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Intermittence Phenomena in the Burgers Equation Involving Thermal Noise

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Abstract—Leading terms of the asymptotic behavior of the pair and higher order correlation functions for finite times and large distances have been calculated for the Burgers equation involving thermal noise. It is shown that an intermittence phenomenon occurs, whereby certain correlation functions are much greater than their reducible parts. © *2000 MAIK "Nauka/Interperiodica".*

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Considerable deviation of the statistics of fluctuating fields from the Gaussian form is usually referred to as the intermittence. This property is typical of the hydrodynamic systems in the state of developed turbulence [1–3]. Under such strongly nonequilibrium conditions, the intermittence is manifested, in particular, as the dominance of the irreducible parts of the fourth-order correlation functions for certain quantities over the corresponding reducible parts. As a rule, the simultaneous correlation functions were considered in the papers cited above.

In thermodynamic equilibrium, the simultaneous correlation functions for the local fluctuating fields, as the functions of distances between points, are of the order of their reducible parts even in the critical region, provided that these reducible parts are nonzero. This is the foundation for the renormalization group method taking into account interaction between fluctuations through renormalization of the local fields and the effective Hamiltonian [4].

Recently, Lebedev [5] showed that the behavior can be substantially different for different-time correlation functions for the equilibrium fluctuating quantities. He demonstrated that different-time correlation functions for the density of vortex charges may be much larger than their Gaussian parts in the low-temperature phase of two-dimensional Berezinskii–Costerlitz–Towless systems. The following physical explanation for such behavior was proposed in [5]: different-time correlation functions of all orders in the vicinity of a given space point at low temperature are determined by a single rare fluctuation. This interpretation concludes that the intermittence phenomena must be manifested by features in the equilibrium dynamics of a wide class of systems.

In this study, we consider evolution of a one-dimensional velocity field $u(t, x)$ according to the following Burgers equation involving thermal noise:

$$u_t + uu_x - \nu u_{xx} = \xi(t, x). \quad (1)$$

Here, ν is the dissipation constant that is assumed to be small and $\xi(t, x)$ is the random noise described by the Gaussian statistics and by the pair correlation function

$$\langle\xi(t, x)\xi(t_1, x_1)\rangle = -\nu\beta^{-1}\delta''(x-x_1)\delta(t-t_1). \quad (2)$$

The parameter β plays the role of inverse temperature and the simultaneous steady-state velocity distribution function $\mathcal{P}[u]$ has the form

$$\mathcal{P}[u] = \mathcal{N}\exp\{-\beta\mathcal{F}[u]\}, \quad \mathcal{F}[u] = \int dx u^2(x), \quad (3)$$

where $\mathcal{N}$ is the normalization constant. The equality

$$\langle u(t, x)u(t, x')\rangle = (2\beta)^{-1}\delta(x-x'), \quad (4)$$

which follows from expression (3), corresponds to the total absence of correlation between the velocity values in different points at the same time instant. We have calculated certain asymptotes of the different-time pair, triple, and quadruple correlation functions for the field $u(t, x)$. The obtained results indicate that the intermittence phenomena in fact occur in the equilibrium dynamics of a system described by equation (1).

A dynamic scaling exponent of $z = 3/2$ for problem (1)–(2) was found in [6] based on dimensional estimates and Galilean invariance. Considering the spectrum $\omega \propto k^{3/2}$, Lebedev and L'vov [7] demonstrated the absence of logarithmic divergences in each order of the perturbation theory with respect to translation. Therefore, the ratio $\beta x^3/T^2$ is a dimensionless argument of the function $F_2(T, x) = \langle u(T, x)u(0, 0)\rangle$. First, we determine an unknown leading term of the asymptotic behavior F_2 at $\beta x^3/T^2 \gg 1$ and $\nu \longrightarrow 0$. It follows from the latter

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relationship that contribution of the diffusion mechanism to establishing correlation between points 0 and x during the time interval T is negligible. The fact that T is also small means that we can neglect the effect of noise on the dynamics in the time interval $(0, T)$. In this case, $u(0, y)$ is a functional $u(T, x)$ (or vice versa) and a Gaussian form of statistics of the velocity field at the time instant T makes it possible to represent the different-time pair correlation function in the form

$$F_2(T, x) = (2\beta)^{-1}\left\langle\frac{\delta u(0,0)}{\delta u(T,x)}\right\rangle. \quad (5)$$

For $\nu \longrightarrow 0$, the variational derivative $\Theta(t, y) = \delta u(t, u)/\delta u(T, x)$ satisfies the continuity equation

$$\Theta_t + u\Theta_y + u_y\Theta = 0 \quad (6)$$

and the condition $\Theta(T, y) = \delta(x - y)$. A solution to this Cauchy problem is found by the method of characteristic curves and the correlation function $F_2(T, x)$ is determined in the form [8]

$$\begin{aligned} F_2(T, x) &= (2\beta)^{-1}\langle\Theta(0,0)\rangle \\ &= (2\beta)^{-1}\left\langle\delta(x - y(T))\left(\frac{\partial y(T,\zeta)}{\partial\zeta}\right)_{\zeta = 0}\right\rangle. \end{aligned} \quad (7)$$

Here, $y(T, \zeta)$ is the position of a Lagrangian particle leaving the point with the coordinate ζ at the moment $t = 0$

$$\dot{y} = u(t, y), \quad y(0, \zeta) = \zeta, \quad (8)$$

where $y(T) = y(T, 0)$. If $u(t, y)$ is discontinuous, equation (8) requires an extension of the definition. Bauer and Bernard [9] formally justified a physically obvious condition that the velocity of a Lagrangian particle at a discontinuity is equal to the velocity of motion of the discontinuity itself.

We conclude from expression (7) that the correlation function F_2 in the limit under consideration is determined by the most probable initial fluctuation of the velocity $u_0(y)$ that, evolving, transfers the particle from the point 0 to the point x in a time T. The probabilities of the initial distributions of the velocity field are specified by functional (3). We demonstrate that the desired optimum fluctuation $u_0(y)$, minimizing $\mathcal{F}[u_0]$ under the condition $y(T) = x$, has the form of a linear profile

$$\begin{aligned} u_0(y) &= u_0^* \equiv x/T - y/T, \quad 0 < y < x, \\ u_0(y) &= 0, \quad y < 0, y > x. \end{aligned} \quad (9)$$

Indeed, it is obvious that the function $u_0(y)$ must attain the maximum at $y = 0$. The zero value of the function $u_0(y)$ at $y < 0$ and $y > x$ is easily explained: nonzero values of the function $u_0(y)$ beyond the interval $(0, x)$ do not affect the trajectory $y(t)$, but increase the $\mathcal{F}[u_0]$ value. The left edge of the derived distribution $u(t, x)$ is a straight line characterized by the slope $\sigma = 1/t$ at any time instant in the limit $\nu \longrightarrow 0$. This time dependence is easily verified by direct substituting into the Burgers equation (see also [10]). The coordinate of the fastest point of the profile at $t = T$ is equal to x. The coordinates of all other points become the same. Therefore, for the initial data $u_0(y)$ belonging to the class described above, the plot of the function $u(T, y)$ has the form of the triangle

$$\begin{aligned} u(T, y) &= y/T, \quad 0 < y < x, \quad u_0(y) = 0, \\ & y < 0, \quad y > x. \end{aligned} \quad (10)$$

Note now that the Burgers equation leads to the relation

$$d\mathcal{F}[u(t, y)]/dt = -2\nu\int dy u_y^2 \le 0, \quad (11)$$

which yields

$$\mathcal{F}[u_0(y)] \ge \mathcal{F}[u(T, y)]. \quad (12)$$

This inequality is strict even in the limit $\nu \longrightarrow 0$, provided that shock waves are formed during the evolution. Therefore, the minimum value of the functional $\mathcal{F}$ is

$$\mathcal{F}[u(T, y)] = x^3/3T^2. \quad (13)$$

The functional $\mathcal{F}$ value for the function $u_0^*(y)$ coincides with value (13) and the condition of forbidded formation of the shock waves during the time interval from 0 to T provides that expression (9) is the only possible form for $u_0(y)$.

The probability of initial fluctuation (9), being equal to $\exp(-\beta\mathcal{F}[u_0(y)])$, determines the exponential part of the asymptotic behavior of the pair correlation function F_2

$$F_2(T, x) \sim \exp\left(-\frac{\beta x^3}{3T^2}\right). \quad (14)$$

Note that the factor $(\partial y(T, \zeta)/\partial\zeta)_{\zeta = 0}$ entering into formula (7) at the δ function vanishes for configuration (9), but it is nonzero at a small variation of $u_0(y)$. In other words, this factor, as well as the unknown preexponential factor in expression (14) as a whole, is determined by integration over the deviations δu of the initial velocity field with respect to $u_0^*(y)$. The typical δu values are small as compared to $u_0^*(y)$ with respect to the parameter $\beta x^3/T^2$. Nevertheless, the integration over δu is not reduced to the Gaussian form even in the limit $\beta x^3/T^2 \gg 1$. The reason is that the functional $\mathcal{F}[u]$ is nonanalytic in the limit $\nu \longrightarrow 0$ for the class of functions $u(y)$ such that $y(T) = x$. The variation $\delta\mathcal{F}$ is of the first-order smallness in δu although the inequality $\delta\mathcal{F} \ge 0$ is still fulfilled. The functional $\mathcal{F}[u]$ can be expanded in the functional Taylor series in terms of δu only for $\delta u \ll \nu/x$. Corresponding analysis will be performed elsewhere and we restrict this consideration to exponential accuracy.

Noting that linear profile (9) transfers all points belonging to the interval $(0, x)$ to the point x by the time instant $t = T$, we obtain to within the exponential accuracy that

$$F_{n+2} = \left\langle u(T, x)\prod_{j=1}^{n} u(0, y_j)u(0, 0)\right\rangle \sim F_2(T, x) \sim \exp\left(-\frac{\beta x^3}{3T^2}\right), \quad (15)$$

where $0 < y_1 < y_2 \ldots < y_n$. The reducible part of this correlation function at $n \geq 1$ is obviously equal to zero. The same fluctuation $u_0^*(y)$ determines the following leading asymptotic behavior of the correlation function $\Phi_4 = \langle u(T, x)u(T, x + a_1)u(0, a)u(0, 0)\rangle$ for $0 < a < x$ and $0 < a_1 \ll a$

$$\Phi_4 \sim \exp\left(-\frac{\beta x^3}{3T^2}\right) \gg \Phi_{4,\,\mathrm{Gauss}} \sim \exp\left(-\frac{2\beta x^3}{3T^2}\right). \quad (16)$$

Here, $\Phi_{4,\,\mathrm{Gauss}}$ is the reducible part of the correlation function Φ_4. To find the correlation function Φ_4 as a function of the parameter a, it is necessary to consider evolution of the perturbed linear profile; during this evolution, the reversal inevitably occurs and the problem becomes substantially more complex. We note also that dependence of the correlation function Φ_4 on a is related directly to the distribution function of the velocity field gradients; the latter function is determined, as was shown in [11, 12], by the shock waves being formed. Coincidence of the leading asymptotic terms of the pair and higher order correlation functions is typical of the turbulence problems, as was originally indicated for these problems in [13].

The correlation functions for the filed $u(t, x)$ can be represented in the form of functional integrals (see, for example, [14]). These integrals are calculated, in essence, by the saddle point method, where the saddle-point parameter $\beta x^3/T^2 \gg 1$ is determined by the averaged quantity rather than by the action. This approach was originally proposed by Lifshits [15]. More recently, it was generalized to determine higher order correlation functions for both equilibrium [16] and substantially nonequilibrium systems [12, 17–22]. The optimal fluctuation is also referred to as instanton, the term generally accepted in the quantum field theory. The long-term asymptotic behavior of the autocorrelation function of the current through the disordered contact was calculated in [23], where large observation time interval was used as a saddle-point parameter determining the instanton.

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