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Intermediate Asymptotic Expressions for High-Order Spin Correlation Functions in a Two-Dimensional Classical Ferromagnet

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Different-point spin correlation functions are calculated for a two-dimensional classical ferromagnet in a perturbative range of distances r: $a < r \ll m^{-1}$, where a is the lattice parameter and m^{-1} is the correlation length. The expressions for the four- and higher-order correlation functions are presented. © *2000 MAIK "Nauka/Interperiodica".*

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The long-wave static statistical characteristics of a two-dimensional classical Heisenberg ferromagnet are specified by Gibbs measure

$$\prod_{\mathbf{x}} \mathcal{D}\mathbf{n}(\mathbf{x})\delta(\mathbf{n}^2(\mathbf{x})-1)\exp\left(-\frac{1}{2g_0}\int d^2\mathbf{x}(\partial_\mu\mathbf{n})^2\right), \quad (1)$$

where $\mathbf{n}$ is a three-component vector and the coupling constant g_0 is proportional to temperature. The lattice with parameter a is assumed to exist at small distances. The limit $g_0 \ll 1$ is considered.

Although expression (1) has long been examined (see references in [1]) under the name of the two-dimensional nonlinear O(3)–σ model, actual progress was made only in Polyakov work [2], where it was demonstrated that the theory (1) is renormalizable and that the effective interaction between the fluctuations of the $\mathbf{n}$ field increases with distance. This result was extended in [3] to the equilibrium dynamics of two-dimensional ferromagnets. There is the exact statement (the Mermin–Wagner theorem) that Eq. (1) does not allow for the spontaneous breaking of symmetry about the global rotations of spins $\mathbf{n}$. It follows from this statement that the pair spin–spin correlation function should decrease with increasing distance. The effective interaction ceases to be weak starting at the scale $r \geq m^{-1} \simeq a\exp(2\pi/g_0)$. The hypothesis that m^{-1} is the correlation length in the system (see [4]) was later confirmed by rigorous results [5–8]. At small distances $a \leq r \ll m^{-1}$, the spin correlation functions can be obtained by summation of leading logarithmic terms of the perturbation series. In this case, $g_0\ln(r/a)$ may be ~1 and the spin correlation functions may vary as strongly as is desired: from the one-site values of order 1 to the asymptotically small (proportional to the positive powers of g_0) values. The two-spin result has long been known [4, 9] and was proved by both rigorous calculations and numerical simulations [10].

In this work, the anomalous dimensionalities of arbitrary tensor operators constructed from the products of vector $\mathbf{n}$ components are calculated in the leading logarithmic approximation. The result proves to be rather simple: a tensor $T^{(l)}$ belonging to the irreducible representation of group O(3) with angular momentum l transforms as

$$T^{(l)} \longrightarrow \left(\frac{g_0}{g(s)}\right)^{l(l+1)/2} T^{(l)}, \quad (2)$$

upon passing from the scale a to the scale $\exp(2\pi s/g_0)$. In Eq. (2), $g(s) = g_0(1-s)^{-1}$ is the running value of the coupling constant on the scale $a\exp(2\pi s/g_0)$. Although this statement can also be proved within the framework of the Polyakov approach [2], a different formalism, which is more suitable for the perturbative treatment, is used in this work.

Let us begin with the exact statement: for a single spin $\mathbf{n}$ the averaging in measure $d\mathbf{n}\delta(\mathbf{n}^2-1)$ is equivalent to the averaging in measure $d\psi^+d\psi^-$ defined on the $\psi^+\psi^- \leq 4$ disk in the complex plane of variable $\psi^+ = (\psi^-)^*$. When averaging, the components n_z and $n^{\pm} \equiv n_x \pm in_y$ are expressed through $\psi^{\pm}$ as follows:

$$n^+ = \psi^+, \quad n^- = \psi^- - \frac{1}{4}(\psi^-)^2\psi^+, \qquad n_z = 1 - \frac{1}{2}\psi^+\psi^-. \quad (3)$$

To prove this theorem, it is sufficient to evaluate the generating function $\mathcal{Z}(\mathbf{h}) = \langle \exp(\mathbf{hn})\rangle$ by two methods and make sure that the results coincide. Representation (3) can be considered as a formal classical limit of the Dyson–Maleev representation [11, 12] for a quantum spin. However, it should be emphasized that Eq. (3) is not the parametrization of the sphere points in the strict sense; the equality sign in Eq. (3) merely implies that the corresponding means coincide with each other. One can also state that Eq. (3) is the change of variables followed by the deformation of the integration surface (cf. an analogous construction for quantum spins and its use in [13– 15]).

By passing from the $\mathbf{n}$ components to the $\psi^{\pm}$ variables for each spin on the lattice sites, one obtains that the spin fluctuation statistics in a two-dimensional classical ferromagnet is specified by the measure

$$\prod_{\mathbf{x}} \mathcal{D}\psi^{\pm}(\mathbf{x}) \times \exp\left\{-\frac{1}{2g_0}\int d^2\mathbf{x}\left[\partial_\mu\psi^+\partial_\mu\psi^- + \frac{1}{4}(\psi^+)^2(\partial_\mu\psi^-)^2\right]\right\}, \tag{4}$$

with the constraint $\psi^+\psi^- \le 4$ in each point of the space. Note that this constraint (as distinct, say, from $\mathbf{n}^2 = 1$) does not explicitly manifest itself in the perturbation theory.

The fact that the nonlinearity in ψ^+ and ψ^- is asymmetric in Eq. (4) provides the renormalizability of the action and the massless fluctuations in the perturbation theory. After integration in the one-loop approximation for the Fourier components of the $\psi^{\pm}$ field with wave vectors from a^{-1} to $a^{-1}\exp(-2\pi s/g_0)$, the effective measure takes the form

$$\prod_{\mathbf{x}} \mathcal{D}\psi^{\pm}(\mathbf{x}) \times \exp\left\{-\frac{1}{2g_0}\int d^2\mathbf{x}\left[\partial_\mu\psi^+\partial_\mu\psi^- + \frac{g(s)}{4g_0}(\psi^+)^2(\partial_\mu\psi^-)^2\right]\right\}, \tag{5}$$

where $g(s) = g_0(1 - s)^{-1}$. By redetermining the field $\psi^{\pm} \longrightarrow (g_0/g(s))^{1/2}\psi^{\pm}$, the Lagrangian in Eq. (5) is reduced to the initial form, in which g_0 should be replaced by $g(s)$.

The $\psi^+\psi^- < 4$ constraint provides the infrared regularization of the theory because it eliminates infinitely large contributions from the large-scale fluctuations to the means. However, an appropriate detailed analysis is essentially nonperturbative and is beyond the scope of this work. It is merely assumed here that the m^{-1} value (or $s = 1$) can be used as the infrared cutoff parameter of theory (4). In other words, it is assumed, in line with the exact solution [5, 7], that the $\psi^+\psi^- \le 4$ constraint is "processed" into the mass m for the fluctuations with wavelengths $\ge m^{-1}$, so that no logarithmic contributions of such scales arise to the local means. The correlation functions depend logarithmically on m at distances $r \ll m^{-1}$, while the contributions from the fluctuations with $s \longrightarrow 1$ do not contain singularities and have a negligible integral effect at $g_0 \longrightarrow 0$ [see below Eq. (7)]. All this allows the perturbation theory for the nonlinearity in Eq. (4) to be considered local, so that the strong interaction on the confinement scale $\ge m^{-1}$ does not impede the calculations for the intermediate distances with an accuracy indicated above.

The value of $m = a^{-1}\exp(-2\pi/g_0)$ taken as the smallest momentum in the loop integrations corresponds to the recovery of the O(3) symmetry in the leading logarithmic approximation: $\langle n_z\rangle = 1 - \langle\psi^+\psi^-\rangle/2 = 0$. This relationship provides the invariance of means with respect to the infinitesimal rotations about the z axis

$$\delta\psi^+ = \alpha\left(1 - \frac{1}{2}\psi^+\psi^-\right), \qquad \delta\psi^- = \frac{\alpha}{4}(\psi^-)^2, \quad \alpha \longrightarrow 0, \tag{6}$$

for the infrared-regularized theory as well. It follows from this that $\langle(\psi^+\psi^-)^m\rangle = 4^m(m + 1)^{-1}$, which is equivalent to the moments of the isotropically distributed vector $\langle n_z^{2m}\rangle = (2m + 1)^{-1}$.

The main property of theory (4) in the calculation of the different-point spin correlation functions is that the operator $(\psi^+)^m$ is covariant with respect to the transformation of the renormalization group: on averaging over the fluctuations in the momentum layer $a^{-1}\exp(-2\pi s/g_0) < k < a^{-1}$, the $(\psi^+)^l$ operator transforms as

$$(\psi^+)^l \longrightarrow \left(\frac{g_0}{g(s)}\right)^{l(l+1)/2}(\psi^+)^l. \tag{7}$$

The $l(l + 1)/2$ exponent is equal to the number of ways of pairing $(\psi^+)^l$with vertex $(\psi^+)^2(\partial_\mu\psi^-)^2$ plus the $l/2$ term due to the rescaling of the $\psi^{\pm}$ fields. From the viewpoint of the O(3) group elements acting on $\mathbf{n}$, the operator $(\psi^+)^l$ is a vector of highest weight in the irreducible representation with angular momentum l. Because of the invariance about transformation (6), the transformation properties associated with the operations of the renormalization group are uniform throughout the irreducible representation. Equation (2) follows precisely from this fact.

The spin correlation functions of an arbitrary order can be obtained by decomposing the products of $\mathbf{n}$ components into the irreducible tensors of the rotation group. In this case, the dependence on the distances can be determined by the standard summation over the parquet diagrams [16, 17]. Below, the explicit expressions are given for the four-point correlation function $K_4 = \langle n_z(r_1)n_z(r_2)n_z(r_3)n_z(r_4)\rangle$; to the accuracy adopted in this work, only the following asymptotic geometries are relevant ($r_{jl} \equiv |r_j - r_l|$, $K_4 \equiv (g_0/2\pi)^4\mathcal{K}_4$):

$$r_{12}, r_{34} \ll r_{13} \simeq r_{24} \simeq R$$

$$\mathcal{K}_4 = \frac{4}{45}\frac{\ln^6(mR)}{\ln(mr_{12})\ln(mr_{34})} + \frac{1}{9}\ln^2(mr_{12})\ln^2(mr_{34}), \tag{8}$$

$$r_{12} \ll r_{13} \ll r_{34},$$

$$\mathcal{K}_4 = \frac{4}{45}\frac{\ln^2(mr_{34})\ln^3(mr_{13})}{\ln(mr_{12})} + \frac{1}{9}\ln^2(mr_{12})\ln^2(mr_{34}), \tag{9}$$

$$r_{12} \simeq r_{13} \simeq r_{34} \simeq R, \quad \mathcal{K}_4 = \frac{1}{5}\ln^4(mR). \tag{10}$$

Note in conclusion that $l(l+1)$ in the transformation (2) for the N-component field $\mathbf{n}$ should be replaced by the eigenvalues of the $O(N)$ group Casimir operator (angular part of the N-dimensional Laplacian) divided by $N - 2$.

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