Intermittency of Burgers' Turbulence

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We consider the tails of probability density function (PDF) for the velocity that satisfies Burgers equation driven by a Gaussian large-scale force. The saddle-point approximation is employed in the path integral so that the calculation of the PDF tails boils down to finding the special field-force configuration (instanton) that realizes the extremum of probability. For the PDFs of velocity and its derivatives $u^{(k)} = \partial_x^k u$, the general formula is found: $\ln \mathcal{P}(|u^{(k)}|) \propto -(|u^{(k)}|/\mathrm{Re}^k)^{3/(k+1)}$. [S0031-9007(97)02444-7]

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Nonlinear systems usually demonstrate non-Gaussian output even if the input (forcing or initial conditions) is Gaussian. In turbulence, the deviation of the PDF tails from Gaussian is regarded as a manifestation of intermittency. It has been realized recently that those tails can be found by considering the saddle-point configurations (we call them instantons) in the path integral determining the probability density function (PDF) [1]. Note the simultaneous revival of the method of optimal fluctuation in the description of rare events in disordered metals [2].

We consider the forced Burgers equation

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = \phi\,, \tag{1}$$

which describes the evolution of weak one-dimensional acoustic perturbations in the reference frame moving with the sound velocity [3]. It is natural to assume the external force ϕ to be δ correlated in time in that frame: $\langle\phi(t_1,x_1)\phi(t_2,x_2)\rangle = \delta(t_1 - t_2)\chi(x_1 - x_2)$. Then the statistics of ϕ is Gaussian and is completely characterized by χ. We are interested in turbulence excited by a large-scale pumping with some correlation length L so that χ does not essentially change at $x \lesssim L$ and goes to zero where $x > L$. Besides L, the correlation function χ may be characterized by the parameter $\omega = [-(1/2)\chi''(0)]^{1/3}$ having the dimensionality of frequency. Then, e.g., $\chi(0) \sim L^2\omega^3$. Developed turbulence corresponds to a large Reynolds number $\mathrm{Re} = L^2\omega/\nu \gg 1$.

The physical picture of Burgers turbulence is quite clear: arbitrary localized perturbation evolves into shock wave with the viscous width of the front, which gives k^{-2} for the energy spectrum at $\mathrm{Re} \gg kL \gg 1$ [3,4]. The presence of shocks leads to a strong intermittency, PDF of velocity gradients is substantially non-Gaussian [5], and there is an extreme anomalous scaling for the moments of velocity differences $w = u(\rho) - u(-\rho)$: $\langle w^n\rangle \propto (\rho/L)$ for $n > 1$ [6]. Simplicity of the equation and transparency of underlying physics make it reasonable to hope that a consistent formalism for the description of intermittency could be developed starting from Burgers equation [7–9]. The instanton formalism has been applied to the Burgers equation first by Gurarie and Migdal [10] who found the right tail $\ln \mathcal{P}(w) \sim -[w/(\omega\rho)]^3$ determined by inviscid behavior of smooth ramps between the shocks. That cubic right tail has been earlier predicted by Polyakov from the conjecture on the operator product expansion [7]; it corresponds to the same right tail for the gradients $\ln \mathcal{P}(u') \sim -(u'/\omega)^3$ derived by Gotoh and Kraichnan [11]. The PDFs $\mathcal{P}(u')$ and $\mathcal{P}(w)$ are not symmetric; the asymmetry is due to the simple fact that positive gradients are smeared while the steepening of negative gradients could be stopped only by viscosity. Here we describe the viscous instantons that give the left tails of PDFs and single-point velocity PDF.

Even though some calculations are lengthy, the simple picture appears as a result. Since white forcing pumps velocity by the law $w^2 \propto \phi^2 t$, while the typical time of growth is restricted by the breaking time $t \sim L/w$, the Gaussianity of the forcing $\ln \mathcal{P}(\phi) \propto -\phi^2/\chi(0)$ leads to $\ln \mathcal{P}(w) \sim -[|w|/(L\omega)]^3$. At a shock, $w^2 \simeq -\nu u'$ so that $\ln \mathcal{P}(u') \sim -[-u'/(\omega \mathrm{Re})]^{3/2}$. These simple estimates are confirmed below by consistent calculations.

Following [1], we write the high-order moments of the velocity derivatives $u^{(k)} = \partial_x^k u$ (including velocity for $k = 0$) and of the difference w as the path integrals:

$$\langle [u^{(k)}]^n\rangle = \int \mathcal{D}u\mathcal{D}p \exp\Big\{iI + n\ln[u^{(k)}(0,0)]\Big\}, \tag{2}$$

$$\langle w^n\rangle = \int \mathcal{D}u\mathcal{D}p \exp\{iI + n\ln[u(0,\rho) - u(0,-\rho)]\}. \tag{3}$$

Here $I = \int dt\, \mathcal{L}(u,p)$ is the effective action with the Lagrangian $\mathcal{L}$ determined by the equation (1) [12,13]:

$$\mathcal{L} = \int dx\,(p\partial_t u + pu\partial_x u - \nu p\partial_x^2 u) + \frac{i}{2}\int dx_1 dx_2 p_1\chi_{12}p_2\,.$$

The main idea implemented here is that the high-order moments (for $n \gg 1$) are determined by the saddle-point

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configurations of the path integrals (2) and (3). The corresponding saddle-point equations are

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = -i\int dx'\,\chi(x - x')p(x')\,, \quad (4)$$

$$\partial_t p + u\partial_x p + \nu\partial_x^2 p = \delta(t)\lambda(x)\,, \quad (5)$$

$\lambda = in[\delta(x + \rho) - \delta(x - \rho)]/w$ for differences, $\lambda = in\delta'(x)/\partial_x u(t,0)$ for gradients. The solution of (4) and (5) gives the moments in the saddle-point approximation:

$$\langle[u^{(k)}]^n\rangle \sim [u^{(k)}(0,0)]^n e^{i\mathcal{I}_{\rm extr}}, \quad \langle w^n\rangle \sim [w(0)]^n e^{i\mathcal{I}_{\rm extr}}. \quad (6)$$

An instanton solution describes the field-force configuration that corresponds to the rare fluctuation giving the main contribution into the high-order moment. This configuration can be called also optimal fluctuation.

The initial conditions for the instanton equations were formulated in [1]: $u \to 0$ at $t \to -\infty$ and $p = 0$ at $t > 0$. The role of the last term in (5) is then reduced to the final condition $p(x) = -\lambda(x)$ imposed on p at $t = -0$. Viscosity smears p to move backwards in time so that at large negative time both fields u and p are zero.

In this Letter, we describe general properties of instantons and find their dependence on n, which gives the form of PDF tails. Complete analytic descriptions of the instanton solutions will be published elsewhere.

Since the Lagrangian does not explicitly depend on time, then the "energy" E is conserved by (4) and (5):

$$E = i\int dx(pu\partial_x u + \nu\partial_x p\partial_x u) - \frac{1}{2}\int dx_1 dx_2 p_1\chi_{12}p_2\,.$$

Since $\mathcal{L}$ does not explicitly depend on coordinates then the "momentum" J is conserved as well: $iJ = \int dx\, p\partial_x u$. Because of the conservation laws we should treat solutions of (4) and (5) with $E = J = 0$ since they are zero at $t \to -\infty$. That gives the following saddle-point value of the effective action in (6): $\mathcal{I}_{\rm extr} = \int dt\, dx\, p\partial_t u$.

Conservation laws help to understand general properties of the solutions. We consider $t = 0$, substitute $p(x) = -\lambda(x)$, and analyze the balance of different terms in E. For the gradient $a(t) = \partial_x u(t,0)$, $E = -na(0) + \omega^3 n^2/a^2(0) - n\nu\partial_x^3 u(0,0)/a(0) = 0$. There is a difference between the cases of positive and negative a. For $a(0) > 0$, the viscous contribution to the energy is unessential and two first terms can compensate each other (see below). On the contrary, the instanton that gives $a(0) < 0$ cannot exist without viscosity. For the velocity, $J = \partial_x u(0,0) = 0$ and $E = \chi(0)n^2/u^2(0) - \nu\partial_x^2 u(0,0)$. Without viscous term, energy cannot be zero. Note that the answer we shall obtain for the velocity PDF does not contain viscosity, while its consistent derivation requires the account of the viscous terms in the equations.

Let us first describe the essentially inviscid instantons producing the right tails of the PDFs for gradients and differences [7,10,11]. At $t = 0$, the field p is localized near the origin. A positive velocity slope "compresses" the field p so that one can expect that at negative time the width of p remains much smaller than L despite a viscous spreading. Then it is possible to formulate the closed system of equations for the quantities $a(t)$ and $c(t) = -i\int dx\, x\, p(t,x)$ since for narrow p and small x we can put $\int dx'\chi(x - x')p(t,x') \to -i\partial_x\chi(x)c(t) \approx 2i\omega^3 xc(t)$:

$$\partial_t c = 2ac\,, \quad \partial_t a = -a^2 + 2\omega^3 c\,. \quad (7)$$

The instanton is a separatrix solution of (7). The initial condition $a(0)c(0) = n$ by virtue of the energy conservation gives $a(0) = \omega^3 c^2(0)/n = \omega n^{1/3}$. For differences, $w = 2a(0)\rho$. One can check that $\mathcal{I}_{\rm extr} = n$, which is negligible in comparison with $n\ln[a(0)]$, so that $\langle(u')^n\rangle \sim [a(0)]^n \sim \omega^n n^{n/3}$, which gives the right cubic tails of the PDFs $\ln\mathcal{P}(u') \sim -(u'/\omega)^3$ [11] and $\ln\mathcal{P}(w) \sim -[w/(\rho\omega)]^3$ [7,10]. The width of p is much less than L through the time of evolution $T \sim n^{-1/3}\omega^{-1}$ giving the main contribution into the action [10]. The right tails of $\mathcal{P}(u')$ and $\mathcal{P}(w)$ are thus universal, i.e., independent of the large-scale properties of the pumping. Above consideration does not imply that the instanton is completely inviscid, it may well have viscous shock at $x \sim L$: this has no influence on the instanton answer (since p is narrow), while it may influence the fluctuation contribution, i.e., pre-exponential factor in the PDF.

The main subject of this paper is the analysis of the instantons that give the tails of $\mathcal{P}(u)$ and the left tails of $\mathcal{P}(u')$ and $\mathcal{P}(w)$ corresponding to negative a, w. Even though the field p is narrow at $t = 0$, we cannot use the simple system (7) to describe those instantons. The reason is that sweeping by a negative velocity slope provides for stretching (rather than compression) of the field p at moving backwards in time. As a result, the support of $p(x)$ stretches up to L so that one has to account for the given form of the pumping correlation function $\chi(x)$ at $x \simeq L$. This leads to a nonuniversality of $\mathcal{P}(u)$ and of the left tails of $\mathcal{P}(u')$ and $\mathcal{P}(w)$, which depend on the large-scale properties of the pumping. As we shall see, the form of the tails is universal, nonuniversality is related to a single constant in PDF. Additional complication in analytical description is due to the shock forming from negative slope near the origin. The shock cannot be described in terms of the inviscid equations so that we should use the complete system (4) and (5) to describe what can be called viscous instantons.

Apart from a narrow front near $x = 0$, the velocity field has L as the only characteristic scale of change. The life time T of the instanton is then determined by the moment when the position of p maximum reaches L due to sweeping by the velocity u_0: $T \sim L/u_0$. Such a velocity u_0 itself has been created during time T by the forcing so that $u_0 \sim |c|_{\max}TL\omega^3$. To estimate the maximal value of $|c(t)|$, let us consider the backward evolution from $t = 0$. We first notice that the width of p (which was zero at $t = 0$) is getting larger than the width of the velocity front $\simeq u_0/a$ already after the short time $\simeq a^{-1}$. After that time, the values of c and a are of order of their values at $t = 0$. Then one may consider that $p(t,x)$ propagates (backwards in time) in the almost homogeneous velocity field u_0 so that

$$\partial_t c = -i \int_{-\infty}^{\infty} dx\, xup_x \approx 2iu_0 \int_0^{\infty} dx\, p\,.$$

The (approximate) integral of motion $i\int dx\, p$ can be estimated by its value at $t = 0$, which is $n/2u_0$. Therefore we get $c_{\max} \simeq nT$ so that $T \simeq n^{-1/3}\omega^{-1}$ and $u_0 \simeq L\omega n^{1/3}$. At the viscosity-balanced shock, the velocity u_0 and the gradient a are related by $u_0^2 \simeq \nu a$ so that $a(0) \simeq \omega \mathrm{Re}\, n^{2/3}$.

Let us briefly describe now the consistent analytic procedure of the derivation of the function $c(t)$ that confirms above estimates. We use the Cole-Hopf substitution [3] for the velocity $\partial_x\Psi = -u\Psi/2\nu$ and introduce $P = 2\nu\partial_x p/\Psi$. The saddle-point equations for Ψ and P

$$\partial_t\Psi - \nu\partial_x^2\Psi + \nu F\Psi = 0\,, \qquad (8)$$

$$\partial_t P + \nu\partial_x^2 P - \nu FP - 2\nu\lambda'(x)\delta(t)\Psi^{-1} = 0 \quad (9)$$

contain F determined by $\partial_x F(t,x) = -i\int dx'\chi(x - x')p(t,x')/2\nu^2$ and fixed by the condition $F(t,0) = 0$. Calculations are straightforward if one passes to Heisenberg representation for (8) introducing the evolution operator $\hat{U}(t)$ which satisfies the equation $\partial_t\hat{U} = \hat{H}\hat{U}$ with $\hat{H}(t) = \nu(\partial_x^2 - F)$. Then one can develop the closed description in terms of two operators $\hat{A} = \hat{U}^{-1}x\hat{U}$ and $\hat{B} = \hat{U}^{-1}\partial_x\hat{U}$:

$$\partial_t\hat{A} = -2\nu\hat{B}\,, \partial_t\hat{B} = -\nu F_x(t,\hat{A})\,. \qquad (10)$$

We note that all the moments of p [and therefore $F(t,x)$] can be expressed in terms of $\hat{A}$. Since we study the time interval when $p(t,x)$ is narrow, it is enough for our purpose to consider $x \ll L$ where $F(t,x) = c(t)x^2\omega^3/2\nu^2$. Further simplification can be achieved in this case and the closed equation for $c(t) = -(1/2)\int dx\lambda'(x) \times \Psi^{-1}(0,x)\hat{A}^2(t)\Psi(0,x)$ can be derived from (10):

$$(\partial_t c)^2 = 4\omega^3c^3 + 16\xi_2^2 + 4\omega^3\xi_1^3\,.$$

Here $\xi_1 = i\int dx\lambda(x)x$ and $4\xi_2 = -i\int dx\lambda(x)\partial_x \times [xu(0,x)]$. Integrating we get

$$t = \frac{1}{2}\int_{c(0)}^{c} \frac{dx}{\sqrt{\omega^3x^3 + 4\xi_2^2 + \omega^3\xi_1^3}}\,, \qquad (11)$$

which describes $c(t)$ in an implicit form. Further analysis depends on the case considered. For the gradients, we substitute $\xi_1 = n/a_0$ and $\xi_2 = -n/2$ and see that, as time goes backwards, negative $c(t)$ initially decreases by the law $c(t) = c(0) + 2nt$ until $T = \omega^{-1}(n/2)^{-1/3}$ then it grows and the approximation loses validity when $c(t)$ approaches zero and the account of the pumping form $\chi(x)$ at $x \simeq L$ is necessary. For self-consistency, we require the width of $p(x)$ at this time to be of order L and get the estimate $a(0) \simeq \omega\mathrm{Re}\, n^{2/3}$ and thus confirm the above picture. The main contribution to the saddle-point value (6) is again provided by the term $[\partial_x u(0,0)]^n$ and we find $\langle (u')^n\rangle \simeq [a(0)]^n \simeq (\omega\mathrm{Re})^n n^{2n/3}$, which corresponds to the following left tail of PDF at $u' \gg \omega\mathrm{Re}$

$$\mathcal{P}(u') \propto \exp[-C(-u'/\omega\mathrm{Re})^{3/2}]\,. \qquad (12)$$

For higher derivatives $u^{(k)}$, by using (11) we get initial growth $c(t) = c(0) + n(k+1)t$, which gives $u^{(k)}(0,0) \sim N^{k+1}L^{1-k}\omega\mathrm{Re}^k$ leading to $\langle [u^{(k)}]^n\rangle \sim \omega\mathrm{Re}^k L^{1-k} \times n^{(k+1)/3}$, which can be rewritten in terms of PDF:

$$\mathcal{P}(|\mathcal{U}^{(\mathcal{K})}|) \propto \exp[-C_k(|u^{(k)}|L^{k-1}/\omega\mathrm{Re}^k)^{3/(k+1)}]\,. \quad (13)$$

Note that the non-Gaussianity increases with increasing k. On the other hand, the higher k the more distant is the validity region of (13): $u^{(k)} \gg u^{(k)}_{\mathrm{rms}} \sim L^{1-k}\omega\mathrm{Re}^k$.

For the differences, $\xi_1 = 2n\rho_0/w$ and $4\xi_2 = -n[1 + 2\rho_0 u_x(0,\rho_0)/w]$ and we get $\langle w^n\rangle \simeq (L\omega)^n n^{n/3}$, which corresponds to the cubic left tail

$$\mathcal{P}(w) \propto \exp\{-B[w/(L\omega)]^3\} \qquad (14)$$

valid at $w \gg L\omega$. In the intermediate region $L\omega \gg w \gg \rho\omega$, there should be a power asymptotics, which is the subject of current debate [7,11,14]. It is natural that the ρ dependence of $\mathcal{P}(w)$ cannot be found in a saddle-point approximation; as a pre-exponential factor, it can be obtained only at the next step by calculating the contribution of fluctuations around the instanton solution. This is consistent with the known fact that the scaling exponent is n independent for $n > 1$: $\langle w^n(\rho)\rangle \propto \rho$.

For the velocity, $\lambda(x) = -in\delta(x)/u(0,0)$ is an even function so that F is a linear (rather than quadratic) function of x for narrow p: $F(x) = \chi(0)bx/2\nu^2$ with $b = -i\int dx p(x)$. Direct calculation shows that energy and momentum conservation makes b time independent: $b = n/u(0,0)$. It is easy then to get the n dependence of $u(0,0)$: Velocity stretches the field p so that the width of p reaches L at $T \simeq L/u(0,0)$, while the velocity itself is produced by the pumping during the same time: $u(0,0) \simeq \chi(0)bT = \chi(0)nT/2u(0,0) \simeq n\chi(0)L/u(0,0)$. That gives $u(0,0) \simeq L\omega n^{1/3}$ and

$$\mathcal{P}(u) \propto \exp\{-D[u/(L\omega)]^3\}.$$

The product $L\omega$ plays the role of the root-mean square velocity u_{rms}. The numerical factors C, B, and D are determined by the evolution at $t \simeq T$, i.e., by the behavior of pumping correlation function $\chi(x)$ at $x \simeq L$.

We thus found the main exponential factors in the PDF tails. Complete description of the tails requires the analysis of the fluctuations around the instanton, which will be the subject of future detailed publications. Here our aim is to show that fluctuation contributions are not infinite and the saddle-point approximation is meaningful. The account of the fluctuations in the Gaussian approximation is straightforward and leads to the shift of $\mathcal{I}_{\mathrm{extr}}$ insignificant at $n \gg 1$. However, the terms of the perturbation theory with respect to the interaction of fluctuations are infrared divergent (proportional to the observation time). That means that there is a soft mode which is to be taken into account exactly. A soft mode usually corresponds to a global symmetry with a continuous group: if one allows the slow spatiotemporal variations of the parameters of the transformation then small variations of the action appear. Our instantons break Galilean invariance so that the respective Goldstone mode has to be taken into account. Namely, under the transformation

$$x \to x - r, u(x) \to u(x - r) + v, r = \int_t^0 v(\tau) d\tau , \tag{15}$$

the action is transformed as $I \to I - i \int dx dt p \partial_t v$. The source term $\int dx dt \lambda u$ is invariant with respect to (15) for antisymmetric $\lambda(x)$. To integrate exactly along the direction specified by (15) in the functional space we use the Faddeev-Popov trick inserting the additional factor

$$1 = \int \mathcal{D} v(t) \delta\left[u\left(t, \int_t^0 v(\tau) d\tau\right) - v(t)\right] J \tag{16}$$

into the integrand in (2) and (3). Jacobian J is determined by a regularization of (15) according to our choice of the retarded regularization for the initial integral: at discretizing time we put $\partial_t u + u \partial_x u \to (u_n - u_{n-1})/\epsilon + u_{n-1} u'_{n-1}$ (otherwise, some additional u-dependent term appears in the effective action [13]). The discrete version of (15), $p_n(x) \to p_n(x - \epsilon \sum_{j=n}^{N-1} v_j)$, $u_n(x) \to u_n(x - \epsilon \sum_{j=n}^{N-1} v_j) - v_n$, $u_N(x) \to u_N(x) - v_N$ gives

$$J = \exp\left[\int_{-T}^0 dt u'\left(t, \int_t^0 v(\tau) d\tau\right)\right].$$

Substituting (16) into (2),(3) and making (15) we calculate $\int \mathcal{D} v$ as a Fourier integral (the saddle-point method is evidently inapplicable to such an integration) and conclude that after the integration over the mode (15) the measure $\mathcal{D} u \mathcal{D} p e^{iI}$ acquires the additional factor

$$\prod_t \delta\left[\int \partial_t^2 p(t,x) dx \right] \delta[u(t,0)] \exp\left[\int_{-T}^0 u'(t,0) dt \right].$$

The last (Jacobian) term here exactly corresponds to the term $u' \mathcal{P}(u')$ in the equation for $\mathcal{P}(u')$ derived in [5,11]. This term makes the perturbation theory for the fluctuations around the instanton to be free from infrared divergences; the details will be published elsewhere.

Let us summarize. At smooth almost inviscid ramps, velocity differences and gradients are positive and linearly related $w(\rho) \approx 2\rho u'$ so that the right tails of PDFs have the same cubic form [7,10,11]. Those tails are universal, i.e., they are determined by a single characteristics of the pumping correlation function $\chi(r)$, namely, by it's second derivative at zero $\omega = [-(1/2)\chi''(0)]^{1/3}$. On the contrary, the left tails found here contain a nonuniversal constant which depends on a large-scale behavior of the pumping. The left tails come from shock fronts where $w^2 \simeq -\nu u'$ so that the cubic tail for velocity differences (14) corresponds to a semicubic tail for gradients (12). The formula (14) is valid for $w \gg u_{\rm rms} \simeq L\omega$, where $\mathcal{P}(w)$ should coincide with a single-point $\mathcal{P}(u)$ since the probability is small for both $u(\rho)$ and $u(-\rho)$ being large simultaneously. Indeed, we saw that the tails of $\ln \mathcal{P}(u)$ at $u \gg u_{\rm rms}$ are cubic as well. Note that (13) is the same as obtained for decaying turbulence with white (in space) initial conditions by a similar method employing the saddle-point approximation in the path integral with time as large parameter [15]. That probably means that white-in-time forcing corresponds to white-in-space initial conditions. Note that if the pumping has a finite correlation time τ then our results, strictly speaking, are valid for $u, w \ll L/\tau$ and $u' \ll 1/\tau$.

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