Instanton and Superconductivity in Supersymmetric CP^{N-1} Model

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memory of Larkin

- Topological excitations (Pokrovskii talk)

  - instanton discussion in the first visit(1979)

- ”hic sunt leones”

  (Theory of fluctuation in superconductor by Varlamov-Larkin, P. 465 russian edition)

  - discussions of magnetoresistance in high Tc, τ_φ(1988)

Fluctuation of superconductor;

Ginzburg-Landau Hamiltonian (N=1)

$$H_{GL} = \frac{1}{2}|\partial_\mu \phi_i + ieA_\mu \phi_i|^2 + g|\phi_i|^4 + \frac{1}{4}F_{\mu\nu}^2$$

(i=1,...,N) renormalization group does not give a stable fixed point at $d = 4 - \epsilon$ except for large N (first order transition).

For d=3, another Hamiltonian, CP^{N-1} model,

$$H_{CP^{N-1}} = \frac{1}{2t}(\partial_\mu \bar{Z}_i \partial_\mu Z_i + (\bar{Z}_i \partial_\mu Z_i)^2)$$

$(\bar{Z}_i Z_i = 1)$

(renormalization group works for $2 < d$)

If there is a fixed point, then the bare couplings become infinite, $g \to \infty$, $e \to \infty$,

$$H_{GL} \to H_{CP^{N-1}}$$

$$\beta(t) = \epsilon t - N t^2 + \cdots$$

($\epsilon = d - 2$) At d=3, a nontrivial critical exponent is obtained for N=1 (second order transition for a superconductor). (For N=2, it is same as classical Heisenberg model)

This gauge invariant CP^{N-1} nonlinear σ model is a proto-type of the Anderson localization problem. (1979)

Grassmanian manifold;

$$\frac{U(N)}{U(N-1)\times U(1)} \to \frac{U(N)}{U(N-p)\times U(p)}$$

$$CP^{N-1} = \frac{U(N)}{U(N-1)\times U(1)}$$

Vaks-Larkin four-Fermi interation model

mass generation; mass = superconductor order parameter

gap equation

Renormalization group β function is asymptotic free at d=2. There is no long range order in d=2 (Mermin-Wagner theorem). The average of the generated mass (superconductor gap Δ) has to be vanishing; due to kink excitation.

supersymmetric generalization of these two models

$CP^{N-1} \to$ supersymmetric CP^{N-1} model

$$S = \int d^2\sigma d^2\theta \sqrt{g} g^{\mu\nu} G_{\alpha\beta}(X) D_\mu X^\alpha D_\nu X^\beta$$

$$D = \partial_\theta + \theta\partial_z, \;\; \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$$

$$\beta_{ij} = \epsilon t - R_{ij} t^2 + \frac{1}{2}\zeta(3)(a_3 - a_6)t^5 + \cdots$$

(1) The two dimensional supersymmetric CP(1) model

(2) Boundary of the strong coupling and the weak coupling regions: a curve of marginal stability (CMS)

(3) Pseudo gap phase in the high T_c superconductor

(4) Charged deformed instanton

(5) Vortex core state

(6) Equivalent one dimensional spin model

The supersymmetric O(3)/O(2) (=CP^1) non-linear σ model

$$\mathcal{L} = \frac{1}{g}\int d^2x[\frac{1}{2}(\partial_\mu n)^2 + \frac{1}{2}\bar{\psi} i\gamma_\mu\partial_\mu\psi + \frac{1}{8}(\bar{\psi}^a\psi^a)^2] \quad (1)$$

(a=1,2,3)

$$n^2 = 1, \quad n\cdot\psi = 0 \qquad (2)$$

Fermionic part: Vaks-Larkin, Gross-Neveu model

β-function shows asymptotic free behavior ($\beta > 0$) in one loop order, and no higher order corrections

generating mass $\sim$ the gap of BCS theory; $\Delta = e^{-c/g}$

There is no long range order in d=2

supersymmetric CP^{N-1} model

$$\mathcal{L} = \int d^4\theta K(\Phi, \Phi^\dagger)$$
$$= G_{ij}(\partial^\mu \phi^{\dagger j}\partial_\mu \phi^i + i\bar{\psi}^j \gamma^\mu D_\mu \psi^i) - \frac{1}{2} R_{i\bar{j}k\bar{l}} \bar{\psi}^{\bar{j}} \psi^i \bar{\psi}^{\bar{l}} \psi^k \tag{3}$$

$$\begin{aligned}\mathcal{L}(CP^1) &= G(\partial_\mu \phi^\dagger \partial^\mu \phi - |m|^2 \phi^\dagger \phi + i\bar{\psi}\gamma^\mu \partial_\mu \psi \\ &-\frac{1-\phi^\dagger\phi}{1+\phi^\dagger\phi}\bar{\psi}M\psi \\ - &\frac{2i}{1+\phi^\dagger\phi}\phi^\dagger \partial_\mu \phi \bar{\psi}\gamma^\mu \psi + \frac{1}{(1+\phi^\dagger\phi)^2}(\bar{\psi}\psi)^2)\end{aligned} \tag{4}$$

with $M = m(1+\gamma_5)/2 + \bar{m}(1-\gamma_5)/2$, $\psi^T = (\psi_R, \psi_L)$. $G = 2/g(1+\phi^\dagger\phi)^2$

or equivalently,

$$\begin{aligned}
\mathcal{L}(CP^1) &= G(\partial_\mu \phi^\dagger \partial^\mu \phi - |m|^2 \phi^\dagger \phi \\
&+ \frac{i}{2}(\psi_L^\dagger \partial_R \psi_L + \psi_R^\dagger \partial_L \psi_R) \\
- \quad & i\frac{1-\phi^\dagger\phi}{1+\phi^\dagger\phi}(m\psi_L^\dagger \psi_R + \bar{m}\psi_R^\dagger \psi_L) \\
- \quad & \frac{2}{(1+\phi^\dagger\phi)^2}\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R \\
- \quad & \frac{i}{1+\phi^\dagger\phi}[\psi_L^\dagger \psi_L(\phi^\dagger \bar{\partial}_R \phi) + \psi_R^\dagger \psi_R(\phi^\dagger \bar{\partial}_L \phi)]) \quad (5)
\end{aligned}$$

with $\partial_L = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}$, $\partial_R = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}$.

Twisted mass m

Superfield has a mass term m (Higgs mass). The presence of this mass term does not break the $\mathcal{N} = 2$ supersymmetry. This mass term gives the easy axis anisotropy (Ising like).

bosonic part

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{n} \partial^\mu \vec{n} - \frac{1}{2} m^2 (\xi - n_3^2) \tag{6}$$

with $\xi = 1/g$.

Time dependent dyon solution

$$\phi_{cl} = e^{\sqrt{|m|^2 - \omega} x + i\omega t}$$

$$\phi = \tan(\frac{\varphi}{2})e^{i\alpha}$$

$$n = (\sin\varphi\cos\alpha, \sin\varphi\sin\alpha, -\cos\varphi)$$

$$\phi = \frac{n_1 + in_2}{1 - n_3}$$

Charge Q

instanton with charge Q: (ϕ is time dependent)

$$Q = i\xi \int d^2x \frac{\bar{\phi}\dot{\phi} - \phi\dot{\bar{\phi}}}{(1+\bar{\phi}\phi)^2} \tag{7}$$

instanton number n:

$$n = -\frac{i}{4\pi}\int d^2x \epsilon_{ij} \frac{\partial_i\bar{\phi}\partial_j\phi - \partial_j\bar{\phi}\partial_i\phi}{(1+\bar{\phi}\phi)^2} \tag{8}$$

tension T in BPS form

$$T = \xi \int d^2x [|\frac{\partial_t\phi \pm im\phi}{1+\bar{\phi}\phi}|^2 + \frac{1}{2}\ \frac{\partial_i\phi \pm i\epsilon_{ij}\partial_j\phi}{1+\bar{\phi}\phi}|^2 + 2\pi\xi n + mQ \qquad (9)$$

(The first two terms have to vanish.)

$$\phi(z,t) = (\sum_{i=1}^{n} \frac{c_i}{z-z_i})e^{imt} \qquad (10)$$

In the four dimensional SQCD, the massive excitations of monopole, dyon, and Noether charges are evaluated by the elliptic integrals. (Seiberg-Witten)

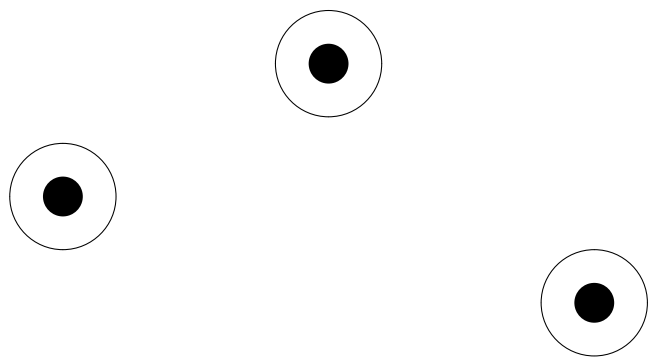
In the two dimensional $CP(1)$ case, the electric charge Q and topological charge T, are combined with the (twisted) mass m and the dual mass m_D to form the central charge $Z_{Q,T}$

$$Z_{Q,T} = mQ + m_D T \tag{11}$$

(Q,T) is pseudo particle (charged deformed instanton). Q,T are integers

Figure of the magnetic flux and the charge

—– bamboo structure —–



The effective twisted superpotential $\tilde{W}$: the twisted chiral superfield Σ: $\partial \tilde{W}/\partial \Sigma = 0$

$$\prod_{i=1}^{N}(\sigma + m_i) - \tilde{\Lambda}^N = \prod_{i=1}^{N}(\sigma - e_i) = 0. \tag{12}$$

$$\begin{aligned} Z_{12} &= 2[\tilde{W}(e_1) - \tilde{W}(e_2)] \\ &= \frac{1}{2\pi}[N(e_1 - e_2) - \sum_{i=1}^{2} m_i \ln(\frac{e_1 + m_i}{e_2 + m_i})]. \end{aligned} \tag{13}$$

$m_1 = -m_2 = -m/2$, $\sigma^2 - \frac{m^2}{4} = \tilde{\Lambda}^2$. m_D (dual mass)

$$m_D = \frac{i}{\pi}[\sqrt{m^2 + 4\tilde{\Lambda}^2} + \frac{m}{2}\ln(\frac{m - \sqrt{m^2 + 4\tilde{\Lambda}^2}}{m + \sqrt{m^2 + 4\tilde{\Lambda}^2}})] \tag{14}$$

θ term the vacuum angle θ: the coupling constant g:

$$\tau = -\frac{i}{g} + \frac{\theta}{2\pi}, \tag{15}$$

$\tilde{\Lambda}$, modified by the θ value,

$$\frac{\tilde{\Lambda}}{m} = \frac{1}{2} e^{-1+i\pi\tau} = \frac{1}{2} e^{-1+\frac{\pi}{g}+\frac{i}{2}\theta}. \tag{16}$$

a critical point at $g = \theta = \pi$, $4(\frac{\tilde{\Lambda}}{m})^2 = -1$. The twisted chiral superfield :

$$\Sigma = \sigma + \sqrt{2}\vartheta^{\alpha}\tilde{\chi}_{\alpha} + \vartheta^{\alpha}\vartheta_{\alpha} S. \tag{17}$$

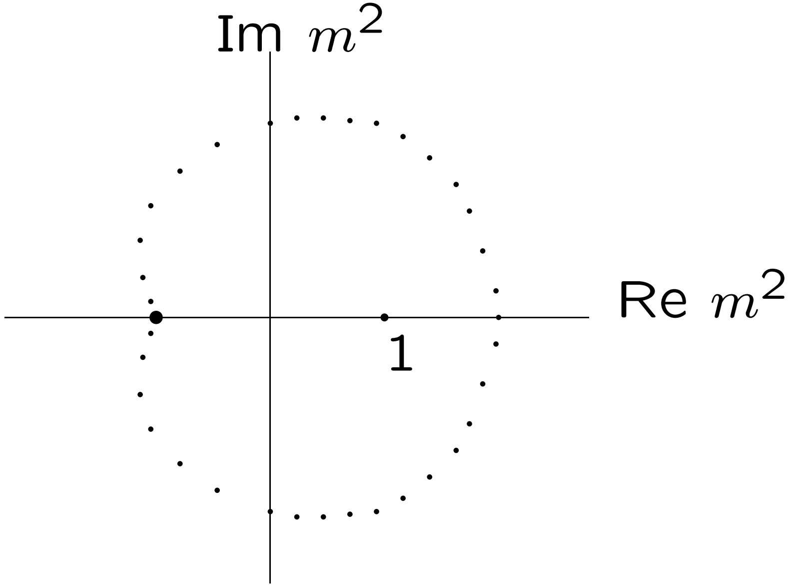
Curve of Marginal Stability (CMS)

In the strong coupling region $|m| \ll \Lambda$, BPS sates becomes only two states $(Q = 0, T = 1)$ and $(Q = 1, T = 1)$ $(m \simeq 0)$. In the weak coupling region $|m| \gg \tilde{\Lambda}$, these two states are bounded, and other (Q, T) BPS states appear, and whole instanton contributions appear.

There is a boundary, called as the curve of the marginal stability (CMS) in the complex mass parameter m^2, where the restructuring of the BPS states occurs. The curve of marginal stability is expressed as the following equation (Dorey, Shifman)

$$\mathrm{Re}[\ln \frac{1+\sqrt{1+4\tilde{\Lambda}^2/m^2}}{1-\sqrt{1+4\tilde{\Lambda}^2/m^2}} - 2\sqrt{1+4\tilde{\Lambda}^2/m^2}] = 0. \tag{18}$$

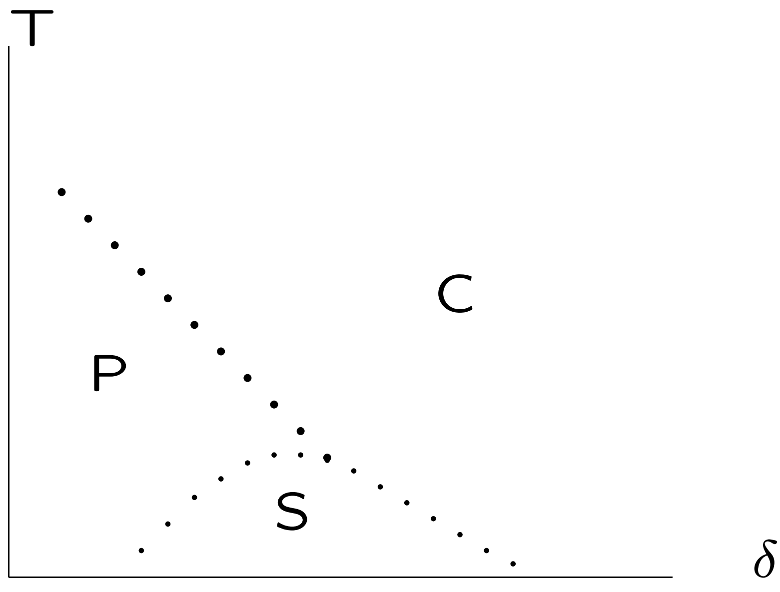
a singular point $4\frac{\tilde{\Lambda}^2}{m^2} = -1$, the value of m_D in (14) becomes vanishing. This critical point is realized for $g = \theta = \pi$.



Curve of marginal stability (CMS)

$(4\Lambda^2 \to 1)$

phase diagram of high T_c superconductor



Pseudo gap phase

PGP = Soliton (monopole) + Dyon (q,T) (weak coupling phase)

Coulomb Phase = Soliton, Dyon [(0,1), (1,1)] $m \to 0$ (strong coupling phase)

Ising like easy axis anisotropy

Correlation function is described by Painleve III, same as 2D Ising model for the spin-spin crrelation function (T.T. Wu, Cecotti-Vafa)

$\frac{\Lambda}{m} \gg 1$

Weak coupling phase and strong coupling phase is separated by a curve of marginal stability (CMS).

In two dimension, there is no long-range order similar to the disorder phase of 2D Ising model.

kink walls give strip structures or bamboo structure.

Vortex core excitation

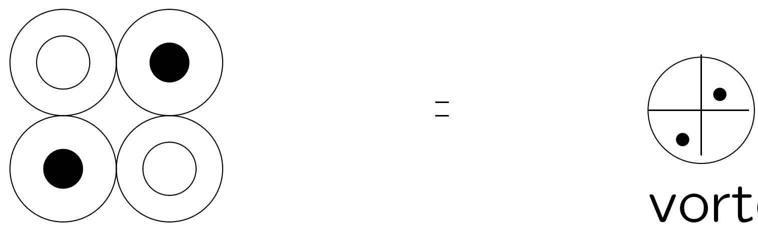
Inside the vortex, the excitation of the pseudo-gap phase can be seen in the underdoped region.

Charged deformed instanton (dyon, (Q,T))

recent STM observations of the antisymmetrical charging

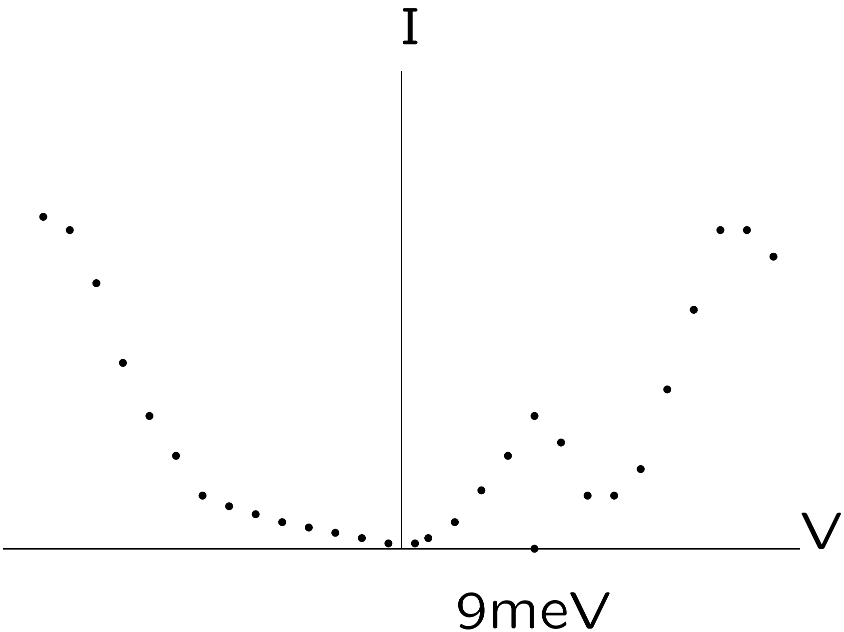
Figure of the vortex structure

—- Cross cap (Satsuma) structure —-



vortex core
cross cap

• : charge



STM, Nishida et al (2007)

Equivalent one dimensional spin model

The Hamiltonian $\mathcal{H}$ of n-vector model is

$$\mathcal{H} = -J \sum_{i=1}^{M-1} \vec{S}_i \cdot \vec{S}_{i+1}, \tag{19}$$

with a condition,

$$|\vec{S}_i|^2 = \sum_{m=1}^{n} S_i^2(m) = n. \tag{20}$$

$$Z = [(\frac{nJ}{2kT})^{1-\frac{n}{2}} \Gamma(\frac{n}{2}) I_{\frac{n}{2}-1}(\frac{nJ}{kT})]^{M-1} \tag{21}$$

$I_\nu(z)$: modified Bessel function. $\nu = \frac{n}{2}$ and $Y = \frac{2J}{kT}$

$$I_\nu(\nu Y) = \frac{1}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}(\frac{\nu Y}{2})^\nu 4^\nu e^{-\nu Y} \int_0^\infty e^{-F(t)} dt, \tag{22}$$

with

$$F(t) = (\nu + \frac{1}{2})t - (\nu - \frac{1}{2})\ln(1 - e^{-t}) - 2\nu Y e^{-t}. \tag{23}$$

$\nu \to \infty$ $\partial F(t)/\partial t = 0$ (saddle point equation) The two saddle points t_+ and t_-

$$e^{-t_\pm} = \frac{Y - 1 \pm \sqrt{1 + Y^2}}{2Y}. \tag{24}$$

$$F(t_\pm) = \nu \left(1 - Y \mp \sqrt{1+Y^2} - \ln \frac{-1 \pm \sqrt{1+Y^2}}{2Y^2} \right). \tag{25}$$

The difference becomes

$$F(t_-) - F(t_+) = 2\nu \left(\frac{1}{2} \ln \frac{1-\sqrt{1+Y^2}}{1+\sqrt{1+Y^2}} + \sqrt{1+Y^2} \right). \tag{26}$$

This give the dominant contribution to the large order behavior;instanton contribution (Hikami-Brezin 1979)

In the $\frac{1}{n}$ expansion, we obtain

$$I_\nu(\nu Y) = \frac{1}{2\pi\nu}\frac{e^{\nu\eta}}{(1+Y^2)^{\frac{1}{4}}}(1+\sum_{k=1}^{\infty}\frac{u_k(t)}{\nu^k}), \qquad (27)$$

where we obtain η, by collecting the coefficients in (22),

$$\eta = \sqrt{1+Y^2} + \frac{1}{2}\ln\frac{\sqrt{1+Y^2}-1}{\sqrt{1+Y^2}+1}. \qquad (28)$$

The parameter t is $(1+Y^2)^{-1/2}$. It is an asymptotic expansion, which glows like $(k-1)!/[F(t_+) - F(t_-)]^k$.

If we identify $\frac{J}{kT} = \frac{\tilde{\Lambda}}{m}$, $(Y^2 = \frac{4\tilde{\Lambda}^2}{m^2})$, we find the expression for m_D.

This one dimensional n-vector model is solved by the transfer matrix method. The transfer matrix is

$$T = e^{\frac{nJ}{kT}\vec{S}\cdot\vec{S}'}. \tag{29}$$

The eigenvalues are given by

$$T_l = C I_{\frac{n}{2}-1+l}(\frac{nJ}{kT}), \tag{30}$$

where $l = 0, 1, 2, ...$, and C is a constant. The spin-spin correlation function for the distance r is given by

$$< \vec{S}(0) \cdot \vec{S}(r) >= \left(\frac{T_1}{T_0}\right)^r = \left(\frac{I_\nu(\nu Y)}{I_{\nu-1}(\nu Y)}\right)^r, \tag{31}$$

with $\nu = \frac{n}{2}$.

When Y is a real, a correlation length ξ is finite. $I_\nu(\nu Y) < I_\nu(\nu Y)$; is no phase transition at finite temperature.

$$\xi^{-1} = \ln(\frac{I_{\nu-1}(\nu Y)}{I_\nu(\nu Y)}). \qquad (32)$$

When Y is a pure imaginary number $Y = -i|Y|$, there appears a phase transition with the infinite correlation length by the degeneracy of the eigenvalues of the transfer matrix.

When Y is imaginary, the modified Bessel function is expressed by the Bessel function J as $I_\nu(\nu Y) =$

$e^{-\nu\pi i/2} J_\nu(\nu|Y|)$. We find in the large ν limit, there appears a crossing at $Y = \pm i$,

$$J_\nu(\nu|Y|) = J_{\nu-1}(\nu|Y|). \tag{33}$$

The point $Y = \pm i$ is the critical point, which corresponds to $4\frac{\tilde{\Lambda}^2}{m^2} = -1$. Since the Bessel function $J_\nu(z)$ is an oscillating function, there appear successive degeneracies for ν ($\nu = \frac{n}{2} - 1 + l$).

The succesive transitions due to the degeneracy of the angular quantum numbers $l = 0, 1, 2, ...$, which represent s,p,d,f,....,states.

Such successive transitions give a cut in the large n limit beyond $|Y| > 1$, which is a low temperature phase. The mass of the inverse of the correlation length ξ, which is finite in the high temperature region, becomes zero below the transition temperature T_c for $\theta = \pi$.

Note: such phase transitions also appear for the one dimensional n-vector model, with a real positive $\frac{J}{kt}$, and for $n < 1$, (Balian and Toulose).

Ising-like correlation function (Painlevé III equation)

P II $\rightarrow$ Airy, Gross-Witten model; $u'' = 2u^3 + xu$

P III $\rightarrow$ chiral random matrix model (Bessel kernel);

$$u'' = \frac{1}{u}(u')^2 - \frac{1}{x}u' + \frac{1}{x}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}$$

Cecotti-Vafa tt^* equation = Painlevé III equation = T.T. Wu Ising solution; quantum cohomology

$$\partial_z \partial_{\overline{z}} q_i + e^{(q_{i+1} - q_i)} - e^{(q_i - q_{i-1})} = 0$$

$$< \bar{x} | x >^{-1} = \sum_{n=0}^{\infty} |\beta|^{2n} P_n$$

P_n: polynomial of $(-log|\beta|-2\gamma)$ with 2n+1 degree.

Ising model $t = \frac{R}{\xi}, R = \sqrt{M^2 + N^2} \rightarrow PIII$.

Random matrix model (Painlevé III)

$$M = \begin{pmatrix} 0 & C^\dagger \\ C & 0 \end{pmatrix}$$

Bessel kernel:

$$K(z_1, z_2) \;=\; N \int_{-\infty}^{\infty} \frac{dt}{2\pi} \oint \frac{du}{2\pi i} \frac{1}{u - it} e^{\frac{i}{t} + \frac{1}{u} - iN^2 t z_1 + N^2 u z_2 + N(z_1 - z_2)} \tag{34}$$

Fredholm determinant; probability of level spacing; H(q,p);

$$\psi'' + \frac{1}{x}\psi' = \frac{1}{2}\sin(2\psi)$$

$$q(s) = \cos(\psi(s)), x^2 = s$$

$$(\psi \to i\psi) \to \text{Ising case}$$

Reduction to O(n) vector model

supersymmetric $CP^1 \rightarrow$ O(n) vector model ($n \rightarrow \infty$)

AdS_2 (2d Liouville theory) $\rightarrow$ 2d supersymmetric QCD :(gauge/string correspondence)

Polyakov (conjecture for higher spin operators)

$AdS_{d+1} \rightarrow$ d-dimensional O(n) vector model ($n \rightarrow \infty$)

Conclusion

Supersymmetric CP^1 model in d=2 with a twisted mass can be solved, and can be applied to the superconducting fluctuation phase.

Interesting topological excitations, instantons with a charge, solitons and dyons are obtained. (Exact CMS)

The correlation function is described by 2d Ising model.

Applications may be for high temperature superconductor in the pseudo gap phase, or quantum Hall effect with chiral symmetry ($\mathcal{N} = 2$ supersymmetry).