and Eq. (63) reproduces Eq. (55). For the nearest-neighbor tunneling,

$$\underline{\beta}(k) = (3m/2) + \lambda_+ e^{-ik} + \lambda_- e^{ik}, \tag{64}$$

such that Eqs (63) reproduces the results (45) and (46). Therefore, all previously discussed limits can be seen as special cases of the general expression (63).

It is instructive to introduce two parameters $\epsilon = -2i\underline{\beta}'(0)/\underline{\beta}(0)$ and $\phi = 2|\underline{\beta}''(0)|/\underline{\beta}(0)$. Then, the expressions (63) can be rewritten as follows:

$$\bar{g} = N\sqrt{\phi - \epsilon^2}, \quad \bar{g}_H = N(1 - \epsilon). \tag{65}$$

By choosing the function $\underline{\beta}(k)$, one can change the parameters ϕ and ϵ, and, thus, control the starting point for the two-parameter renormalization group flow governed by the NLσM action. Parameter ϵ is responsible for the deviation of the spin Hall conductance from the integer value. Parameter ϕ controls the bare value of the spin conductance. Provided the function $\underline{\beta}(k)$ satisfies the inequality $\mathrm{Re}\underline{\beta}(k) > 0$ [89], one can prove that the parameters ϕ and ϵ satisfy the following inequality (see Appendix B 2):

$$\phi > \epsilon^2. \tag{66}$$

This inequality guarantees that the longitudinal spin conductance is well-defined for $\mathrm{Re}\underline{\beta}(k) > 0$. We note that the necessary condition, $|\underline{\beta}(k)| > 0$, for the absence of zero eigenvalues of the matrix $\beta^{(0)}$, does not guarantee that $\phi > \epsilon^2$ (see Appendix B 2). We note that the regime $\phi < \epsilon^2$ is realized for nearest-neighbor tunneling [see Eq. (64)], provided that $3m/(2(\lambda_+ + \lambda_-)) < 2(\lambda_+ - \lambda_-)^2/(\lambda_+ + \lambda_-)^2 - 1$. This condition requires a relatively strong anisotropy in the tunneling between even and odd links, namely, $|\lambda_+ - \lambda_-|/(\lambda_+ + \lambda_-) > 1/\sqrt{2}$. The case when $\phi < \epsilon^2$ requires a separate analysis. In particular, for $\pi_x < 0$, the saddle point given in Eq. (31) does not correspond to a minimum of the action. However, a detailed investigation of this regime lies beyond the scope of the present manuscript.

V. THE EFFECT OF THE TRIPLET MODES

So far, we have ignored the triplet matrix fields $\tilde{Q}^{(c)}_j$, with $c = 1, 2, 3$, in the action (25). In this section, we discuss their effect on the NLσM action. First, we assume that these triplet modes are massive on the level of the Gaussian action (32). Second, we discuss the situation where the Gaussian action (32) may have some zero modes.

A. Corrections to the NLσM due to the presence of the triplet modes

Under the assumption that the Gaussian action (32) has no zero modes, our aim is to integrate out the triplet modes and find corrections to the NLσM action. Using Eq. (32), we find the following propagator for the $\tilde{Q}^{(c)}_j$ modes:

$$\begin{aligned}&\big\langle \mathrm{tr}_s s_s\big[\tilde{Q}^{(c)}_j(y)\big]^{\alpha\beta}_{\varepsilon_1\varepsilon_2} \mathrm{tr}_s s_{s'}\big[\tilde{Q}^{(c')}_{j'}(y')\big]^{\mu\nu}_{\varepsilon_4\varepsilon_3}\big\rangle_0 \\ &\quad= 2\delta(y - y')\delta_{ss'}\big[\delta^{\alpha\nu}\delta^{\beta\mu}\delta_{\varepsilon_1\varepsilon_3}\delta_{\varepsilon_2\varepsilon_4} - \mathsf{v}_s\delta^{\alpha\mu}\delta^{\beta\nu}\delta_{\varepsilon_1,-\varepsilon_4}\delta_{\varepsilon_2,-\varepsilon_3}\big] \\ &\quad\times \delta^{cc'}[(\theta(\varepsilon_1)\theta(\varepsilon_2) + \theta(-\varepsilon_1)\theta(-\varepsilon_2))[(\beta^{(c)})^{-1}]_{jj'} \\ &\quad+ (\theta(\varepsilon_1)\theta(-\varepsilon_2) + \theta(-\varepsilon_1)\theta(\varepsilon_2))[(\tilde{\beta}^{(c)})^{-1}]_{jj'}]. \end{aligned} \tag{67}$$

Here, we introduce the parameter $\mathsf{v}_s = -\mathrm{tr}_s s_s s_2 s_s^T s_2/2$, and $\theta(\varepsilon)$ denotes the Heaviside step function. tr_s is a partial trace over the Nambu space. Using Eq. (67), one can derive the contraction rule:

$$\begin{aligned}&\int dy' \big\langle \mathrm{tr}A(y', y)\tilde{Q}^{(c)}_j(y)B(y, y')\tilde{Q}^{(c)}_{j'}(y')\big\rangle_0 \\ &\quad= (\check{\beta}^{(c)}_+)^{-1}_{jj'}[\mathrm{tr}A(y, y)\mathrm{tr}B(y, y) - \mathrm{tr}A(y, y)CB^T(y, y)C] \\ &\quad+ (\check{\beta}^{(c)}_-)^{-1}_{jj'}[\mathrm{tr}\Lambda A(y, y)\mathrm{tr}\Lambda B(y, y) \\ &\quad+ \mathrm{tr}\Lambda A(y, y)\Lambda CB^T(y, y)C], \end{aligned} \tag{68}$$

where $(\check{\beta}^{(c)}_\pm)^{-1} = [(\beta^{(c)})^{-1} \pm (\tilde{\beta}^{(c)})^{-1}]/2$, and matrix C is defined in Eq. (23). Next, we use the relation (68) to compute the term that obtained after the second-order expansion of tr ln in $\tilde{Q}^{(c)}_{j'}$. We note that such an expansion must be performed carefully to preserve the invariance of the action (25) with respect to global rotations of the $Q^{(0)}_j$ matrix. Therefore, we proceed as follows: (i) we perform a rotation t_j inside the $\operatorname{tr}\ln$ operator, and then (ii) we change variables from $\tilde{Q}^{(c)}_{j'}$ to $t^{-1}_{j'}\tilde{Q}^{(c)}_{j'}t_{j'}$ in the action (25). Overall, we expand the $\operatorname{tr}\ln$ operator to second order in the quantity $t_j t^{-1}_{j'}\tilde{Q}^{(c)}_{j'}t_{j'}t^{-1}_j$. The result reads

$$\frac{N}{2}\int dydy'\sum_{jj_1j_2,c}\beta^{(c)}_{jj_1}\beta^{(c)}_{jj_2}\operatorname{tr}\langle t_j(y)t^{-1}_{j_1}(y)\tilde{Q}^{(c)}_{j_1}(y)t_{j_1}(y)t^{-1}_j(y)\hat{G}_j(y,y')t_j(y')t^{-1}_{j_2}(y')\tilde{Q}^{(c)}_{j_2}(y')t_{j_2}(y')t^{-1}_j(y')\hat{G}_j(y',y)\rangle_0, \tag{69}$$

where we introduce the inverse Green's function

$$\begin{aligned}\hat{G}^{-1}_j &= \underline{G}^{-1}_j + iv_jL^{(y)}_j + i\frac{N}{2v}\sum_{j'}\beta^{(0)}_{jj'}t_jt^{-1}_{j'}[\Lambda,t_{j'}]t^{-1}_j\\ &\simeq \underline{G}^{-1}_j + iv_jL^{(y)}_j + i(-1)^j a\frac{NA_1}{2v}\big[L^{(x)}_j,\Lambda\big] + ia^2\frac{NA_2}{4v}\big(\big[\partial_xL^{(x)}_j,\Lambda\big]+\big[L^{(x)}_j,\big[L^{(x)}_j,\Lambda\big]\big]\big).\end{aligned} \tag{70}$$

Next, using the contraction rule (68), we obtain from Eq. (69) the following result:

$$\begin{aligned}&-\frac{N}{2}\int dy\sum_{j,j_1,j_2,c}\beta^{(c)}_{jj_1}\beta^{(c)}_{jj_2}\big\{(\check{\beta}^{(c)}_+)^{-1}_{j_1j_2}\big[\big(\operatorname{tr}t_{j_1}(y)t^{-1}_j(y)\hat{G}_j(y,y)t_j(y)t^{-1}_{j_2}(y)\big)^2+\operatorname{tr}\big(t_{j_1}(y)t^{-1}_j(y)\hat{G}_j(y,y)t_j(y)t^{-1}_{j_2}(y)\big)^2\big]\\ &+(\check{\beta}^{(c)}_-)^{-1}_{j_1j_2}\big[\big(\operatorname{tr}\Lambda t_{j_1}(y)t^{-1}_j(y)\hat{G}_j(y,y)t_j(y)t^{-1}_{j_2}(y)\big)^2-\operatorname{tr}\big(\Lambda t_{j_1}(y)t^{-1}_j(y)\hat{G}_j(y,y)t_j(y)t^{-1}_{j_2}(y)\big)^2\big]\big\}.\end{aligned} \tag{71}$$

Here, we use the fact that $t^T_jC = Ct^{-1}_j$ and $C\hat{G}^T_j(y,y)C = -\hat{G}_j(y,y)$. The term (71) is an additional contribution to the NLσM action. It produces corrections to the gradient terms in the action (61). In particular, we find the following corrections [cf. Eq. (62)]:

$$\delta\pi_y = \frac{N}{8}\sum_{c=1}^3\tau^2_j\beta^{(c)}_{jj},\quad \delta\pi_H = \frac{N^2A_1a}{2v^2}\sum_{c=1}^3\tau^2_j\beta^{(c)}_{jj}, \tag{72}$$

and

$$\begin{aligned}\delta\pi_x =&\frac{Na^2}{16v^2}\sum_{c=1}^3\sum_{j_1,j_2}(2j-j_1-j_2)^2\beta^{(c)}_{jj_1}(\check{\beta}^{(c)}_-)^{-1}_{j_1j_2}\beta^{(c)}_{j_2j}\\ &-\frac{N^3A_1^2a^2}{8v^4}\sum_{c=1}^3\tau^2_j\beta^{(c)}_{jj}.\end{aligned} \tag{73}$$

We note that the contribution to $\delta\pi_x$ in the first line of Eq. (73) comes from the expansion of the products $t_{j_1}t^{-1}_j$ in gradients in Eq. (71). All the other contributions in Eqs. (72) and (73) are the result of the expansion of the Green's function $\hat{G}_j(y,y)$. We emphasize that the corrections (72) and (73) are smaller than the expressions (62) by a factor of $1/N$. Therefore, the role of the triplet modes $\tilde{Q}^{(c)}_j$, with $c=1,2,3$, is similar to the massive modes in the singlet sector. Indeed, the massive modes in the singlet sector can be studied in a similar manner. Formally, one can use similar expressions with $\tilde{\beta}^{-1}$ replaced by zero. Also, there is an additional contribution due to the Jacobian [8]. Since the singlet massive modes are similar to the massive modes in the case of the NLσM for class A [35], we do not discuss them in the present work. In addition, we note that the corrections (72) and (73) exist even in the case $\gamma_j = \Delta_j$. This is due to the diagonal part of the matrix $\beta^{(c)}$, cf. Eq. (21). We recall that for $N>1$, one cannot exclude the triplet modes $D^{(c)}_j$, where $c=1,2,3$, from the theory even for $\Delta_j=\gamma_j$.

B. Possible softening of the triplet modes

In the previous section, we used Eq. (67) to derive corrections to the NLσM action due to the triplet modes. We assumed that the matrices $\beta^{(c)}_{jj'}$ and $\tilde{\beta}^{(c)}_{jj'}$ are invertible, meaning they have no zero eigenvalues. This assumption is reasonable for the matrix $\beta^{(c)}_{jj'}$ as it is involved in the Hubbard-Stratonovich decoupling in Eq. (20). If the matrix $\beta^{(c)}_{jj'}$ had zero eigenvalues, the Hubbard-Stratonovich decoupling would have to be modified. Below we assume that this is not the case. In contrast, the matrix $\tilde{\beta}^{(c)}_{jj'}$ is not directly involved in the Hubbard-Stratonovich decoupling and, therefore, may occasionally have zero eigenvalues. Therefore, it is important to analyze this case in more detail.

According to Eq. (32), the matrix $\tilde{\beta}^{(c)}_{jj'}$ determines the Gaussian action for the part of the matrix $\tilde{Q}^{(c)}_j$ that anticommutes with Λ, $\tilde{Q}^{(c)}_{j,-} = \Lambda[\Lambda,\tilde{Q}^{(c)}_j]/2$. We diagonalize the symmetric matrix, $\tilde{\beta}^{(c)}$, by an orthogonal rotation, R, such that $\tilde{\beta}^{(c)}_{jj'} = \sum_r R_{rj}\tilde{\beta}^{(c)}_rR_{rj'}$. We assume that $\tilde{\beta}^{(c_0)}_{r_0}\equiv 0$ for some indices c_0 and r_0. The corresponding zero mode is given by $Z^{(c_0)}_{r_0} = \sum_j R_{r_0j}\tilde{Q}^{(c_0)}_{j,-}$. To demonstrate that this mode is a soft mode rather than a true zero mode, we expand the action (25) to fourth order in $\tilde{Q}^{(c)}_j$ with $c=1,2,3$ (neglecting the singlet mode $\tilde{Q}^{(0)}_j$). The resulting contribution to the action is as follows:

$$\begin{aligned}\mathcal{S}^{(4)}_{\rm t} =& -\frac{N}{16v}\sum_{j,j_k}\tau^3_j\sum_{c_k=1}^3\beta^{(c_1)}_{jj_1}\beta^{(c_2)}_{jj_2}\beta^{(c_3)}_{jj_3}\beta^{(c_4)}_{jj_4}\\ &\times\big(\delta_{c_1c_2}\delta_{c_3c_4}-\delta_{c_1c_3}\delta_{c_2c_4}+\delta_{c_1c_4}\delta_{c_2c_3}\big)\\ &\times\int dy\operatorname{tr}\big[\tilde{Q}^{(c_1)}_{j_1}\tilde{Q}^{(c_2)}_{j_2}\tilde{Q}^{(c_3)}_{j_3}\tilde{Q}^{(c_4)}_{j_4}\\ &-4\Lambda\tilde{Q}^{(c_1)}_{j_1}\Lambda\tilde{Q}^{(c_2)}_{j_2}\tilde{Q}^{(c_3)}_{j_3}\tilde{Q}^{(c_4)}_{j_4}-2\Lambda\tilde{Q}^{(c_1)}_{j_1}\tilde{Q}^{(c_2)}_{j_2}\Lambda\tilde{Q}^{(c_3)}_{j_3}\tilde{Q}^{(c_4)}_{j_4}\\ &+5\Lambda\tilde{Q}^{(c_1)}_{j_1}\Lambda\tilde{Q}^{(c_2)}_{j_2}\Lambda\tilde{Q}^{(c_3)}_{j_3}\Lambda\tilde{Q}^{(c_4)}_{j_4}\big)\big].\end{aligned} \tag{74}$$

Here, we neglect all spatial derivatives with respect to the coordinate y. Projecting the above lengthy expression onto the mode $Z^{(c_0)}_{r_0}$, we obtain

$$\mathcal{S}^{(4)}_{\rm t} \to -\frac{N}{2v}\sum_j \tau^3_j \left(\sum_{j'} \beta^{(c_0)}_{jj'} R_{j' r_0}\right)^4 \int dy \,\mathrm{tr}\big(Z^{(c_0)}_{r_0}\big)^4. \quad (75)$$

Since $R_{j'r_0}$ cannot simultaneously be a zero eigenvector for both matrices $\tilde{\beta}^{(c)}_{jj'}$ and $\beta^{(c)}_{jj'}$, we conclude that there is a quartic term for the mode $Z^{(c_0)}_{r_0}$; i.e., this mode is a soft mode rather than a true zero mode. The quartic action (75) indicates that the contribution of the soft mode $Z^{(c_0)}_{r_0}$ to the correlation function $\langle \tilde{Q}^{(c_0)}_{j,-}\tilde{Q}^{(c_0)}_{j',-}\rangle$ is enhanced by the factor $N\sqrt{mL_y/v}$, where L_y is the system size in the y direction. This enhancement may result in corrections similar to those in Eqs. (72) and (73), which are not small in the parameter $1/N$. However, a detailed analysis of this case is beyond the scope of the present work.

VI. DISCUSSION AND CONCLUSIONS

We demonstrated that the continuum limit of the quantum network model at large N is equivalent to the NLσM in the weak coupling regime. Interestingly, for the general formulation of the model, the triplet sector of the theory does not decouple from the singlet sector. Since the form of the NLσM is dictated solely by symmetry considerations, the inclusion of triplet modes does not modify the structure of the NLσM action. However, they have two important effects. First, the triplet modes renormalize the bare value of the longitudinal spin conductance in the NLσM. In the large-N limit, these corrections to $\bar{g}$ are suppressed as $1/N$; see Eq. (73). Second, for certain choices of tunneling between links, the triplet modes may become massless at the Gaussian level, although they remain massive beyond the Gaussian approximation. This indicates that the triplet modes are anomalously soft and, as a consequence, modify the diffusive length scale of the NLσM. Additionally, in this case, the correction to $\bar{g}$ produced by the triplet modes may not be small in $1/N$ and, thus, it may also be significant. We note that such choices of tunneling between links should be avoided in numerical simulations of the quantum network model, as the required system sizes needed to observe universal effects may be increased compared to the standard scenario.

The quantum network model considered in this work is a generalization of the model proposed for the sqHe in Ref. [55] and extended to a large number of channels N per link. Although this model is similar to the quantum network model studied in Ref. [81] for the iqHe, there is a crucial difference in the bare conductances. In the case of the iqHe, both bare conductances are controlled by a single parameter. In contrast, in the considered case of the sqHe, the bare longitudinal spin conductance is governed by two independent parameters, one of which also enters the bare spin Hall conductance. This fact makes the considered quantum network model convenient for numerical studies of the sqHe flow diagram (see Fig. 3), since one can change the bare values of the conductances independently.

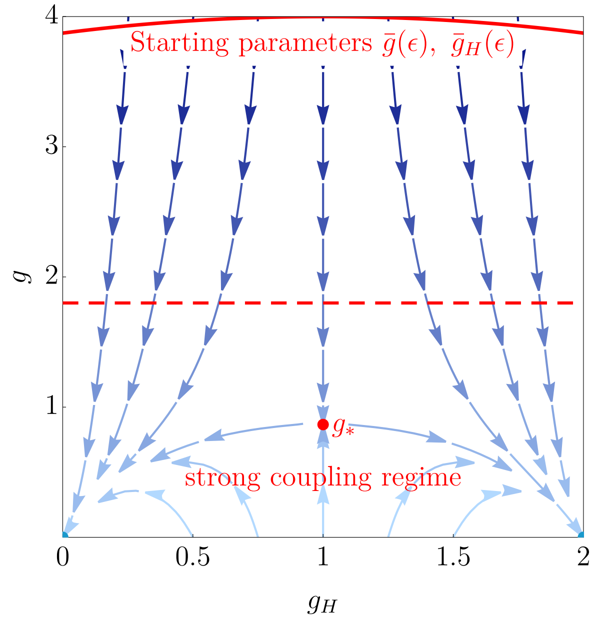

FIG. 3. Sketch of the renormalization group flow for class C NLσM (adopted from Ref. [69]). Arrows indicate flow towards infrared. Red curve at the top demonstrates a change of the bare parameters $\bar{g}$ and $\bar{g}_H$ with ϵ at fixed ϕ.

Perhaps, the most striking result of our analysis is that the absence of zero eigenvalues in the tunneling matrix $\beta^{(0)}$, i.e., the inequality $|\underline{\beta}(k)| > 0$, does not guarantee that $\phi > \epsilon^2$ in Eq. (65). The case $\phi < \epsilon^2$ implies that the saddle point (31) does not correspond to a minimum of the action. Since fulfillment of the inequality $\phi < \epsilon^2$ requires sufficient anisotropy in the tunneling between even and odd links, perhaps it implies that in such a situation the quantum network effectively splits into pairs of neighboring links decoupled from each other. We leave a detailed analysis of this possibility for future work.

We also considered the effect of the Zeeman field acting on fermions in each link. As expected, we found that the Zeeman splitting leads to terms in the NLσM action that break SU(2) symmetry; see Eq. (43). These terms are relevant and responsible for the crossover from the sqHe to the iqHe [58,70,90].

The NLσM for class C has been derived as an effective description in the large N limit of a system of $2N$ species of 2D Dirac fermions with class C symmetry, as described in Ref. [91]. However, for the case of isotropic disorder, it was found [91] that the bare value of the spin Hall conductance, $\bar{g}_H$, is equal to $2N$. This implies that NLσM is on a line where all bulk states are localized, and the physical spin Hall conductance is also equal to $2N$. This situation contrasts with the quantum network model we consider in this work, where for odd N and symmetric tunneling ($\epsilon = 0$), the NLσM lies on the critical line.

We emphasize the difference between the quantum network model we consider and the SU(2) Chalker-Coddington network model. The former has random tunneling amplitudes, while the latter has two fixed tunneling amplitudes. Their asymmetry determines the parameter ϵ in the bare value of the spin Hall conductance. Simultaneously, in the Chalker-Coddington model, the parameter ϕ [see Eq. (65)] is always equal to 1. Since the NLσM for the Chalker-Coddington model is derived from a mapping to a spin chain, it is interesting to understand which type of spin chain corresponds to the NLσM action derived in our work and what the meaning of ϕ is for the spin chain Hamiltonian.

Recently, there have been indications that the Chalker-Coddington network model may be too simplistic to accurately describe the critical behavior of the iqHe. Numerical studies have shown that geometric disorder in the Chalker-Coddington model for the iqHe leads to a modification of the critical exponent of the correlation length [28–32]. In the case of the sqHe, a modification of the critical exponents has recently been shown for random networks [33]. In this context, it would be interesting to extend the quantum network model considered in this work to include the effects of geometric disorder. One possible step in this direction would be to introduce finite correlation lengths along the y direction for the tunneling amplitudes, see Eq. (12).

Finally, we comment on an interesting feature of the NLσM action for the quantum network model considered in this work, which provides an interesting connection to the experimental platforms mentioned in the Introduction. Twisted $Bi_2Sr_2CaCu_2O_{8+x}$ bilayers with twist angles near 45° also demonstrate the Josephson diode effect [71]. As is known, the superconducting diode effect requires the breaking of time-reversal and inversion symmetries (see Ref. [92] for a review). We note that the quantum network model we consider in this work does not have time-reversal symmetry or inversion symmetry for a given realization of random tunneling. After disorder averaging, the inversion symmetry is restored unless the Zeeman splitting is taken into account. In this case, the NLσM action, see Eq. (43), includes a term that explicitly breaks inversion symmetry [93]. Therefore, it would be interesting to investigate non-reciprocal effects associated with broken inversion symmetry in the quantum network model.

To summarize, in this study we examined a quantum network model with N channels per chiral link, adhering to the symmetries of the spin quantum Hall effect. We established that the triplet and singlet sectors remain coupled, though triplet modes may influence the derived large-N NLσM only when they become soft under specific conditions. Calculations of bare conductances in the NLσM revealed that the standard saddle-point approximation fails in the case of sufficiently large asymmetry in tunneling between even and odd links. Furthermore, the inclusion of a Zeeman field was shown to break not only the SU(2) symmetry but also the inversion symmetry within the NLσM action.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: BALLISTIC RENORMALIZATION GROUP

In this appendix, we present the ballistic renormalization of the disorder-averaged action (20). This renormalization, which should not be confused with that due to diffusive collective modes, is performed before the diffusive regime sets in. Its role is to describe how the bare tunneling matrices $\beta^{(c)}$ are modified upon integrating out fermionic modes with momenta much larger than the inverse elastic mean free path q_Λ. In particular, even if the bare matrix $\beta^{(c)}$ is tridiagonal, the ballistic renormalization generates nonzero matrix elements corresponding to non-nearest-neighbor tunneling.

We neglect the Zeeman term in this appendix. We split the fermionic fields into slow and fast modes in momentum space by introducing a separating momentum scale q_Λ. The correction to the disorder-averaged action, Eq. (20), comes from the second order in $\mathcal{S}_{\rm av}$, where four of the eight fermionic fields are contracted as fast fields. The relevant term is

$$\frac{1}{2}\langle \mathcal{S}_{\rm av}^2\rangle_0 = \frac{1}{2}\int dydy' \sum_{jj'j_1j_1'}\sum_{c,c'} \beta^{(c)}_{jj'}\beta^{(c')}_{j_1j_1'} \sum_{z_k} \big[D^{(c)}_j(y)\big]_{z_1;z_2} \times \big[D^{(c')}_{j_1'}(y')\big]_{z_3;z_4} \big\langle \big[D^{(c)}_{j'}(y)\big]_{z_2;z_1} \big[D^{(c')}_{j_1}(y')\big]_{z_4;z_3}\big\rangle_0. \tag{A1}$$

Here z_k denotes the combined index $\{\varepsilon_n, \alpha, s\}$. The average $\langle \dots\rangle_0$ is taken with respect to $\mathcal{S}_0$, Eq. (17), with $\check{\Upsilon}^{(Z)} = 0$. In what follows, we keep only the Wick contractions that are leading in the number of flavors N. The omitted contractions are of order $O(N^0)$ and are therefore negligible in the large N limit.

Let us now evaluate the average entering Eq. (A1). For two D matrices, one has

$$\big\langle \big[D^{(a)}_{j''}(y)\big]_{\varepsilon_m,\beta,s'';\varepsilon_n,\alpha,s} \big[D^{(b)}_{j_1}(y')\big]_{\varepsilon_{m_1},\beta_1,s_1';\varepsilon_{n_1},\alpha_1,s_1}\big\rangle_0 = ① + ②. \tag{A2}$$

The first contribution corresponds to a normal-type contraction:

$$① = -\sum_{\sigma,\sigma',\sigma_1,\sigma_1',f,f_1} \big\langle \Xi_{\varepsilon_m\beta j'f\sigma's'}(y)\bar{\Xi}_{\varepsilon_{n_1}\alpha_1j_1f_1\sigma_1s_1}(y')\big\rangle_0 \times (\sigma_a)_{\sigma\sigma'}(\sigma_b)_{\sigma_1\sigma_1'}\big\langle \Xi_{\varepsilon_{m_1}\beta_1j_1f_1\sigma_1's_1'}(y')\bar{\Xi}_{\varepsilon_n\alpha j'f\sigma s}(y)\big\rangle_0. \tag{A3}$$

The second contribution is the anomalous-type contraction:

$$② = \sum_{\sigma,\sigma',\sigma_1,\sigma_1',f,f_1} \big\langle \Xi_{\varepsilon_m\beta j'f\sigma's'}(y)\Xi_{\varepsilon_{m_1}\beta_1j_1f_1\sigma_1's_1'}(y')\big\rangle_0 \times (\sigma_a)_{\sigma\sigma'}(\sigma_b)_{\sigma_1\sigma_1'}\big\langle \bar{\Xi}_{\varepsilon_{n_1}\alpha_1j_1f_1\sigma_1s_1}(y')\bar{\Xi}_{\varepsilon_n\alpha j'f\sigma s}(y)\big\rangle_0. \tag{A4}$$

Evaluating the normal contraction, we obtain

$$① = \frac{\mathsf{N}}{2}\delta_{ab}\delta_{j_1j'}\delta_{\varepsilon_{n_1}\varepsilon_m}\delta_{\varepsilon_{m_1}\varepsilon_n}\delta_{\alpha_1\beta}\delta_{\beta_1\alpha} \times \big[g_{\varepsilon_m,j_1}(y,y')\big]_{s's_1}\big[g_{\varepsilon_n,j_1}(y',y)\big]_{s_1's}. \tag{A5}$$

Using the symmetry relation (7), the anomalous contribution can be written as

$$② = \frac{\mathsf{N}}{2}(1-2\delta_{a0})\delta_{ab}\delta_{j_1j'}\delta_{\varepsilon_{n_1},-\varepsilon_n}\delta_{\varepsilon_{m_1},-\varepsilon_m}\delta_{\alpha_1\alpha}\delta_{\beta_1\beta} \times \left[g_{\varepsilon_m,j_1}(y,y')s_2\right]_{s's_1}\left[s_2 g_{\varepsilon_n,j_1}(y',y)\right]_{s_1's}. \tag{A6}$$

Here, $g_{\varepsilon_m,j}(y,y')$ denotes the single-particle Green's function in $d=1$. In the Nambu space, it takes the following form:

$$g_{\varepsilon_m,j}(y,y') = \begin{pmatrix} \langle y|(i\varepsilon_m + iv_j\partial_y)^{-1}|y'\rangle & 0 \\ 0 & -\langle y'|(-i\varepsilon_m + iv_j\partial_y)^{-1}|y\rangle \end{pmatrix} = s_0\int\frac{dq}{2\pi}\frac{e^{-iq(y-y')}}{i\varepsilon_m - v_j q}. \tag{A7}$$

Combining the two contributions, we find that the leading term in the large-N limit becomes

$$① + ② = \frac{\mathsf{N}}{2}\delta_{j_1j'}\delta_{ab}\big[\delta_{\varepsilon_{n_1}\varepsilon_m}\delta_{\varepsilon_{m_1}\varepsilon_n}\delta_{\alpha_1\beta}\delta_{\beta_1\alpha}\delta_{s's_1}\delta_{s_1's} + (1-2\delta_{a0})\delta_{\varepsilon_{n_1},-\varepsilon_n}\delta_{\varepsilon_{m_1},-\varepsilon_m}\delta_{\alpha_1\alpha}\delta_{\beta_1\beta}(s_2)_{s's_1'}(s_2)_{s_1s}\big] \times \int\frac{dqdk}{(2\pi)^2}\frac{e^{-ik(y-y')}}{(i\varepsilon_m - v_{j_1}q)(i\varepsilon_n - v_{j_1}(q+k))}. \tag{A8}$$

In the low-energy and long-wavelength limit, the kernel in Eq. (A8) is short-ranged with respect to $y-y'$. More explicitly, the fields $D_j^{(c)}$ left uncontracted in Eq. (A1) are slow fields, while the contracted fermions carry momenta $|q|\geqslant q_\Lambda$. Therefore, in the operator product one can set $D_{j_1'}^{(c')}(y')\simeq D_{j_1'}^{(c')}(y)$. The integral over the relative coordinate $y-y'$ then sets the momentum k in (A8) to zero. Neglecting external Matsubara frequencies compared to vq_Λ, one obtains

$$\int d(y-y')\int\frac{dqdk}{(2\pi)^2}\frac{e^{-ik(y-y')}}{\left(i\varepsilon_m - v_{j_1}q\right)\left(i\varepsilon_n - v_{j_1}(q+k)\right)} \simeq \frac{1}{2\pi v^2}\int_{q_\Lambda}^{\infty}\frac{dq}{q^2}. \tag{A9}$$

Taking into account the Kronecker symbols in (A8), the disorder matrices in Eq. (A1) combine into a matrix product, while the normal and anomalous Nambu structures in Eq. (A8), together with the symmetry relations for $D_j^{(c)}$, reconstruct the same local operator as in the disorder-averaged action, $\mathrm{tr}D_j^{(c)}D_{j_1'}^{(c)}$. Renaming $j_1'\to j'$, we obtain

$$\frac{1}{2}\langle\mathcal{S}_{\mathrm{av}}^2\rangle_{\mathrm{fast}} \simeq \frac{N}{2\pi v^2}\int_{q_\Lambda}^{\infty}\frac{dq}{q^2}\int dy\sum_{c=0}^{3}\sum_{jj'}(\beta^{(c)}\beta^{(c)})_{jj'} \times \mathrm{tr}D_j^{(c)}(y)D_{j'}^{(c)}(y). \tag{A10}$$

Comparing Eq. (A10) with the original disorder-averaged action, Eq. (20), gives the ballistic renormalization of the tunneling matrix. Introducing dimensionless momentum-dependent disorder strength $\bar\beta^{(c)} = N\beta^{(c)}/(\pi v^2 q_\Lambda)$, we obtain the following renormalization group equations:

$$-\frac{d\bar\beta^{(c)}}{d\ln q_\Lambda} = \bar\beta^{(c)} - (\bar\beta^{(c)})^2. \tag{A11}$$

As one can see, the singlet ($c=0$) and triplet ($c=1,2,3$) channels are not mixed under renormalization group flow (at least within the one-loop approximation). Next, $\bar\beta^{(c)}=0$ is the unstable fixed point of Eq. (A11) at $q_\Lambda\to 0$. Additionally, even if one starts with the bare matrix $\beta^{(c)}$, which corresponds to tunneling only between nearest-neighbor links, see Eq. (21), the nonzero elements connecting non-nearest-neighbor links are generated during the renormalization group flow. We note that a similar picture holds for the renormalization of tunneling action at the superconductor–normal metal boundary [94]. Formally, Eq. (A11) has a stable fixed point at the unit matrix $\bar\beta^{(c)}=1$. However, this fixed point lies outside the controlled weak-coupling regime of the one-loop calculation. Indeed, when $\bar\beta^{(c)}$ becomes of order of unity, higher loop corrections, which generate terms of order $(\bar\beta^{(c)})^3$ and higher, are no longer parametrically small compared with the one-loop terms retained in Eq. (A11). Therefore, Eq. (A11) should be used for $||\bar\beta^{(c)}||\ll 1$. Its main consequence is that renormalization at the ballistic scales generates non-nearest-neighbor elements of the tunneling matrix $\bar\beta^{(c)}$ before the diffusive regime is reached.

APPENDIX B: EVALUATION OF DETERMINANTS OF MATRICES $\beta^{(a)}$ AND $\tilde\beta^{(a)}$

Throughout the derivation of the NLσM, we assumed that the matrices β^a do not possess zero eigenvalues, since such eigenvalues would generate additional zero modes. This assumption is equivalent to requiring that $\det\beta^{(a)}\neq 0$. Here we analyze these conditions in more detail.

1. The case of the nearest neighboring tunneling

We begin with the tridiagonal matrix $\beta^{(a)}$ introduced in Eq. (21). For the singlet sector, the determinant of the $M\times M$ matrix $\beta^{(0)}$ can be written as

$$\det\beta_M^{(0)} = d_M = \begin{vmatrix} \frac{3}{2}m & \lambda_+ & 0 & 0 & \dots \\ \lambda_+ & \frac{3}{2}m & \lambda_- & 0 & \dots \\ 0 & \lambda_- & \frac{3}{2}m & \lambda_+ & \dots \\ 0 & 0 & \lambda_+ & \frac{3}{2}m & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}. \tag{B1}$$

Using standard properties of determinants, we find the following recursion relation:

$$d_{n+2} = (9m^2/4 - \lambda_+^2 - \lambda_-^2)d_n - \lambda_+^2\lambda_-^2 d_{n-2}. \tag{B2}$$

Since this is a linear recurrence, its solution can be expressed in terms of the roots $x_\pm$ of the characteristic equation:

$$x^2 - (9m^2/4 - \lambda_+^2 - \lambda_-^2)x + \lambda_+^2\lambda_-^2 = 0, \tag{B3}$$

which has two roots:

$$x_\pm = \frac{1}{4}\left(\sqrt{\frac{9}{4}m^2 - (\lambda_+ + \lambda_-)^2} \pm \sqrt{\frac{9}{4}m^2 - (\lambda_+ - \lambda_-)^2}\right)^2. \tag{B4}$$

Then the odd and even determinants are given by

$$d_{2k-1}=\frac{d_2-x_-}{d_1(x_+-x_-)}(\lambda_+^2+x_+)x_+^{k-1}+\frac{x_+-d_2}{d_1(x_+-x_-)}(\lambda_+^2+x_-)x_-^{k-1},$$
$$d_{2k}=\frac{d_2-x_-}{x_+-x_-}x_+^k+\frac{x_+-d_2}{x_+-x_-}x_-^k, \tag{B5}$$

with the initial conditions $d_1=3m/2$ and $d_2=\frac{9}{4}m^2-\lambda_+^2$. The conditions for the odd and even determinants to vanish are therefore

$$\left(\frac{x_-}{x_+}\right)^{k-1}=\frac{d_2-x_-}{d_2-x_+}\cdot\frac{\lambda_+^2+x_+}{\lambda_+^2+x_-},\quad \left(\frac{x_-}{x_+}\right)^{k}=\frac{d_2-x_-}{d_2-x_+}. \tag{B6}$$

Several cases are possible:

(1) If $3m/2>\lambda_++\lambda_-$, the roots are real and positive, with $d_2>x_+>x_->0$. In this regime, Eq. (B6) cannot be satisfied.

(2) If $3m/2<|\lambda_+-\lambda_-|$, both roots are negative, $x_-<x_+<0$. In this case, Eq. (B6) may be satisfied for a finite matrix size M. However, in the infinite matrix limit, $M\to\infty$, the contribution proportional to x_+^k vanishes, and the remaining term cannot be tuned to zero.

(3) In the intermediate regime, $|\lambda_+-\lambda_-|<3m/2<\lambda_++\lambda_-$, the roots are complex and can be written as $x_\pm=|x|e^{\pm i\phi}$. Then the condition $d_{2k}=0$ can be rewritten as $d_2\sin(\phi k)=|x|\sin(\phi(k-1))$, and an analogous, slightly more complicated relation could be obtained for $d_{2k+1}=0$. Thus, zero value of the determinants are possible.

We therefore conclude that to exclude zero modes of $\beta^{(0)}$, it is sufficient to impose the condition $m>2(\lambda_++\lambda_-)/3=2(\gamma_++\gamma_-)+2(\Delta_++\Delta_-)/3$. In the limit of infinitely many links, $M\to\infty$, this condition can be relaxed by additionally allowing the region $m<2|\lambda_+-\lambda_-|/3$.

For the spin-triplet sector, the analysis is identical, with the substitutions $3m/2\to m/2$ and $\lambda_\pm\to\gamma_\pm-\Delta_\pm$. The corresponding positivity condition would be $m>2|\gamma_+-\Delta_+|+2|\gamma_--\Delta_-|$. A sufficient set of conditions ensuring positivity of $\beta_{jj'}^{(a)}$ for $a=0,1,2,3$ is

$$m>2(\gamma_++\gamma_-)+\tfrac{2}{3}(\Delta_++\Delta_-),$$
$$m>2|\gamma_+-\Delta_+|+2|\gamma_--\Delta_-|. \tag{B7}$$

Under these conditions, both coefficients π_x and π_y in Eq. (37) are positive and the inequality $\phi>\epsilon^2$ holds.

2. Tunneling matrix of a general form

We now consider a general tunneling matrix $\beta_{jk}^{(a)}$ corresponding to the function

$$\underline{\beta}(k)=\beta_0+\sum_{n=1}^{\infty}(\beta_n^{(+)}e^{-ikn}+\beta_n^{(-)}e^{ikn}), \tag{B8}$$

with $\beta_n^{(\pm)}>0$ and $\beta_0>0$. It is convenient to introduce $\beta_n=\beta_n^{(+)}+\beta_n^{(-)}$ and $\delta_n=(\beta_n^{(+)}-\beta_n^{(-)})/(\beta_n^{(+)}+\beta_n^{(-)})$, so $|\delta_n|\leqslant 1$. Let us assume that the diagonal element is sufficiently large:

$$\beta_0>\sum_{n=1}^{\infty}\beta_n=\sum_{n=1}^{\infty}(\beta_n^{(+)}+\beta_n^{(-)}). \tag{B9}$$

Under this assumption, $\mathrm{Re}\underline{\beta}(k)>0$ for all k and the matrix $\beta_{jj'}^{(a)}$ is positive definite. Simultaneously, the inequality $\phi>\epsilon^2$ or, equivalently, $\pi_{x,y}>0$, is also satisfied. Indeed, the inequality $\phi>\epsilon^2$ is equivalent to

$$\left(\beta_0+\sum_{n=1}^{\infty}\beta_n\right)\sum_{n=1}^{\infty}n^2\beta_n>2\left(\sum_{n=1}^{\infty}n\delta_n\beta_n\right)^2. \tag{B10}$$

To prove that this relation holds, we use the inequality

$$\sum_{n=1}^{\infty}\beta_n\sum_{n=1}^{\infty}n^2\beta_n\geqslant\left(\sum_{n=1}^{\infty}n\beta_n\right)^2, \tag{B11}$$

which follows from the Cauchy-Schwarz inequality $\boldsymbol{a}^2\boldsymbol{b}^2\geqslant(\boldsymbol{ab})^2$ upon setting $a_n=\sqrt{\beta_n}$ and $b_n=n\sqrt{\beta_n}$. Therefore, we find

$$\left(\beta_0+\sum_{n=1}^{\infty}\beta_n\right)\sum_{n=1}^{\infty}n^2\beta_n>2\sum_{n=1}^{\infty}\beta_n\sum_{n=1}^{\infty}n^2\beta_n$$
$$\geqslant 2\left(\sum_{n=1}^{\infty}n\beta_n\right)^2\geqslant 2\left(\sum_{n=1}^{\infty}n\delta_n\beta_n\right)^2. \tag{B12}$$

Hence, the condition (B9) is sufficient to guarantee both the absence of zero eigenvalues and the positivity of the NLσM coefficients.

However, the condition (B9) is not necessary. To demonstrate this fact, it is enough to consider the following example: $\underline{\beta}(k)=\beta_0+\beta_1\cos k-i\delta_1\beta_1\sin k$ with $1/\sqrt{2}<|\delta_1|<1$. Then except for the special case $\beta_0=\beta_1$, one has $|\underline{\beta}(k)|>0$. Nevertheless, for $\beta_0/\beta_1<2\delta_1^2-1<1$, one finds $\phi<\epsilon^2$.

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higher-frequency modes are nondiffusive and merely renormalize the bare parameters of the action. Consequently, this truncation does not affect the long-distance, low-energy results, provided all external frequencies remain well below $1/\tau$. For a detailed discussion of this cutoff procedure, see Ref. [97].

[83] We note that one has to preserve the relation (7) between $\bar{\Xi}$ and Ξ. Thus, the relation $\hat{u} = s_2\hat{u}^* s_2$ should hold. Indeed, it satisfies for $\hat{u} \in \mathrm{SU}(2)$.

[84] In the derivation of Eq. (20), we used the following Fierz identities involving $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$: $\mathrm{tr}_\sigma \boldsymbol{\sigma} A \boldsymbol{\sigma} B = 2\mathrm{tr}_\sigma A \mathrm{tr}_\sigma B - \mathrm{tr}_\sigma AB$ and $\mathrm{tr}_\sigma \boldsymbol{\sigma} A \mathrm{tr}_\sigma \boldsymbol{\sigma} B = 2\mathrm{tr}_\sigma AB - \mathrm{tr}_\sigma A \mathrm{tr}_\sigma B$, where the trace tr_σ operates in the spin space.

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[88] We note that these boundary conditions for the matrix Q are not general. For a detailed review on the boundary conditions in the qHe-type problems, see Refs. [69,41].

[89] We note that the inequality $\mathrm{Re}\underline{\beta}(k) > 0$ guarantees that $|\underline{\beta}(k)| > 0$ and, consequently, the matrix $\beta^{(0)}$ has no zero eigenvalues. However, the absence of zero eigenvalues is possible even if $\mathrm{Re}\underline{\beta}(k) < 0$.

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