LECTURE 3

The Ginzburg and Landau theory. Surface Energy, type-I and type-II superconductors. Quantization of magnetic flux.

The derivation of Ginsburg-Landau equations

Preamble In the previous lecture we discussed the origin of the inter-electron attraction in superconducting metals and derived the binding energy of the Cooper pair, which results from this attraction. On the other hand, considering the size of the Cooper pair as infinitesimally small (coarse graining) and neglecting the fermionic excitations in the system one can describe some important thermodynamic characteristics of the metal in the superconducting state solely on the basis of the study of collective behavior of the Cooper pairs condensate. This task was successfully accomplished by Ginzburg and Landau (1950). In the contemporary theoretical approach, that does not neglect the fermionic excitations, the thermodynamics is most elegantly (though not always rigorously) described using the functional integral representation of the partition function of the many-body electronic system followed by its mean-field "decoupling" using the Hubbard-Stratonovich identity and formal integrating out of the fermionic degrees of freedom. While the final outcome is the Ginzburg-Landau (GL) free energy functional (Gor'kov 1959), the important "transient observables" at the half-way of such a standard derivation is the Bogoliubov-de Gennes (BdG) equations.

The BdG equations describe fermionic quasi-particles in the superconductor in the presence of the two fields: the vector potential of the magnetic field, ***A*** (where rot ***A***=***H***) and the field $\Psi(\vec r)$ of the superconducting Bose-condensate of the Cooper pairs:

$$\hat\xi u_\nu(\vec r)-\Psi(\vec r)\mathrm{v}_\nu(\vec r)=\varepsilon_\nu u_\nu(\vec r);\quad \hat\xi_c\mathrm{v}_\nu(\vec r)+\Psi^*(\vec r)u_\nu(\vec r)=-\varepsilon_\nu\mathrm{v}_\nu(\vec r),\qquad(3.1)$$

where:

$$\hat\xi=\frac{1}{2m}\left(\hat{\vec p}-\frac{e}{c}\vec A\right)^2-\mu;\ \hat\xi_c=\frac{1}{2m}\left(\hat{\vec p}+\frac{e}{c}\vec A\right)^2-\mu\qquad(3.2)$$

and $u_\nu(\vec r),\mathrm{v}_\nu(\vec r)$ are the amplitudes of the quasi-electron and quasi-hole, respectively, in the state with the quantum numbers set by ν. The physical meaning of the complex field $\Psi(\vec r)$ is that it represents the wave-function of the center of mass of the Cooper pair in the superconducting Bose-condensate of the pairs. Solution of BdG equations (3.1) with the homogeneous field: $\Psi(\vec r)\equiv\Delta$, gives the spectrum of the fermionic excitations in the superconducting state:

$$\varepsilon(\vec p)=\sqrt{|\Delta|^2+\xi^2(\vec p)};\quad \xi(\vec p)\equiv\mathrm{v}_F|p-p_F|,\qquad(3.3)$$

which is sketched with the dashed line in Fig. 8 of the Lecture 2. It is obvious then, that -2Δ is the binding energy of the Cooper pair derived in Lecture 2, Eq. (2.25).

Remark₁ While the Bogoliubov-de Gennes equations are derived from the microscopic Hamiltonian of the electron system in the superconductor, the complex field $\Psi(\vec{r})$, the so-called *order parameter*, enters the Ginzburg-Landau free energy functional (1950), which was written before the microscopic BCS theory was proposed, based on the general conditions of the guage-invariance of the free energy of the electron system, and re-derived by Gor'kov (1959) on the basis of procedure sketched above. The GL free energy functional of the superconductor is:

$$\int \Omega_s \, dV = \int \Omega_n^{(0)} \, dV + \int \left\{ \alpha\tau |\Psi|^2 + (1/2) b |\Psi|^4 + (4m)^{-1} \left| \left(i\hbar\nabla - (2e/c) A \right) \Psi \right|^2 + H^2/(8\pi) \right\} dV \,, \qquad (3.4)$$

here $\hbar = 1$, $\tau \equiv T - T_c/T_c$, $\alpha, b > 0$, and the charge $2e$ and mass $2m$ of the Cooper pair is substituted instead of the general charge e* and mass m* in the original paper. Gor'kov (1959) demonstrated that the GL theory was exact limit of the microscopic theory under the conditions: a) $T_c - T << T_c$, b) $\delta_L \gg \xi_0 \sim \hbar v_F / \Delta(T=0)$. The state with $\boldsymbol{p} = 0$ is the ground state of the Bose gas of the Cooper pairs under the homogeneous conditions. Below the Bose-condensation temperature there is a finite number of particles (in the thermodynamic limit) having the wave function $\Psi = const \cdot \exp(i\vec{p}\vec{r}/\hbar + i\alpha)$ with $\vec{p} = 0$ for all particles, thus manifesting the coherence of the charged superfluid. It is assumed that the coherence is preserved under a weak (long wave) breakdown of the homogeneity related e.g. with applied external magnetic field, and that the function $\Psi(\vec{r}) \neq const$ characterizes all the particles in the condensate. Since $|\Psi|^2 = n_s/2$ is the density of the Cooper pairs in the coherent Bose-condensate, Ψ is small near the critical temperature of 2nd order phase transition and the free energy Ω_s can be expanded as the power series in $|\Psi|^2$.

The gradient term in (3.4) describes the "rigidity" of the order parameter with respect to its modulus change or a "phase-twist" along the sample. Zero value, $\Psi = 0$, should describe the minimum of Ω_s above T_c, whereas below T_c : $\Psi \neq 0$, thus the coefficient $\alpha\tau$ must change its sign at the transition point T_c, with $\alpha > 0$. The condition that $\Psi = 0$ provides the minimum of Ω_s at the transition point as well, implies that $b \approx b(T_c) > 0$. Some important relations between GL coefficients and measurable characteristics of the superconductor are derived by variation of the free energy Ω_s with respect to the order parameter. At first, this is done approximately, by neglecting the size of the London penetration depth relative to the dimensions of a bulk sample, hence, assuming $\Psi = \text{const}$:

$$\Psi(\alpha\tau + b|\Psi|^2) = 0. \qquad (3.5)$$

and:

$$\Psi = 0, \; T > T_c;$$

$$|\Psi|^2 = -\alpha\tau/b \equiv \Psi_0^2, \; T < T_c. \qquad (3.6)$$

Substituting the equilibrium value $|\Psi|^2 = \Psi_0^2$ in (3.4) under zero magnetic field, we have

$$\Omega_n - \Omega_s = (\alpha\tau)^2/(2b) = H_{cm}^2/(8\pi) \qquad (3.7)$$

Here notation H_{cm} stays for the critical field of the bulk superconductor. The GL theory gives the correct temperature dependence for H_{cm} near T_c. The following microscopic formulas for the combination of the coefficients α and b stem from the comparison of (3.7) with the results of the BCS microscopic theory:

$$\alpha^2/b = \left[4/7\xi(3)\right]T_c^2 mp_0/\hbar^3 = \left[4\pi/7\xi(3)\right]\nu(\mu)T_c^2 . \qquad (3.8)$$

In the magnetic field the field inside the superconductor and the order parameter Ψ depend on the coordinates. The energy of the magnetic field per unit volume is $H^2/(8\pi)$. Besides, since the order parameter Ψ has the meaning of the wave function of the Cooper pairs, its phase changes with the vector and scalar potentials. The guage invariance of the free energy under the guage shift of the vector potential of magnetic field: $\boldsymbol{A} \to \boldsymbol{A} + \nabla\varphi$ is fulfilled provided that the "momentum" operator $-i\hbar\nabla$ enters Ω_s in the combination: $-i\hbar\nabla - (2e/c)\boldsymbol{A}$. This is the logics behind the expression $[-i\hbar\nabla - (2e/c)\boldsymbol{A}]\Psi$ in Eq. (3.4).

In order to find the functions $\Psi(\vec{r})$, $\vec{A}(\vec{r})$ that minimize the total free energy Ω_s of the superconducting sample, one varies Ω_s in Eq. (3.4) with respect to Ψ^* at fixed $\vec{A}(\vec{r})$ and *vice versa*, and equates obtained first variational derivatives to zero. The external magnetic field is taken as the boundary condition. The variation with respect to Ψ^* at fixed $\vec{A}(\vec{r})$ leads to the following result:

$$\int\{\alpha\tau\Psi\delta\Psi^* + b|\Psi|^2\delta\Psi^* + (4m)^{-1}[-i\hbar\vec{\nabla} - (2e/c)\vec{A}]\Psi[i\hbar\vec{\nabla} - (2e/c)\vec{A}]\delta\Psi^*\}dV = 0. \qquad (3.9)$$

After integrating by parts the term containing $\nabla\delta\Psi^*$ and using the Gauss theorem, one rewrites (3.9) in the equivalent form:

$$(i\hbar/4m)\int\delta\Psi^*\vec{n}\cdot[-i\hbar\vec{\nabla} - (2e/c)\vec{A}]\Psi dS + (4m)^{-1}\int\delta\Psi^*[-i\hbar\vec{\nabla} - (2e/c)\vec{A}]^2\Psi dV = 0, \qquad (3.10)$$

here the integration in the first term is over the surface, S, of the superconductor (***n*** is the unit vector normal to the surface).

Since the variation $\delta\Psi^*$ is arbitrary in the bulk, condition (3.9) is satisfied identically when the factor behind $\delta\Psi^*$ in the integrand entering the integral over the volume is set to zero:

$$(4m)^{-1}[-i\hbar\vec{\nabla} - (2e/c)\vec{A}]^2\Psi + \alpha\tau\Psi + b|\Psi|^2\Psi = 0 \qquad (3.11)$$

Besides, considering $\delta\Psi^*$ as being arbitrary at the surface of the superconductor in the integrand of the surface integral in (3.10) leads to the boundary condition:

$$\vec{n}\cdot[-i\hbar\vec{\nabla} -(2e/c)\vec{A}]\Psi|_S = 0\,. \qquad (3.12)$$

Nothing new is obtained by varying (3.4) in $\delta\Psi$ instead of $\delta\Psi^*$, except the resulting equations become complex conjugates of the equations (3.11) and (3.12).

The next step is variation of (3.4) with respect to the vector potential $\boldsymbol{A}$. First of all, variation of $\boldsymbol{H}^2$ gives: 2rot$\boldsymbol{A}$rot$\delta\boldsymbol{A}$. Using the well known relation of vector calculus: div[$\boldsymbol{ab}$] = $\boldsymbol{b}$rot$\boldsymbol{a}$ – $\boldsymbol{a}$rot$\boldsymbol{b}$, one finds:

$$2\mathrm{rot}\vec{A}\cdot\mathrm{rot}\delta\vec{A} = 2\delta\vec{A}\,\mathrm{rot}\,\mathrm{rot}\vec{A} + 2\mathrm{div}[\delta\vec{A}\times\mathrm{rot}\vec{A}]\,. \qquad (3.13)$$

Next, the volume integral of *div* transforms into the integral over the sample surface, S, where $\boldsymbol{A}$ is fixed and hence, $\delta\boldsymbol{A}$ vanishes. Equating the bulk variation of (3.4) with respect to $\delta\boldsymbol{A}$ to zero is equivalent to the combination of the following two equations:

$$\nabla\times\nabla\times\vec{A} = (4\pi/c)\vec{j} \qquad (3.14)$$

$$\vec{j} = -(ie\hbar/2m)(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*) - (2e^2/mc)|\Psi|^2\vec{A}\,. \qquad (3.15)$$

Equation (3.14) has the form of the Maxwell equation. Equation (3.15) is essentially the quantum mechanical current in the magnetic field, provided that the particle is characterized with the wave function Ψ, and the charge $2e$ and mass $2m$, being the attributes of the Cooper pair in the superconducting state. The boundary condition fixes the magnetic field at the superconductor's surface.

Remark$_2$ Equation (3.15) differs from the Londons' theory equation (1.4a) of Lecture 1 by the first term. The latter vanishes when the order parameter Ψ is real. The physical meaning of Ψ is the wave function of the Bose-condensed Cooper pair, and the wave function acquires phase shift when the guage is changed. This phase shift is then taken care of in the expression for the current density (3.15), so that the current density remains invariant under the change of the guage field $\vec{A}$.

Remark$_3$ Equations (3.11)-(3.15) contain two length-scales, this situation may be alternatively described by a single length-scale and a dimensionless scalar of the theory, called the Ginzburg-Landau prameter. The latter description scheme is achieved by expressing all the lengths in units of the London penetration depth, δ, of the magnetic field into the superconductor:

$$\delta = \left[2mc^2/4\pi\Psi_0^2(2e)^2\right]^{1/2};\quad \text{where:}\quad 2\Psi_0^2 \equiv n_S\,. \qquad (3.16)$$

here n_S is the density of the "superconducting electrons". Namely, after changing the variables:

$$\Psi' = \Psi/\Psi_0\,,\; H' = H/\left(H_{cm}\sqrt{2}\right),\; \Psi_0^2 = \alpha|\tau|/b \qquad (3.17a)$$

$$r' = r/\delta,\; A' = A/\left(H_{cm}\sqrt{2}\delta\right),\; H_{cm} = 2\sqrt{\pi}\alpha\tau/b^{1/2} \qquad (3.17b)$$

the GL equations expressed in the new variables acquire the form (we will not write primes in the new variables):

$$(-i æ^{-1}\vec{\nabla} - \vec{A})^2\Psi - \Psi + |\Psi|^2\Psi = 0\,, \tag{3.18}$$

$$\vec{n}\cdot(-i æ^{-1}\vec{\nabla} - \vec{A})^2\Psi|_S = 0\,, \tag{3.19}$$

$$\text{rot rot}\vec{A} = -(i/2æ)(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*) - |\Psi|^2\vec{A}\,. \tag{3.20}$$

The single scalar æ that enters equations (3.18) – (3.20) is the Ginsburg-Landau parameter, which then equals to:

$$æ = 2^{3/2} e H_{cm}\delta^2/(\hbar c). \tag{3.21}$$

The physical meaning of the second length scale $\xi=\delta/æ$ becomes most transparent from the calculation of the order parameter spatial evolution near the boundary between the superconductor and the normal metal. This phenomenon was neglected in the London theory, making impossible description of the surface energy at the normal-superconducting interface.

The Landau correlation length ξ

Consider the case when both Ψ and $\boldsymbol{A}$ depend only on one coordinate, say x, and $\boldsymbol{A}$ lies in the *{yz}* plane perpendicular to x-axis. In this geometry the general equations (3.18) – (3.20) simplify:

$$æ^{-2} d^2\Psi/dx^2 + \Psi(1 - A^2) - \Psi^3 = 0, \tag{3.22}$$

$$d\Psi/dx|_S = 0, \tag{3.23}$$

$$d^2A/dx^2 - \Psi^2 A = 0. \tag{3.24}$$

Thus, it is easy to see, that the sought for solution Ψ(x) can be taken real, because the imaginary unity, i, drops out from the equations above. This , inturn, leads to the fact, that equations (3.22) and (3.24) can be solved in quadratures. Namely, one multiplies both (3.22) by $d\Psi/dx$, and (3.24) by dA/dx, and then sums the resulting expressions and integrates them over x. Taking into account the boundary condition in the bulk of the superconductor: $A(x\to\infty)=0$, $\Psi(x\to\infty)=1$, one has:

$$æ^{-2}(d\Psi/dx)^2 + (dA/dx)^2 + \Psi^2(1 - A^2) - \Psi^4/2 = \text{const} = 1/2 \tag{3.25}$$

Suppose that superconducting phase is at $x\to\infty$, and the normal phase is at $x\to-\infty$. We take into account the conditions $\boldsymbol{H}||z$, $\boldsymbol{A}||y$, $H = dA/dx$. In this case the boundary conditions are:

$$x\to\infty:\ \Psi = 1,\ H = A = 0,\ d\Psi/dx = 0;$$

$$x\to-\infty:\ \Psi = 0,\ H = H_0 = 1/\sqrt{2}\,,\ d\Psi/dx = 0 \tag{3.26}$$

($\Psi = 1$ corresponds to Ψ_0, and $H_0 = 1/\sqrt{2}$ corresponds to H_{cm} in usual units). Equations (3.22) – (3.24), in general, cannot be integrated in analytic form, except in the limiting case æ << 1 considered below. <u>This case corresponds</u> to: ξ >> δ. It is obvious then that magnetic field $\vec{H}$ and vector potential $\vec{A}$ vanish well inside the major part of the superconducting region of the thickness ~ ξ near the boundary with the normal metal. Setting accordingly $\vec{A} = \vec{H} = 0$ in (3.25) we obtain the equation:

$$d\Psi/dx = \pm(æ/\sqrt{2})(1-\Psi^2)\,. \qquad (3.27)$$

The solution of this equation satisfying the boundary condition Ψ = 1 at $x \to \infty$ and decreasing in the direction of the boundary (i.e. $x\to 0$) is:

$$\Psi = \mathrm{th}(æx/\sqrt{2}). \qquad (3.28)$$

It is necessary to take into account that this solution becomes incorrect in the region near *x~0* of width δ, where the field $\vec{H}$ penetrates inside, but this region is small due to adopted above condition: ξ >> δ. Hence, we choose the origin at $x = 0$, where $\Psi \approx 0$ for simplicity.

Remark₄ Result (3.28) is remarkable, since it reveals the meaning of the second length in the GL theory introduced above as:

$$\xi = \delta/æ\,. \qquad (3.28a)$$

Namely, ξ is the correlation length of the Cooper pairs wave function that describes the Bose-condensate. Hence, e.g. near the boundary with the normal metal the wave function Ψ changes from its bulk value to zero on the length scale ξ.

It is important to clarify relation between ξ and the Cooper pair size:

$$\xi_0 = \hbar v_F/\pi\Delta(T = 0^\circ K)\,, \qquad (3.28b)$$

as well as their relation to the London penetration depth δ. As it follows from Eqs. (3.16), (3.17a,b) and (3.21), (3.28a,b) the GL parameter æ has the temperature independent limit near the superconducting temperature T_c: $æ \propto H_{cm}\delta^2 \propto \tau\delta^2\big|_{T\to T_c} \approx \mathrm{const}$. Hence, it is a characteristic of the superconducting material, and so is the Cooper pair size too: $\xi_0 \approx \mathrm{const}$. On the other hand, both the London penetration depth, δ, and Landau correlation length, ξ, diverge as $\sim 1\big/\sqrt{(T-T_c)/T_c}$. In the limit $T \to 0^\circ K$ δ has finite value, that according to the BCS theory, is obtained from Eq. (3.16) by substituting the "superconducting density" n_s by the complete electron density n_e. Hence, the GL parameter æ can be formally calculated also at $T = 0^\circ K$, and the result is : $æ(0) \approx (1.2 \div 1.3)æ(T_c)$.

Remark₅ *The Cooper pair size,* ξ_0*, is temperature independent and characterizes correlation between electrons <u>inside a single</u> pair, while the Landau correlation length,* ξ*, characterizes correlation between the centers of mass of the <u>different Cooper pairs</u> in the superconducting Bose-condensate, and diverges at* T_c.

Remark₆ *The BCS theory is valid in the <u>weak coupling</u> limit:* $k_B T_c \ll \varepsilon_F$*, and in this limit the binding energy is proportional to the density of the Bose-condensate of the Cooper pairs and, therefore, they* *both vanish at T_c: $-2\Delta(T=T_c) \propto \Psi_0^2(T=T_c)=0$. In the* <u>*strong coupling*</u> *limit, $k_BT_c \sim \varepsilon_F$, the binding energy of the electron pairs is already non-zero at the superconducting (Bose-condensation) temperature T_c, and hence, the pairs are called "preformed". The binding energy of the preformed pairs vanishes at a temperature $T^* > T_c$, that is, possibly, the case in the underdoped high-T_c cuprates.*

The surface energy at the boundary between the normal and the superconducting phases

Remark₇ *The GL theory, by treating the possible inhomogeneity of the superconducting order parameter Ψ(x), provides an explanation of the origin of the surface energy at the boundary between the superconducting and normal metal parts of the sample.* To see this, we first transform expression (3.4) for the free energy using dimensionless variables defined in (3.17):

$$\int\Omega_s dV = \int\Omega_n^{(0)} dV + \frac{H_{cm}^2}{4\pi}\int\{-|\Psi|^2 + |\Psi|^2/2 + |(-i\nabla/æ - \boldsymbol{A})\Psi|^2 + H^2\}dV. \tag{3.29}$$

Then, we integrate by parts the term containing $\nabla\Psi^*$ as we did earlier. In this case the surface integral disappears because of the boundary condition (3.19). The remaining bulk integral is rewritten assuming that Ψ satisfies already the equation (3.18). As a result we have

$$\int\Omega_s\, dV = \int\Omega_n^{(0)}\, dV + \frac{H_{cm}^2}{4\pi}\int\left(H^2 - \frac{|\Psi|^4}{2}\right)dV. \tag{3.30}$$

Consider the transition from the normal to the superconducting state under the influence of the external magnetic field. For the cylindrical geometry the free energy in the given external field can be obtained by subtracting of $\boldsymbol{H}_0\boldsymbol{B}/(4\pi)$, where $\boldsymbol{H}_0$ is the external field, and $\boldsymbol{B}$ is the magnetic induction, which is equal to the average field in the sample:

$$\boldsymbol{B} = V^{-1}\int\boldsymbol{H}dV. \tag{3.31}$$

For the normal phase: $\Omega_{nH} = \Omega_n^{(0)} - H_0^2/(8\pi)$.

Hence, we obtain (in the reduced units):

$$\int(\Omega_{sH} - \Omega_{nH})dV = \frac{H_{cm}^2}{4\pi}\int\left[(H-H_0)^2 - \frac{|\Psi|^4}{2}\right]dV. \tag{3.32}$$

Expression (3.32) enables us to find the surface energy between the normal and superconducting phases. Suppose that superconducting phase is at $x \to \infty$, and the normal phase is at $x \to -\infty$. We take into account the conditions $\boldsymbol{H}||z$, $\boldsymbol{A}||y$, $H = dA/dx$. In this case we have the boundary conditions

$$x \to \infty: \Psi = 1,\ H = A = 0,\ d\Psi/dx = 0;$$

$$x \to -\infty: \Psi = 0,\ H = H_0 = 1/\sqrt{2},\ d\Psi/dx = 0 \tag{3.33}$$

(Ψ = 1 corresponds to Ψ_0, and $H_0 = 1/\sqrt{2}$ corresponds to H_{cm} in usual units). We consider the limiting case æ << 1 solved above in (3.28). The integrand in (3.32) goes to zero both at $x \to \infty$ and $x \to -\infty$.

Notice, that in the $x \to \infty$ limit the above statement is justified, because: $H=0$, $H_0=1/\sqrt{2} = H_{cm}$, and $\Psi = 1$. In the $x \to -\infty$ limit: $H - H_0 = 0$, $\Psi = 0$, and the statement above is valid again. Thus, only the transition region contributes. The contribution corresponds to the excess energy associated with the boundary, i.e. to the σ_{ns}. Substituting $H_0 = 1/\sqrt{2}$, $H = 0$ (as long as æ << 1) and Ψ from (3.28) in (3.32) we obtain:

$$\sigma_{ns} = \frac{H_{cm}^2}{8\pi}\int_0^\infty \left(1-\Psi^4\right)dx = \frac{H_{cm}^2}{8\pi}\int_0^\infty\left[1-\mathrm{th}^4\frac{x æ}{\sqrt{2}}\right]dx =$$

$$= \frac{H_{cm}^2}{8\pi}\int_0^\infty ch^2\frac{x æ}{\sqrt{2}}\left[1+\mathrm{th}^2\frac{x æ}{\sqrt{2}}\right]dx = \frac{H_{cm}^2}{8\pi}4\sqrt{2}/(3æ). \quad (3.34)$$

The lower limit of the integral over x is chosen to be at the point where Ψ vanishes and the field penetrates, i.e. . $x = 0$. In the usual units we get:

$$\sigma_{ns} = \frac{H_{cm}^2}{8\pi}(4\sqrt{2}\,\delta/3æ) = \frac{H_{cm}^2}{8\pi}\,\xi 4\sqrt{2}/3. \quad (3.35)$$

Remark₇ The calculation of σ_{ns} for æ ≈ 1 is possible only numerically. In this case σ_{ns} =0 at æ = $1/\sqrt{2}$, and σ_{ns} becomes negative at larger æ:

$$\sigma_{ns} \sim -\frac{H_{cm}^2}{8\pi}\delta\ . \quad (3.36)$$

The small values of æ are obtained for the pure superconductors: æ = 0,16 for mercury; æ = 0,15 for tin; æ = 0,026 for aluminum. The change of sign of σ_{ns} when æ passes through *1* is sketched in Fig. 10 a,b.

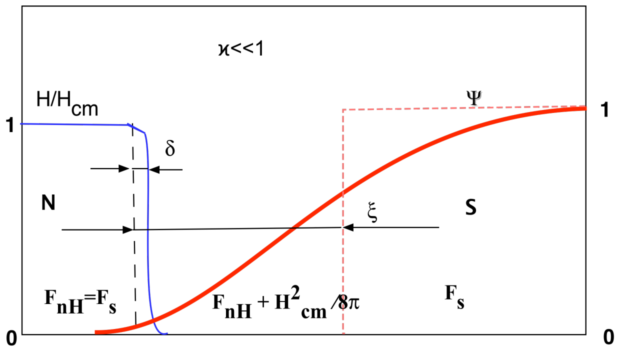

Fig. 10 a.

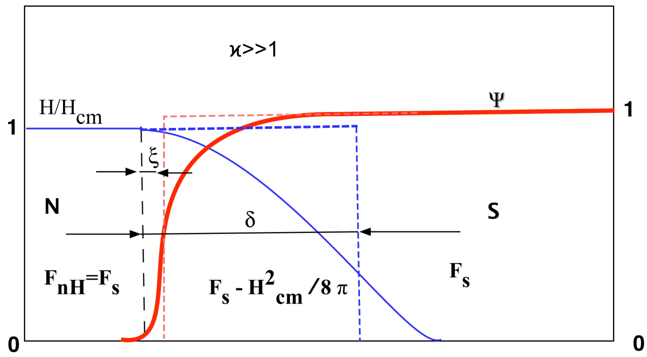

Fig. 10 b.

Remark₈ *The origin of the surface energy σ_{ns} in Fig. 10a and Fig.10b: it is clearly seen that Eqs. (3.35) and (3.37) adequately describe the situation in the both cases of* æ ≪ 1, and æ ≫ 1. *The superconductors with the positive surface energy: $\sigma_{ns} > 0$, æ<$1/\sqrt{2}$, and with the negative surface energy: $\sigma_{ns} < 0$, æ>$1/\sqrt{2}$, are called the type-I and type-II superconductors, respectively.* As it is clear from the discussion above, the surface energy of the boundary between the normal and superconducting phases in the equilibrium arises due to the inhomogeneity of the Cooper pairs wave function (the superconducting order parameter) Ψ(x).

Influence of impurities on the GL parameter **æ**

If there is impurity scattering of the electron with the mean free path l , the diffusion coefficient D~lv$_F$ and the correlation length $\xi_0' \sim \sqrt{D\xi_0/\mathrm{v}_F} \approx \sqrt{l\xi_0}$, and $\delta' \sim \delta\sqrt{\xi_0/l}$, hence:

$$æ' \approx \delta'/\xi' \sim \delta/l \approx \delta/\xi_0\left(\xi_0/l\right) \approx æ\left(\xi_0/l\right) \gg æ. \qquad (3.37)$$

Therefore, adding impurities may change the sign of the surface energy from e.g. positive to the negative, and e.g. turn a superconductor from the I to II type.

Validity and limitations of the GL theory: fluctuations

To conclude this Lecture, let's estimate the size of the region where GL theory based on the mean-field approximation for the thermodynamics of the superconducting phase transition becomes not applicable because of the fluctuation effects. The domain of applicability of the GL approximation is defined by the condition:

$$1 >> |\tau| >> m^2b^2T_c^2/(\alpha\hbar^2), \qquad (3.38)$$

where, as before, $\tau = (T - T_c)/T_c$. The dimensionless parameter $m^2b^2T_c^2/(\alpha\hbar^2)$ of the GL theory is called *Levanuk-Ginsburg parameter*, it measures the relative strength of the mean-field and fluctuations contributions to the specific heat of the superconductor near the second order phase transition at T$_c$ (Levanuk 1959, Ginsburg 1960).

Substituting α/b^2 from Eq. (3.8) for the pure superconductor, we obtain:

$$1 >> |\tau| >> (T_c/\varepsilon_F)^4, \qquad (3.39)$$

here $\varepsilon_F \propto p_F^2/m$ is the Fermi energy. Generally (but not for the high-T$_c$ cuprates), T_c/ε_F doesn't exceed 10^{-3}. Hence, the relative temperature interval around the T$_c$, where fluctuations are strong and mean-field fails, is of the order of 10^{-12}. In the case of dirty superconductors, i.e. when $l << \xi$, the relative interval of the strong fluctuations is:

$$1 >> |\tau| >> T_c\left(\hbar/\tau t_{tr}\right)^2/\varepsilon_F^4. \qquad (3.40)$$

This requirement is weaker than (3.39), but in practice it excludes the fluctuation region as well, except in the new high-T$_c$ superconducting materials discovered since 1986. The physical reason for this difference is as follows. In the "common" superconductors the Landau correlation length ξ is several orders of magnitude greater than the crystal lattice constant, and hence, the overlap between the Cooper pairs is syrong. On the contrary, in some of the high-T$_c$ superconducting materials ξ reaches a few lattice constants, and hence, the fluctuations become important. Nevertheless the cases can exist when the role of the fluctuations increases noticeably even in the "common" superconductors with big

ξ, and hence, the fluctuations can lead to the observable effects. This takes place for the kinetic effects (e.g. electric conductivity) in the superconducting samples with small dimensions, i.e. thin films or filaments.

The limitations of the Ginsburg and Landau theory from the low temperature side come from the demands: $|\tau| << 1$, $\delta >> \xi_0$, same as in the London theory. The latter condition fails in the type-I superconductors at low enough temperatures. Therefore, both requirements can be expressed via a single condition, that is derived using both of the facts: i.e., that $\xi_0 \propto \xi(0) \propto \delta_L(0)/æ$, and that $\delta(T) \propto \delta_L(0)|\tau|^{-1/2}$ at $|\tau| << 1$. The restriction $\delta(T) >> \xi_0$ can be re-written as $|\tau|^{1/2} << æ$. Thus, the final condition for the applicability of the GL theory is:

$$\min(æ^2, 1) \;\; >> |\tau| >> m^2 b^2 T_c^2 / (\alpha \hbar^2) \tag{3.41}$$

The formula (3.41) is applicable for the superconductors with impurities as well.

Magnetic flux quantization

The phenomenon of the flux quantization can be readily deduced from the GL equations (3.14), (3.15) applied in the case of e.g. the hollow cylinder with the walls having thickness more than the London's δ. Consider the expression (3.15) for the superconducting current and express the function Ψ via modulus and phase: $\Psi = |\Psi|\exp(i\chi)$. Then the current acquires the form:

$$\boldsymbol{j} = (\hbar e/m)|\Psi|^2[\nabla\chi - (2e/\hbar c)\boldsymbol{A}]. \tag{3.42}$$

Next, we divide $\boldsymbol{j}$ by $|\Psi|^2$ and integrate along the closed contour which encircles the hollow and lies inside the cylinder wall away from the outer and inner surface layers of thickness δ. Thus, we obtain:

$$\oint \frac{j}{|\Psi|^2} dl = \frac{\hbar e}{m}\left[\oint \nabla\chi dl - \frac{2e}{\hbar c}\oint A dl\right]. \tag{3.43}$$

Since the contour lies inside the tube's wall, the supercurrent $\boldsymbol{j}$ on the contour equals zero. On the other hand, the second integral in the right hand side of Eq. (3.43) equals in accord with the Stocks theorem to:

$$\oint A dl = \int \mathrm{rot} A dS = \int H dS = \Phi\,. \tag{3.44}$$

As to the first integral it is not necessarily zero. Since the function $\Psi = |\Psi|\exp(i\chi)$ is single-valued the phase χ , when going around the closed contour, may change only by $2\pi n$, where n is integer. Thus, we have:

$$\oint \nabla\chi dl = 2\pi n\,. \tag{3.45}$$

Combining (3.44) with (3.45) one finds, while equating (3.43) to zero, that *flux through the cylinder is quantized*:

$$\Phi = n\Phi_0, \tag{3.46}$$

$$\Phi_0 = \frac{1}{2}(hc/e) \equiv \frac{1}{2}\tilde{\Phi}_0 = 2{,}07 \cdot 10^{-7}\ \text{Oe} \cdot cm^2 \tag{3.47}$$

The quantity Φ_0 is called the superconducting flux quantum.

Remark₉ The quantum of the magnetic flux in the superconductor is half the quantum of the magnetic flux $\tilde{\Phi}_0$ ascribed to a single electron in a Landau level state in the external magnetic field: see Lecture 1, Section 2, Eq. (1.12).

LECTURE 4

Mixed state of type-II superconductors. Abrikosov vortex lattice. Surface superconductivity.

Magnetic properties of the type-II superconductors

Preamble *The famous topological defect of the GL superconducting order parameter Ψ(**r**) is the Abrikosov vortex (1957) predicted and discovered in the type-II superconductors in the external magnetic field. Being the macroscopic quantum object, the vortex in the superfluid liquid helium was first discovered by Onsager, 1949 and Feynman, 1955. Importantly, the pinning of the Abrikosov vortices in the type-II superconductors provides the possibility of use of superconducting devices in strong magnetic fields much greater than bulk critical field H_{cm}.*

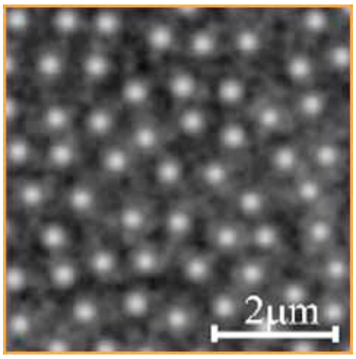

Fig. 11. Magnetic-force microscopy of Abrikosov vortex lattice. A. Volodin et al., Katholieke Universiteit Leuven, Europhys. Lett. 58, 582 (2002).

The peculiarity of the type-II superconductors is that they are characterized with GL parameter æ $> 1/\sqrt{2}$ and, hence, possess negative surface energy $\sigma_{ns} < 0$. Allowing for the latter, one concludes that the first order phase transition into the normal state at $H=H_{cm}$ is impossible in the type-II superconductor, since condition $\sigma_{ns} < 0$ makes energetically favorable splitting into normal and superconducting layers parallel to the magnetic field. But the critical field of the thin superconducting layer, e.g. of thickness d, is proportional to $H_{cm}\delta/d$, and hence, can considerably exceed H_{cm}. Thus (unlike in the type-I superconductors with $\sigma_{ns} > 0$), in the type-II superconductors it is energetically favorable for such layers remain superconducting in the fields higher than H_{cm}. The analytical solution of the GL equations found by Abrikosov (1957) indicated, that in reality, the type-II superconductors possess normal-core vortices rather than thin layers in magnetic field, and the vortices prove to be energetically stable in the fields that exceed $H_{c1} < H_{cm}$, called the lower critical field, up to field H_{c2} $> H_{cm}$, called the upper critical field, at which the normal cores coalesce and superconductivity vanishes via the second order phase transition.

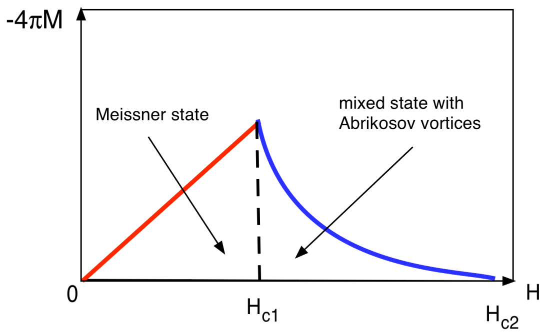

Fig. 12. Diamagnetic moment M of the type-II superconductor as function of the external magnetic field H.

Analytical solution for æ>>1

Consider the limiting case of the type-II superconductor with GL parameter æ>>1, and hence: $\delta(T) \gg \xi$, one can solve the GL equations describiun a single Abrikosov vortex analutically. Consider the vortex centered at the origin of the cylindrical coordinate system with coordinates r, θ, z .

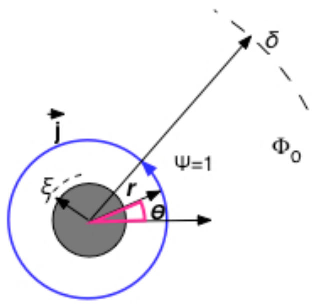

Fig. 13. Polar coordinates in the plane perpendicular to the symmetry axis z of the Abrikosov's vortex.

Consider region r>> ξ, so that $\Psi(r) \equiv |\Psi| \exp\{i\theta\} \approx \Psi_0 \exp\{i\theta(r)\}$, where Ψ_0 is order parameter in the bulk. Then, applying curl to both sides of the Eqs. (3.14) and (3.15) of Lecture 3 one finds:

$$\vec{H} + \delta^2 \nabla \times \nabla \times \vec{H} = \frac{\Phi_0}{2\pi} \nabla \times \nabla \theta = \Phi_0 n \delta(r) \hat{z}; \quad n = 1, \ldots \tag{4.1}$$

where the Dirac delta-function $\delta(r)$ arises when we neglect length scale ~ξ , and $\hat{z}$ is unit vector along the z-axis of the vortex. In the above the integer n on the right hand side of Eq. (4.1) manifests the magnetic flux quantization and will be put to 1 from the minimal energy considerations. Then, solution of (4.1), vanishing at $r \to \infty$, is :

$$H = \frac{\Phi_0}{2\pi\delta^2} K_0(r/\delta), \; \xi \ll r < \infty \tag{4.2}$$

where $K_0(x)$ is the Macdonald function:

$$K_0(x) \approx \begin{cases} \ln\left[2/(\gamma x)\right]; & x \ll 1 \\ \sqrt{\pi}\exp(-x)/\sqrt{2x}; & x \gg 1 \end{cases} \tag{4.3}$$

where $\gamma \simeq 1.78$ is the Euler's constant. Hence, magnetic field H behaves logarithmically in the region $\xi \ll r \ll \delta$, and decreases exponentially (Meissner effect) in the region $r \gg \delta$. At the vortex core, in the limit $x \sim 1$ of $K_0(x)$, we find from Eq. (4.2):

$$H(0) \approx \frac{\Phi_0}{2\pi\delta^2}\left(\ln æ - 0.18\right), \tag{4.4}$$

where the last term in the braces is the correction to the approximate result (4.2), which allows for $|\Psi| \to 0$, inside the vortex core: $r \leq \xi$. Next, using the Maxwell equation in the polar coordinates:

$$-\frac{dH}{dr} = \frac{4\pi}{c} j \tag{4.5}$$

we derive the radial distribution of the supercurrent expressed via the magnetic field from Eq. (4.2):

$$j = -\frac{c\Phi_0}{8\pi^2\delta^2}\frac{dK_0(r/\delta)}{dr} \approx \begin{cases} \dfrac{c\Phi_0}{8\pi^2\delta^2 r}; & \xi \ll r \ll \delta \\ \dfrac{c\Phi_0}{2^{7/2}\pi^{3/2}\delta^2\sqrt{\delta r}}\exp(-r/\delta); & r \gg \delta \end{cases} \tag{4.6}$$

Finally, using the first GL equation (3.18) in the polar coordinates and substituting in it the solution for the supercurrent from (4.6) in the region $\xi \ll r \ll \delta$ extended to the region $\xi_0 \ll r \ll \delta$, we find equation for the order parameter modulus $|\Psi|$ of the Cooper pairs condensate expressed in the dimensionless form as $|\Psi| \equiv \Psi_0 f(r)$:

$$(æ\delta)^{-2}\left[r^{-1}\frac{d}{dr}\left(r\frac{df}{dr}\right) - r^{-2}f\right] = f^3 - f \tag{4.7}$$

Solution of Eq. (4.7) is :

$$f(r) \approx \begin{cases} 1-(\xi/r)^2; & \xi \ll r \ll \delta \\ const \cdot r/\xi; & \xi_0 \ll r \ll \xi \end{cases} \tag{4.8}$$

Actually, solution in the region $\xi_0 \ll r \ll \xi$ obtained in (4.8) has used somewhat "illegally" the solution for the supercurrent from Eq. (4.6) obtained for the interval $\xi \ll r \ll \delta$. Nevertheless, obtained result proves to be qualitatively correct, as it fits smoothly with the solution in the region $\xi \ll r \ll \delta$, provided the constant is chosen properly, and, importantly, gives finite kinetic energy of the supercurrents in the vortex core, as we'll see below.

Remark$_1$ *Summarizing the results derived for the radial distributions of the magnetic field, order parameter modulus and supercurrent in the Abrikosov vertex, we arrive at the following "anatomic" picture of this object, see Fig. 14.*

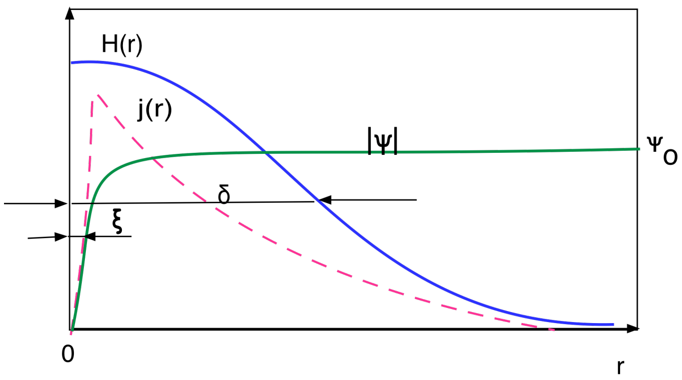

Fig. 14. Morphology of the Abrikosov vortex in the type-II superconductor: the radial distributions of the thermodynamic parameters.

Free energy of the Abrikosov vortex: the lower and the upper critical fields H_{c1} and H_{c2}

Using now Abrikosov vortex characteristics (4.2)-(4.8) we are able to consider the thermodynamics of the origination of a single vortex in the Meissner state. The free energy of a single vortex line consists of the energy associated with the magnetic field (4.2) that penetrates in the superconductor with the vortex, and of the kinetic energy of the supercurrents (4.6):

$$F_v = \frac{1}{8\pi}\int \vec{H}^2 dV + \frac{\delta^2}{8\pi}\int \left(\nabla \times \vec{H}\right)^2 dV \qquad (4.9)$$

In this expression the second integral logarithmically diverges in the interval $\xi \ll r \ll \delta$, thus producing the "big logarithm" – the fame of XX-th century theoretical physics, while the first integral Gives relatively small contribution:

$$\varepsilon_0 = F_v/L \approx \left(\frac{\Phi_0}{4\pi\delta}\right)^2 \ln\frac{\delta}{\xi} \equiv \left(\frac{\Phi_0}{4\pi\delta}\right)^2 \ln æ \qquad (4.10)$$

where ε_0 is the energy per unit length of the vortex line. The second integral in (4.9) producing the dominant contribution (4.10) can be rewritten equivalently as:

$$\frac{\delta^2}{8\pi}\int \left(\nabla \times \vec{H}\right)^2 dV \equiv \int \frac{\rho_s \vec{v}_s^2}{2} dV; \quad \rho_s \vec{v}_s \equiv \frac{m}{e}\vec{j} \qquad (4.11)$$

with $\vec{j}$ given in (4.6). Thus, (4.10) is indeed, the kinetic energy of the Cooper pairs providing the suprcurrent in the vortex body. Despite the superfluid velocity $|\mathrm{v}_s|$ still diverges $\propto 1/r$,when $r \to 0$, the density of Cooper pairs vanishes in the vortex core even faster, i.e. in accord with (4.8):

$$\rho_s \mathrm{v}_s^2 \sim 2m\left(\Psi_0 f(r)\right)^2 \mathrm{v}_s \propto r^2 \cdot r^{-1} = r; \quad \xi_0 \ll r \ll \xi \qquad (4.12)$$

and hence, the density of the kinetic energy vanishes as well.

Remark₂ Relation (4.10) explains why Abrikosov's vortices carry single quantum of magnetic flux Φ_0: in case a single vortex would carry flux $n\Phi_0$ its linear density of energy would be $\varepsilon_0 n^2$, which is greater than the energy $n\varepsilon_0$ of n vortices, each carrying flux Φ_0.

The lower critical field H_{c1}

Besides the increase in the free energy (per unit of length), ε_0, see Eq. (4.10), the vortex also brings a decrease of the free energy F_M due to the paramagnetic effect of the Abrikosov vortex in the external magnetic field H_0:

$$F_M/L = -H_0 M/L = -H_0 \frac{1}{2c}\int_\xi^\delta [j_s r] 2\pi r dr \approx -H_0 \frac{1}{2c}\int_\xi^\delta \frac{c\Phi_0}{8\pi^2\delta^2} 2\pi r\, dr = -\frac{H_0\Phi_0}{16\pi} \qquad (4.13)$$

Hence, the total balance of $F_v + F_M$ becomes negative when the external field H_0 exceeds the lower critical field H_{c1}, at which $F_v + F_M = 0$:

$$H_{c1} = \frac{\Phi_0}{\pi\delta^2}\ln æ \approx H_{cm}\frac{\ln æ}{æ}. \qquad (4.14)$$

The upper critical field H_{c2} *(an estimate)*

In the phase diagram the boundary between the normal and superconducting state of the type-II superconductor defines the *upper critical field* H_{c2} (Abrikosov, 1957).

The order of magnitude of the field H_{c2} can be estimated qualitatively. The Cooper pairs coherence in the superfluid Bose-condensate breaks down as a consequence of their rotation in the magnetic field. Obviously, the Cooper pair can sustain coherence in a field, at which the Larmor radius is larger than Landau correlation length, ξ, i.e. :

$$r_L \propto cp_\perp/(eH) > \xi . \qquad (4.15)$$

Here $p_\perp$ is the component of the pair center of mass momentum in the plane perpendicular to the magnetic field. Hence $p_\perp \le p \propto m\mathrm{v}_s$, where v_s is the superfluid velocity. It follows from the Landau criterion that $\mathrm{v}_s < \Delta/p_\perp$, else the superfluidity breaks down. Allowing for the Heisenberg uncertainty of the center of mass motion: $p_\perp < \hbar/\xi$, we obtain the chain of estimates:

$$\xi < cp_\perp/(eH) < c\hbar/(\xi eH), \qquad (4.16)$$

or

$$\mathrm{H} < c\hbar/(\mathrm{e}\xi^2) \sim \Phi_0/\xi^2 \sim \mathrm{H}_{c2}. \qquad (4.17)$$

The field H_{c2} can be expressed in terms of H_{cm}:

$$\mathrm{H}_{c2} \sim \Phi_0/\xi^2 \sim æ\Phi_0/(\xi\delta) \sim æ\mathrm{H}_{cm}. \qquad (4.18)$$

So far we considered the pure superconductor. If it contains the impurities and $l << \xi$, then ξ should be substituted by $\xi' \propto (\xi l)^{1/2}$ in the formula (4.18). In this case we obtain

$$H_{c2} \propto H_{cm}\delta/l \propto H_{cm}æ. \qquad (4.19)$$

Therefore the field H_{c2} has the order of magnitude of H_{cm}æ in both cases, but the estimate (4.19) is important. It follows from (4.19) that one can increase considerably H_{c2} , and hence, the region of fields sustainable for the superconducting wire may be increased by means of raising the impurity concentration and thus decreasing the mean free path ($l \propto n_i^{-1}$). The mean free path of order of interatomic distances would lead to the upper critical fields, that taking into account the ordinary values $H_{cm} \propto 10^2 - 10^3$ Oe, and $\delta \propto 10^{-5} - 10^{-3}$ cm, would be of order of:

$$H_{c2} < 10^3(10^{-5}/10^{-8}) \text{ Oe} \propto 10^6 \text{ Oe}=10^2\text{T} \tag{4.20}$$

These fields are of order of the so-called Klogston paramagnetic limit, i.e. field that orients both spins in the Cooper pair parallel to the magnetic field:

$$\mu_B H_{cp} \approx 2\Delta , \tag{4.21}$$

where μ_B is the Bohr magneton. Taking maximal T_c about 100 K, we have that H_{cp} and H_{c2} are of the same order of magnitude of 100 tesla.

Quantitative theory of the upper critical field H_{c2}

Remark₃ *The superconducting transition at H_{c2} is second order, hence, at $H = H_{c2}$ the stationary infinitesimally small superconducting nuclei can exist. Hence, the GL equation (3.11) (or dimensionless Eq. (3.18)) can be linearized with respect to the order parameter $\Psi(r)$. Another important consequence is that magnetic field can be considered as uniform in the sample at H_{c2} and coincident with the external field in the absence of the superconductor.*

The above observations lead to a conclusion, that farther simplification is possible, namely, the solution of the GL equations could be sought for in the form of a function of the single argument (one-dimensional solution):

$$æ^{-2} d^2\Psi / dx^2 + \Psi(1 - A^2) = 0 , \tag{4.22}$$

$$d^2 A / dx^2 = 0 . \tag{4.23}$$

It follows from (4.23) that $A = H_0 x$ (we choose: $\vec{H} \parallel \hat{z}$, and $\vec{A} \parallel \hat{y}$), the origin of the coordinate system can be taken anywhere inside the superconductor. Substituting this $A(x)$ in (4.22) we have:

$$-d^2\Psi / dx^2 + æ^2 x^2 \Psi = æ^2\Psi. \tag{4.24}$$

We have to find solution that remains finite at $x \to \pm\infty$. Equation (4.24) has the form of the Schrödinger stationary equation for a harmonic oscillator:

$$-(\hbar^2/2m) d^2\Psi/dx^2 + (k/2)x^2\Psi = \varepsilon\Psi. \tag{4.25}$$

As it is well known from the quantum mechanics the latter equation has vanishing at $x \to \pm\infty$ solution, provided that:

$$\varepsilon = \hbar\omega(n + 1/2), \tag{4.26}$$

where $\omega = (k/m)^{1/2}$. Comparing (4.24) with (4.25) we obtain: $\hbar\omega = 2æH_0$, and hence, the sought for solution exists when $æ^2 = 2æH_0(n + 1/2)$, or:

$$H_0 = æ/(2n + 1). \quad (4.27)$$

Thus, the maximal field, which provides nonvanishing solution $\Psi \neq 0$ corresponds to $n = 0$ and equals:

$$H_{c2} = æ, \quad (4.28)$$

or in the dimensionful units:

$$H_{c2} = æ\sqrt{2}\, H_{cm}. \quad (4.29)$$

This result agrees with the estimate (4.18). Note that for $æ > 1/\sqrt{2}$ the condition $H_{c2} > H_{cm}$ is satisfied.

LECTURE 5

Tunnel junctions. Stationary and non-stationary Josephson effect. International standard of volt. SQUID.

Preamble *Quantum particle can tunnel through the potential barrier while possessing the energy which is below the barrier height. The current-voltage (I-V) characteristic of the tunnel junctionof two conductors is widely used as an indicator of electron correlations reflected in the electron density of states (DOS): information about electronic DOS provide curves of dI/dV* vs *V. As well, the electron-lattice inelastic scattering in solids can be also investigated: information about bosonic modes coupled to electronic system provide curves of* d^2I/dV^2 vs *V. The research using scanning tunneling microscopy (STM) is based on the tunnel junction theory. Besides, the superconducting tunnel junction introduced by Giaever stimulated theoretical prediction of the Josephson effect.*

Tunnel junction: *single-particle tunneling current*

The tunneling probability (transmission coefficient) through the barrier in the quasi-classical approximation is given by the exponential factor:

$$W \propto \exp\left[-\frac{2}{\hbar}\int \mathrm{Im}(p_x)dx\right] = \exp\left\{-\frac{2(2m)^{1/2}}{\hbar}\int_{x_1}^{x_2}\left[U(x)-E\right]^{1/2}dx\right\}. \quad (5.1)$$

The integration is performed over the region where $U(x) > E$, see Fig. 15. The probability of the tunneling through the barrier decreases with increase of both its height ($U_{\max} - E$) and width ($x_2 - x_1$), as well as the mass of the particle, m.

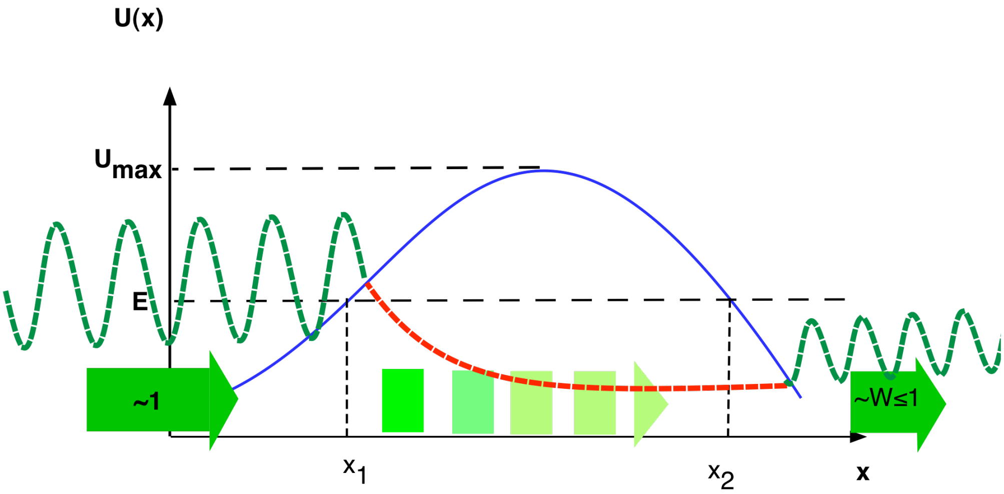

Fig. 15. Schematic drawing of the potential barrier in the problem of quantum tunnelling.

Quantitative estimates suggest that in the common experimental setup, tunneling is plausible event for electrons, and if the barrier is the dielectric layer, its thickness should be no more than several inter-atomic distances. The most simple is to use the natural oxide layer arising on the surface of various metals such as aluminum, tin, lead, etc. By evaporating either another or the same metal onto the surface of a given metal one obtains the so called tunnel junction (Giaever, 1960), this simple device is a very useful experimental tool for investigation of the superconductivity.

Tunnel junction between two metals in the normal state (N-N)

The chemical potentials of the metals connected via the tunnel junction (contact) are the same in the equilibrium, Fig. 16 a. When a potential difference is applied to the contact, all this potential difference falls only over the dielectric layer due to the large resistance of the latter. Therefore, the chemical potentials of both metals are at different levels (the levels difference is eV, Fig. 16 *b*). Considering the height of the barrier to be much greater than eV, we conclude that the number of electrons which are able to enter the free states in the other metal is proportional to eV, and all of these electrons pass through the barrier with the same probability. Therefore the current is proportional to the potential difference V, that agrees with the Ohm's law (Fig. 17, curve 1).

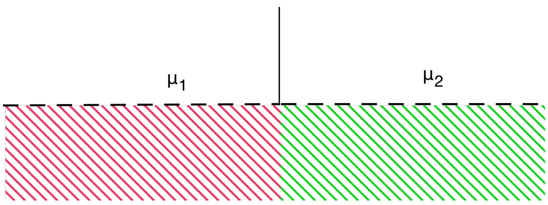

Fig. 16 a. The N-N tunnel junction

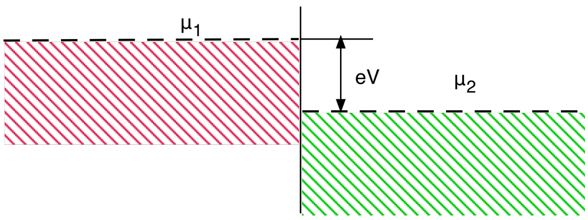

Fig. 16 b. The N-N tunnel junction under the potential difference V.

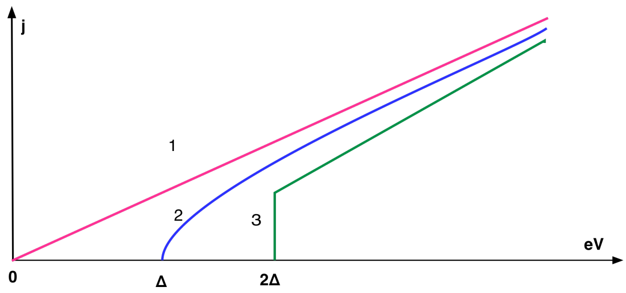

Fig. 17. I-V characteristics of the N-N, N-S, and S-S tunnel junctions.

The quasi-particle current density through the tunnel junction can be expressed as follows:

$$J(V) \propto \int W\nu_1(\varepsilon - eV)\nu_2(\varepsilon)\{n_1(\varepsilon - eV)[1 - n_2(\varepsilon)] - n_2(\varepsilon)[1 - n_1(\varepsilon - eV)]\} =$$
$$= \int W\nu_1(\varepsilon - eV)\nu_2(\varepsilon)[n_1(\varepsilon - eV) - n_2(\varepsilon)]d\varepsilon =$$
$$= W\nu_{1n}\nu_{2n}\int|\varepsilon - eV|[(\varepsilon - eV)^2 - \Delta^2]^{-1/2}[n_1(\varepsilon - eV) - n_2(\varepsilon)]d\varepsilon. \quad (5.2)$$

Here $\nu_1(\varepsilon)$ and $\nu_2(\varepsilon)$ are the densities of quasi-particle states on the left (1) and right (2) sides of the tunnel junction, and the energy is counted from the Fermi-level; $n(\varepsilon)$ is the Fermi-Dirac distribution of the quasi-particles, W is transmission coefficient through the potential barrier.

Remark$_1$ *Eq. (5.2) signifies that the current flowing from one metal to another must be proportional to the tunneling probability, the number of the occupied states in the first metal, the number of the vacant states in the second metal and the product of the densities of states in the both metals.*

Tunnel junction between normal metal and superconductor (N-S)

Now let one metal to be in the normal state, and another one - in the superconducting state. Consider zero temperature case: $T = 0$. In the equilibrium, at zero potential difference $V=0$, the chemical potentials are equal: $\mu_1 = \mu_2$, Fig. 18 a.

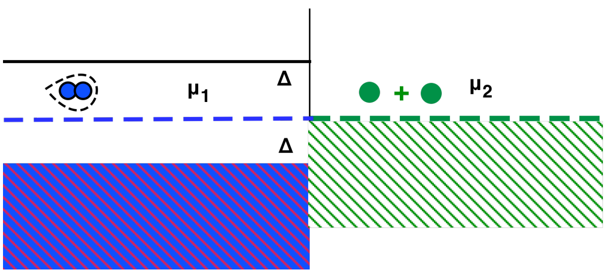

Fig. 18 a. S-N junction at *V=0*.

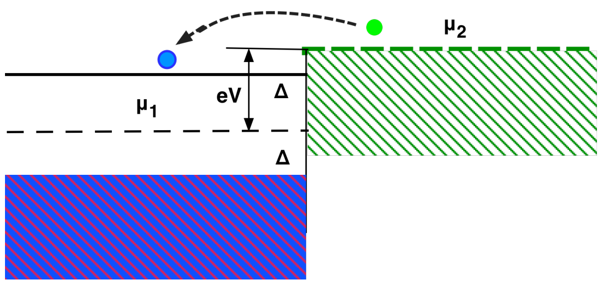

Fig. 18 b. S-N junction at $V \neq 0$.

At the same time the electrons in the superconductor are bound in the Cooper pairs, the latter are in the Bose condensate. Therefore, the chemical potential μ_1 is just the chemical potential of the pair, see Fig. 18 a. Hence, fermionic quasi-particles may eneter the superconductor only when applied to the junction potential difference eV exceeds the binding energy per electron: $eV > \Delta$, Fig. 18 b and Fig. 17 (curve 2). This means that to transmit an electron from the normal metal to the superconductor it is necessary supply it with the additional energy that exceeds Cooper pair binding energy per electron, Δ. The same is true for a transition of the electron from the superconductor to the normal metal. First, one has to break the Cooper pair, i.e. spend the amount of energy Δ per one electron, and after that the electron can pass to the normal metal.

Remark₂ *According to Eq. (5.1), and allowing for the facts that: 1) the Cooper pair has the double charge 2e, which increases U(x), and 2) it also has double mass, 2m; one concludes that, comparing with the single electron, the tunneling probability of the Cooper pair as a whole is vanishingly small, unless the quasiparticle current is blocked by the insufficient potential difference at low temperatures. This conclusion is not true for the* <u>*weak links*</u> *exhibiting the Josephson effect, which is considered farther.*

I-V characteristic of the N-S tunnel junction

The voltage threshold for the quasi-particle current through the N-S tunnel junction, see Fig. 17 and Fig. 18 b, is:

$$eV = \Delta. \tag{5.3}$$

Here it is assumed that the potential difference is applied as is shown on Fig. 18 b. The total current flowing from the superconductor to the normal metal is the difference between the currents flowing in the forward and backward directions. The zero of the energy is counted from the Fermi-level of the superconductor μ_1. The I-V characteristic is the dependence $J(V)$. To integrate over the energy in (5.2) one has to know the density of states. The density of states in the normal metal is equal to:

$$\nu_2 \equiv \nu_n = p_0 m/(\pi^2 \hbar^3) = \text{const} \tag{5.4}$$

The density of the quasi-particle states in the superconductor is:

$$\nu_1 \equiv \nu_s = \begin{cases} \nu_n |\varepsilon| / \left(\varepsilon^2 - \Delta^2\right)^{1/2}; & |\varepsilon| > \Delta \\ 0; & |\varepsilon| < \Delta \end{cases} \tag{5.5}$$

Here $\varepsilon > 0$ corresponds to the quasi-particles created in the superconductor, e.g. due to the incoming electrons from the normal metal side, and $\varepsilon < 0$ correspond to the fermionic quasi-particles created by Cooper pair-breaking, that may then tunnel into the metal. The significant fact is that $\nu_s(|\varepsilon| \to \Delta) \to \infty$, though this divergence is integrable, i.e. the total number of states in the small energy interval near $|\varepsilon| = \Delta$ is small .

Substituting (5.4) and (5.5) into (5.2) and assuming, for simplicity, zero temperature limit $T=0$, we may integrate, allowing for the simple conditions: $n(\varepsilon) = 0$ at $\varepsilon > 0$ and $n(\varepsilon) = 1$ at $\varepsilon < 0$. Therefore, the difference $n_1 - n_2$ equals unity in the interval $0 < \varepsilon < eV$ and zero outside this interval. The density of states entering the integrand is non-zero when $|\varepsilon - eV| > \Delta$, i.e. in this case $\varepsilon < eV - \Delta$. Hence, the current is indeed absent when $eV < \Delta$. When $eV > \Delta$ one finds:

$$j \propto W \nu_{1n} \nu_{2n} \int_0^{eV-\Delta} (eV-\varepsilon)\left[(eV-\varepsilon)^2 - \Delta^2\right]^{-1/2} d\varepsilon = W \nu_{1n} \nu_{2n} \left[(eV)^2 - \Delta^2\right]^{1/2}. \tag{5.6}$$

The difference between the normal metal and the superconductor must disappear when $eV >> \Delta$, i.e. the current must be the same as it is in the contact of two normal metals. Thus we have

$$j/j_n = [(eV)^2 - \Delta^2]^{1/2}/(eV). \tag{5.7}$$

The result is shown on Fig. 17 (the curve *2*).

I-V characteristic of the S-S tunnel junction

Consider now the tunnel contact between the two superconductors, that for simplicity we assume to be the same on the both sides of the tunnel junction. We find analogously to the derivation path from Eq. (5.2) to Eq. (5.6) :

$$J(V) \propto W \nu_n^2 \int_\Delta^{eV-\Delta} (eV-\varepsilon)\left[(eV-\varepsilon)^2 - \Delta^2\right]^{-1/2} \varepsilon\left(\varepsilon^2 - \Delta^2\right)^{-1/2} d\varepsilon \tag{5.8}$$

Now the quasi-particle current through the junction is different from zero provided that $eV > 2\Delta$. Calculation in (5.8) gives the following answer:

$$J/J_n = E\left(\left[(eV)^2 - (2\Delta)^2\right]^{1/2}/eV\right) - 2(\Delta/eV)^2 K\left(\left[(eV)^2 - (2\Delta)^2\right]^{1/2}/eV\right) \tag{5.9}$$

where *K* and *E* are complete elliptic integrals of the first and second kind respectively:

$$\begin{aligned} \mathrm{K(k)} &= \int_0^{\pi/2} \left(1 - k^2 \sin^2\phi\right)^{-1/2} d\phi, \\ E(\mathrm{k}) &= \int_0^{\pi/2} \left(1 - k^2 \sin^2\phi\right)^{1/2} d\phi. \end{aligned} \tag{5.10}$$

Again, in the limit $eV >> \Delta$, $J/J_n \to 1$. At $eV = 2\Delta$:

$$J/J_n = \pi/4, \tag{5.11}$$

and $J=0$ when $eV < 2\Delta$.

Finally, we mention that besides the fermionic quasi-particle current, that exists due to tunneling from one metal to another and that vanishes at zero bias voltage, (single-particle current), there is also superconducting current through the contact. In the next section we study this phenomenon.

The Josephson effect

Remark₃ *So far we considered tunneling of the fermionic excitations between both sides of the contact neglecting correlations between electrons (holes) in the Cooper pairs. In fact the electron transition through the barrier is itself the result of the electron wave function propagation over the contact and, therefore, the consistent theory should consider the coherent state forming in the whole electronic system. Hence, the Cooper pairs formed of electrons belonging to different sides of the junction may carry supercurrent with the transmission coefficient of the order of the one assigned to the single-particle tunneling, but at zero voltage across the junction. This effect was first proposed by Josephson in 1962 and since that is called the Josephson effect.*

To calculate the Josephson supercurrent we use the modified boundary conditions at the junction surface *S*, that differ from Eq. (3.12) by the non-vanishing right hand side allowing for extended over the junction Cooper pairs:

$$\vec{n}\cdot[-i\hbar\vec{\nabla} - (2e/c)\vec{A}]\Psi\Big|_{S1} = \frac{\Psi}{\lambda}\Big|_{S2}; \quad \vec{n}\cdot[-i\hbar\vec{\nabla} - (2e/c)\vec{A}]\Psi\Big|_{S2} = \frac{\Psi}{\lambda}\Big|_{S1} \qquad (5.12)$$

Specify the coordinate system, as is shown in Fig. 19, and choose the magnetic field $\vec{H} = \{0,0,H\}$ along the z-axis (parallel to the junction plane) and the vector potential along the x-axis $\vec{A} = \{A_x(y),0,0\}$. Then, rewrite the boundary conditions (with $1/\lambda$ being the electron (hole)

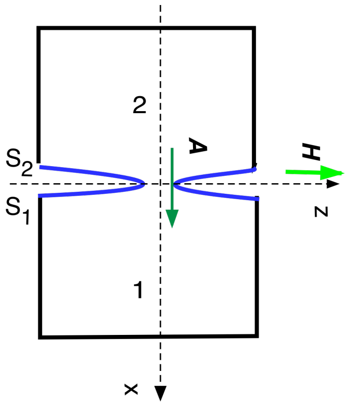

Fig. 19. *A sketch of the Josephson junction realized via "weak link" between the superconducting sides 1 and 2.*

transmission coefficient between the sides 1 and 2), and the supercurrent equation (3.15) in the following form:

$$\frac{\partial\Psi_1}{\partial x} - \frac{2ie}{\hbar c}A_x\Psi_1 = \frac{\Psi_2}{\lambda}; \quad \frac{\partial\Psi_2}{\partial x} - \frac{2ie}{\hbar c}A_x\Psi_2 = \frac{\Psi_1}{\lambda}; \qquad (5.13)$$

$$j = -\frac{ie\hbar}{2m}\left(\Psi_1{}^*\frac{\partial\Psi_1}{\partial x} - \Psi_1\frac{\partial\Psi_1{}^*}{\partial x}\right) - \frac{2e^2}{mc}|\Psi_1|^2 A_x. \qquad (5.14)$$

Then, express derivatives of Ψ_1 and Ψ^*_1 in algebraic form via Ψ_2 from the boundary conditions in Eq. (5.13), and then substitute those into Eq. (5.14). This "pathway" terminates in expression for the stationary Josephson current:

$$j_J = -\frac{ie\hbar}{2m\lambda}\left(\Psi^*_1\Psi_2 - \Psi_1\Psi^*_2\right) = \frac{e\hbar}{m\lambda}|\Psi|^2 \sin\left(\theta_2 - \theta_1\right), \tag{5.15}$$

where in the last equality in the right hand side of (5.15) it is assumed that the order parameters Ψ_1 and Ψ_2 differ only by the phase angles: $\Psi_{1,2} = |\Psi|\exp\left\{i\theta_{1,2}\right\}$.

Remark$_4$ *It is natural, that since the Josephson current depends on the phase (difference) of the order parameter, it should be periodic function of this difference, since the phase itself is defined modulo* 2π *. The amplitude of the Josephson current, though finite at zero voltage bias V=0, vanishes at* T_c *linear with* T_c*-T, since it is proportional to the density of the superconducting Bose-condensate,* $|\Psi|^2$ *.*

Remark$_5$ The contacts with resistance less than 0,1 Ohm·cm^2 are commonly used to observe experimentally the Josephson effect. In practice, one can make junctions with even smaller resistances of 10^{-4} Ohm·cm^2. The corresponding critical current density can reach $10^2 \div 10^3$ A/cm^2. Comparing this value with the current density of 10^8 A/cm^2 destroying the Cooper pairs in the massive superconductor we see that maximal Josephson current is relatively small. Thus, the Josephson effect and the associated phenomena are called sometimes the "weak superconductivity".

When a finite bias voltage V is applied to the junction, the wave functions $\Psi_{1,2}$ acquire different time-dependent factors, e.g.:

$$\Psi_1 \propto \exp\left\{-i2eV_1t/\hbar\right\}; \quad \Psi_2 \propto \exp\left\{-i2eV_2t/\hbar\right\}; \quad V_1 - V_2 = V \tag{5.16}$$

where the charge of the Cooper pair is *2e*. Substituting (5.16) into (5.15) (keeping the constant part of the difference of phases intact) one finds expression for the non-stationary Josephson current:

$$j_J = -\frac{ie\hbar}{2m\lambda}\left(\Psi^*_1\Psi_2 - \Psi_1\Psi^*_2\right) = \frac{e\hbar}{m\lambda}|\Psi|^2 \sin\left(\theta_2 - \theta_1 + \frac{2eVt}{\hbar}\right) \equiv j_c \sin\left(\Delta_{21} + \frac{2eVt}{\hbar}\right), \tag{5.17}$$

where j_c is the amplitude of the Josephson current, and the phase difference without the external magnetic field is denoted with Δ_{21}.

International standard of volt.

Since 1990 the volt has been maintained internationally for *practical measurement using the non-stationary Josephson effect*, where a conventional value is used for the *Josephson constant,* K_J, fixed by the 18th General Conference on Weights and Measures as:

$$K_{J\text{-}90} = 2e/h = 0.4835979 \text{ GHz}/\mu\text{V}. \tag{5.18}$$

This is typically used with an array of several thousand or tens of thousands of junctions, excited by microwave signals between 10 and 80 GHz (in the several array designs).

The Josephson effect in magnetic field

Finally, we introduce the external magnetic field applied e.g. along the z-axis and, hence, the corresponding vector potential aligned along the x-axis, the latter being perpendicular to the plane of the junction {e,z}. Allowing for fulfillment of the guage invariance condition the phase of the order parameter Ψ and the vector potential $\vec{A}$ should enter in the combination:

$$\vec{\nabla}\theta - (2e/\hbar c)\vec{A} \tag{5.19}$$

This allows to integrate the above combination over x from the point 1 in the depth of the left superconductor till the point 2 in the depth of the right one, see Fig. 19, and obtain

$$\tilde{\Delta}_{12} = \theta_2 - \theta_1 - \frac{2e}{c\hbar}\int_1^2 A_x\, dx \approx \frac{2e}{c\hbar} H_z y 2\delta; \quad j_J = j_c \sin\left(\frac{2e}{c\hbar} H_z y 2\delta\right), \tag{5.20}$$

where we neglect the width of the junction in comparison with the London penetration depth of magnetic field, while the latter eneters in the region of the junction parallel to {y,z} plane, along the z-axis. Substituting the result in (5.19) into (5.17) one finds that the Josephson current in the external magnetic field acquires the coordinate dependence along the y-axis perpendicular to the applied magnetic field H. Then, averaging the expression (5.20) for the Josephson current over tha junction's {y,z} plane we find:

$$\bar{j} = L^{-1}\int_0^L j(y)\, dy = j_c \frac{c\hbar}{2eHL2\delta}\left\{-\cos\left[-\Delta_{21} + \frac{2e}{c\hbar} H_z L 2\delta\right] + \cos(\Delta_{21})\right\} \tag{5.21}$$

where $HL2\delta = \Phi$ is magnetic flux through the junction. Using this equality we rewrite Eq. (5.21) in the final form, that represents an expression for the *Josephson current in the external magnetic field parallel to the plane of the tunnel junction*:

$$\bar{j} = j_c \frac{\Phi_0}{\pi\Phi} \sin\left(\frac{\pi\Phi}{\Phi_0}\right) \sin\left(-\Delta_{21} + \frac{\pi\Phi}{\Phi_0}\right) \tag{5.22}$$

where Φ_0 is the quantum of magnetic flux in Landau-Ginzburg theory, see Fig. 20.

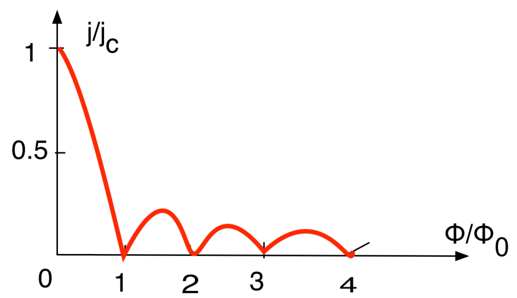

Fig. 20. Dependence of the Josephson current on the external magnetic field.

Thus, choosing the bare phase difference Δ_{21} such, that the second sine-factor in (5.22) becomes ± 1, we conclude that the amplitude of the Josephson current oscillates according to the following expression:

$$\overline{j}_{\max} = j_c \left| \frac{\Phi_0}{\pi\Phi} \sin\left(\frac{\pi\Phi}{\Phi_0}\right) \right| \tag{5.23}$$

The above result in Eq. (5.23) provides the possibility to measure the London penetration depth entering the definition of the magnetic flux: $HL2\delta = \Phi$.

As the Josephson current is very weak we can omit the magnetic field, which is created by this current.

SQUID

One of the important practical applications of the Josephson effect is the superconducting quantum interference device (SQUID), used as, e.g. a quantum magnetometer with precision reaching 10^{-10} Oe. The schematic drawing of SQUID in Fig. 21 explains the basic physical principle lying behind this device: the interference between the Josephson currents through the parallel Josephson junctions in the external magnetic field ***H***, that produces the magnetic flux Φ through the hollow inside the contour C drawn within the bulk of the superconducting circuit. The total supercurrent ***I*** flowing through the device equals the sum of the supercurrents flowing in the two parallel legs:

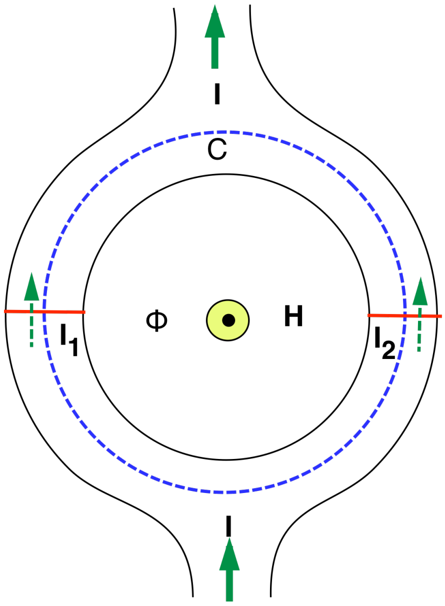

Fig. 21. Schematic drawing of the SQUID.

$$I = I_{c1} \sin\theta_1 + I_{c2} \sin\theta_2 \tag{5.24}$$

Here θ_1 and θ_2 are the jumps of the phase across the corresponding Josephson junctions (JJ). Assuming that external magnetic field is directed perpendicular to the plane of the SQUID, and choosing the corresponding vector potential ***A*** in the plane perpendicular to the field, one can write the following induced extra phase gradient of the superconducting order parameter:

$$\vec{\nabla}\theta = \frac{2e}{\hbar c}\vec{A} \tag{5.25}$$

Hence, the total phase-shift along the contour *C* drawn in the bulk of the superconducting circuit, Fig. 21, equals the multiple of the angle period 2π, necessary for well defined order parameter Ψ as function of the coordinate:

$$\theta_1 - \theta_2 + \frac{2e}{\hbar c}\Phi \equiv \theta_1 - \theta_2 + 2\pi\frac{\Phi}{\Phi_0} = 2\pi n\,. \tag{5.26}$$

Using Eq. (5.26) one may accordingly assign the effective phase shifts to the angles entering the supercurrents through the JJ:

$$\theta_1 = \theta - \pi\frac{\Phi}{\Phi_0} + 2\pi n, \quad \theta_1 = \theta + \pi\frac{\Phi}{\Phi_0}\,. \tag{5.27}$$

Then, the total current through the SQUID equals:

$$I = I_{c1}\sin\left(\theta - \pi\frac{\Phi}{\Phi_0}\right) + I_{c2}\sin\left(\theta + \pi\frac{\Phi}{\Phi_0}\right) \tag{5.28}$$

Hence, the external magnetic field shifts the phases of the enterfering JJ. Assuming for simplicity that both JJ are identical, $I_{c1} = I_{c2} = I_c$, one finds:

$$I = 2I_c\cos\left(\pi\frac{\Phi}{\Phi_0}\right)\cdot\sin\theta \equiv \tilde{I}_c\sin\theta \tag{5.29}$$

Hence, we see that the effective critical current of the interferometer $\tilde{I}_c$ oscillates with the flux of the external magnetic field: the total supercurrent through the SQUID turns into zero when:

$$\Phi = (n + 1/2)\Phi_0\,. \tag{5.30}$$

Since the flux quantum is small: $\Phi_0 = 2\cdot 10^{-7}Oe\cdot cm^2$, with the SQUID one can achieve measurement accuracy of the order of 10^{-10} Oe. In reality the measurement of the gradient of the external magnetic field by the SQUID gives the most accurate results weakly dependent on the noise.