Coexistence of Weak and Strong Wave Turbulence in a Swell Propagation

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By performing two parallel numerical experiments—solving the dynamical Hamiltonian equations and solving the Hasselmann kinetic equation—we examined the applicability of the theory of weak turbulence to the description of the time evolution of an ensemble of free surface waves (a swell) on deep water. We observed qualitative coincidence of the results. To achieve quantitative coincidence, we augmented the kinetic equation by an empirical dissipation term modeling the strongly nonlinear process of white capping. Fitting the two experiments, we determined the dissipation function due to wave breaking and found that it depends very sharply on the parameter of nonlinearity (the surface steepness). The onset of white capping can be compared to a second-order phase transition. The results corroborate the experimental observations of Banner, Babanin, and Young [J. Phys. Oceanogr. **30**, 3145 (2000)].

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Wave turbulence is realized in plasmas, liquid helium, magnetohydrodynamics, nonlinear optics, etc. A perfect example of wave turbulence is a wind-driven sea. The major conceptual difference between wave turbulence and "classical" turbulence in an incompressible fluid is the presence of a characteristic dimensionless parameter μ, characterizing the level of nonlinearity. Turbulence is considered to be "weak" if $\mu \ll 1$; otherwise it is "strong." In classical hydrodynamic turbulence, $\mu \to \infty$.

A more rigorous definition of weak turbulence is the following: this is the turbulence which is well described by the kinetic equation for waves. These equations are the quantum kinetic equations for bosons in the limit of very high occupation numbers. They were derived in statistical physics in the late 1920s by Nordheim and Peierls and rediscovered in nonlinear wave dynamics in the 1960s. The kinetic equation, describing four-wave resonant interaction of gravity waves, was named after K. Hasselmann, who derived it in 1962–1963 [1].

The theory of weak turbulence is well developed [2]. The kinetic equation has rich families of Kolmogorov-Zakharov (KZ) and self-similar solutions, which can be efficiently used for explaining a wide range of experimental data [3,4]. However, today we have a clear understanding of the following fact: even for small values of μ, the theory of weak turbulence may be incomplete. In many important physical situations weak and strong turbulences coexist.

Even if the weak turbulent resonant interaction effects dominate in the greater part of space, strongly nonlinear effects could appear as rare localized coherent events. If they are smooth and regular, they are solitons, quasisolitons, or vortices. However, they could be catastrophic, in which case they are wave collapses, similar to self-focusing in nonlinear optics or Lagmuir collapses in plasma. Even rare sporadic collapse events can essentially affect the physical picture of wave turbulence.

There are two main types of wave collapse events in a wind-driven sea. The first is the formation of freak waves; this is not a subject of our study. The second, which is much more common, is wave breaking or white capping, which is an essential mechanism of wave energy dissipation. It would be hopeless to develop an efficient operational model of wave forecasting without an understanding and a proper parametrization of this fundamental effect. Meanwhile, a reliable analytical theory of this phenomenon is still not developed, while field and laboratory experimental data are scarce. The most promising approach to resolving this problem is a massive numerical experiment.

The most informative experiment would be one that could provide a direct numerical solution of the primitive dynamic equations describing the wave ensemble. In 1992, Dyachenko, Pushkarev, Newell, and Zakharov numerically solved 2D focusing NLSE and observed the coexistence of self-focusing collapses with weak turbulence [5]. Simulation of the surface gravity waves turbulence for the first time was done simultaneously by Tanaka [6] and Onorato *et al.* [7]. Because of limitations of calculation performance of that time computers simulations were limited to dynamical equations and quite short simulation times. Later on, the 1D MMT (Maida, McLaughlin, and Tabak) model and its generalizations were solved numerically by different authors. Again, the coexistence of wave collapses and weak turbulence was verified. In our Letter, we present the results of a far more detailed experiment. We performed the numerical simulation of the evolution of an ocean swell using two different approaches.

In the first, we solved the Euler equations for the 3D potential flow of an ideal incompressible fluid with a free surface in the presence of gravity. We used the Hamiltonian form of these equations [8,9]. For gravity waves, the parameter of nonlinearity is the average steepness μ. We expanded the Hamiltonian in powers of μ up to order μ^4. In the second experiment, we solved the Hasselman kinetic equation.

The comparison of the results demonstrates qualitative accordance. Both experiments describe expected effects, such as the downshift of the spectrum peak, the angular spreading of the spectrum, and the formation of Zakharov-Filonenko spectral tails $F_\omega \sim \omega^{-4}$ [2,10]. To obtain quantitative coincidence of the results, we have to augment the Hasselmann equation by an empirical dissipation term $S_{\rm diss}$, modeling white capping effects. We tried several versions of this term. The versions of $S_{\rm diss}$ used in the operational wave-predicting models WAM3 and WAM4 essentially overestimate the dissipation for a moderate steepness. The comparison with dynamical computations shows that white capping dissipation decreases dramatically with decreasing steepness and that it is probably a threshold phenomenon, similar to a second-order phase transition. Similar results were earlier obtained in the field experiment by Banner, Babanin, and Young [11].

Dynamical model.—In this part of our experiment, the surface of the liquid is described by two functions of the horizontal variables x, y, and the time t: the surface elevation $\eta(x, y, t)$ and the velocity potential on the surface $\psi(x, y, t)$. In our approximation, they satisfy the following equations [8,12,13]:

$$\begin{aligned} \dot\eta &= \hat k\psi - [\nabla(\eta\nabla\psi)] - \hat k[\eta\hat k\psi] + \hat k(\eta\hat k[\eta\hat k\psi]) \\ &\quad + \tfrac{1}{2}\nabla^2[\eta^2\hat k\psi] + \tfrac{1}{2}\hat k[\eta^2\nabla^2\psi] + \hat F^{-1}[\gamma_k\eta_k], \\ \dot\psi &= -g\eta - \tfrac{1}{2}[(\nabla\psi)^2 - (\hat k\psi)^2] - [\hat k\psi]\hat k[\eta\hat k\psi] \\ &\quad - [\eta\hat k\psi]\nabla^2\psi + \hat F^{-1}[\gamma_k\psi_k]. \end{aligned} \tag{1}$$

Here $\hat k$ is the linear integral operator $\hat k = \sqrt{-\nabla^2}$; $\hat F^{-1}$ corresponds to the inverse Fourier transform.

Equations (1) are nowadays widely used in numerical experiments and are solved by different versions of the spectral code [6,7,12–16]. In the present experiment, we solved the equations in the real space domain $2\pi \times 2\pi$ using the finest currently possible rectangular grid 512×4096, putting $g = 1$. The dissipative terms $\hat F^{-1}[\gamma_k\eta_k]$ and $\hat F^{-1}[\gamma_k\psi_k]$ are taken in the form of pseudoviscous high frequency damping. We put

$$\gamma_k = \begin{cases} 0, & k < k_d, \quad k_d = 1024, \\ -\gamma(k-k_d)^2, & k \ge k_d, \quad \gamma = 5.65\times 10^{-3}. \end{cases} \tag{2}$$

In accordance with recent results [17], the dissipation term should be included in both equations.

The distribution of the wave action is described by the function $n(k, t) = |a_{\vec k}(t)|^2$, where

$$a_{\vec k} = \sqrt{\frac{\omega_k}{2k}}\eta_{\vec k} + i\sqrt{\frac{k}{2\omega_k}}\psi_{\vec k} \tag{3}$$

are complex normal variables. Here $\omega_k = \sqrt{gk}$.

As the initial condition, we used a Gaussian-shaped distribution in the Fourier space:

$$\begin{aligned} |a_{\vec k}| &= A_i \exp\left(-\frac{1}{2}\frac{|\vec k - \vec k_0|^2}{D_i^2}\right), \qquad |\vec k - \vec k_0| \le 2D_i, \\ |a_{\vec k}| &= 10^{-12}, \qquad |\vec k - \vec k_0| > 2D_i, \\ A_i &= 0.92\times 10^{-6}, \qquad D_i = 60, \qquad \vec k_0 = (0; 300). \end{aligned} \tag{4}$$

The initial phases of all harmonics were random. The average steepness of this initial condition, defined as $\mu = \sqrt{2\langle|\nabla\eta|^2\rangle}$, was $\mu \simeq 0.176$.

The period of the most intensive wave was $T_0 = 2\pi/\sqrt{300} = 0.362$. Calculations continued until $t = 3378T_0$. We observed an angular spreading of the initial spectral distribution together with a downshift of the spectral peak. Level lines of the initial and the final spectra are presented on Figs. 1 and 2. We observed the following indications of wave-turbulent behavior. (1) The statistics of energy-capacity spectral modes is close to the Rayleigh distribution (see Fig. 3). We observed the presence of a few very intensive harmonics (so-called oligarchs [15]), which did not obey the Rayleigh statistics, but their contribution to the total balance of the wave action is small (no more than 5%). This means that we almost overcame negative effects caused by the finite size of our system (see [15,16]) and that our grid is fine enough. (2) As in many other papers [7,13–16], we observed the formation of the Zakharov-Filonenko spectral tail in the wave numbers spectrum $|\eta_{\vec k}|^2$. Although anisotropy of wave number grid gives us no opportunity to determine the exact exponent due to short effective inertial range after angle averaging, the qualitative correspondence is quite good.

At the same time, we observed an indication of strong-turbulent effects. They are manifested by the formation of "fat tails" on the probability distribution function (PDF) for surface elevations and especially for its gradients (see

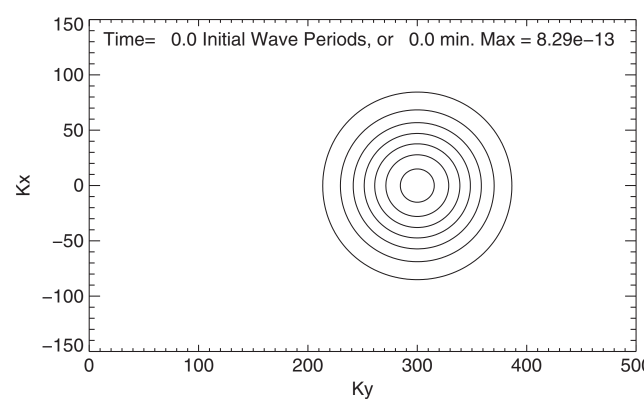

FIG. 1. Initial spectrum $|a_{\vec k}|^2$. $t = 0$.

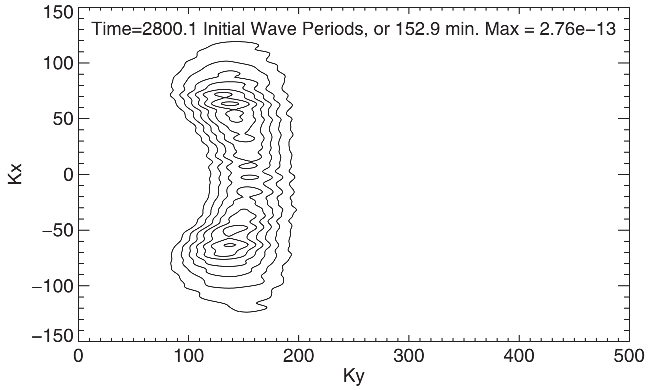

FIG. 2. Final spectrum $|a_{\vec{k}}|^2$. Dynamic equations. $t \simeq 2800T_0$.

Fig. 4). The presence of these tails indicates that the surface has a tendency to become rough and to produce white capping. In our model, wave breaking is arrested by the strong pseudoviscosity.

Statistical experiments.—In the second experiment, we solved the Hasselmann kinetic equation for $n_{\vec{k}} = \langle |a_{\vec{k}}|^2 \rangle$ [1]

$$\begin{aligned} \frac{\partial n_{\vec{k}}}{\partial t} &= S_{\rm nl}[n] + S_{\rm diss} + 2\gamma_k n_{\vec{k}}, \\ S_{\rm nl}[n] &= 2\pi g^2 \int |T_{\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3}|^2 (n_{\vec{k}_1} n_{\vec{k}_2} n_{\vec{k}_3} \\ &\quad + n_{\vec{k}} n_{\vec{k}_2} n_{\vec{k}_3} - n_{\vec{k}} n_{\vec{k}_1} n_{\vec{k}_2} - n_{\vec{k}} n_{\vec{k}_1} n_{\vec{k}_3}) \\ &\quad \times \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \\ &\quad \times \delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3) d\vec{k}_1 d\vec{k}_2 d\vec{k}_3. \end{aligned} \tag{5}$$

Here γ_k is the pseudoviscosity and $S_{\rm diss}$ is the phenomenological dissipation term modeling the white capping process.

Equation (5) was solved on the grid 71×36 in polar coordinates on the frequency-angle plane by the Resio-Tracy code [18], modified in [3,4]. We first performed the experiment with $S_{\rm diss} = 0$. We observed good qualitative coincidence with the dynamical experiment. We observed a downshift of the spectral peak, angular spreading, and the formation of ω^{-4} spectral tails. But the quantitative agreement of the experiments was not good: it was clear that the inclusion of some phenomenological dissipation is necessary.

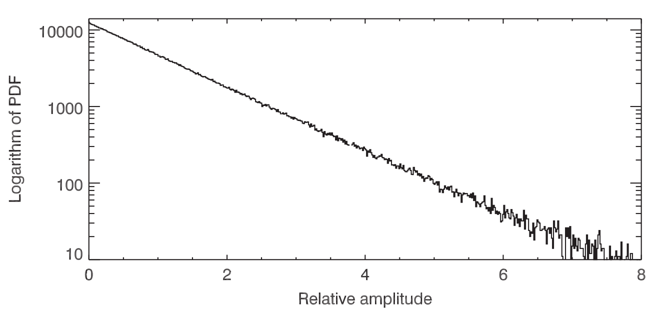

FIG. 3. Probability distribution function for relative squared amplitudes $|a_k|^2 / \langle |a_k|^2 \rangle$. $t \simeq 925T_0$.

We examined the standard form of $S_{\rm diss}$ used in the operational models of wave forecasting—wave action model (WAM) Cycle 3 and WAM Cycle 4 (hereafter WAM3 and WAM4) [19]:

$$S_{\rm diss} = C_{\rm ds} \tilde{\omega} \frac{k}{\tilde{k}} \left[(1 - \delta) + \delta \frac{k}{\tilde{k}} \right] \left(\frac{\tilde{S}}{\tilde{S}_{\rm PM}} \right)^p n_k, \tag{6}$$

where k and ω are the wave number and the frequency, the tilde denotes the mean value; $C_{\rm ds}$, δ, and p are tunable coefficients; $\tilde{S} = \tilde{k}\sqrt{H}$ is the overall steepness; $\tilde{S}_{\rm PM} = (3.02 \times 10^{-3})^{1/2}$ is the value of $\tilde{S}$ for the Pierson-Moscowitz spectrum (note that the characteristic steepness is $\mu \simeq \sqrt{2}\tilde{S}$). It is worth noting that according to [11], the theoretical value of the steepness for the Pierson-Moscovitz spectrum is $\tilde{S}_{\rm PM} \simeq (4.57 \times 10^{-3})^{1/2}$, which gives us $\mu \simeq 0.095$. The values of tunable coefficients in the WAM3 case are

$$C_{\rm ds} = 2.36 \times 10^{-5}, \qquad \delta = 0, \qquad p = 4 \tag{7}$$

and in the WAM4 case are

$$C_{\rm ds} = 4.10 \times 10^{-5}, \qquad \delta = 0.5, \qquad p = 4. \tag{8}$$

The evolution of the total wave action is presented on Fig. 5. One can see that in the long run, the models WAM3 and WAM4 overestimate white capping dissipation. To achieve better agreement of both experiments, we used the following form of the dissipative term:

$$C_{\rm ds} = 1.00 \times 10^{-6}, \qquad \delta = 0, \qquad p = 12. \tag{9}$$

The total wave action curve corresponding to this new dissipation term is shown on Fig. 5 by the thick solid line and displays excellent correspondence with the dynamical model. One can compare the final spectrum obtained in the framework of dynamical equations (Fig. 2) with the result of simulations using kinetic equation (Fig. 6). The similarity of angle structures and shapes of spectra is obvious.

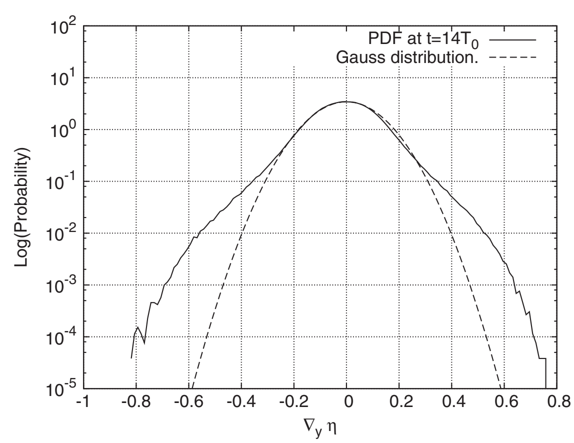

FIG. 4. PDF for $(\nabla\eta)_y$ at the moment of the maximum surface roughness. $t \simeq 14T_0$.

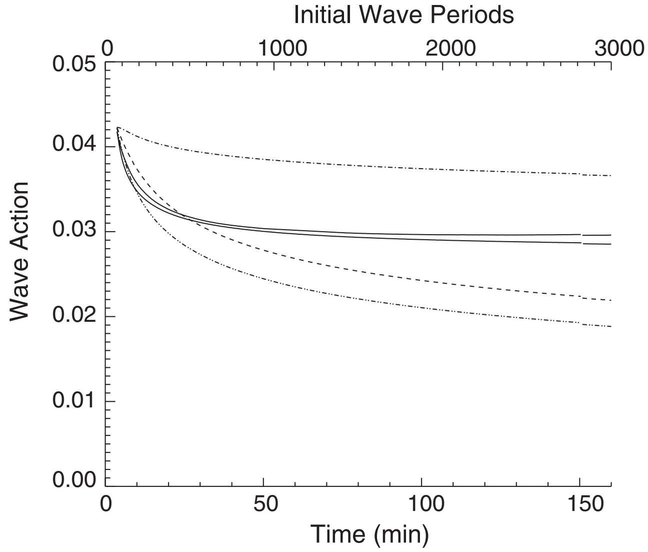

FIG. 5. Total wave action as a function of time. The solid line corresponds to the dynamical equations, the dashed-dotted line—to the kinetic equation with artificial viscosity, the dashed line—to the kinetic equation with the WAM3 damping term, the dotted line—to the kinetic equation with the WAM4 damping term, and the thick solid line—to the kinetic equation with the new damping term.

Conclusion.—Our experiments can be interpreted as a confirmation of the theory of weak turbulence augmented with additional dissipation term. We got qualitative correspondence of kinetic and dynamic equations even without any artificial dissipation. However, even at moderate values of the parameter of the nonlinearity μ, the strongly nonlinear effects of white capping are essential. They manifest themselves as fat tails of the PDF and lead to additional dissipation of wave energy. This dissipation demonstrates a very strong dependence on the steepness. At steepness $\mu = 0.176$ they dominate; at steepness $\mu = 0.09$ they are negligibly small. The results of our experiments are in good qualitative agreement with the field experiment of Banner, Babanin, and Young [11]. We stress that the dependence (9) is much sharper than it is usually stated. So

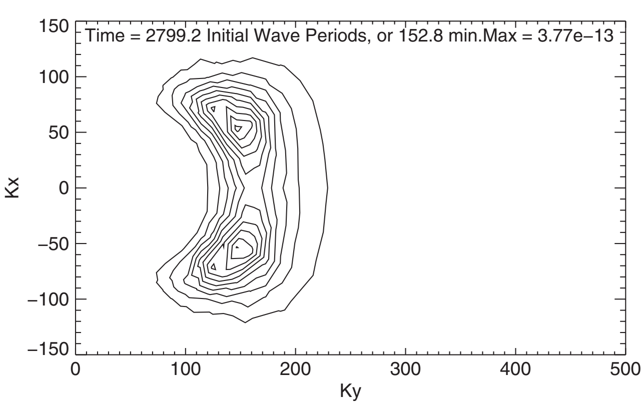

FIG. 6. Final spectrum $|a_{\vec{k}}|^2$. Kinetic equations. $t \simeq 2800T_0$.

far, the sharpest dependence $p = 5$ was given by Donelan [20]. We can guess that the real dependence of $S_{\rm diss}$ on μ is even stronger, and that the onset of the wave breaking is a threshold-type phenomenon like a second-order phase transition.

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