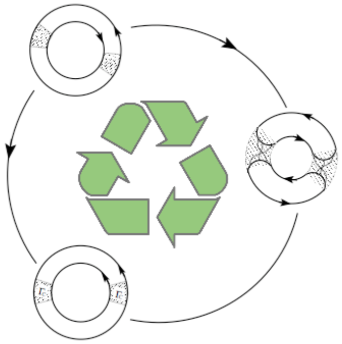
Branch-cut singularities in the thermodynamics of Fermi liquid systems



A. Shekhter, and A. M. Finkel’stein
UC RIVERSIDE UNIVERSITY OF CALIFORNIA

Department of Condensed Matter Physics, the Weizmann Institute of Science
Rehovot 76100 Israel

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Phys. Rev. B 74, 205122 (2006).

why the spin susceptibility of 2D electron gas decreases strongly with temperature

★an experiment (Technion group, 2003) has found a puzzling temperature behavior of the spin susceptibility (at T>2K) which contradicts the standard Fermi-liquid phenomenology

★ motivated by this experiment, we develop a theory which extends the existing microscopic theory of the Fermi-liquid systems by inclusion of branch-cut singularities

★the extended theory resolves the puzzle of the observed behavior (including the ***sign***) of the spin susceptibility in Si-MOSFETs

★the work shows that the theory of the Fermi liquid systems is incomplete unless the branch-cut singularities are included

★**non Fermi liquid corrections within the Fermi liquid state**

Motivation of the work:
Measurements of the spin susceptibility in silicon **MOSFETs**

SdH oscillations clearly indicate the presence of the Fermi surface: i.e. according Landau (1956) this is a **Fermi Liquid system.**

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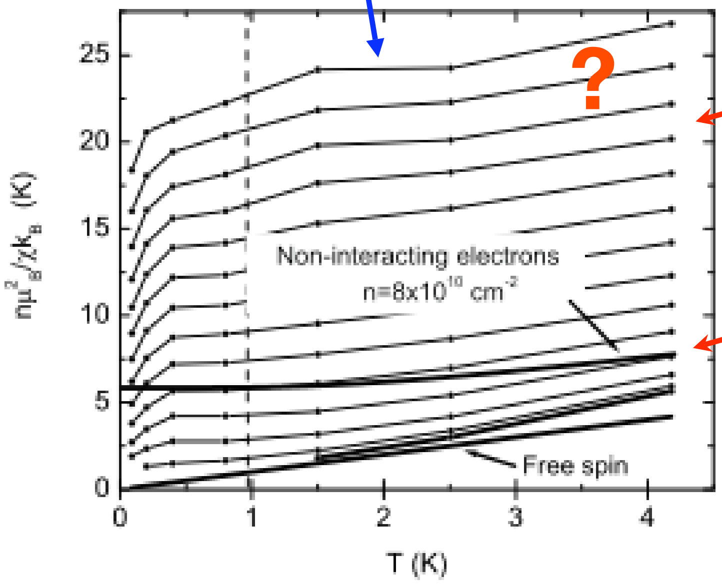
Pauli

$$\epsilon_F \simeq 30K \div 40K$$

$(T/\epsilon_F)^2$ **will be invisible !**

Prus et.al. (2003)

$$n\mu_B^2/\chi k_B(K)$$

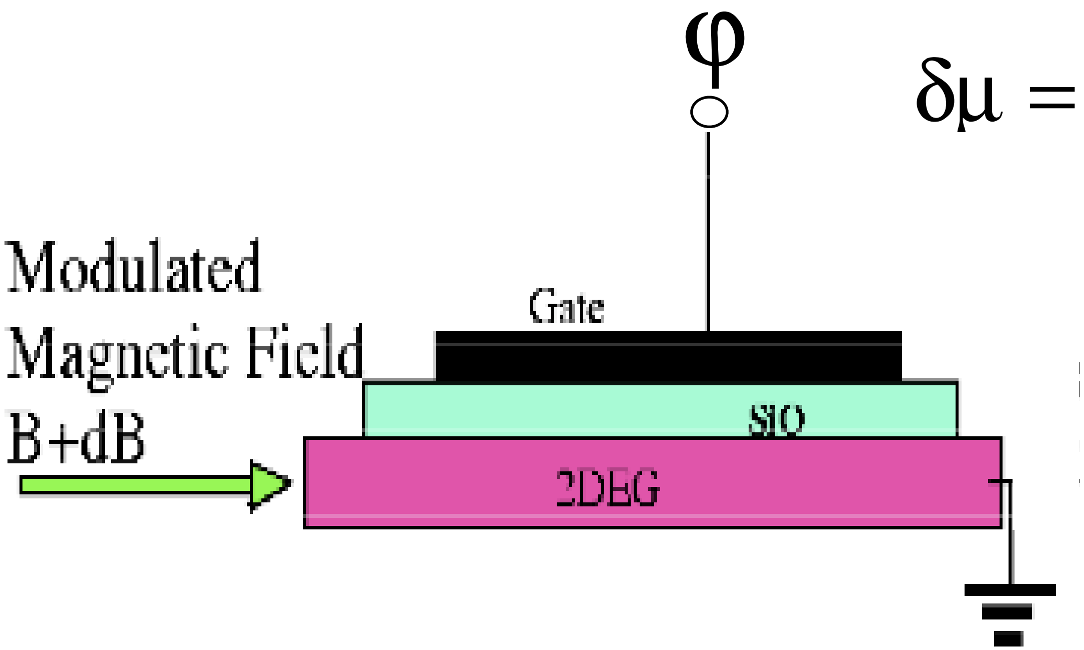


Thick line is **Fermi gas:**

$$\delta\chi/\chi = -(T/\epsilon_F)^2$$

FIG. 4: Inverse susceptibility as determined from $M(B)$ at $B = 0.7$ T. Experimental points from bottom to top correspond to densities $0.8 \div 6 \times 10^{11}$ cm^{-2} in 4×10^{10} cm^{-2} steps. The thick straight line depicts Curie law and the dashed line marks $T = (g\mu_B/k_B) \times 0.7$ T. The experimental points at $n = 8 \times 10^{10}$ are connected by a thick line for comparison with the expectation for non-interacting electrons of the same density.

The principle of the measurement:



$$\delta\mu = e\delta\varphi$$

$$-\partial\mu / \partial B = \frac{1}{C}\partial n / \partial B$$

measured in the experiment

$F(n,B)$- free energy $\quad \mu = \dfrac{\partial F}{\partial n}, \quad \mathrm{M} = -\dfrac{\partial F}{\partial B} \quad \Rightarrow \quad \boxed{\dfrac{\partial\mu}{\partial \mathrm{B}} = -\dfrac{\partial M}{\partial n}}$

$$M(B,T,n) = const + \int_{n_0}^{n} dn\, \partial M / \partial n$$

Search for non analyticity:

$$\int d\omega \ \coth(\beta\omega/2)\ f(\omega)$$

If ***f*** is smooth and regular in the vicinity of ***f=0***,
the standard Sommerfeld expansion will involve
even powers of ***T*** only.

$$(T/\epsilon_F)^2$$

How to avoid Sommerfeld expansion?

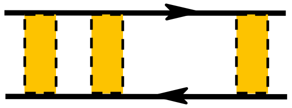
Theory of the Fermi Liquid systems

Pole singularities

Landau quasiparticles
appear as poles of a single-particle correlation function

$$\frac{Z}{\epsilon-\epsilon_{\mathbf{p}\sigma}+\mu+i\delta}$$

Collective modes
appear as poles of a two-particle correlation function



$$-\frac{\chi\,\omega_{\mathbf{q}}^2}{\omega^2-\omega_{\mathbf{q}}^2+i\delta}$$

Collective modes
have small phase space.

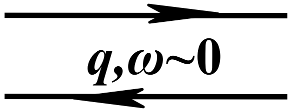
what else remains in a Fermi liquid system?

Theory of the Fermi Liquid systems (extension of the Fermi-liquid theory)

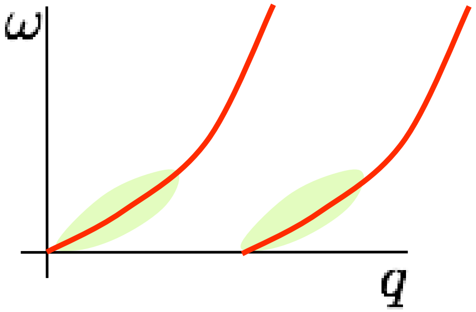
what else remains?

Branch cut (edge) singularities

“Edge modes”
appear as the **threshold**
to the particle-hole continuum.



$$-\frac{\omega}{\sqrt{\omega^2-(qv_F)^2}}$$



Interaction allows for a *rescattering* of pairs of the quasiparticles

As a result of the rescattering induced by interactions, the branch-cut singularities generate non-analyticities in the thermodynamic potential which reveal themselves via anomalously strong temperature dependences.

States near the thresholds of p-h continuum are highly sensitive to temperature smearing and therefore are important for temperature dependences

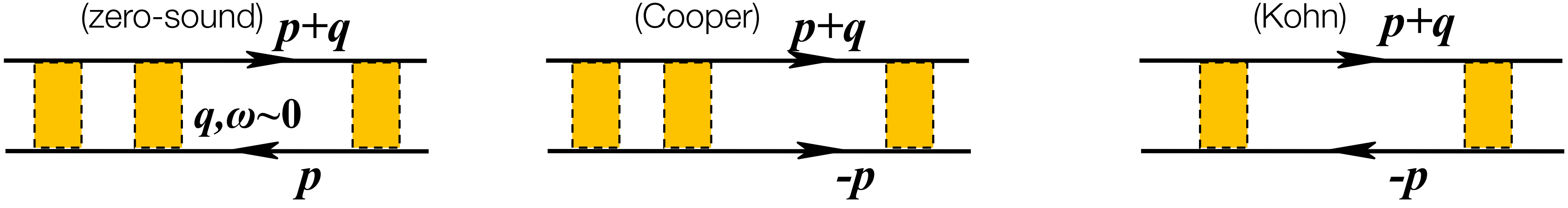
Three channels of rescattering

branch-cut singularities appear in each channel

particle-hole pairs (zero-sound)

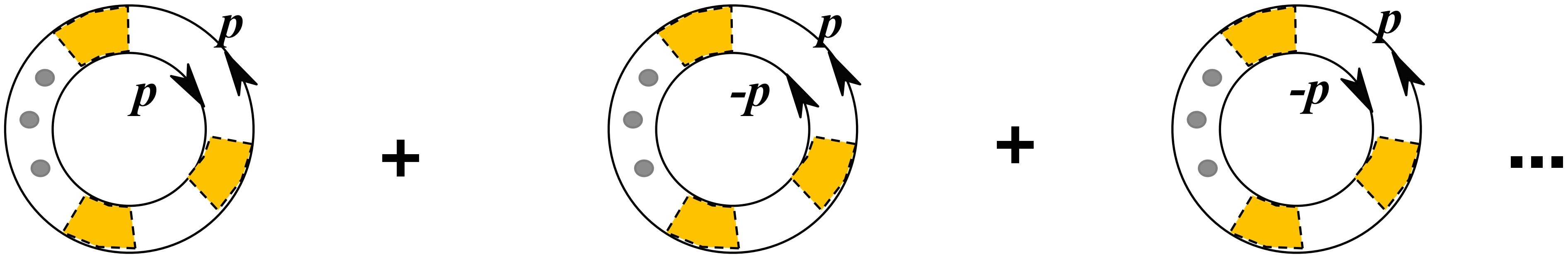
particle-particle pairs (Cooper)

$2p_F$ - pairs (Kohn)



In thermodynamics **the non-analyticities associated with branch-cuts** enter via **ring diagrams,** i.e., ladders which are closed onto themselves

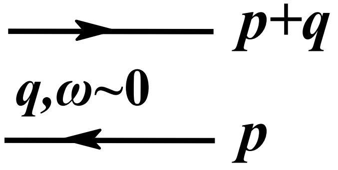
$$\delta\Omega(\Delta) =$$



For the ladder diagrams, the constraints imposed by the conservation of the momentum and energy are most effective because they are applied to a minimal number of quasiparticles. In this way, the dominant terms are generated in the thermodynamic potential. **In ladders the non-analyticities associated with branch-cuts are not smeared out by subsequent integrations**

Particle-hole ring in the thermodynamic potential (harmonics)

Section of rescattering



$$[GG]_{q,\omega,\Delta} = \overbrace{-\nu}^{\text{static}} + \underbrace{\nu S(\theta)_{q,\omega,\Delta}}_{\text{dynamic}}$$

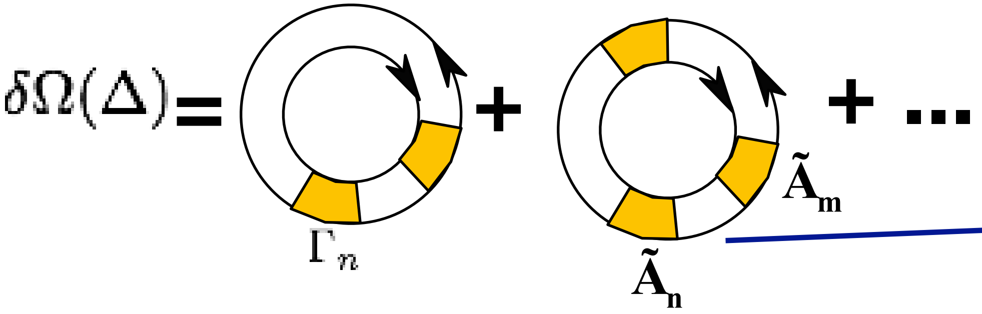
Dynamic particle-hole section

$$S(\theta)_{q,\omega,\Delta} = \frac{\omega}{\omega + \Delta - v_F q \cos\theta}$$

$$\tilde{\omega} = \omega + \Delta$$

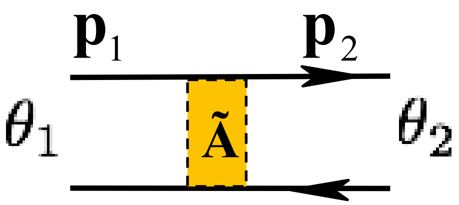
$$\underline{\Delta = 2(g\mu_B/2)H}$$

In fermi-liquid we use **angular harmonics** of the interaction amplitudes

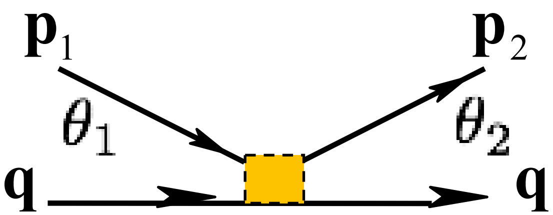


angular harmonics of the dynamic particle-hole propagation function

$$S_{n-m} = \int (d\theta / 2\pi) S(\theta) e^{i(n-m)\theta}$$

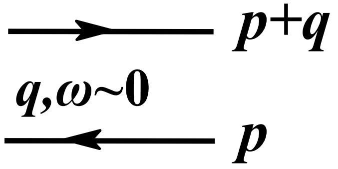


$$\Gamma(\theta_1 - \theta_2) \rightarrow \textit{harmonics} \rightarrow \sum \Gamma_n e^{in(\theta_1 - \theta_2)}$$



Particle-hole (zero-sound) ring in the thermodynamic potential

Section of rescattering



static

$$[GG]_{q,\omega,\Delta} = -\nu + \nu S(\theta)_{q,\omega,\Delta}$$

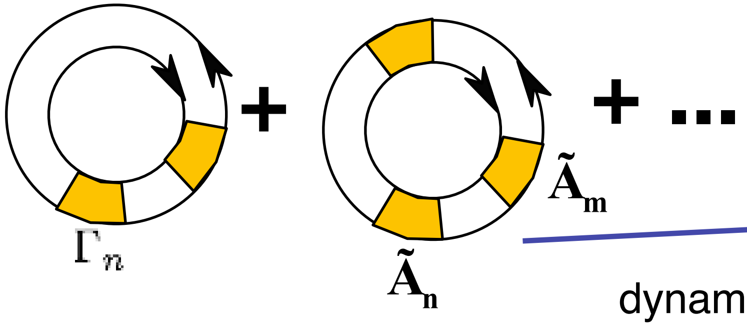
dynamic

Dynamic particle-hole section

$$S(\theta)_{q,\omega,\Delta} = \frac{\omega}{\omega + \Delta - v_F q\cos(\theta)}$$

$$\tilde{\omega} = \omega + \Delta$$
$$\underline{\Delta = 2(g\mu_B/2)H}$$

$\delta\Omega(\Delta) =$



dynamic particle-hole propagation functions

$$S_{n,m} = \int (d\theta/2\pi) S(\theta) e^{i(n-m)\theta}$$

$$S_{nn} = S_0 = \frac{\omega}{\sqrt{\tilde{\omega}^2 - q^2}}$$

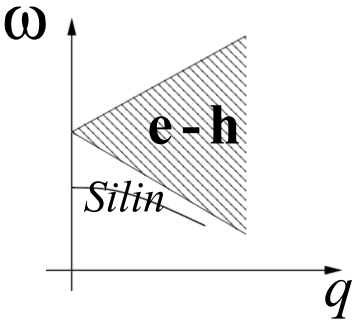
$$S_{n\neq m} = \left(\frac{\tilde{\omega} - \sqrt{\tilde{\omega}^2 - q^2}}{q}\right)^{|n-m|} S_0$$

For a single **dominant** harmonics, e.g., $\gamma = \Gamma_0$

$$\delta\Omega(\Delta) = -(\nu/2\epsilon_F)\int d\omega \coth\frac{\beta\omega}{2}\,\frac{qdq}{2\pi}\,\mathrm{Im}\ln\frac{1}{1+\gamma S_0(q,\omega,\Delta)}$$

Pole in the **complex-q** plane (“*Regge pole*”)

A “cocktail” of pairs of quasi-particles and the collective mode (spin wave; Silin) can be effectively described by a pole in the complex momentum plane (like in the high-energy physics).



The anomalous part of the magnetization is $M = -\frac{\partial\Omega}{\partial H}$

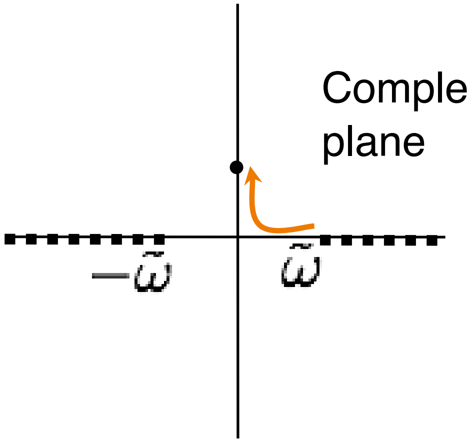
$$\delta M = \int \underline{\frac{d\omega}{2\pi}\coth\frac{\beta\omega}{2}} \quad \mathrm{Im}\int_0^\infty \frac{q\,dq}{\pi}\, \frac{\tilde{\omega}}{\tilde{\omega}^2-(qv_F)^2}\, \frac{\gamma\omega}{\gamma\omega+\sqrt{\tilde{\omega}^2-(qv_F)^2}}$$

The q-integration is non-vanishing when the **pole** in the ***complex-q*** plane is on the imaginary axis

$$\gamma\omega + \sqrt{\tilde{\omega}^2 - q^2} = \mathbf{0}$$

Pole **enters** into the **imaginary** axis of the complex *q-plane* when

$$-\Delta < \omega < -\Delta/(1+\gamma)$$



Complex-***q*** plane

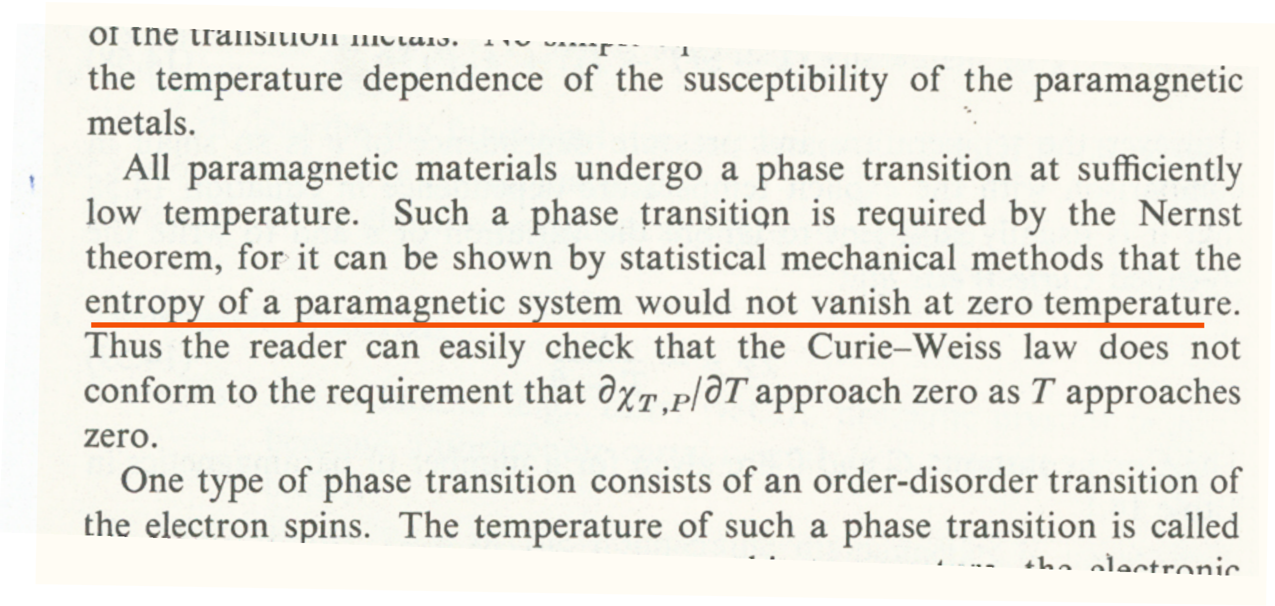
Taking the residue we obtain:

$$\delta M = -\frac{\nu}{2\epsilon_F}\int_{-\Delta}^{-\Delta/1+\gamma}(\omega+\Delta)\coth\frac{\beta\omega}{2}d\omega$$

$$\delta\chi_{\mathrm{e-h}} = -2\nu\frac{T}{\epsilon_F}\left(\ln\frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma}\right)$$

! non analytic contribution >> than in the regular FL
? problems with 3rd law of thermodynamics
? opposite sign compared to experiment

Does the linear-T term contradict **the third law of the thermodynamics?**



of the transition metals. … the temperature dependence of the susceptibility of the paramagnetic metals.

All paramagnetic materials undergo a phase transition at sufficiently low temperature. Such a phase transition is required by the Nernst theorem, for it can be shown by statistical mechanical methods that the entropy of a paramagnetic system would not vanish at zero temperature. Thus the reader can easily check that the Curie–Weiss law does not conform to the requirement that $\partial\chi_{T,P}/\partial T$ approach zero as T approaches zero.

One type of phase transition consists of an order-disorder transition of the electron spins. The temperature of such a phase transition is called

Callen, (1967), p252

“paramagnetic behavior”

$$\chi^{-1} \sim T + \text{const}$$

Third law:

$$S_{T\to 0} = 0$$

Maxwell’s relation:

$$\left(\frac{\partial M}{\partial T}\right)_H = \left(\frac{\partial S}{\partial H}\right)_T$$

it is indirectly assumed above that the thermodynamic potential has a **regular expansion** near ***H,T = 0***

However, for **non-analytic** function the paramagnetic behavior can survive to zero temperature $\delta M = THm_\gamma(H/T)$

$$H,T \to 0 \qquad \begin{array}{ll} \delta M \propto HT & T > H \\ \delta M \propto H^2 & T < H \end{array}$$

(Thermal expansion coefficient)

Misawa (1999) emphasized the non-analyticity of $\delta\Omega(H,T)$ and guessed (incorrectly) its non-analytic form.

$$\delta M = -\frac{\nu}{2\epsilon_F}\int_{-\Delta}^{-\Delta/1+\gamma}(\omega+\Delta)\coth\frac{\beta\omega}{2}d\omega$$

Partial list of references (theory)

The early work in 3d: G.M. Eliashberg (logT-corrections in the specific heat).

Works in 2D

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G. Y. Chitov and A. J. Millis, Phys. Rev. Lett. **86**, 5337 (2001);
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A. V. Chubukov and D. L. Maslov, Phys. Rev. B **68**, 155113 (2003);
B **69**, 121102R (2004).
V. M. Galitski, A. V. Chubukov, and S. Das Sarma, Phys. Rev. B **71**,201302R (2005).
A. V. Chubukov, D. L. Maslov, S. Gangadharaiah, and L. I. Glazman
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Phys. Rev. Lett. **95**, 026402 (2005).
J. Betouras, D. Efremov, and A. Chubukov, Phys. Rev. B **72**, 115112 (2005).
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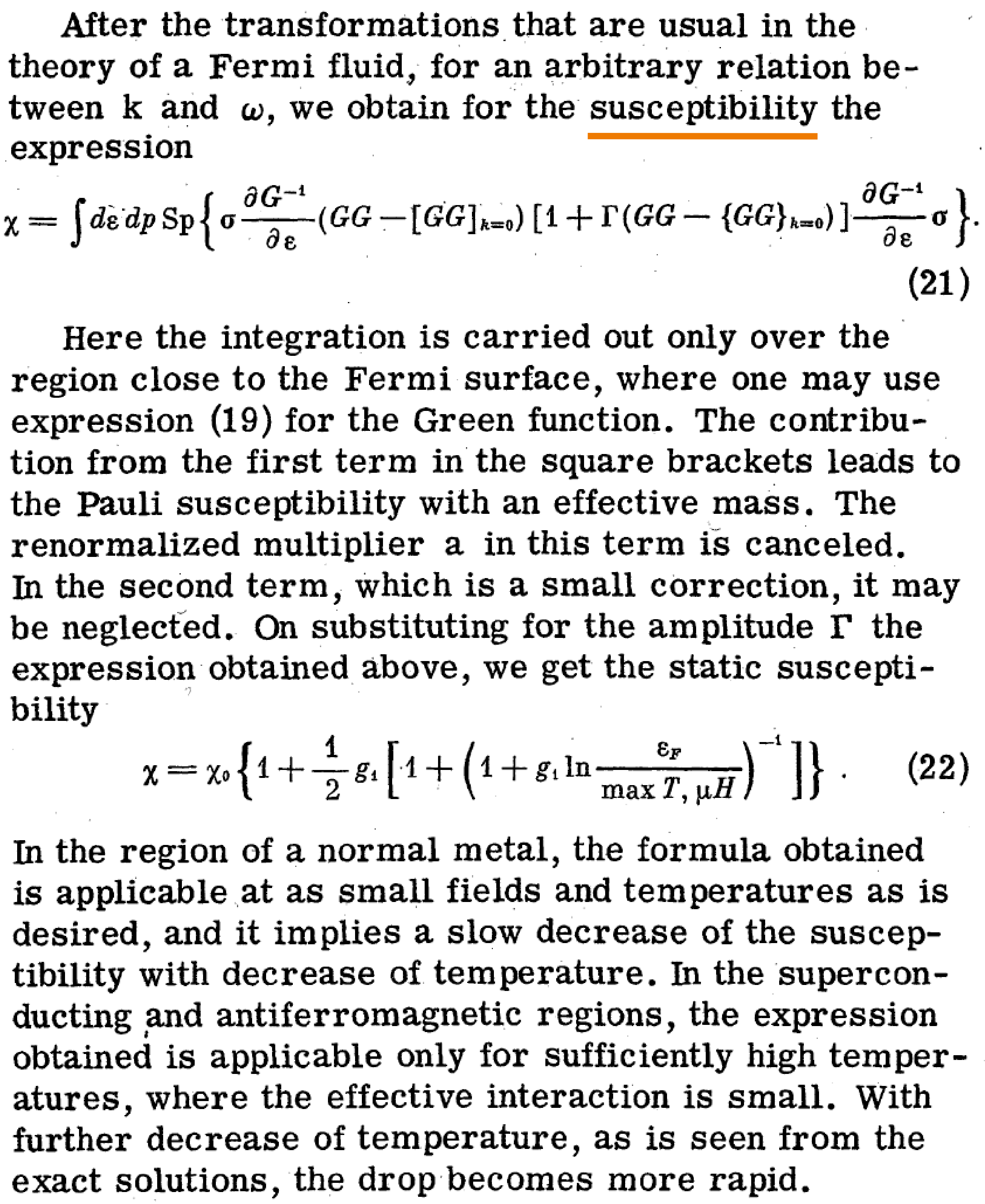
I. L. Aleiner and K. B. Efetov, Phys. Rev. B **74**, 075102 (2006).

D. L. Maslov, A. V. Chubukov, R. Saha, cond-mat/0609102
Nonanalytic Magnetic Response of Fermi- and non-Fermi Liquids



After the transformations that are usual in the theory of a Fermi fluid, for an arbitrary relation between k and ω, we obtain for the susceptibility the expression

$$\chi = \int d\varepsilon\, dp\, \mathrm{Sp}\left\{\sigma \frac{\partial G^{-1}}{\partial \varepsilon}(GG - [GG]_{k=0})\,[1 + \Gamma(GG - \{GG\}_{k=0})]\frac{\partial G^{-1}}{\partial \varepsilon}\sigma\right\}. \tag{21}$$

Here the integration is carried out only over the region close to the Fermi surface, where one may use expression (19) for the Green function. The contribution from the first term in the square brackets leads to the Pauli susceptibility with an effective mass. The renormalized multiplier a in this term is canceled. In the second term, which is a small correction, it may be neglected. On substituting for the amplitude Γ the expression obtained above, we get the static susceptibility

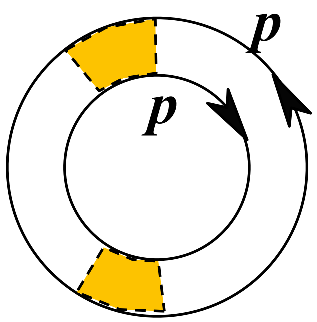
$$\chi = \chi_0\left\{1 + \frac{1}{2}g_1\left[1 + \left(1 + g_1 \ln\frac{\varepsilon_F}{\max T, \mu H}\right)^{-1}\right]\right\}. \tag{22}$$

In the region of a normal metal, the formula obtained is applicable at as small fields and temperatures as is desired, and it implies a slow decrease of the susceptibility with decrease of temperature. In the superconducting and antiferromagnetic regions, the expression obtained is applicable only for sufficiently high temperatures, where the effective interaction is small. With further decrease of temperature, as is seen from the exact solutions, the drop becomes more rapid.

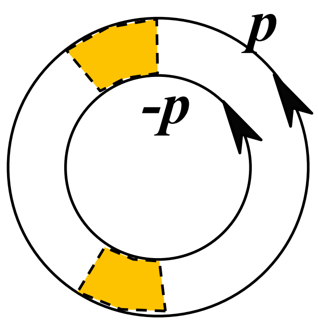
"Possible States of Quasi One Dimensional Systems"
Dzyaloshinskii and Larkin
JETP 34, 422 (1972)
(ZhETF 61, 791 (1971))

Calculations (in harmonics) of two-section terms, $\delta\chi_{(2)}$

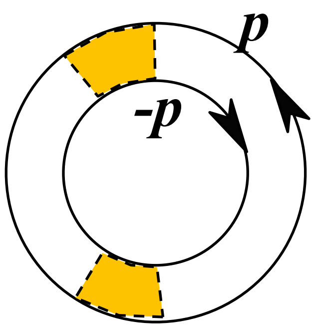
p-h (zero-sound)



p-p (Cooper)



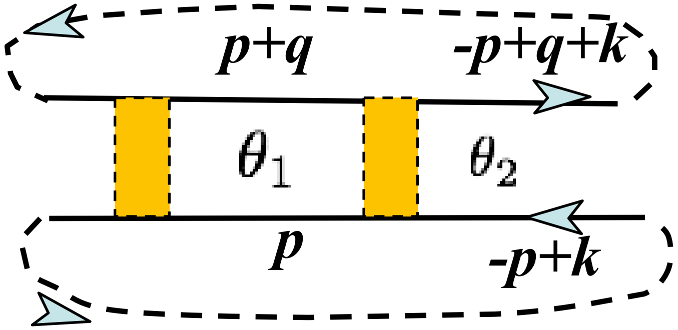
2p_F (Kohn)



Calculations (**in harmonics**) can be done in the zero-sound and Cooper channels **separately** and demonstrate (surprisingly) that:
in the spin susceptibility the **two-section terms** in **all** channels are **dominated** by the scattering sharply peaked near the **backward direction,** . $\Gamma(\pi)$

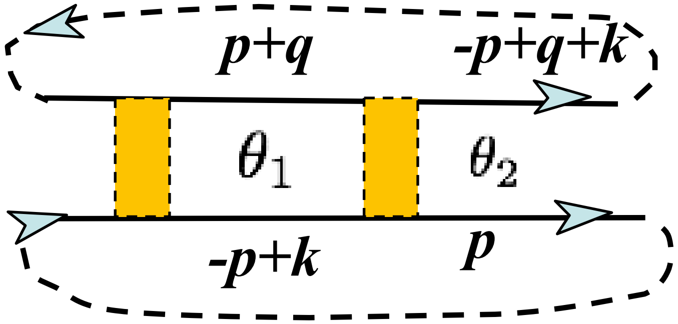
$$\delta\chi_{(2)} = \nu\frac{T}{\epsilon_F}\left|\sum(-1^n\Gamma_n)\right|^2 \;;$$

$$\sum(\Gamma_n e^{in(\theta_1-\theta_2)}) \to \sum(-1^n\Gamma_n) = \Gamma(\pi)$$



$$\theta_1 - \theta_2 \approx \pi$$

$|p|>>|q|,|k|$



in all channels $\delta\chi_{(2)}$ is controlled by $\Gamma(\pi)$

Backward scattering amplitude $\Gamma(\pi)$ can be read in three different ways

two-section terms in the spin susceptibility are **dominated** by the scattering sharply peaked near the **backward direction:** $\theta_1 - \theta_2 \approx \pi$

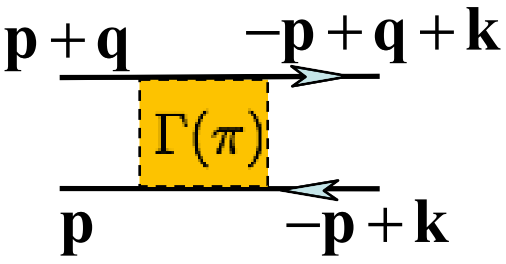
$$\delta\chi^{\mathrm{zs}}_{(2)} = \nu\frac{T}{\epsilon_F}|\sum(-1^n\Gamma^{\mathrm{zs}}_n)|^2$$

$$\delta\chi^{\mathrm{C}}_{(2)} = \nu\frac{T}{\epsilon_F}|\sum(-1^n\Gamma^{\mathrm{C}}_n)|^2$$

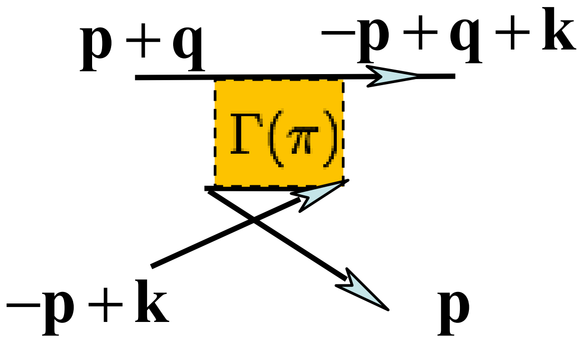
A. Shekhter and AF
Phys. Rev. B 74 (2006).

$|p| >> |q|, |k|$

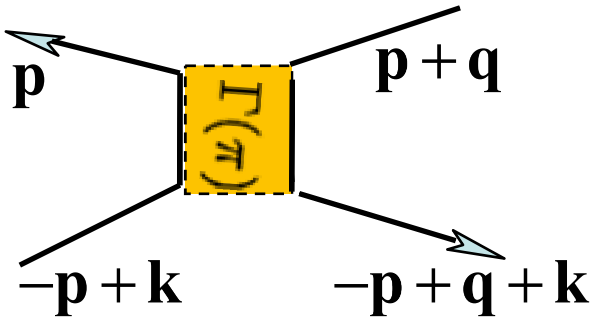
p-h (zero-sound)



p-p (Cooper)

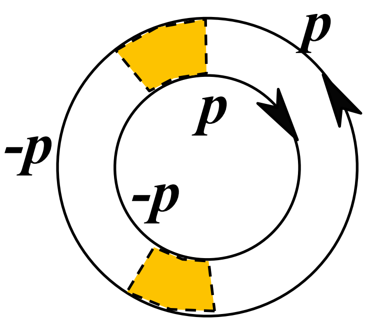


$2p_F$ (Kohn)

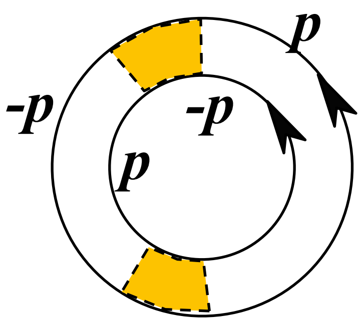


Moment of truth: complete overlapping of two-sections terms

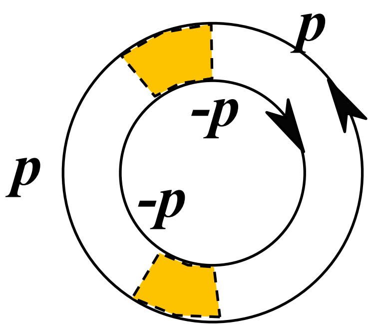
p-h (zero-sound)



p-p (Cooper)

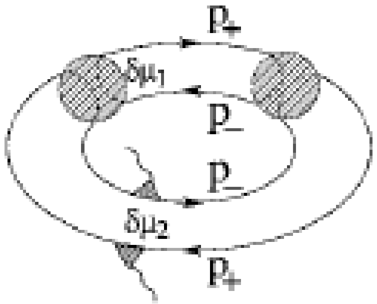


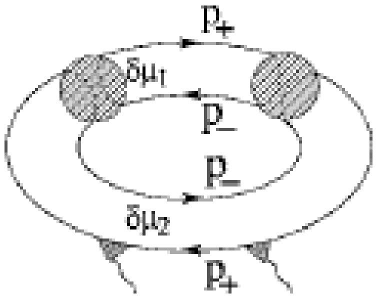
$2p_F$ (Kohn)



★The diagram with **two rescattering sections** dominated by the **backward** scattering can be read in **three different ways**.

★ The diagram in the zero-sound channel can be **twisted** so as to describe the rescattering in the **Cooper** channel or two sections in the $2p_F$-scattering channel.





Calculation can be done in **each channel separately** and demonstrates:

$$\delta\chi^{\mathbf{zs;C}}_{(2)} = \nu \frac{T}{\varepsilon_F} \left|\Gamma(\pi)\right|^2$$

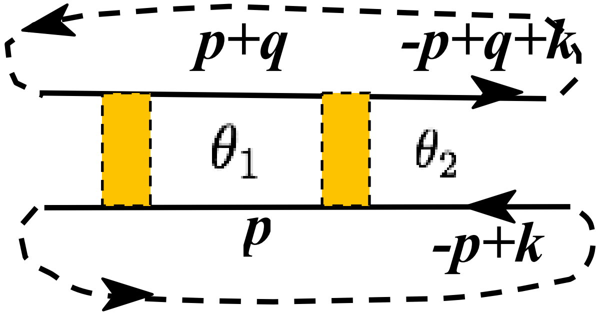
$$\boldsymbol{\Gamma(\pi) = \sum(-1)^n \Gamma_n^{\mathrm{zs;C}}}$$

Backward scattering was a starting point of many previous works discussing linear in ***T*** corrections in χ .

This was by analogy with 1D, e.g., BKV.

Central point: complete overlapping of two-sections terms

two-section term in the spin susceptibility is **dominated** by the scattering sharply peaked near the **backward direction.** The diagram with two rescattering sections dominated by the backward scattering can be read in **three different ways**. This fact leads to far reaching consequences for thermodynamics.



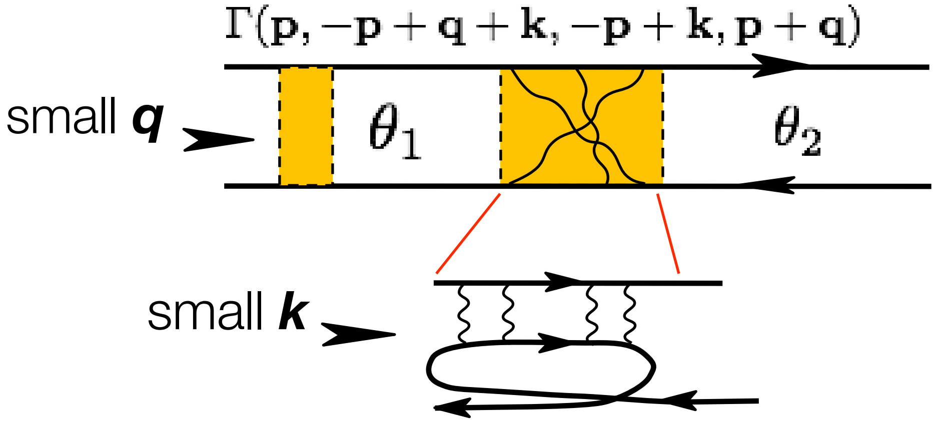
$\theta_1 - \theta_2 \approx \pi$

$|p| >> |q|, |k|$

★**Problems:** a) **double counting**

b) near the backward scattering angle the **rescattering** in the **Cooper** channel may intervene

Only ***dynamic sections*** *are important for non-analytic corrections. (Static parts are used for renormalizations of the analytic amplitudes.) So, in the discussion of* ***two dynamic sections*** *many static can be involved.*



for $\Gamma_n^C > 0$

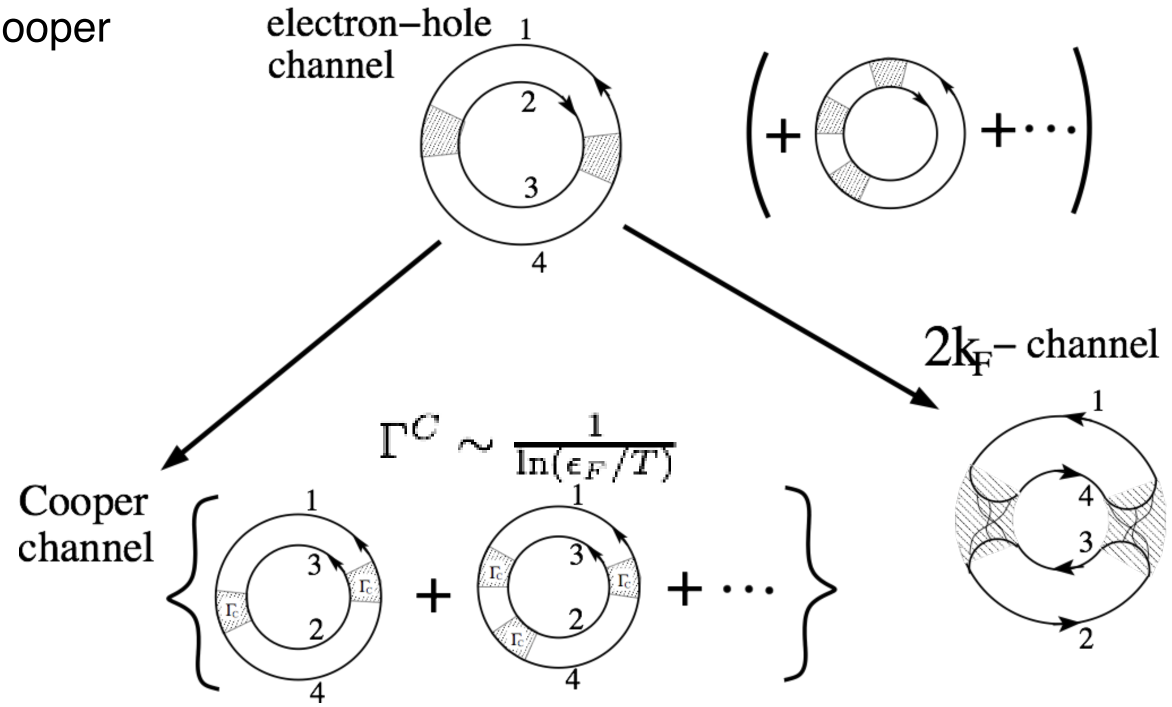
$$\Gamma_n^C(T) = \frac{\Gamma_n^C}{1+\Gamma_n^C \ln \epsilon_F/T} \approx 1/\ln(\epsilon_F/T)$$

Resolution of the problems a) and b)

We resolve the problem of a) **the double counting** and b) **logarithmic corrections** by calculating the term with two rescattering sections within the Cooper channel ladder where the logarithmic renormalizations originate.

We **give** the two-section term to the Cooper channel ladder where it **gets killed** :

$$\delta\chi' = \delta\chi_{\mathrm{e-h}} - \delta\chi_{(2)}$$

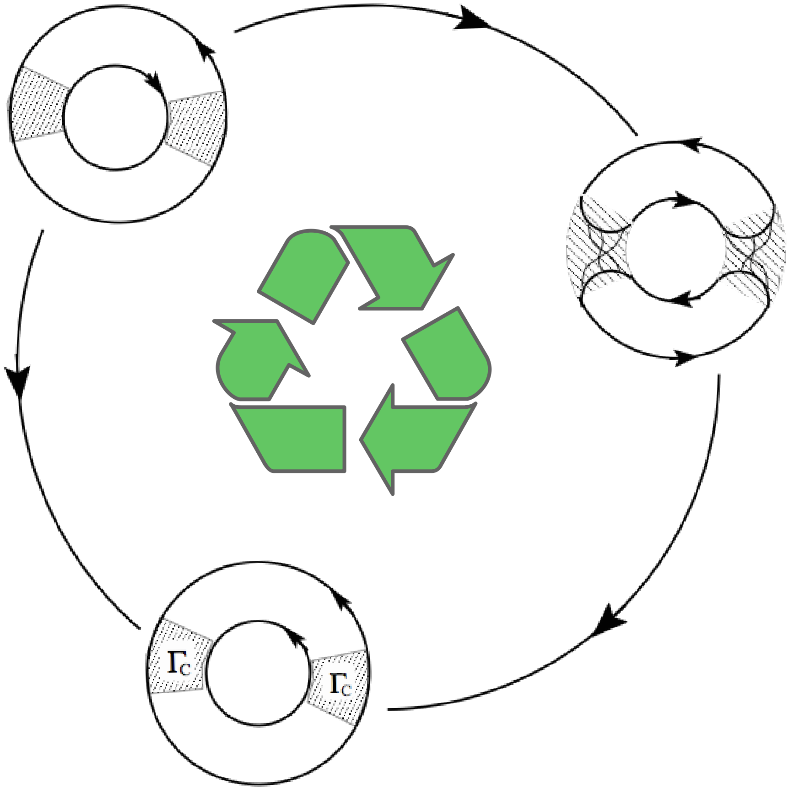


$$\delta\chi = -2\nu\frac{T}{\epsilon_F}\left(\gamma^2/2 + \ln\frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma}\right)$$

resolution of the sign problem

Resolution of the *sign*-problem and avoiding the double counting

The two-section term is sent to and counted in the Cooper channel ladder where it **gets killed (!)**



There is no overlap for **three-section** term

$$\delta\chi_{(3)} = (T/\epsilon_F)\nu \int \alpha(\theta_1\theta_2\theta_3)\Gamma(\theta_1-\theta_2)\Gamma(\theta_2-\theta_3)\Gamma(\theta_3-\theta_1)d\theta_1 d\theta_2 d\theta_3$$

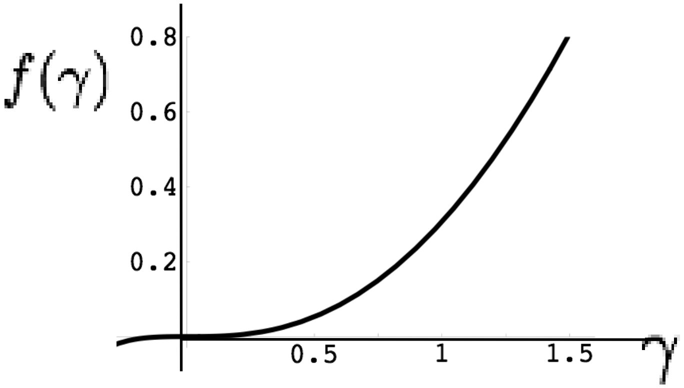
backscattering is inessential

χ decreases with temperature

$$\gamma = \Gamma_0$$

$$\delta\chi = -2\nu \frac{T}{\epsilon_F}\left(\gamma^2/2 + \ln\frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma}\right)$$

$$f(\gamma) = \frac{\gamma^2}{2} + \ln\frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma}$$



$$f(\gamma) \sim 0.3 \quad for \quad \gamma \sim 1$$

$$f(\gamma) \sim 0.7 \quad for \quad \gamma \sim 1.5$$

Previous works:

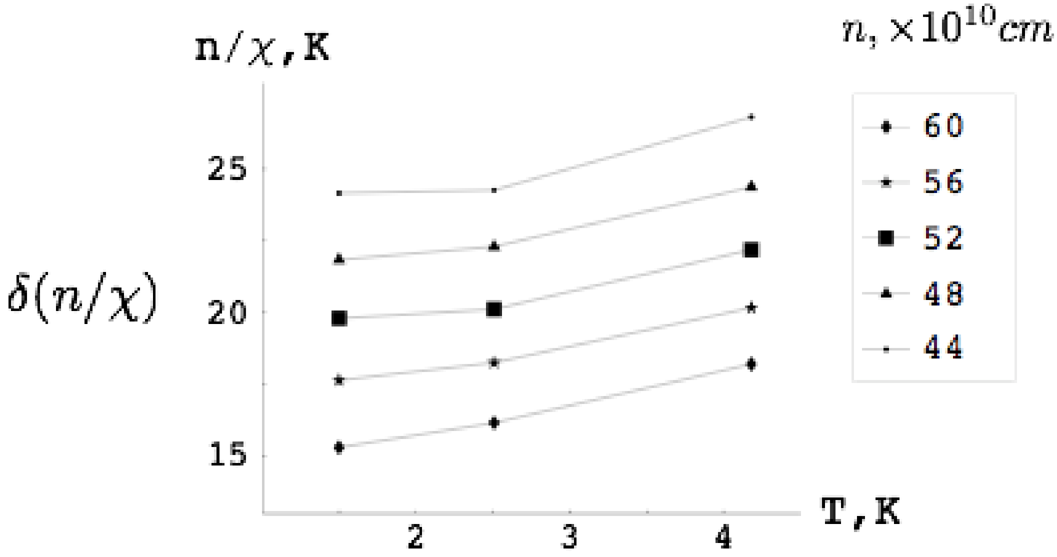
χ **increases** with temperature

$\mathbf{n} \neq 0$ $\quad \gamma = \Gamma_{\mathbf{n}} = \Gamma_{-\mathbf{n}}$

$$\delta\chi_{\mathbf{n}\neq\mathbf{0}} = -2\nu\,(T/\varepsilon_F)[f(\Gamma_{\mathbf{n}}) + f(\Gamma_{-\mathbf{n}}) + \phi_{\mathbf{n}}]$$

discussion of the experiment: Prus et al. Phys Rev B (2003)

$$\partial(n/\chi)/\partial T ; 1$$



$$r_s = 3 \div 4$$

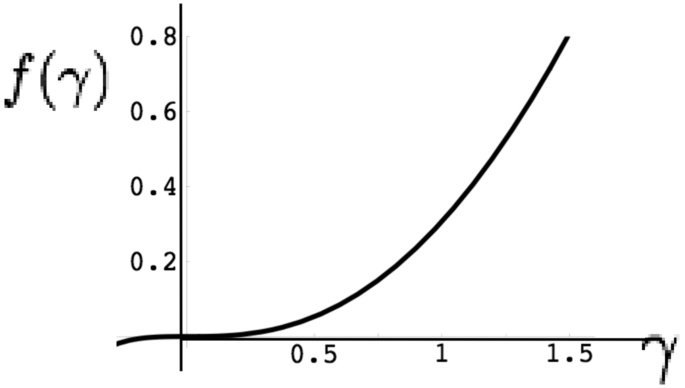
$$\Gamma_{\mathbf{n}} \sim 1$$

$$g\mu_B/2 = 1$$

$$\gamma = \Gamma_{\mathbf{n}} = \Gamma_{-\mathbf{n}}$$

$$\delta\,(n/\chi_{\mathbf{n}\neq 0}) = T[f(\Gamma_{\mathbf{n}}) + f(\Gamma_{-\mathbf{n}}) + \phi_{\mathbf{n}}]$$

$$f(\gamma) = \frac{\gamma^2}{2} + \ln\frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma}$$



$$f(\gamma = 1) \sim 0.3$$

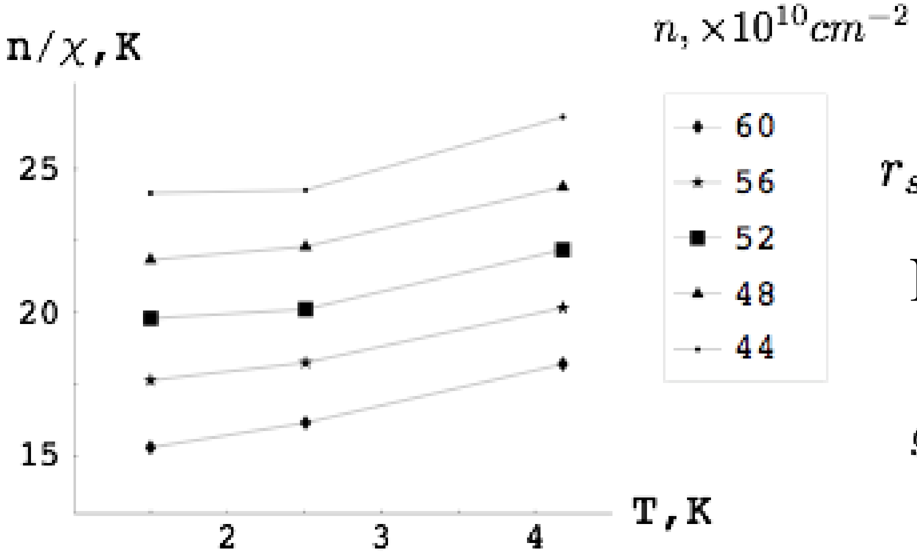
$$\phi_{\mathbf{n}=1}(\gamma = 1) \sim 0.24$$

Few harmonics may be involved

discussion of the experiment: disorder

$\partial(n/\chi)/\partial T\ ;\ 1$

$\delta(n/\chi)$



$r_s = 3 \div 4$

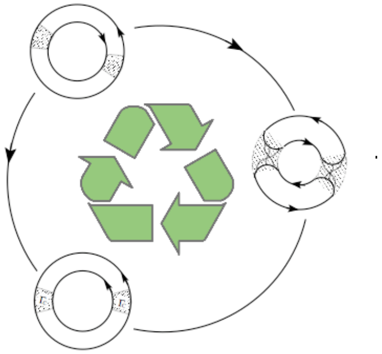
$\Gamma_n \sim 1$

$g\mu_B/2 = 1$

Disorder kills the non-analytic contributions:
a) smears edge singularities
b) harmonics higher than n=0 become ineffective

$1/\tau\ :\ 2K$

Conclusion: abstract, Proc. Natl. Acad. Sci. U.S.A. 103 (2006)



★The recently measured spin susceptibility of the two dimensional electron gas exhibits a strong dependence on temperature, which is incompatible with the standard Fermi liquid phenomenology. Here we show that the observed temperature behavior is inherent to ballistic two dimensional electrons.

★Besides the single-particle and collective excitations, the thermodynamics of Fermi liquid systems includes effects of the branch-cut singularities originating from the edges of the continuum of pairs of quasiparticles. As a result of the rescattering induced by interactions, the branch-cut singularities generate non-analyticities in the thermodynamic potential which reveal themselves in anomalous temperature dependences.

★Calculation of the spin susceptibility in such a situation requires a non-perturbative treatment of the interactions. As in high-energy physics, a "cocktail" of the collective excitations and pairs of quasi-particles can be effectively described by a pole in the complex momentum plane.

★This analysis provides a natural explanation for the observed temperature dependence of the spin susceptibility, both in sign and magnitude.

★The sign-problem which has been analyzed in this work may have consequences for the physics of the quantum critical point near the ferromagnetic instability

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