Intermittent Dissipation of a Passive Scalar in Turbulence

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The probability density function (PDF) of passive scalar dissipation $\mathcal{P}(\epsilon)$ is found analytically in the limit of large Peclet and Prandtl numbers (Batchelor-Kraichnan regime) in two dimensions. The tail of PDF at $\epsilon \gg \langle\epsilon\rangle$ is shown to be stretched exponent $\ln \mathcal{P}(\epsilon) \propto \epsilon^{1/3}$; at $\epsilon \ll \langle\epsilon\rangle$, $\mathcal{P} \propto 1/\sqrt{\epsilon}$. [S0031-9007(98)05479-9]

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Probability distribution of the gradients of turbulent fields is probably the most remarkable manifestation of the intermittency of developed turbulence and related strong non-Gaussianity. A typical plot of the logarithm of gradient's probability density function (PDF) (which would be parabolic for Gaussian statistics) is concave rather than convex, with a strong central peak and slowly decaying tails. This is natural for an intermittent field since rare strong fluctuations are responsible for the tails, while large quiet regions are related to the central peak. In particular, such PDFs were observed for the dissipation field (square gradient) of passive scalar advected by incompressible turbulence which is the subject of the present paper. We consider scalar advection within the framework of the Kraichnan model assuming velocity field to be delta correlated in time [1]. Most of the rigorous results on turbulent mixing have been obtained so far with the help of that model which is likely to play in turbulence the role the Ising model played in critical phenomena. High-order moments of the scalar were treated hitherto by the perturbation theory around Gaussian limits. Clearly, the kind of strongly non-Gaussian PDF observed for gradients cannot be treated by any perturbation theory that starts from a Gaussian statistics as zero approximation.

Since we consider developed turbulence with large Peclet number Pe (measuring relative strength of advection with respect to diffusion at the pumping scale), it is tempting to use Pe^{-1} as a small parameter. Yet any attempt to treat diffusion term perturbatively is doomed to fail because the PDF of the dissipation is a nonperturbative object with respect to the inverse Peclet number: it is zero at $\epsilon \neq 0$ and zero diffusivity yet has nonzero limits as diffusivity goes to $+0$. Indeed, the main contribution into dissipation is given by the scales around the diffusion scale where advection and diffusion are comparable. Still, the presence of a small parameter calls for finding a proper way to simplify the description. Following [1–4], we show here that the dynamical formalism of Lagrangian trajectories is a proper language to describe the probability distribution of the dissipation field. Our goal is to express the unknown (statistics of dissipation) via the known (statistics of pumping). Since the Peclet number is the ratio between pumping and diffusion scales, then any piece of passive scalar has a long way to go between birth and death, and our goal is to describe how statistics is modified along the way. Dynamical formalism explicitly reveals the presence of two different time scales, a short one related to stretching and a long one related to diffusion (which eventually restricts the process of stretching); this time separation has been exploited first in solving a similar problem for one-dimensional compressible velocity [4]. The time scale of stretching fluctuations is of order of inverse Lyapunov exponent while the whole time of stretching is ln Pe times larger. Clear time separation in the dynamical history is guaranteed for (but probably not restricted by) trajectories that correspond to the value of ϵ larger than that produced by the pumping, $\langle\epsilon\rangle/\mathrm{Pe}^2$, yet smaller than $\langle\epsilon\rangle \ln \mathrm{Pe}$ (whether it is possible to extend the time-separation procedure for a wider interval of ϵ will be the subject of future analysis). At $\mathrm{Pe} \to \infty$, we are able to calculate PDF $\mathcal{P}(\epsilon)$ rigorously, exploiting time separation and executing explicitly separate averaging over slow and fast degrees of freedom.

There are three main steps in deriving $\mathcal{P}(\epsilon)$: (1) Presenting the PDF as an average of a functional of the time-ordered exponent of the strain matrix; (2) Reparametrization of that average into a path integral over the measure nonlocal in time; (3) Implementing time separation which makes reduction to two subsequent path integrations both local in time: one describing the long evolution and another describing the fast fluctuations.

Let us define the problem and make the first step of the derivation. Advection of a passive scalar $\theta(t,\mathbf{r})$ by an incompressible flow $\mathbf{v}(t,\mathbf{r})$ is governed by the equation

$$(\partial_t + v_\alpha \nabla_\alpha - \kappa\Delta)\theta = \phi, \qquad \nabla_\alpha v_\alpha = 0, \quad (1)$$

where κ is the diffusivity and $\phi(t,\mathbf{r})$ is the external source assumed to be Gaussian and δ correlated in time: $\langle\phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\rangle = \delta(t_1 - t_2)\chi(r_{12})$. Here $\chi(r_{12})$ as a function of $r_{12} \equiv |\mathbf{r}_1 - \mathbf{r}_2|$ decays on the scale L, and $\chi(0)$ is the production rate of θ^2.

We consider the simplest possible (yet physically realizable) velocity: smooth in space and delta correlated in time. The velocity field is thus a random laminar flow, while the scalar field will be considered fully turbulent and multiscale (Batchelor-Kraichnan problem) [1–3]. That requires the Prandtl number $\mathrm{Pr} = \nu/\kappa$ (viscosity-to-diffusivity ratio) to be large; that is, our consideration is intended for liquids rather than gases. The scalar pumping scale L lies in the viscous interval where we thus presume the velocity statistics to be described by the pair correlation function $\langle v_\alpha(t_1, \mathbf{r}_1) v_\beta(t_2, \mathbf{r}_2)\rangle$ given by

$$\delta(t_1 - t_2)[V_0 \delta_{\alpha\beta} - D(2\delta^{\alpha\beta} r^2/2 - r^\alpha r^\beta)] \quad (2)$$

for the scales less than the velocity infrared cutoff L_u, which is supposed to be the largest scale of the problem. We presume also the inequality $\mathrm{Pe}^2 = dDL^2/2\kappa \gg 1$ which guarantees that the mean diffusion scale $r_d = 2\sqrt{\kappa/D}$ is much less than the pumping scale L. It follows from (2) that the correlation functions of the strain field $\sigma_{\alpha\beta} \equiv \nabla_\beta v_\alpha$ are $\mathbf{r}$ independent. That property means that $\sigma_{\alpha\beta}$ can be treated as a random function of time t only. To exploit that, it is convenient to pass into the comoving reference frame that is moving to the frame with the velocity of a Lagrangian particle of the fluid [3,5],

$$\partial_t \theta + r_j \sigma_{jl}(t) \partial_l \theta - \kappa \Delta \theta = \phi(t, \mathbf{r}). \quad (3)$$

Making spatial Fourier transform and introducing

$$\hat{W}(t) \equiv T \exp \int_0^t \hat{\sigma}(\tau)\, d\tau, \qquad \frac{d}{dt} \hat{W} = \hat{\sigma} \hat{W}, \quad (4)$$

one can write the solution of (3) as follows:

$$\theta_k(t) = \int_0^t dt'\, \phi(t - t', \hat{W}^{-1,T}(t')\mathbf{k}) \times \exp\left\{-\kappa k_\mu \int_0^{t'} [\hat{W}^{-1}(\tau) \hat{W}^{-1,T}(\tau)]_{\mu\nu} d\tau\, k_\nu\right\}. \quad (5)$$

Averaging over ϕ a simultaneous product of the $2n$th replicas of the inverse Fourier transform of $\mathbf{k}\theta_k$, we get the nth moment of the dissipation field $\epsilon = \kappa(\nabla\theta)^2$. The PDF restored from all the moments is

$$\mathcal{P}(\epsilon) = \int_{0^+ - i\infty}^{0^+ + i\infty} \frac{ds\, e^{s\epsilon}}{2\pi^2 i} \int d\mathbf{m}\, e^{-m^2} \langle e^{-s \int_0^\infty dt\, Q} \rangle_\sigma, \quad (6)$$

$$Q = \int d\mathbf{q}\, \chi_q \left[\frac{\mathbf{q}\hat{W}(t)\mathbf{m}}{2\pi \mathrm{Pe}}\right]^2 \exp\left[-\frac{\mathbf{q}\hat{\Lambda}(t)\mathbf{q}}{\mathrm{Pe}^2}\right], \quad (7)$$

$$\hat{\Lambda}(t) \equiv D\hat{W}(t) \int_0^t dt'\, \hat{W}^{-1}(t') \hat{W}^{-1,T}(t') \hat{W}^T(t), \quad (8)$$

where $\mathbf{q} = \mathbf{k}L$ and an extra integration over the auxiliary vector field $\mathbf{m}$ takes care of combinatorics and summation over vector indices. Notice that the direction of time in (7) and (8) is opposite to that in (3). The averaging over the traceless $\hat{\sigma} = \begin{pmatrix} a & b + c \\ b - c & -a \end{pmatrix}$ is according to $\mathcal{D}\hat{\sigma}(t) \exp[-\int_0^\infty dt'(a^2 + b^2 + c^2/2)/2D]$. The whole expression (7) is invariant with respect to the global (time independent) rotation $\mathbf{m} \to \hat{R}\mathbf{m}$, $\hat{\sigma} \to \hat{R}\hat{\sigma}\hat{R}^T$, provided the pumping is statistically isotropic. That allows one to get rid of the angular integration in $d\mathbf{m}$, counting all the dynamical angles from the direction of $\mathbf{m}$.

Let us do step 2 now. The general tool to average a function of $\hat{W}$, particularly (7), is Kolokolov transformation replacing $T \exp$ by a regular function of new fields [6]. We represent $\hat{W} = \hat{R}_\vartheta \hat{V} \hat{\varphi}$, where $\hat{R}_\vartheta$ is the matrix of rotation by the angle $\int_0^t dt'\, \vartheta(t')/2$, $\hat{V}$ is the diagonal matrix with the elements $\exp[\pm \int_0^t dt'\, \eta(t')]$ and $\hat{\varphi} = \begin{pmatrix} 1 & \varphi(t) \\ 0 & 1 \end{pmatrix}$, $\varphi(0) = 0$, to provide for $\hat{W}(0) = \hat{1}$. Our transform $\{a, b, c\} \to \{\eta, \vartheta, \varphi\}$ is a hybrid version of those used in [7] and in Appendix B of [3]. The relation between the old and new variables is obtained by differentiating $\hat{R}_\vartheta \hat{V} \hat{\varphi}$ and comparing with (4),

$$a + ib = \eta \exp\left[i \int_0^t dt'\, \vartheta\right] + i\frac{\dot{\varphi}}{2} \exp\left[\int_0^t dt'(2\eta + i\vartheta)\right], \quad (9)$$
$$2c = -\vartheta + \dot{\varphi} \exp\left[2\int_0^t dt'\, \eta\right].$$

The Jacobian of the transformation $J = J_{ul} J_l$ has a standard product $J_{ul} \sim \prod_{t=0}^T \exp[2\int_0^t dt'\, \eta]$ and nontrivial Jacobian $J_l \sim \exp[\int_0^T dt\, \eta(t)]$ associated with the positive Lyapunov exponent, describing the exponential stretching of trajectories. The rotation matrix $\hat{R}_\vartheta$, which is the same for all the dynamical matrix processes, can be removed by collective transformation of all the external vectors in the problem; i.e., one may explicitly integrate over $\mathcal{D}\vartheta$ since we average ϑ-independent objects in (6). Averaging some function of $\hat{W}(T)$ is then reduced to averaging the same function of $\hat{V}(T)\hat{\varphi}$ with respect to the measure $\exp[-S]\mathcal{D}\eta(t)\mathcal{D}\varphi(t) \prod_{t=0}^T \exp[2\int_0^t dt'\, \eta]$, where

$$S \equiv \frac{1}{2D} \int_0^T dt \left\{\eta^2 - 2D\eta + \exp\left[4\int_0^t dt\, \eta\right] \frac{\dot{\varphi}^2}{4}\right\}. \quad (10)$$

Finally, we apply the time-separation procedure to capture the term in (6) dominant at large Pe. Two different time scales are clearly seen in (6)–(8) at least for $\epsilon \gg \langle\epsilon\rangle/\mathrm{Pe}^2$ when $\int Q\, dt$ has to be large, which is achieved on such realizations of $\hat{\eta}$ where the integrand grows exponentially due to $W(t)$ until the last exponential factor restricts the growth at the time of order $D^{-1} \ln(\mathrm{Pe})$. Well before, on a time scale of the inverse stretching rate D^{-1}, the exponentially growing $W(t')$ makes the integral over dt' in (8) saturated.

A dynamical picture in r space would be as follows: For the Lagrangian trajectories giving the main contribution into $\langle\epsilon^n\rangle$, fluid particles start very close. Diffusion remains dominant until the particles separate by a distance comparable to the diffusion scale r_d. This phase of the dynamics takes place on times of order D^{-1}; notice that this time is diffusion independent since r_d is a scale where diffusion and stretching are comparable. Once the distance between particles outgrows r_d, random multiplicative stretching due to velocity becomes dominant. Because of a multiplicative nature of the dynamics, the time to go from r_d to L is proportional to $D^{-1}\ln(\mathrm{Pe})$. Let us now introduce a separation time t_0 satisfying $1 \ll Dt_0 \ll \ln(\mathrm{Pe})$; that time will disappear from the final answer.

The temporal separation makes it possible to substitute t by t_0 as an upper limit in integration over t' in (8). Also, an important manifestation of time separation is a distinctively different behavior of the φ field (responsible for the "rotation" in the pseudospace) at time intervals smaller and larger than the separation time. The action (10) shows that for $t > t_0 \gg 1/D$ the φ field is frozen—no dynamics at all, $\varphi(t) = \varphi(t_0)$, which has a clear physical meaning since stretching proceeds along one direction. At the smallest times the φ-field dynamics is essential. Indeed, let us denote $\rho(t) = \int_0^t \eta(t')\,dt'$ and explicitly transform $\hat\Lambda(t)$ according to (9),

$$\hat\Lambda(t) = \begin{pmatrix} e^{2\rho} & 0 \\ 0 & 1 \end{pmatrix} D\int_0^t dt' \exp[2\rho(t') - 2\rho(t)] \begin{pmatrix} e^{-4\rho(t')} + [\varphi(t) - \varphi(t')]^2 & \varphi(t) - \varphi(t') \\ \varphi(t) - \varphi(t') & 1 \end{pmatrix} \begin{pmatrix} e^{2\rho} & 0 \\ 0 & 1 \end{pmatrix}. \quad (11)$$

For trajectories with predominantly positive η we replace $\varphi(t)$ by $\varphi(t_0)$ and neglect the integral from 0 to t_0 in (6) at $t \gg t_0$. Since $[\varphi(t) - \varphi(t')]^2\exp[2\rho(t')]$ and $\exp[-2\rho(t')]$ decrease exponentially as t' increases, then the integral over dt' saturates, which allows for time separation too. Finally, nondiagonal and lower diagonal elements in $\hat\Lambda$ are exponentially small. For time separation to be complete, the integral $\rho(t)$ at $t \gg t_0$ has to be counted from t_0; i.e., we replace $\rho(t) \to \rho(t) + \rho(t_0)$, then in the dominant order in the Peclet number we get

$$\int_0^\infty dt\, Q \approx \frac{m^2\beta}{(2\pi\mathrm{Pe})^2}\int_{t_0}^\infty dt \int d\mathbf{q}\,\chi_q q_1^2 e^{2\rho - q_1^2 e^{2\rho}\mu\beta\mathrm{Pe}^{-2}}, \quad (12)$$

$$\mu \equiv D\int_0^{t_0} dt'\, e^{2\rho(t')}\{e^{-4\rho(t')} + [\varphi(t) - \varphi(t')]^2\}, \quad (13)$$

where we denoted $\beta = \exp[2\rho(t_0)]$. Next, one needs to average $\exp[-s\int dt\, Q]$ over the short-time $\mathcal{D}\eta_<\mathcal{D}\varphi$ and a long-time $\mathcal{D}\eta_>$ separately. The corresponding weights of averaging with respect to $\eta_>$ and $\eta_<$ are completely decomposed: $S = S_< + S_>$. The great advantage of (12) is that, in the long-time averaging, both β and μ are just external parameters, depending neither on time t nor on $\eta_>$. Once the average over $\eta_>$ is performed, we are left with a function of μ. The final result for $\mathcal{P}(\epsilon)$ will then be obtained by averaging over $\eta_<$ and φ.

We first average $\exp[-s\int dt\, Q]$ over $\eta_>$ using (10)

$$\mathcal{P}_s^>(\mathbf{m},\mu) = \lim_{T\to\infty} e^{-(T-t_0)/2}\Phi(T - t_0; \rho = 0), \quad (14)$$

$$\Phi(T,\rho) = e^{\rho(T)}\int \mathcal{D}\rho(t)\delta[\rho(0)]e^{-\int_0^T[(2D)^{-1}\dot\rho^2 + sQ]\,dt}.$$

Here $e^{-(T-t_0)/2}$ is a normalization factor (the inverse value of the path integral without $\exp[-s\int dt\, Q]$). Exactly in a way that Schrödinger equation appears from the path-integral representation of quantum mechanics [8], from (14) one gets $(\partial_t - \hat H_>)\Phi(t,\rho) = 0$ with

$$\hat H_> \equiv -\frac{1}{2}\partial_\rho^2 + \frac{sm^2\beta}{(2\pi\mathrm{Pe})^2}\int d\mathbf{q}\,\chi_q q_1^2 e^{2\rho - q_1^2 e^{2\rho}\mu\beta\mathrm{Pe}^{-2}}. \quad (15)$$

Here sQ, with Q from (12), plays the role of potential, the η^2 term of (10) gives ∂_ρ^2, while the linear in the η part of (10) gives the initial condition of $t = 0$: $\Phi(0;\rho) = e^\rho$. Since sQ vanishes at $\rho \to \infty$ and $\Phi(0;\rho)$ does not, one obtains the asymptotic behavior of Φ at $t \to \infty$

$$\Phi(t;\rho) \to e^{t/2}y\mathrm{Pe}\Phi_*(y), \qquad \mathcal{P}_s^> = \Phi_*((\mathrm{Pe}\mu\beta)^{-1}), \quad (16)$$

$$[-y^{-3}\partial_y y^3\partial_y + 2sm^2\mu^{-1}U(y)]\Phi_*(y) = 0, \quad (17)$$

$$U(x) \equiv (2\pi)^{-2}\int d\mathbf{k}\,\chi_k k_1^2\exp(-k_1^2x^2), \quad (18)$$

where the new variable $y = e^\rho(\mathrm{Pe}\mu\beta)^{-1}$ has been introduced. $\Phi_* \to 1$ at $y \to \infty$ and $\Phi_* y$ should vanish at $y \to 0$. $\mathcal{P}_s^>$ is a function of a single argument sm^2/μ. The potential $U(x)$ is everywhere positive; it has to turn into a constant at $x \to 0$ and behave as x^{-3} at large x. Under such a general assumption on the pumping function χ_q, one can show that the only singularities of $\mathcal{P}_s^>(z)$ are poles on the negative semiaxis at a finite distance from zero. The asymptotic condition on $\Phi_*(y)$ gives the normalization $\mathcal{P}_s^>(0) = 1$ and at $z \to \infty\ \ln[1/\mathcal{P}_s^>(z)] \sim \sqrt{z}$. For example, in the particular case

$$U_c(x) = \begin{cases} 1, & x < 1, \\ 1/x^3, & x > 1, \end{cases} \quad (19)$$

one finds $2\mathcal{P}_s^>(z) = z[I_2'(2\sqrt{z})I_1(\sqrt{z}) + I_2(2\sqrt{z}) \times I_1'(\sqrt{z})]^{-1}$ and $\mathcal{P}_s^>(z) \to (z^{3/2}/4\pi)\exp[-3\sqrt{z}]$ at $z \to +\infty$.

Now we have to average $\mathcal{P}_s^>(2sm^2/\mu)$ over $\eta_<$ and φ, which is equivalent to averaging over the random variable μ (responsible for the fluctuations of the diffusion scale due to strain variations): $\langle e^{-sQ}\rangle_\eta = \langle \mathcal{P}_s^>(2sm^2/\mu)\rangle_\mu$. A convenient auxiliary object to calculate is the generating function $\mathcal{P}_\lambda = \langle\exp[-\lambda\mu]\rangle$, which can be expressed via the solution of another differential equation:

$$\mathcal{P}_\lambda = \int_{-\infty}^\infty d\varphi\,\Psi(\rho = 0,\varphi), \qquad (2\hat H_< + 1)\Psi = 0, \quad (20)$$

$$\left[\int_{-\infty}^{\infty} d\varphi\, \varphi^n \Psi e^{-\rho}\right]_{\rho\to+\infty} \to \delta_{n0}, \qquad \Psi|_{\rho\to-\infty} \to 0\,.$$

$$\hat{H}_< \equiv -2e^{-4\rho}\partial_\varphi^2 - \tfrac{1}{2}\partial_\rho^2 + \lambda e^{-2\rho}(1 + \varphi^2 e^{4\rho})\,,$$

where the static equation (20) follows at $t_0 \gg 1$ from the respective dynamical one similar to the way (16) follows from (14). The exact solution of (20), $\Psi\pi\sqrt{1 + \varphi^2 e^{4\rho}} = \sqrt{2\lambda}\, e^{2\rho} K_1\big(e^{-\rho}\sqrt{2\lambda(1 + \varphi^2 e^{4\rho})}\big)$, gives

$$\mathcal{P}(\mu) = (2\pi\mu^3)^{-1/2}\exp(-1/[2\mu])\,. \tag{21}$$

Note that $\mu^{-1/2}$ (which can be interpreted as inverse local diffusion scale) has exactly Gaussian PDF that seems to be a consequence of strain Gaussianity. Integration of $\mathcal{P}_s^>(2sm^2/\mu)$ with $\mathcal{P}(\mu)$ gives the final answer,

$$\mathcal{P}(\epsilon) = \frac{1}{2\pi\sqrt{\epsilon}}\int_{-\infty}^{\infty} ds\, \mathcal{P}_s^>(is) \times \int_0^\infty dx\, \exp\!\left(isx^2 - \frac{\sqrt{\epsilon}}{x}\right). \tag{22}$$

That formula is our main result. It expresses dissipation PDF in terms of the function $\mathcal{P}_s^>$ determined by (16) and (17) with the potential (18) given by the pumping. For any pumping, all that is necessary to get $\mathcal{P}(\epsilon)$ is to solve an ODE of the second order. In particular, one can find $\mathcal{P}_s^>(z)$ as a series in z iterating Φ_* in (17) over U. That expresses dissipation moments directly via the pumping

$$\langle\epsilon^n\rangle = 2^n[(2n-1)!!]^2 n! \times \int_0^\infty \frac{dy_1}{y_1^3}\int_0^{y_1} dy_2\, y_2^3 U(y_2) \cdots \int_{y_{2n-2}}^\infty \frac{dy_{2n-1}}{y_{2n-1}^3}\int_0^{y_{2n}} dy_{2n}\, y_{2n}^3 U(y_{2n})\,. \tag{23}$$

The conservation law identity $\langle\epsilon\rangle = -2\partial_z^2\mathcal{P}_s^>(0) = \chi[0]/2$ is a consistency check of our procedure. The form of the PDF at $\epsilon \simeq \langle\epsilon\rangle$ is pumping dependent while the asymptotics at small and large ϵ have universal forms. At small ϵ (and infinite Pe) the two first terms have the form $\mathcal{P}(\epsilon) \to A\epsilon^{-1/2} + B\ln\epsilon$ so that the gradient's PDF $\mathcal{P}(\nabla\theta) \to A - B|\nabla\theta|/\ln|\nabla\theta|$ tends to a constant; notice the nonanalyticity of the first nonequilibrium correction.

Since the only singularities of $\mathcal{P}_s^>(iz)$ are poles on the imaginary axis, then large-ϵ asymptotics is determined by the pole nearest to the origin in the upper half-plane: $z = i\tau^*$, $\tau^* \sim 1$. In particular, $\tau_* \approx 2.7$ for (19). The dx integration in (22) is done by a saddle-point method,

$$\mathcal{P}(\epsilon) \sim \epsilon^{-1/2}\exp[-3(2\tau^*\epsilon)^{1/3}/2]\,. \tag{24}$$

Stretched-exponential tail is natural for steady PDF of the gradients [2] contrary to log normal unsteady distribution which takes place without diffusion [1,2]. The value $1/3$ for the stretched exponent may be explained as follows: Since the local value of $\nabla\theta$ is proportional to the scalar fluctuation at the diffusion scale $\delta\theta$ (which has exponential PDF tail [2,3,9]) times an inverse local diffusion scale [which is Gaussian according to (21)], then $\langle\epsilon^n\rangle \sim \kappa^n\langle(\delta\theta)^2\rangle^n n^{2n} r_d^{-2n} n^n$ which corresponds to (24). Note that the asymptotics (24) is determined by the dynamics of stretching (not of rotations), thus it is likely to take place in any dimensions. The data of numerics done directly for the Kraichnan model [10] are satisfactorily fitted by the $1/3$ stretched exponent within the window expanding when Pe grows. The exponent $1/3$ agrees also with the values 0.3–0.36 given by numerics [11] and 0.37 by experimental data on air turbulence [12]. Moreover, our exponent derived formally at $\mathrm{Pr} \gg 1$ agrees with the data of [11,12] obtained at $\mathrm{Pr} \simeq 1$ as well. This is surprising because $\mathcal{P}(\epsilon)$ is independent of Pe at large Pr, while at $\mathrm{Pr} \lesssim 1$ the statistics of ϵ is markedly different since $\langle\epsilon^n\rangle \propto \mathrm{Pe}^{\mu_n}$ where μ_n are anomalous exponents of the scalar [13]. It may be, however, that the PDF tail is still given by (24) for any Pr with τ_* and preexponent factor in (24) generally depending on both Pr and Pe; this will be the subject of future studies. To conclude, the agreement of our result with a variety of data suggests that the Kraichnan model is a proper tool for recovering exponents in the theory of turbulent mixing.

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