Topic 2. Kinetic equation. Calculations of electron electrical and thermal conductivity

Plan of the topic 2.

1. **Kinetic equation. Full time derivative of electron distribution function. Approximate form of collision integral in the tau-approximation.**
2. **Kinetic equation in the presence of electric field. Electrical and thermal conductivities of metals in the tau approximation.**
3. **Scattering mechanisms: impurities, electron-electron and electron-phonon interactions. Corresponding temperature dependences of electric and thermal conductivities.**
4. **Thermoelectric phenomena.**
5. **Electronic motion in metals in magnetic field.**
6. **Galvanomagnetic properties of metals. Classical magnetoresistance. Conductivity tensor in the limits of weak and strong magnetic field.**

Kinetic equation $\boxed{\frac{\mathrm{d}f}{\mathrm{d}t}=I(f),}$ is a differential equation on *f*,

where $f=f(p,r)$ is the distribution function in phase space (p,r) and $I(f)$ is the so-called collision integral.

The full time derivative of $f(p,r)$ without magnetic field B is

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial \boldsymbol{r}}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}+\frac{\partial f}{\partial \boldsymbol{p}}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=\frac{\partial f}{\partial t}+\boldsymbol{v}\frac{\partial f}{\partial \boldsymbol{r}}+e\boldsymbol{E}\frac{\partial f}{\partial \boldsymbol{p}}.$$

group velocity of electron wave packet

external force according to the second Newton's law

The collision integral in the so-called tau-approximation is $I(f)=\dfrac{f-f_0}{\tau},$ **where**

f_0 is the distribution function in equilibrium (without external fields), and τ depends on the scattering mechanism (impurity, e-e, phonons, etc.)

Kinetic equation $\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = I(f),}$ where

the full time derivative of distribution function $f(p,r)$ without magnetic field B

$$\frac{df}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \boldsymbol{r}}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \frac{\partial f}{\partial \boldsymbol{p}}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \boldsymbol{v}\frac{\partial f}{\partial \boldsymbol{r}} + e\boldsymbol{E}\frac{\partial f}{\partial \boldsymbol{p}}.$$

The collision integral in the so-called tau-approximation is $I(f) = -\frac{f - f_0}{\tau_{tr}}$, **where** $f=f_0+f_1$ **, and**

f_0 **is the distribution function in equilibrium (without external fields),**

and $f_1(\vec{p}) = \vec{p}\cdot\vec{E}\eta(\varepsilon)$

satisfies the kinetic equation:

$$-\frac{1}{\tau_{tr}(\varepsilon)}\vec{p}\cdot\vec{E}\eta(\varepsilon)$$

Electric current j is simply related to distribution function

$$\vec{j} = 2e\int \vec{\mathrm{v}}_p f_1(\vec{p})\frac{d^3p}{(2\pi\hbar)^3}$$

Substituting f_1 we obtain **electric current**:

$$\boldsymbol{j} = -e^2\int \boldsymbol{v}(\boldsymbol{v}\boldsymbol{E})\tau\frac{\partial f_0}{\partial \varepsilon}\nu(\varepsilon)\,\mathrm{d}\varepsilon\frac{\mathrm{d}\Omega}{4\pi},$$

At T=0 $\partial f_0/\partial\varepsilon \approx -\delta(\varepsilon - \mu)$ **and**

$$\boxed{\boldsymbol{j} = \tfrac{1}{3}e^2\boldsymbol{E}[v^2\tau\nu(\varepsilon)]_{\varepsilon=\mu}}$$

Appendices: some useful mathematical formulas

1. For arbitrary function F(p) we can change the integration variable

$$\int \frac{F(p)d^3p}{(2\pi)^3} = \int F(p)\frac{p^2dpd\Omega}{(2\pi)^3} = \int F(p)\nu(\varepsilon)d\varepsilon\frac{d\Omega}{4\pi},$$

2. At T=0 $\partial f_0/\partial\varepsilon \approx -\delta(\varepsilon-\mu)$ and $\int F(p)\delta(\varepsilon-\mu)d\varepsilon\frac{d\Omega}{4\pi} = \int F(p_F)\frac{d\Omega}{4\pi}.$

3. The angular averaging of the vector product of electron velocity (appears for conductivity): $\int (vE)v\frac{d\Omega}{4\pi} = \frac{1}{3}v^2E.$

To obtain this we take $E=E_x$. Then $\int (vE_x)v\frac{d\Omega}{4\pi} = \int (v_xE_x)v_x\frac{d\Omega}{4\pi} = v_x^2E_x = \frac{v^2}{3}E_x.$

4. Temperature dependence of various quantities can often be calculated using the following formulas with Fermi distribution function $f(\varepsilon)$:

$$\int \frac{\partial f}{\partial\varepsilon}d\varepsilon = -1, \quad \int (\varepsilon-\mu)^2\frac{\partial f}{\partial\varepsilon}d\varepsilon = -(4T)^{-1}\int_{-\infty}^{\infty}\frac{z^2\,dz}{\cosh^2(z/2T)} = -\frac{\pi^2T^2}{3} \quad \Rightarrow$$

$$\int F(\varepsilon)\frac{\partial f}{\partial\varepsilon}d\varepsilon \approx -F(\mu) - \tfrac{1}{6}\pi^2T^2F''(\mu).$$

Electric current at arbitrary temperature $j = -e^2 \int \boldsymbol{v}(\boldsymbol{vE})\tau \frac{\partial f_0}{\partial \varepsilon} \nu(\varepsilon)\, d\varepsilon \frac{d\Omega}{4\pi}$,

It is the same as at T=0 up to the terms $\sim T^2/E_F^2 \ll 1$**, because** $\int F(\varepsilon) \frac{\partial f}{\partial \varepsilon} d\varepsilon \approx -F(\mu) - \frac{1}{6}\pi^2 T^2 F''(\mu)$

The angular averaging of the vector product $\int (vE) v \frac{d\Omega}{4\pi} = \frac{1}{3} v^2 E.$

At T=0 $\partial f_0 / \partial \varepsilon \approx -\delta(\varepsilon - \mu)$ **and** $\boxed{j = \frac{1}{3} e^2 E [v^2 \tau \nu(\varepsilon)]_{\varepsilon = \mu}}$

Here the density of states (DoS) at the Fermi level $\mu = E_F$ **is**

$$\nu(\mu) = 2\int \frac{d^3 p}{(2\pi)^3} \delta(\varepsilon(p) - \mu),$$

factor 2 comes from spin degeneracy

Electrical conductivity

$$\boxed{\sigma = \frac{1}{3} e^2 [v^2 \tau \nu(\varepsilon)]_\mu .}$$

For Fermi gas with quadratic dispersion $\sigma = (1/3) e^2 \upsilon_F^2 \tau_{tr}(\mu) \nu(\mu)$

(as far as $\mu = m\upsilon_F^2/2$)

or $\sigma = n_e e^2 \tau_{tr}/m$ **where electron concentration** $n_e = \frac{N_c}{V} = 2\frac{4\pi}{3}\frac{p_F^3}{(2\pi\hbar)^3}$

Thermal conductivity (general remarks)

In contrast to the electric current, i.e., the transfer of electric charge, energy transfer involves not only electrons but also phonons, and therefore the thermal conductivity has two parts, an electron and a phonon part. We shall here consider only the electronic thermal conductivity. Usually, it far exceeds the phonon thermal conduc-

There is one significant difference between the mechanisms of charge and energy transfer. The charge is transferred by real electrons. Since the density of particles of the gas model ("electrons") is equal to the density of real electrons, we may consider the electric current that arises in the gas model under the action of the field, as has been done in the preceding section. As for the energy transfer, it is achieved by quasiparticles with the energy spectrum $\varepsilon_{qp} = |\xi|$ and therefore we will use the quasiparticle model here*.

First of all, we have to change the form of the kinetic equation slightly. Since the charge of "antiparticles" is opposite to that of "particles", instead of e we must write $e\,\mathrm{sign}\,\xi$, where $\mathrm{sign}\,\xi = \xi/|\xi|$. Further, the velocity of quasiparticles is

$$\frac{\partial|\varepsilon|}{\partial \boldsymbol{p}} = \frac{\partial|\xi|}{\partial\xi}\frac{\partial\xi}{\partial \boldsymbol{p}} = \boldsymbol{v}\,\mathrm{sign}\,\xi, \quad \text{where } \boldsymbol{v} = \partial\varepsilon/\partial\boldsymbol{p} = \boldsymbol{p}/m^*. \qquad \varepsilon(p)\approx\xi(p) = v_F(p-p_F)$$

The kinetic equation for quasiparticles is

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\,\mathrm{sign}\,\xi\,\frac{\partial f}{\partial \boldsymbol{r}} + e\boldsymbol{E}\,\mathrm{sign}\,\xi\,\frac{\partial f}{\partial \boldsymbol{p}} = -\frac{(f-f_0)}{\tau}$$

Thermal conductivity (calculation)

The kinetic equation for quasiparticles is

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\,\mathrm{sign}\,\xi\,\frac{\partial f}{\partial \boldsymbol{r}}+e\boldsymbol{E}\,\mathrm{sign}\,\xi\,\frac{\partial f}{\partial \boldsymbol{p}}=-\frac{(f-f_0)}{\tau}$$

In the presence of temperature gradient, the second term is nonzero:

$$\boldsymbol{v}\,\mathrm{sign}\,\xi\,\frac{\partial f_0}{\partial \boldsymbol{r}}=\frac{\partial f_0}{\partial T}\,\mathrm{sign}\,\xi\,(\boldsymbol{v}\nabla T)=-\frac{|\xi|}{T}\,\frac{\partial f_0}{\partial|\xi|}\,\mathrm{sign}\,\xi\,(\boldsymbol{v}\nabla T)=\frac{-\xi}{T}\,\frac{\partial f_0}{\partial|\xi|}(\boldsymbol{v}\nabla T).$$

The kinetic equation gives the solution for distribution function $f=f_0+f_1$ **:**

$$f_1=\tau\frac{\xi}{T}\,\frac{\partial f_0}{\partial|\xi|}(\boldsymbol{v}\nabla T)$$

The energy flux is

$$q=2\int \underset{\uparrow}{\xi}\boldsymbol{v}f_1\,\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3}=\int \boldsymbol{v}(\boldsymbol{v}\nabla T)\tau\nu(\varepsilon)\,\frac{\xi^2}{T}\,\frac{\partial f_0}{\partial|\xi|}\,\mathrm{d}\xi\,\frac{\mathrm{d}\Omega}{4\pi}.$$

In addition to electric charge, the quasi-particle transfer energy ξ

The integrals are the same, as in the calculation of electric current:

$$\int F(\varepsilon)\,\frac{\partial f}{\partial \varepsilon}\,\mathrm{d}\varepsilon\approx-F(\mu)-\tfrac{1}{6}\pi^2T^2\,F''(\mu).$$

Finally, we get for thermal flux

$$\boxed{\boldsymbol{q}=-\tfrac{1}{9}\pi^2T\nabla T(\nu v^2\tau)_\mu}$$

Thermal conductivity is defined as $\boldsymbol{q}=-\varkappa\nabla T.$ => $\varkappa=\tfrac{1}{9}\pi^2T(\nu v^2\tau)_\mu$

∝ temperature.

Temperature dependence of thermal conductivity in metals

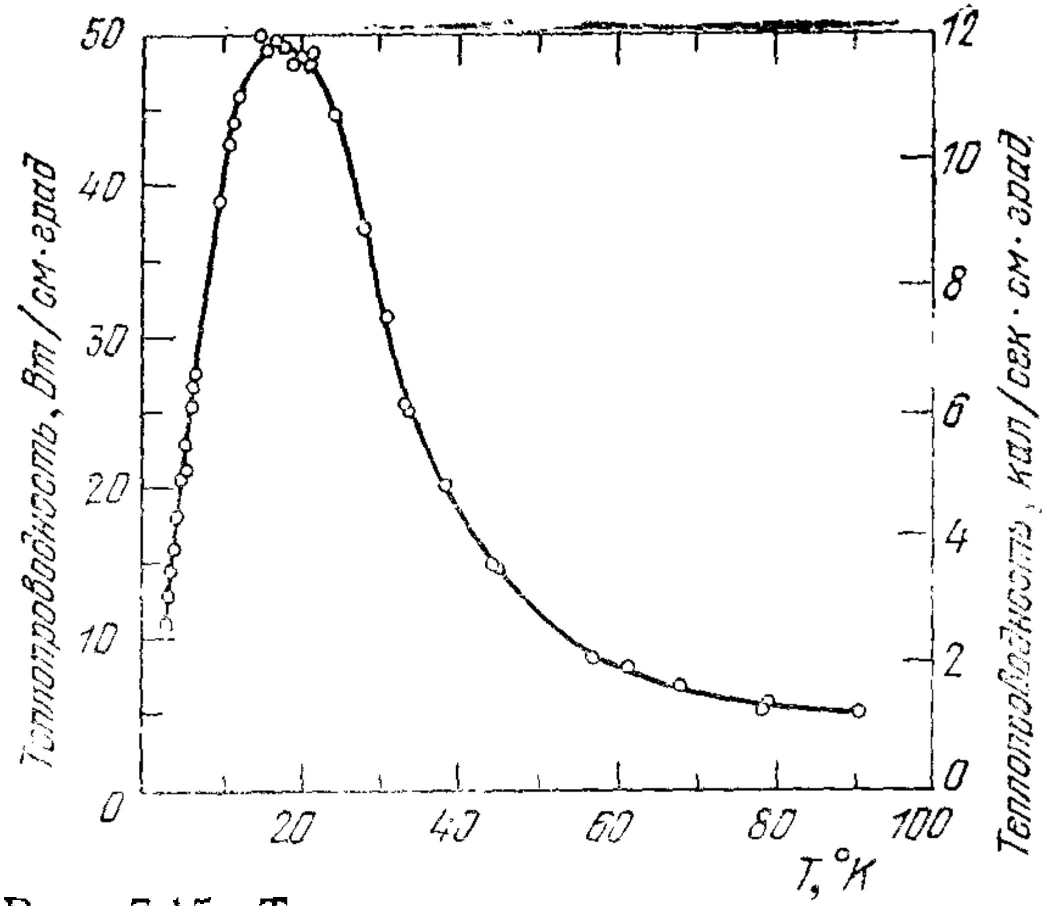


Рис. 7.15. Температурная зависимость теплопроводности меди [11].

Thermal conductivity of ideal gas

$$K = {}^{1}/_{3} C v l,$$

where heat capacity of metal C(T) ∞ T, and the mean free path *l=l(T)*:

$$l(T) \begin{cases} = const, \text{ at low } T \\ \propto 1/T \text{ , at high } T>>\theta_D \end{cases}$$

Thermal conductivity of metals

$$K(T) \begin{cases} \propto T \quad \text{at low } T \\ = const \text{ at high } T>>\theta_D \end{cases}$$

Wiedemann – Franz law

Electric conductivity

$$\sigma = \tfrac{1}{3}e^2[v^2\tau\nu(\varepsilon)]_\mu .$$

Thermal conductivity

$$\varkappa = \tfrac{1}{9}\pi^2 T(\nu v^2\tau)_\mu .$$

Electric σ and thermal κ conductivities are related by the Wiedemann – Franz law :

$$\frac{\varkappa}{\sigma T} = \frac{\pi^2}{3e^2} .$$

This law is valid for arbitrary ε(p).

Thermoelectric phenomena in metals

Under a small temperature gradient combined with external electric field in the metal the following current and heat flows arise:

$$\vec{j} = \sigma \vec{E} + \beta \vec{\nabla} T; \vec{q} = \gamma \vec{E} + \varsigma \vec{\nabla} T$$

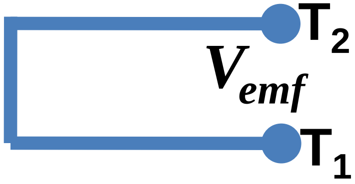
The Onsager symmetry principle relates the coefficients β and γ ; $\gamma = -\beta T$

In practice, it is always easier to control the electric current than the electric field inside a metal. Then one can use the system

$$\begin{cases} \boldsymbol{E} = \rho \boldsymbol{j} + Q \nabla T, \\ \boldsymbol{q} = \Pi \boldsymbol{j} - \kappa \nabla T, \end{cases}$$

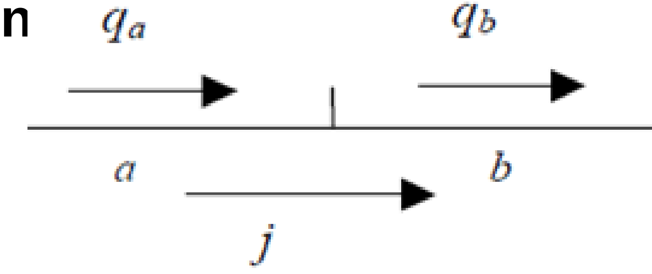
here $\rho = \sigma^{-1}$, $Q = -\beta/\sigma$, $\Pi = \gamma/\sigma$, $\kappa = \gamma\beta/\sigma - \xi$. **Also** $\Pi = QT$, $\kappa = -(E\beta^2/\sigma + \xi)$.

The coefficient Q is called the <u>Seebeck coefficient</u>. It gives thermopower (potential) per unit of temperature change (also called differential thermo-emf).



$$V_{emf} = \int Edl = \int \frac{EdT}{dT/dl} = \int QdT = Q(T_2 - T_1)$$

***<u>Peltier effect.</u>* A current flows through the junction of two metals: $j_a = j_b$. Since $\Pi_a \neq \Pi_b$, the thermal currents $q_a \neq q_b$, => the heat is absorbed or released on the junction: $W_{ab} = q_a - q_b = (\Pi_a - \Pi_b)j$.**



Seebeck effect

The Seebeck effect ***is the conversion of heat directly into electricity*** **at the junction of different types of wire.** Originally discovered in 1794 by Italian scientist Alessandro Volta, it is named after the Baltic German physicist Thomas Johann Seebeck, who in 1821 independently rediscovered it. It was observed that a compass needle is deflected by a closed loop formed by two different metals joined in two places, with a temperature difference between the joints. This was because the electron energy levels in each metal shift differently and a potential difference between the junctions creates an electrical current and therefore a magnetic field around the wires. Seebeck did not recognize that there was an electric current involved and called the phenomenon "thermomagnetic effect". Danish physicist Hans Christian Ørsted rectified the oversight and coined the term ***"thermoelectricity"***.

The Seebeck effect is a classic example of an electromotive force (emf) and leads to measurable currents or voltages in the same way as any other emf. The local current density is given by $J = \sigma\,(-\nabla V + E_{emf})$, where V is the local voltage, and σ is the local conductivity.

The Seebeck effect is described locally by the creation of electromotive field $E_{emf} = -S\,\nabla T$, where S is the ***Seebeck coefficient*****.**

Seebeck coefficient

If the temperature difference ΔT between the two ends of a material is small, then the Seebeck coefficient of a material is defined as:

$$S = -\Delta V / \Delta T = -(V_{left} - V_{right})/(T_{left} - T_{right}) ,$$

where ΔV is the thermoelectric voltage seen at the terminals.

Thus, if S is positive, the end with the higher temperature has the lower voltage, and vice versa. The voltage gradient in the material will point against the temperature gradient.

The Seebeck effect is generally dominated by the contribution from charge carrier diffusion which tends to push charge carriers towards the cold side of the material until a compensating voltage has built up. As a result, in p-type semiconductors (havinge only positive mobile charges, electron holes), S is positive. Likewise, in n-type semiconductors (which have only negative charge carriers, electrons), S is negative. In most conductors, however, the charge carriers exhibit both hole- and electron-like behavior and the sign of S depends on which of them predominates.

Seebeck effect is often used to determine the sign of main charge carriers

Peltier effect

When an electric current is passed through a circuit of a thermocouple, heat is evolved at one junction and absorbed at the other junction. This is known as Peltier effect. The Peltier effect is the presence of heating or cooling at an electrified junction of two different conductors. It is named after French physicist Jean Charles Athanase Peltier, who discovered it in 1834.

When a current is made to flow through a junction between two conductors, A and B, heat may be generated or removed at the junction. The Peltier heat generated at the junction per unit time is $dQ/dt = (\Pi_A - \Pi_B)\, I$, where Π_A and Π_B are the *Peltier coefficients* of conductors A and B, and I is the electric current (from A to B). The total heat generated is not determined by the Peltier effect alone, as it may also be influenced by Joule heating and thermal-gradient effects.
The Peltier coefficients represent how much heat is carried per unit charge. Since charge current must be continuous across a junction, the associated heat flow will develop a discontinuity if Π_A and Π_B differ.

Thermoelectric heat pumps exploit this phenomenon, as do thermoelectric cooling devices found in refrigerators.

Thomson effect

In different materials, the Seebeck coefficient is not constant in temperature, and a spatial gradient in temperature can result in a gradient in the Seebeck coefficient. If a current is driven through this gradient, then a continuous version of the Peltier effect will occur. This Thomson effect was predicted and subsequently observed in 1851 by Lord Kelvin (William Thomson). It describes the heating or cooling of a current-carrying conductor with a temperature gradient. If a current density J is passed through a homogeneous conductor, the Thomson effect predicts a heat production rate per unit volume $dq/dt = -KJ \cdot \nabla T$, where ∇T is the temperature gradient, and K is the *Thomson coefficient*. The Thomson coefficient is related to the Seebeck coefficient as $K = TdS/dT$.

Collision integral in the Boltzmann kinetic equation for the scattering by impurities (1)

The scattering probability of electron on impurity centers in the Born approximation is:

$$\frac{1}{\tau} = w = \frac{2\pi}{\hbar}\int \left|V_{p'p}\right|^2 \delta\left(\varepsilon(p)-\varepsilon(p')\right)\frac{d^3p'}{(2\pi\hbar)^3}V$$

here $V_{p'p}$ is the transition matrix element of the electron interaction $V(r)$ with all impurity atoms, occupying positions R_i:

$$V(\vec{r}) = \sum_i \upsilon\left(\vec{r}-\vec{R}_i\right)$$

The matrix element of interaction potential $V(r)$ is taken on the Bloch wave functions:

$$\Psi_p(\vec{r}) = V^{-1/2}e^{i\vec{p}\vec{r}/\hbar}u_p(\vec{r})$$

$$V_{p'p} = V^{-1}\sum_i \int e^{-ip'r/\hbar}u^*_{p'}(\vec{r})\,\upsilon\left(\vec{r}-\vec{R}_i\right)e^{ipr/\hbar}u_p(\vec{r})\,d^3\vec{r} =$$

$$= V^{-1}\sum_i e^{-i(\vec{p}'-\vec{p})\vec{R}_i/\hbar}\int \upsilon(\vec{r})\,u^*_{p'}(\vec{r})\,u_p(\vec{r})\,e^{-i(\vec{p}'-\vec{p})\vec{r}/\hbar}d^3\vec{r} = V^{-1}\sum_i e^{-i(\vec{p}'-\vec{p})\vec{R}_i/\hbar}\upsilon_{p'p}$$

and

$$w = \frac{2\pi}{\hbar}V^{-2}\int |\upsilon_{p'p}|^2 \sum_{i,k} e^{-i(p'-p)(R_i-R_k)/\hbar}\delta\left(\varepsilon(p)-\varepsilon(p')\right)\frac{d^3p'}{(2\pi\hbar)^3}V$$

Collision integral for scattering by impurities (2)

In Born approximation the scattering probability of electron on impurities

$$w=\frac{2\pi}{\hbar}V^{-2}\int|\upsilon_{p'p}|^2\sum_{i,k}e^{-i(p'-p)(R_i-R_k)/\hbar}\delta\left(\varepsilon(p)-\varepsilon(p')\right)\frac{d^3p'}{(2\pi\hbar)^3}V$$

Positions of the impurity atoms in the crystal are assumed to be random, and the averaging over impurity positions reduces the sum over impurities to diagonal terms only, i.e. with coinciding positions of the impurity atoms $R_i=R_k$, because other terms strongly oscillate and vanish. Hence, the sum turns into the number of impurities in the crystal N_i and w finally takes the form (after cancellation of one power of V) :

$$w=\frac{2\pi}{\hbar}\frac{N_i}{V}\int|\upsilon_{p'p}|^2\,\delta\left(\varepsilon(p)-\varepsilon(p')\right)\frac{d^3p'}{(2\pi\hbar)^3}$$

, where $N_i/V=n_i$ is the impurity concentration

The collision integral $I=I(f)$ depends on the filling numbers (Fermi-Dirac distribution function) in the initial and final states:

$$I(\mathrm{f})=\frac{2\pi}{\hbar}n_i\int|\upsilon_{p'p}|^2\left[\mathrm{f}(p')\left(1-\mathrm{f}(p)\right)-\mathrm{f}(p)\left(1-\mathrm{f}(p')\right)\right]\delta\left(\varepsilon(p)-\varepsilon(p')\right)\frac{d^3p'}{(2\pi\hbar)^3}$$

It is zero for the equilibrium (zero-field) distribution function $f_0=f_0(\varepsilon)$.

Collision integral (3)

The kinetic equation is $$\frac{d\boldsymbol{f}}{d\boldsymbol{t}}=\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{t}}+\frac{\partial \boldsymbol{f}}{\partial \mathbf{r}}\frac{d\mathbf{r}}{d\boldsymbol{t}}+\frac{\partial \boldsymbol{f}}{\partial \mathbf{p}}\frac{d\mathbf{p}}{d\boldsymbol{t}}=\boldsymbol{I}(\boldsymbol{f}),$$

In a stationary state the distribution function $f=f_0+f_1$ is spatially uniform and does not explicitly depend on time: $\partial f/\partial t=0$ and $\partial f/\partial r=0$. The last term $\partial f/\partial p\approx(\partial f_0/\partial\varepsilon)\,(\partial\varepsilon/\partial p)=v_F(\partial f_0/\partial\varepsilon)\neq 0$ and $dp/dt=eE$. Then in the first order in the small perturbation eE the kinetic equation becomes:

$$e\vec{E}\cdot\vec{\mathrm{v}}_p\frac{\partial \mathrm{f}_0}{\partial\varepsilon}=\frac{2\pi}{\hbar}n_i\int|\upsilon_{p'p}|^2\left[\mathrm{f}_1(p')-\mathrm{f}_1(p)\right]\delta\left(\varepsilon(p)-\varepsilon(p')\right)\frac{d^3p'}{(2\pi\hbar)^3}$$

here $\vec{\mathrm{v}}_p\equiv\partial\varepsilon(\vec{p})/\partial\vec{p}$ is the group velocity of the particle.

The first-order correction f_1 to distribution function is linear in force eE. There is only one additional vector quantity p to make a scalar product to form f_1: $\mathrm{f}_1(\vec{p})=\vec{p}\cdot\vec{E}\eta(\varepsilon)$ **where the function $\eta(\varepsilon)$ is unknown yet.**

Integration over the energy is performed in two steps:

$$\delta\left(\varepsilon(p)-\varepsilon(p')\right)\frac{2d^3p'}{(2\pi\hbar)^3}\overset{step1}{\Longrightarrow}\delta(\varepsilon-\varepsilon')\frac{2d\varepsilon' dS_{p'}}{(2\pi\hbar)^3\left|\vec{\mathrm{v}}_{p'}\right|}\overset{step\,2\,def\ \nu(\varepsilon')}{\Longrightarrow}\nu(\varepsilon')d\varepsilon'\frac{d\Omega'}{4\pi}$$

Here ν(ε’) is the density of electron states at the energy ε′ (including the factor of 2 allowing two possible projections of the electron spin 1/2).

Collision integral for scattering by impurities (4)

Then the kinetic equation is

$$\frac{e\vec{p}\cdot\vec{E}}{m}\frac{\partial f_0}{\partial\varepsilon} = \eta(\varepsilon)\int W\left(\theta_{p,p'}\right)\left[\vec{p}'\cdot\vec{E}-\vec{p}\cdot\vec{E}\right]\frac{d\Omega'}{4\pi}$$

where the dependence of the scattering amplitude *v* on the scattering angle θ is taken into account via the function

$$W\left(\theta_{p,p'}\right) = \frac{\pi}{\hbar}n_i\left|\upsilon\left(\theta_{p,p'}\right)\right|^2\nu(\varepsilon)$$

After all the manipulations, integral kinetic equation reduces to a simply algebraic equation:

definition of the transport scattering time

$$\frac{e\vec{p}\cdot\vec{E}}{m}\frac{\partial f_0}{\partial\varepsilon} = \vec{p}\cdot\vec{E}\eta(\varepsilon)\int W\left(\theta_{p,p'}\right)\left[\cos\left(\theta_{p,p'}\right)-1\right]\frac{d\Omega'}{4\pi} \equiv -\frac{1}{\tau_{tr}(\varepsilon)}\vec{p}\cdot\vec{E}\eta(\varepsilon)$$

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calculation details:

$$I(f) = pE\eta(\varepsilon)\int W(\theta)\left(\cos(\widehat{\mathbf{p}',\mathbf{E}})-\cos(\widehat{\mathbf{p},\mathbf{E}})\right)\frac{d\Omega}{4\pi}.$$

We choose the polar axis *z* along *p*. Then $\mathbf{p}'\mathbf{E} = p'_z E_z + p'_\perp E_\perp$, i.e.,

$$\cos(\widehat{\mathbf{p}',\mathbf{E}}) = \cos(\widehat{\mathbf{p}',\mathbf{p}})\cos(\widehat{\mathbf{E},\mathbf{p}}) + \sin(\widehat{\mathbf{p}',\mathbf{p}})\sin(\widehat{\mathbf{E},\mathbf{p}})\cos\phi_{p'E}$$

where $\phi_{p'E}$ is the angle between the projections $\mathbf{p}'$ and $\mathbf{E}$ onto the plane $\perp p$.

This term disappears after the integration over the angle φ.

Why impurities in metals can be described as short-range?

The typical range of impurity potential is of the order of atomic radius a_B. If impurity is uncharged (another atom), its range is ~ a_B . If the impurity is charged, in metals any charge is screened on the Debye radius r_D.

Then the impurity potential is given by $\varphi = -Ze\frac{e^{-\varkappa_D r}}{r}$, where $-Ze$ is the charge of the ion and $\varkappa_D = r_D^{-1}$ is the inverse Debye radius

This impurity potential can be derived from $\Delta\varphi = 4\pi e n_e'$ **where the change of electron density**

$$n_e' = n_e(\mu + e\varphi) - n_e(\mu) \approx e\varphi\frac{\partial n_e}{\partial \mu}$$

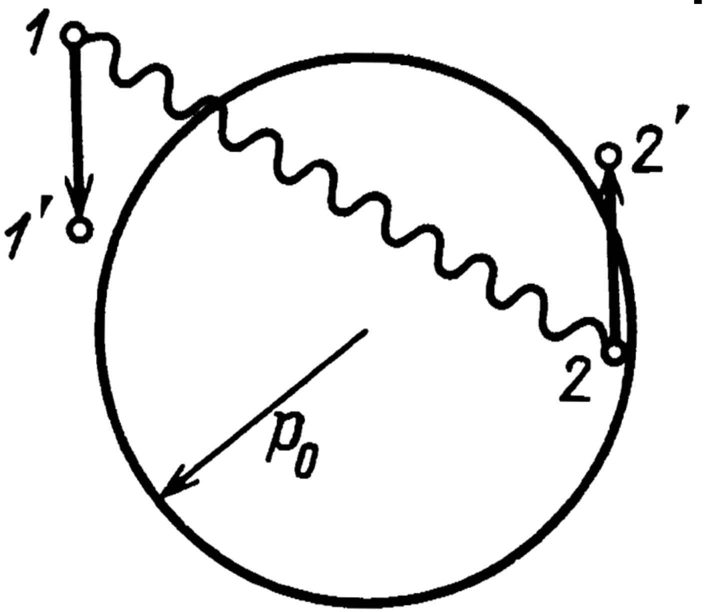
Then the Maxwell equation gives $\Delta\varphi - 4\pi e^2\varphi\frac{\partial n_e}{\partial \mu} = 0.$ **=>** $\varphi = -Ze\frac{e^{-\varkappa_D r}}{r}$,

where the inverse Debye (screening) radius is $\varkappa_D = \left(4\pi e^2\frac{\partial n_e}{\partial \mu}\right)^{1/2}$.

By the order of magnitude $\varkappa_D \sim \left(e^2\frac{p_0^3}{\hbar^3}\frac{m}{p_0^2}\right)^{1/2} \sim \frac{p_0}{\hbar}\left(\frac{e^2 m}{p_0\hbar}\right)^{1/2} \sim \frac{p_0}{\hbar}\left(\frac{e^2}{\hbar v}\right)^{1/2} \sim \frac{p_0}{\hbar}$

the Debye radius is of the order of atomic distance.

Electron-electron scattering rate $1/\tau_{ee} \propto T^2$.



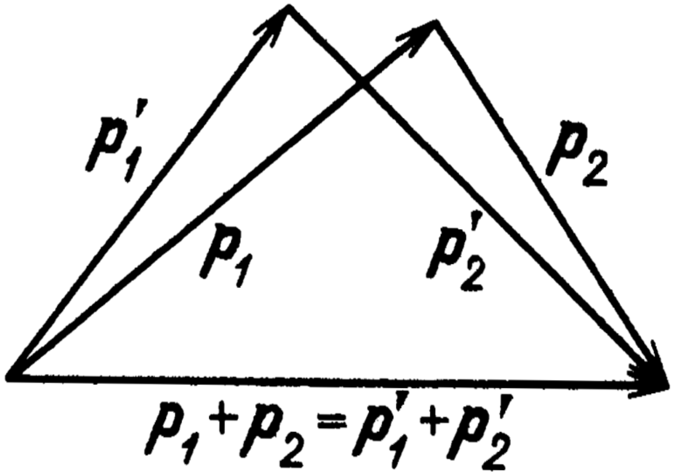
The scattering probability W is proportional to

$$W \propto \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2')\, d^3 p_2\, d^3 p_1'$$

law of energy conservation

p_2' is fixed by law of total momentum conservation

The initial energy of quasiparticle is close to Fermi energy: $|E_1 - E_F| \sim T \ll E_F$, being in a shell of width ~T around Fermi surface (if $E_1 - E_F \ll -T$, the total energy of electron pair is much below $2E_F$, and there is no empty states for them to scatter in at this energy). Then the energies E_1' and E_2 are also close to Fermi energy (in a shell of the width T for same reason), which reduces the scattering probability W by a factor $(T/E_F)^2$.

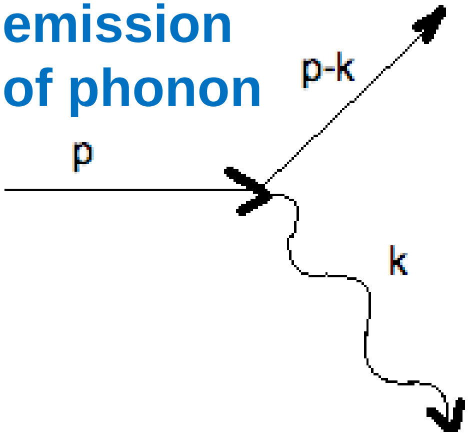


Total momentum relaxation (resistance) appears only from the umklapp processes, + scattering by reciprocal lattice vector:

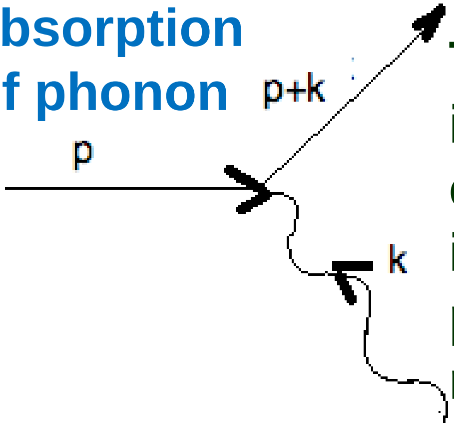
$$p_1 + p_2 = p_1' + p_2' + \hbar K.$$

Electron-phonon interaction.

emission of phonon



absorption of phonon



The origin of e-ph interaction is the Coulomb force between electrons and the charges induced by electric polarization created by ion replacements in their motion.

The scattering probability w is given by golden Fermi rule

$$w^{(1)}_{e-ph}=\frac{2\pi}{\hbar}\int\left|\hat{H}_{e-ph}\left(\vec{p}-\hbar\vec{k},\vec{p}\right)\right|^2\delta\left(\varepsilon\left(\vec{p}\right)-\varepsilon\left(\vec{p}-\hbar\vec{k}\right)-\hbar\omega\left(\vec{k}\right)\right)\frac{d^3\vec{k}}{\left(2\pi\hbar\right)^3}V$$

$$\hat{H}_{e-ph}=e\int Q(\vec{r}-\vec{r}')\mathrm{div}\left[en\vec{u}(\vec{r}')\right]d^3\vec{r}'$$

where u(r) is the phonon (vibration) amplitude,

$Q(r)$ is the screened Coulomb potential of the ions, experienced by conduction electrons, $\mathrm{div}\left[en\vec{u}(\vec{r}')\right]\equiv\mathrm{div}\vec{P}$ is the charge distribution due to crystal-lattice polarization, caused by deformation u(r): $P=neu$. Without screening $Q(r)=1/r$. With screening $Q(r)\approx a^2\delta(r)$, where $a\sim\hbar/p_F$.

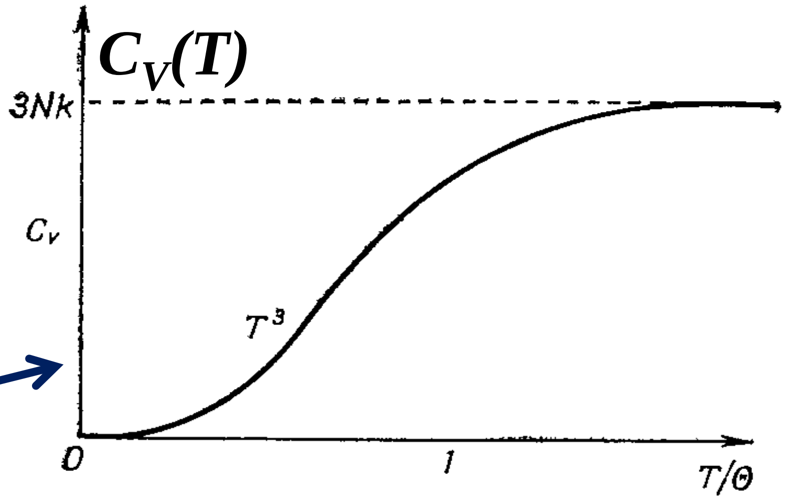
The matrix element

$$V_k\sim V^{-1/2}ie^2a^2n\frac{\omega}{s}u_k\sim -iV^{-1/2}e^2a^2n\left(\frac{\hbar(n_k+1)}{nM\omega}\right)^{1/2}\frac{\omega}{s}\sim -iV^{-1/2}p_0\left(\frac{\hbar\omega(n_k+1)}{nm}\right)^{1/2}.$$

Electron-phonon scattering rate at high T: $1/\tau_{e\text{-}ph} \propto T$.

The integrals are cut off at maximal phonon frequency=Debye energy = phonon frequency at wave length of crystal lattice constant: $\Theta_D=\omega(k=\pi/a)$.

Temperature dependence of heat capacity $C_V(T)$ changes at $T\sim\Theta_D$.



The “high” temperature case: $T >> \Theta_D$

The main contribution to electron scattering comes from phonons with high energy $\hbar\omega\sim\Theta_D$ and with filling numbers (obtained from Bose distribution)

$$n\left(\vec{k}_D\right)=\frac{1}{\exp\left(\hbar\omega\left(\vec{k}_D\right)/k_BT\right)-1}\bigg|_{T>>\Theta_D}\approx\frac{k_BT}{\hbar\omega\left(\vec{k}_D\right)}>>1$$

Then the electron-phonon scattering rate and conductivity:

$$\frac{1}{\tau_{tr}^{eph}}=w_{e-ph}\sim\frac{k_BT}{\hbar}\Rightarrow\sigma_{e-ph}=\mathrm{n_e e^2}\,\tau_{tr}^{eph}/\mathrm{m}\approx\mathrm{n_e e^2}\hbar/\mathrm{m}k_B\mathrm{T}\sim10^{16}\left(\frac{\mu}{T}\right)\mathrm{s}^{-1}$$

Therefore at room temperature $T_R>\Theta_D$, the main part of electron scattering is from phonons, and one observes a linear-T resistivity: $R(T)=\text{const}+\alpha T$.

Electron-phonon scattering rate at low T<< Θ_D : $1/\tau_{e\text{-}ph} \propto T^3$, but $1/\tau_{e\text{-}ph}^{tr} \propto T^5$, and resistivity from phonons $R_{e\text{-}ph}(T) \propto T^5$.

The typical phonon energy at T<< Θ_D is $\hbar\omega\left(\vec{k}\right) = sk \sim k_B T << \Theta_D$

Scattering rate is given by golden Fermi rule:

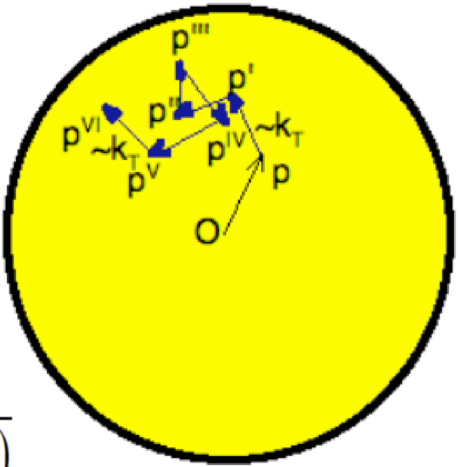
$$w_{e-ph}^{(1)} = \frac{2\pi}{\hbar} \int \left| \hat{H}_{e-ph}\left(\vec{p} - \hbar\vec{k}, \vec{p}\right) \right|^2 \delta\left(\varepsilon(\vec{p}) - \varepsilon\left(\vec{p} - \hbar\vec{k}\right) - \hbar\omega\left(\vec{k}\right)\right) \frac{d^3\vec{k}}{(2\pi\hbar)^3} V$$

The integral is taken over the absolute value of the phonon momentum, *k, and the angle* θ. The integral over θ cancels δ–function, because

$$\varepsilon(\vec{p}) - \varepsilon\left(\vec{p} - \hbar\vec{k}\right) - \hbar\omega\left(\vec{k}\right) = \frac{p^2 - \left(\vec{p} - \hbar\vec{k}\right)^2}{2m} - \hbar\omega\left(\vec{k}\right) = \frac{\hbar p k}{m}\left(\cos(\theta_{p,k}) - \frac{\hbar k}{2p} - \frac{m\omega}{pk}\right) = 0$$

The sound velocity s is much smaller than Fermi velocity: $s/\mathrm{v}_F \sim \sqrt{m_e/M_i}$

Hence, scattering by phonons gives electron diffusion along the Fermi surface, each scattering by a small "thermal" phonon momentum $k_T \sim k_B T/s \approx p_F T/\Theta_D \ll p_F$ **and by a small angle** $\theta_{p,p'} \sim k/p_F \sim T/\Theta_D \ll 1$



Hence, the transport scattering rate $1/\tau_{tr} << 1/\tau$:

$$\frac{1}{\tau_{tr}(\varepsilon)} \equiv \int W\left(\theta_{p,p'}\right)\left[1 - \cos\left(\theta_{p,p'}\right)\right]\frac{d\Omega'}{4\pi} \approx \left(T/\Theta_D\right)^2 \int W\left(\theta_{p,p'}\right)\frac{d\Omega'}{4\pi} \equiv \left(T/\Theta_D\right)^2 \frac{1}{\tau_E(\varepsilon)}$$

Electron-phonon scattering rate at low T<< Θ_D : $1/\tau_{e\text{-}ph} \propto T^3$, but $1/\tau_{e\text{-}ph}^{tr} \propto T^5$, and resistivity from phonons $R_{e\text{-}ph}(T) \propto T^5$.

The transport scattering rate $1/\tau_{tr} << 1/\tau_E$:

$$\frac{1}{\tau_{tr}(\varepsilon)} \equiv \int W(\theta_{p,p'})\left[1-\cos(\theta_{p,p'})\right]\frac{d\Omega'}{4\pi} \approx (T/\Theta_D)^2 \int W(\theta_{p,p'})\frac{d\Omega'}{4\pi} \equiv (T/\Theta_D)^2 \frac{1}{\tau_E(\varepsilon)}$$

The simple e-ph scattering rate is ∝T³: $1/\tau_E(\varepsilon) \sim (T/\Theta_D)^2 \cdot (k_B T/\hbar) \propto T^3$

because it contains $\int d^3k \propto \int_0^{k_{\max}} k^2 dk = k_{\max}^3/3 = \omega_{\max}^3/3s^3 \propto T^3$

Hence,

$$\sim (T/\Theta_D)^2 \frac{k_T^2}{p_F^2}\frac{k_B T}{\hbar} \sim (T/\Theta_D)^4 \cdot (k_B T/\hbar)$$

And we obtain the electric conductivity in pure metals at the low T due to electron-phonon scattering:

$$\boxed{\sigma \sim 10^{16}(\mu/k_B T)(\Theta_D/T)^4 \ \mathrm{s}^{-1}}$$

Simultaneously, for the electron thermal conductivity κ in metals we use the relaxation time τ_E, which gives

$$\kappa \propto C_V l v_F \approx C_V \tau_E v_F^2 \sim 10^8 (\Theta_D/T)^2 \frac{\mathrm{erg}}{\mathrm{cm \cdot s \cdot K}}$$

Hence, the Wiedemann-Frantz law does not hold.

Блоха-Грюнайзена формула (закон)

описывает температурную зависимость той части удельного электросопротивления металлов, которая обусловлена рассеянием, электронов на тепловых колебаниях кристаллич. решётки (фононах:)

$$\rho = \frac{m^*}{ne^2} \cdot \frac{1}{\tau};$$

$$\frac{1}{\tau} = \frac{9\pi^3}{\hbar k} \cdot \frac{c^2}{\theta_D} \cdot \frac{m^*}{M} \cdot \frac{1}{(aK_0)^3} \cdot \left(\frac{T}{\theta_D}\right)^5 F_5\left(\frac{T}{\theta_D}\right);$$

$$F_5(x) = \int_0^x \frac{z^5 dz}{(e^z - 1)(1 - e^{-z})}.$$

Она приводит для T<<θ_D к зависимости $\rho \propto T^5$, а при T>>θ_D к $\rho \propto T$.

Б-Г. ф., не учитывает анизотропию металла и рассеяние электронов на дефектах кристаллич. решётки, => служит для грубых оценок

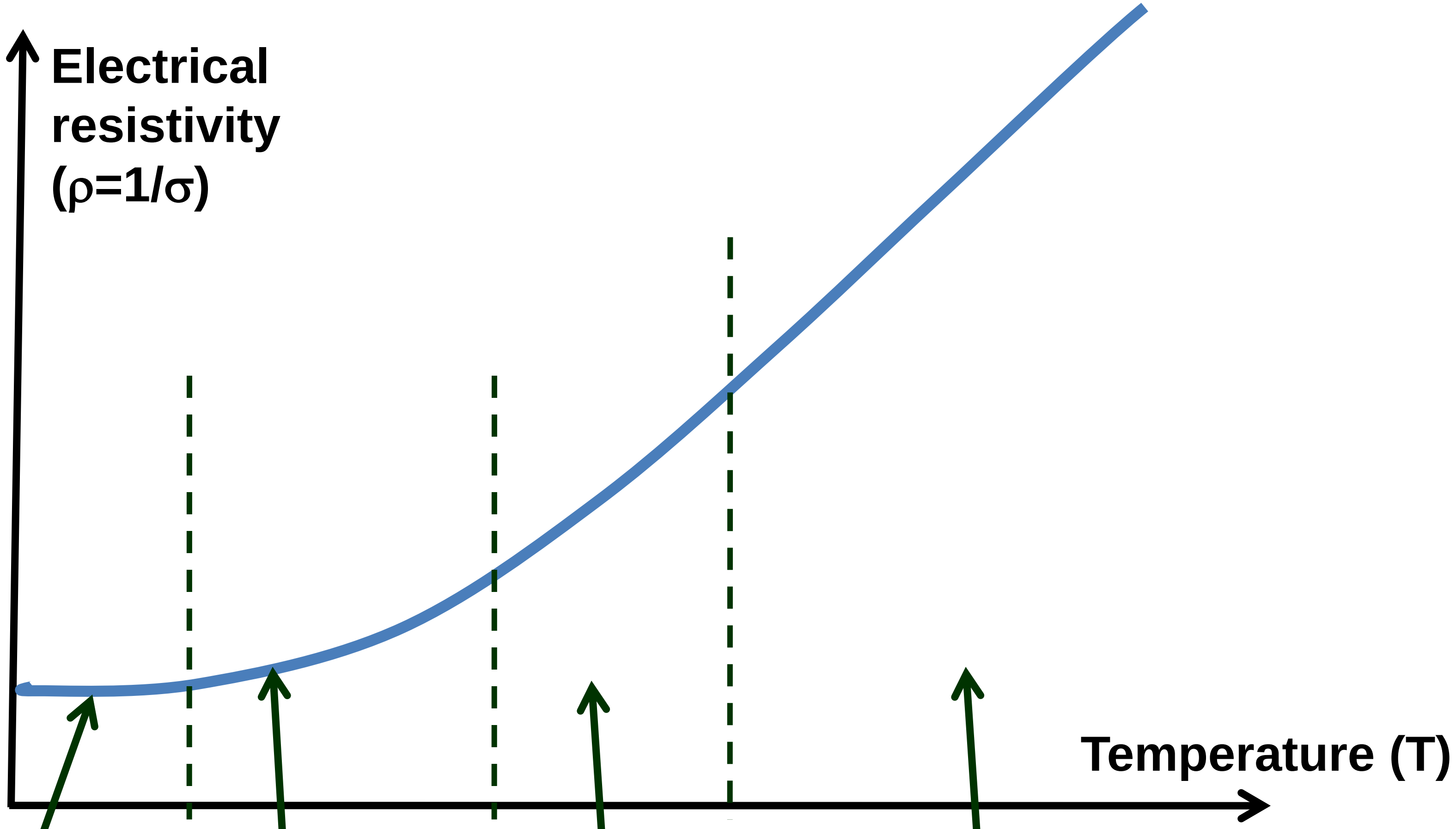
Правило Матиссена

Правило Матиссена — электрическое сопротивление реального металла равно сумме идеального и остаточного сопротивлений. **Идеальным сопротивлением называется электрическое сопротивление идеального кристалла. Остаточным сопротивлением называется электрическое сопротивление реального кристалла при абсолютном нуле температуры, вызванное столкновениями электронов с нарушениями периодичности кристаллической решетки металла.**

Объяснение

Вероятность столкновения электрона в кристаллической решетке с одним из препятствий равна $W = W_{ost} + W_{id}$. Вероятность столкновения обратно пропорциональна длине свободного пробега $W = 1/L$, => $1/L = 1/L_{ost} + 1/L_{id}$. Так как удельная электропроводность σ пропорциональна длине пробега L, а сопротивление $\rho = 1/\sigma$, отсюда следует правило Матиссена: $\rho = \rho_{ost} + \rho_{id}$.

Temperature dependence of electric resistivity in metals (sketch)



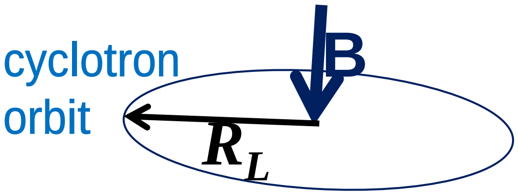
impurity scattering

e-e interaction + impurity scattering

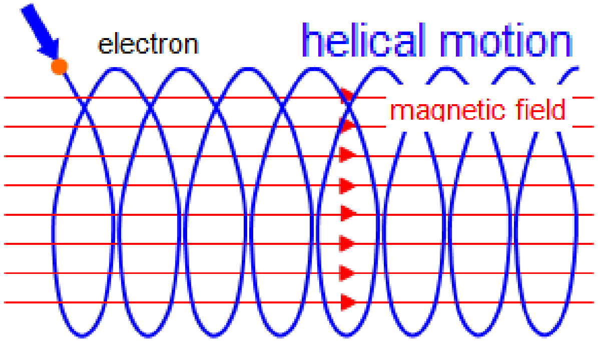
+ phonon scattering
$\rho(T) \propto T^5 + \text{const}$

mainly phonon scattering
$\rho(T) \approx \alpha T + \text{const}$

Electron motion in magnetic field. Lorenz force $F=(e/c)[v\times B]$.

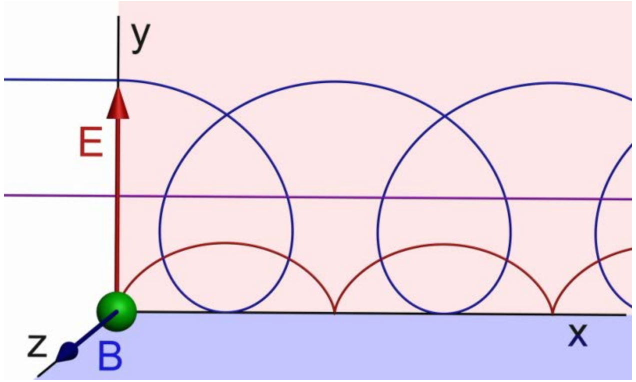


The Lorenz force $F=(e/c)[v\times B]$ is always ⊥ to magnetic field B and to electron velocity v.
In coordinate space free electrons in magnetic field move along a cyclotron circle ⊥B of Larmor radius $R_L=pc/eB$. The frequency of such periodic motion is called cyclotron frequency: $\omega_c=eB/mc$.
The electron momentum p_z along magnetic field is conserved, => electron trajectory is a helix:



In crossed fields (magnetic +weaker electric) this cyclotron orbit drifts ⊥ electric field E, so that the average velocity along electric field is zero, => average kinetic energy of electrons does not change:

$$\frac{d\mathcal{E}_{\text{кин}}}{dt}=\mathbf{v}\frac{d\mathbf{p}}{dt}=e\mathbf{E}\mathbf{v}.$$



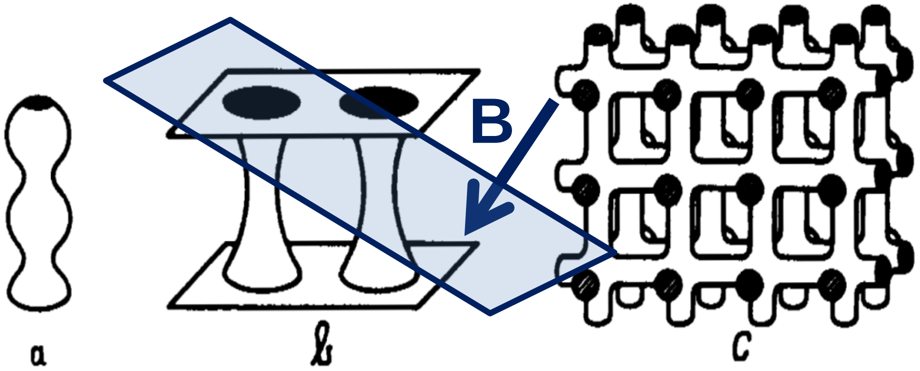
Average electric current along electric field vanishes, => diagonal part of electric conductivity tensor $\sigma_{xx}\perp$ B=B_z vanishes: $\sigma_{xx}=\sigma_{yy}=0$, and σ||B remains unchanged: $\sigma_{zz}(B_z)=\sigma_{zz}(B=0)$. Non-diagonal (Hall) $\sigma_{xy}(B_z)\neq 0$.

Electron motion in magnetic field in metals.

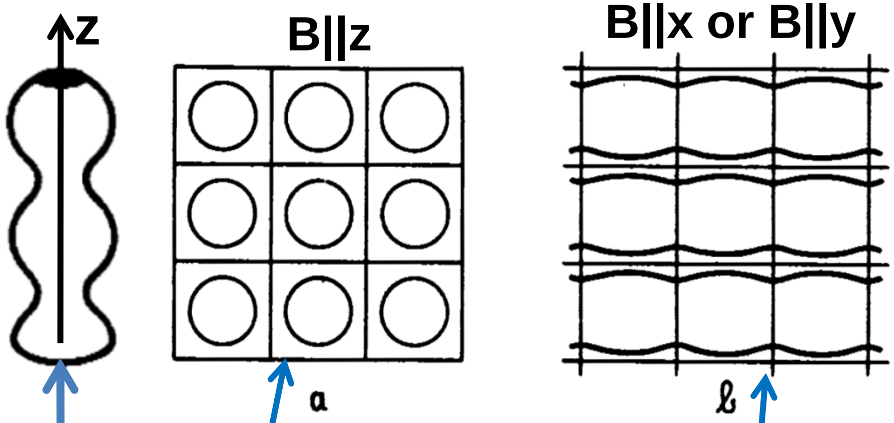
In metals in magnetic field B=H, due to the Lorenz force:

$$d\vec{p}/dt = e/c\left[\vec{v}\times\vec{H}\right] \equiv F_L$$

electrons move in momentum space along the surface ε=const in the plane ⊥B, i.e. p_z=p||B=const.



The electron orbits in momentum space may be closed (i.e. ellipse) or open (i.e. wavy line), depending on if this orbit intersects the boundary of Brillouin zone.



Example: the Fermi-surface is a warped cylinder with axis along z. For B||z the electron trajectories are closed (Fig. a), while for B||x or B||y, electron trajectories are open, i.e. extended out of Brillouin zone (Fig. b).

The conductivity tensor depends on the topology of electron orbit.

$j_i = \sigma_{ij} E_j$, i,j=1,2,3.

$E_i = \rho_{ij}\, j_j$, $\rho_{ij} = (\sigma_{ij})^{-1}$.

ρ_{ij} and σ_{ij} are 3x3 matrices.

Electron motion in magnetic field in metals (add).

The motion in coordinate and momentum space in the plane ⊥B are related by the rotation by 90º, which follows from the condition ε=const:

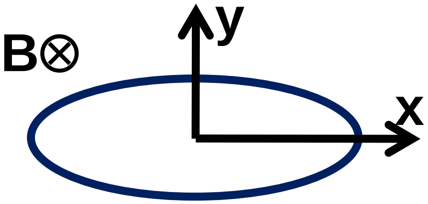
$$d\varepsilon = \left(\partial\varepsilon/\partial p_x\right)dp_x + \left(\partial\varepsilon/\partial p_y\right)dp_y = v_x dp_x + v_y dp_y = 0 \Rightarrow \boxed{dp_x/dp_y = -v_y/v_x}$$

In coordinate space

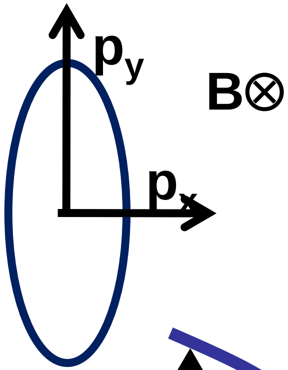
$$dx = v_x dt,\ \ dy = v_y dt;\Rightarrow \boxed{dy/dx = v_y/v_x = -dp_x/dp_y}$$

Examples:

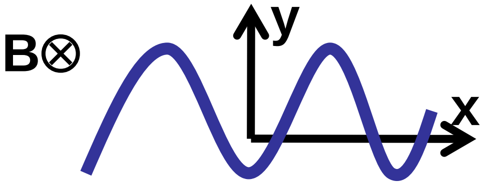
Closed electron orbit in coordinate space



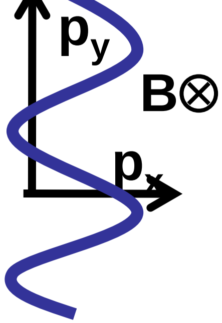
=> Corresponding electron orbit in momentum space:



Open electron orbit in coordinate space



=> Corresponding electron orbit in momentum space:



This property is a consequence of the equations of motion in B

Kinetic equation in the presence of magnetic field $\boxed{\frac{df}{dt}=I(f)}$

The full time derivative of distribution function $f(p,r)$ **in magnetic field B**

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial r}\frac{dr}{dt}+\frac{\partial f}{\partial p}\frac{dp}{dt}=\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial r}+eE\frac{\partial f}{\partial p}+\frac{e}{c}[vH]\frac{\partial f}{\partial p}.$$

The collision integral in the so-called tau-approximation is $I(f)=-\frac{f-f_0}{\tau_{tr}}$, **where** $f=f_0+f_1$ **, and**

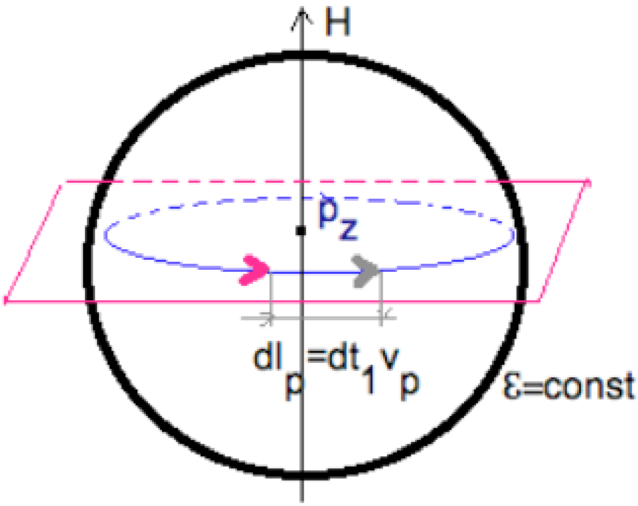
f_0 **is the distribution function in equilibrium (without external fields).**

Without electron scattering (i.e. by impurities) even a weak magnetic field strongly changes (reduces to zero) longitudinal (diagonal) conductivity.

This contradicts the common sense that very weak perturbation cannot strongly affect the observable phenomena. What is solution of paradox?

Answer: **the effect of magnetic field is weak if cyclotron period** $T_L >> \tau$, **i.e. when** $\omega_c\tau = \tau eB/mc << 1$ **and electron trajectories are almost straight lines between scattering events, because their curvature due to B is weak**

Electron motion in magnetic field in metals (2).



Lorentz force ⊥ velocity: $d\vec{p}/dt = e/c\left[\vec{v}\times\vec{H}\right] \equiv F_L$

Therefore kinetic energy is conserved:

$$d\varepsilon/dt = \left(\partial\varepsilon/\partial\vec{p}\right)\cdot d\vec{p}/dt \overset{def\ \vec{v}}{\equiv} \vec{v}\cdot d\vec{p}/dt \equiv e/c\,\vec{v}\cdot\left[\vec{v}\times\vec{H}\right]\equiv 0$$

The projection of momentum on B=H is also conserved $\varepsilon = \text{const}, p_z = \text{const}$

Equations of motion are: $\mathrm{dp}_x/\mathrm{dt} = (\mathrm{e/c})\mathrm{v}_y\mathrm{H}, \mathrm{dp}_y/\mathrm{dt} = -(\mathrm{e/c})\mathrm{v}_x\mathrm{H}$

Taking the square of both the equations and summing up we obtain:

$$\left(dp_x^2 + dp_y^2\right)/dt^2 = (e/c)^2 H^2\left(\mathrm{v}_x^2 + \mathrm{v}_y^2\right)$$ **where** $\left(dp_x^2 + dp_y^2\right) = dl_p^2$

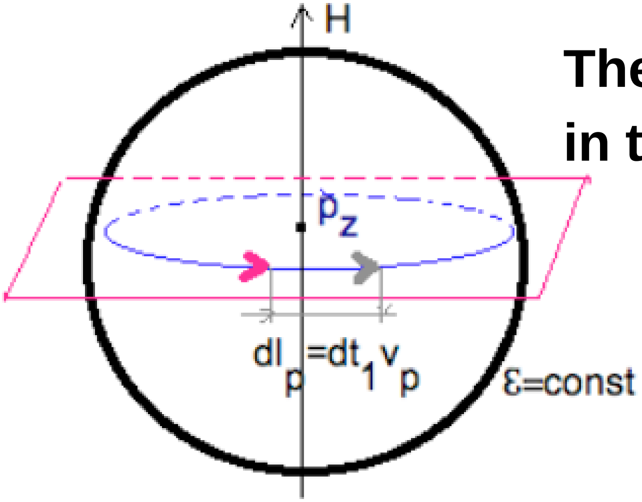
is the element of the length of the path in the momentum space.

Then equations of motion give $dl/dt_1 = (e/c)H\mathrm{v}_\perp;\ \mathrm{v}_\perp \equiv \left(\mathrm{v}_x^2+\mathrm{v}_y^2\right)^{1/2}$

The new variable $dt_1 = \frac{c}{eH}\frac{dl}{\mathrm{v}_\perp},\ t_1 = \frac{c}{eH}\int\frac{dl}{\mathrm{v}_\perp}$ **For closed trajectory the period of motion** $T = \frac{c}{eH}\oint\frac{dl}{v_\perp}$

It is convenient to change variables: $p_x, p_y, p_z \rightarrow t_1, \varepsilon, p_z$

Electron motion in magnetic field in metals (3).



The area inside the contour of closed electron orbit in the plane $p_z = const$ $S(\varepsilon)=\int dp_x\,dp_y=\int d\varepsilon\oint\frac{dl}{v_\perp}$

The period of motion $T = (c/eH)\partial S/\partial\varepsilon$

The "cyclotron mass" $m^* = (2\pi)^{-1}\partial S/\partial\varepsilon$

New variables: $p_x, p_y, p_z \to t_1, \varepsilon, p_z$ **where** $dt_1=\frac{c}{eH}\frac{dl}{v_\perp},\ t_1=\frac{c}{eH}\int\frac{dl}{v_\perp}$

Electric current

$\vec{j} = 2e\int \vec{v}_p f_1(\vec{p})\frac{d^3p}{(2\pi\hbar)^3}$ where $\frac{1}{(2\pi\hbar)^3}dp_xdp_ydp_z=\frac{1}{(2\pi\hbar)^3}\frac{eH}{c}d\varepsilon\,dt_1dp_z$

Kinetic equation in the new variables $\frac{\partial f}{\partial t_1}\dot{t}_1+\frac{\partial f}{\partial p_z}\dot{p}_z+\frac{\partial f}{\partial\varepsilon}\dot{\varepsilon}=I(f)$

Since $dt_1/dt\approx 1$, the kinetic equation is $\frac{\partial f}{\partial t_1}+\frac{\partial f}{\partial p_z}eE_z+\frac{\partial f}{\partial\varepsilon}evE=I(f)$

Introducing the notation $f=f_0-\frac{\partial f_0}{\partial\varepsilon}\psi.$ **we simplify the kinetic equation:** $\frac{\partial\psi}{\partial t_1}-I(\psi)=evE.$

Kinetic equation in magnetic field (2).

Introducing the notation for the distribution function: $f = f_0 - \frac{\partial f_0}{\partial \varepsilon} \psi.$

In the τ-approximation in magnetic field the kinetic equation becomes

$$\frac{\partial \psi}{\partial t_1} + \frac{\psi}{\tau} = e\boldsymbol{v}(t_1)\boldsymbol{E}.$$

The solution of this equation is

$$\psi = \int_c^{t_1} e\boldsymbol{v}(t_2)\boldsymbol{E}\, \mathrm{e}^{-(t_1-t_2)/\tau}\, \mathrm{d}t_2 .$$

where $c = -\infty$ **from boundary condition** $\psi(t_1 + T) = \psi(t_1)$

The electric current is given by the Shockley tube integral (formula):

$$j_\alpha = \frac{2e}{(2\pi h)^3} \int f v_\alpha\, \mathrm{d}^3 \boldsymbol{p} = -\frac{2He^2}{(2\pi\hbar)^3 c} \int \mathrm{d}\varepsilon \frac{\partial f_0}{\partial \varepsilon} \int \mathrm{d}p_z\, \mathrm{d}t_1 v_\alpha \psi$$

$$= \frac{2He^3}{(2\pi\hbar)^3} \int_{-p_0}^{p_0} \mathrm{d}p_z \int_0^T \mathrm{d}t_1\, v_\alpha(t_1) \int_{-\infty}^{t_1} v_\beta(t_2)\, \mathrm{e}^{-(t_1-t_2)/\tau} E_\beta\, \mathrm{d}t_2 .$$

For isotropic closed orbit $v_x = v_\perp \cos \Omega t_1, \qquad v_y = -v_\perp \sin \Omega t_1,$

and this integral gives

$$\begin{Bmatrix} j_x \\ j_y \end{Bmatrix} = \frac{2He^3}{(2\pi\hbar)^3 c} \frac{4}{3} \frac{p_0^3}{m^{*2}} \frac{2\pi c m^*}{2He} \frac{1}{\Omega^2 + \tau^{-2}} \begin{Bmatrix} \tau^{-1} E_x + \Omega E_y \\ -\Omega E_x + \tau^{-1} E_y \end{Bmatrix}$$

Conductivity in weak magnetic field (results)

In weak magnetic field B=B_z, $\sigma_{zz}(B_z)=\sigma_0$, and

$$\begin{Bmatrix} j_x \\ j_y \end{Bmatrix} = \frac{n_e e^2}{m^*}\frac{1}{\Omega^2+\tau^{-2}}\begin{Bmatrix}\tau^{-1}E_x+\Omega E_y \\ -\Omega E_x+\tau^{-1}E_y\end{Bmatrix},$$

The angular frequency $\Omega = 2\pi/T = eH/cm^*$ = ω_c is called Larmor or cyclotron frequency, and τ is mean scattering time.

In weak magnetic field $\Omega\tau$<<1, and $j_x = \frac{n_e e^2\tau}{m^*}E_x$ **Usually in experiments** $j_y=0$ => $E_y=\Omega\tau E_x$

The Hall resistance $R = \frac{E_y}{Hj_x} = \frac{eH}{m^*c}\frac{\tau E_x m^*}{n_e e^2\tau H E_x} = (n_e e c)^{-1}$ can be used to measure electron concentration.

The Shockley tube integral formula for electric conductivity in magnetic field:

$$j_\alpha = \frac{2}{(2\pi\hbar)^3}\frac{e^3H}{c}\int_{-p_F}^{p_F}dp_z\int_0^{T(p_z)}dt_1\,\mathrm{v}_\alpha(t_1)\int_{-\infty}^{t_1}dt_2\,\mathrm{v}_\beta(t_2)\exp\left\{\frac{t_2-t_1}{\tau}\right\}E_\beta\Big|_{\varepsilon=\mu}$$

This formula is suitable both for numerical and analytical calculations of magnetoresistance for arbitrary electron dispersion.

Conductivity tensor in strong magnetic field (**results**)

For closed orbits, in strong magnetic field B_z, **$\Omega\tau$>>1,** and $\sigma_{xx}, \sigma_{yy} \propto 1/\Omega^2 \rightarrow 0$.

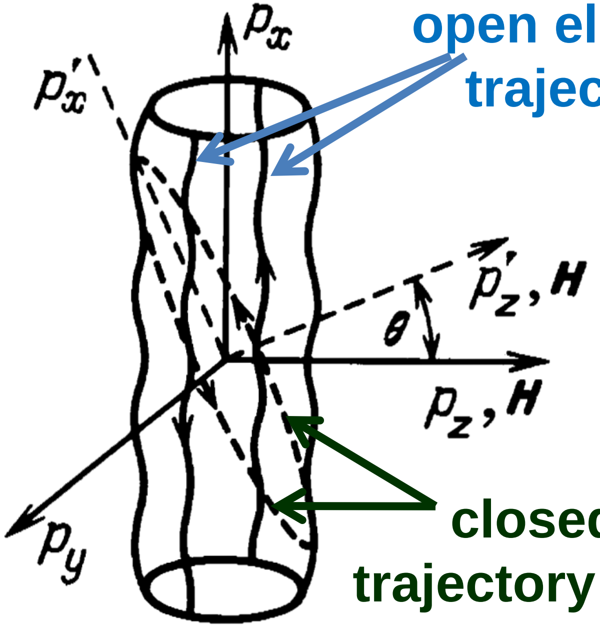
Introducing notation $\gamma = 1/\Omega\tau << 1$, for closed electron trajectories we find

conductivity tensor: $\sigma_{ik} = \begin{pmatrix} \gamma^2 a_{xx} & \gamma a_{xy} & \gamma a_{xz} \\ \gamma a_{yx} & \gamma^2 a_{yy} & \gamma a_{yz} \\ \gamma a_{zx} & \gamma a_{zy} & a_{zz} \end{pmatrix}$

(closed trajectories)

resistivity tensor $\rho_{ij} = (\sigma_{ij})^{-1}$: $\rho_{ik} = \begin{pmatrix} b_{xx} & \gamma^{-1} b_{xy} & b_{xz} \\ \gamma^{-1} b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{pmatrix}$

where $a_{ij} = const$ and all magnetic-field dependence is in $\gamma = 1/\Omega\tau \propto 1/B_z$



open electron trajectory

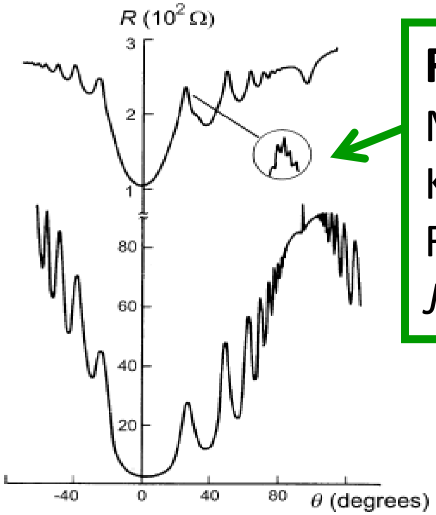
closed electron trajectory (dashed line)

For open trajectories the result is different, e.g., $\sigma_{yy} = const$ and $\rho_{xx} \propto B^2$:

$$\sigma_{ik} = \begin{pmatrix} \gamma^2 a_{xx} & \gamma a_{xy} & \gamma a_{xz} \\ \gamma^2 a_{yx} & a_{yy} & a_{yz} \\ \gamma a_{zx} & a_{zy} & a_{zz} \end{pmatrix}$$

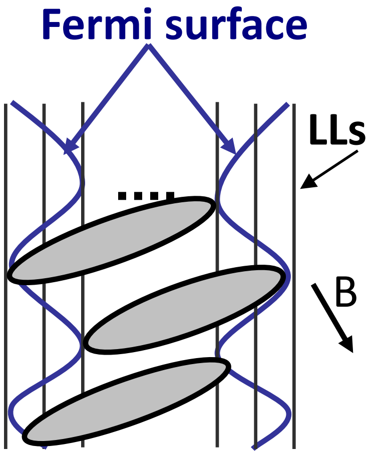
$$\rho_{ik} = \begin{pmatrix} \gamma^{-2} b_{xx} & \gamma^{-1} b_{xy} & \gamma^{-1} b_{xz} \\ \gamma^{-1} b_{yx} & b_{yy} & b_{yz} \\ \gamma^{-1} b_{zx} & b_{zy} & b_{zz} \end{pmatrix}$$

Angle-dependent magnetoresistance oscillations (AMRO) in quasi-2D metals.



First observation:
M.V. Kartsovnik, P. A. Kononovich, V. N. Laukhin, I. F. Schegolev,
JETP Lett. ***48****, 541 (1988).*

First theory:
K.J. Yamaji,
Phys. Soc. Jpn.
58, 1520,
(1989).



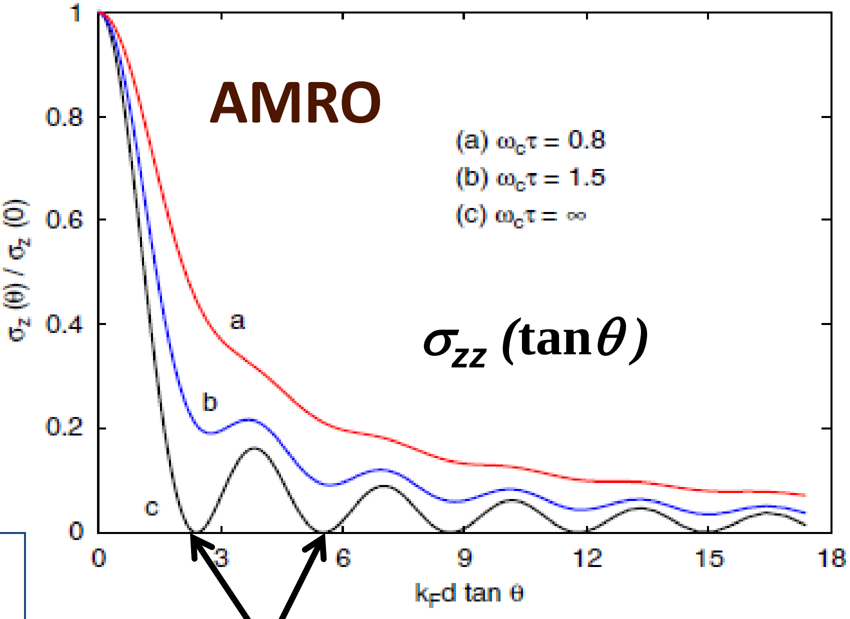
$$\sigma_{zz}^{3D} = e^2\tau\sum_{FS} v_z^2, \quad v_z = \partial\varepsilon/\partial p_z$$

For axially symmetric dispersion and in the first order in t_z the Shockley tube integral gives:
[R. Yagi et al., J. Phys. Soc. Jap. 59, 3069 (1990)]

$$\frac{\sigma_z(\boldsymbol{B})}{\sigma_z(0)} = J_0^2(k_{\mathrm{F}} d\tan\theta) + 2\sum_{j=1}^{\infty}\frac{J_j^2(k_{\mathrm{F}} d\tan\theta)}{1+(j\omega_{\mathrm{c}}\tau)^2}.$$

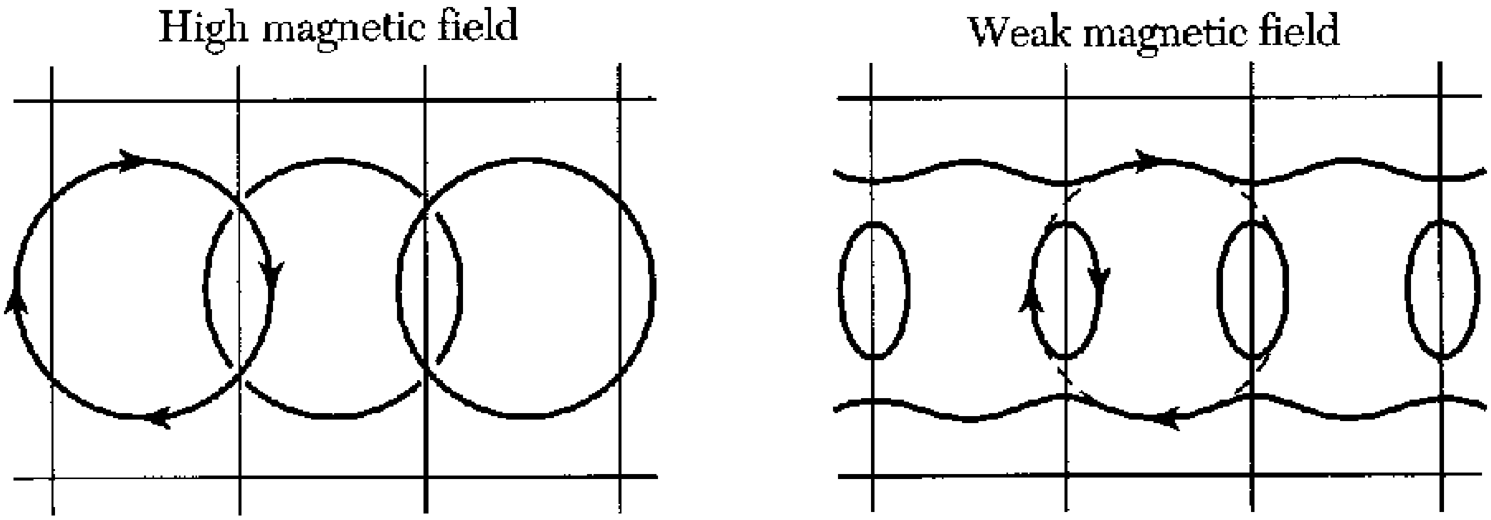
gives AMRO

gives damping of AMRO by disorder



Yamaji angles

Magnetic breakdown



(a) (b)

Figure 33 Breakdown of band structure by a strong magnetic field. Brillouin zone boundaries are the light lines. The free electron orbits (a) in a strong field change connectivity in a weak field (b) to become open orbits in the first band and electron orbits in the second band. Both bands are mapped together.

The probability of magnetic breakdown is given by $P = e^{-H_0/H}$,

where

$$H_0 = \frac{\pi}{4}\frac{\varepsilon}{e\hbar}\frac{\varepsilon_g^2}{v_x v_y} \sim \frac{mc}{e\hbar}\frac{\varepsilon_g^2}{\zeta} = \frac{H}{\hbar\omega_c}\frac{\varepsilon_g^2}{\zeta}$$