Laplacian Growth:

Random Matrices and singularities of growing patterns

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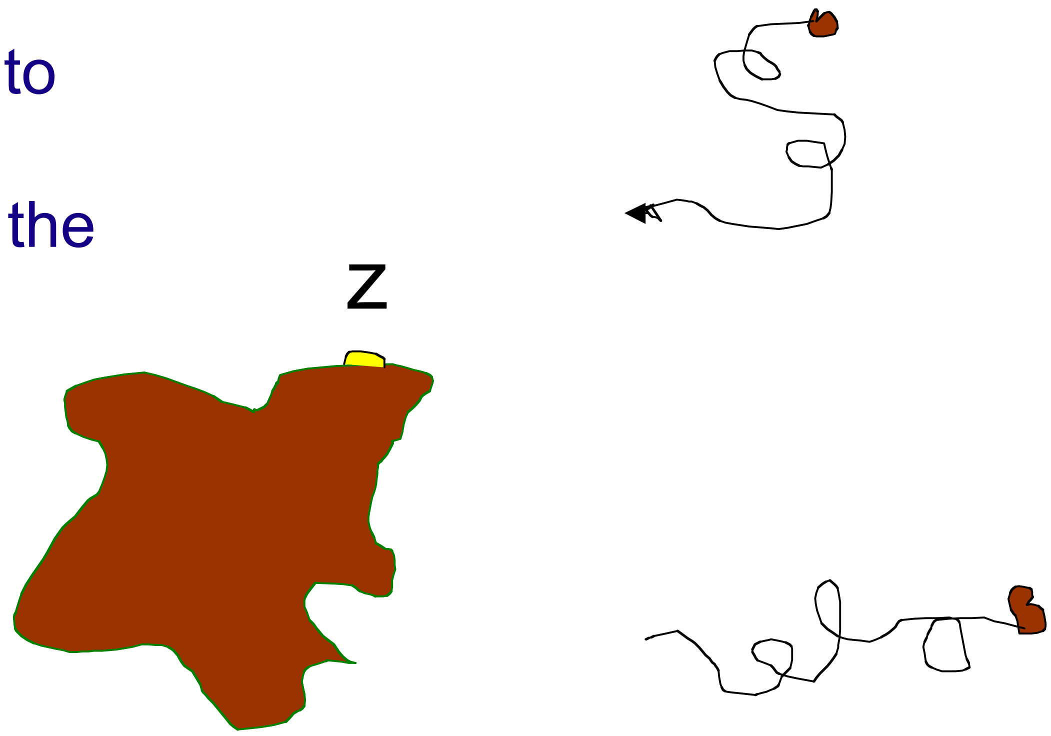
Laplacian growth -

velocity of a moving planar interface =
= a **gradient of a harmonic field**

Laplacian growth - **velocity** of moving planar interface is a **gradient of a harmonic field**

A probability of a Brownian mover to arrive and join the aggregate at a point z is a *harmonic measure* of the domain



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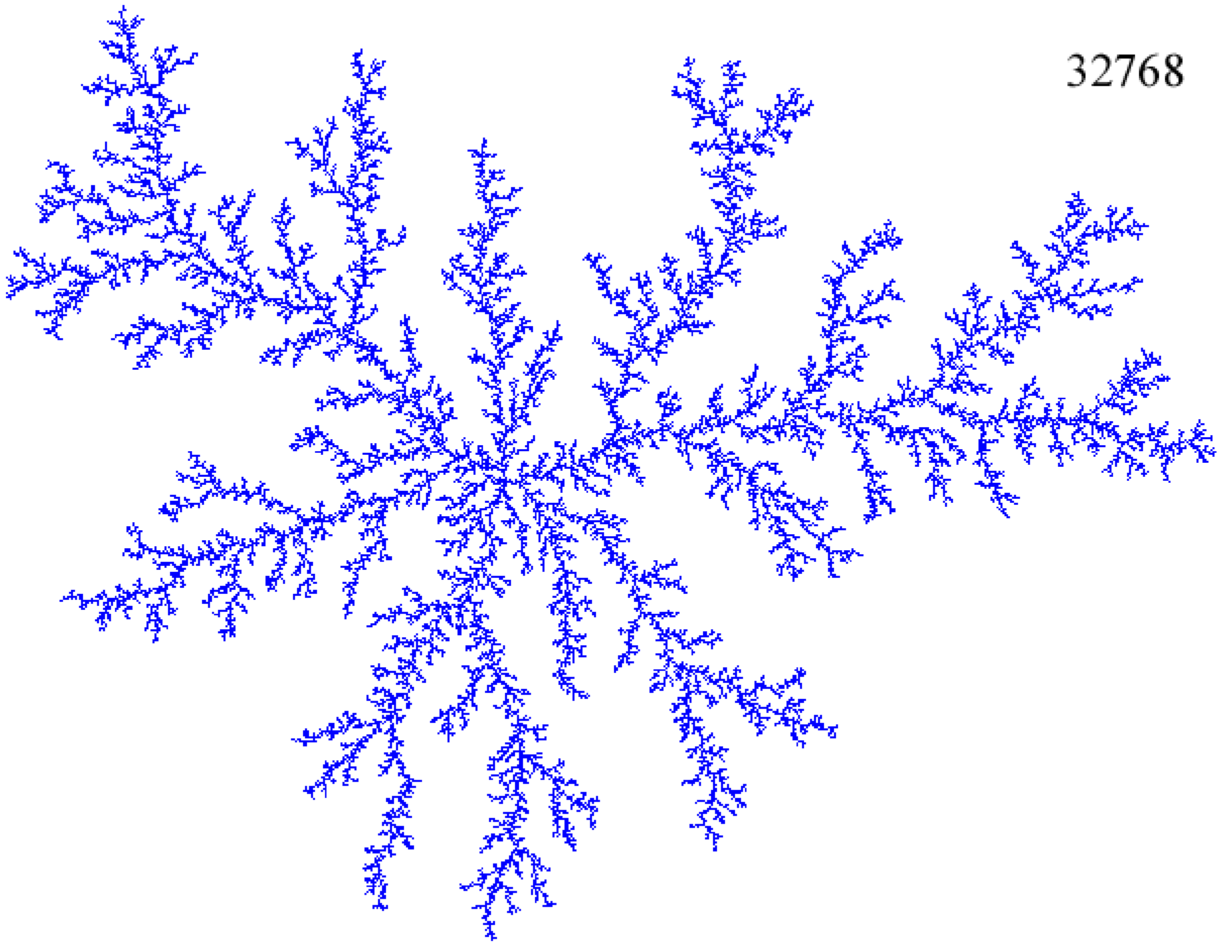


<u>Stochastic Geometry</u>:
statistical ensemble of
fractal shapes

Diffusion-Limited Aggregation, or DLA,

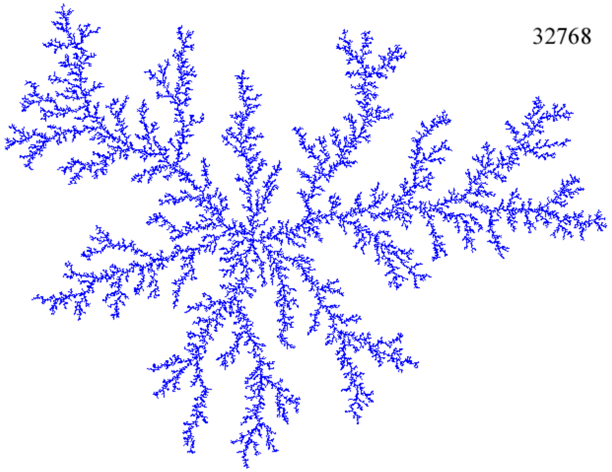
is an extraordinarily simple computer simulation of the formation of clusters by particles diffusing through a medium that jostles the particles as they move.

Hastings&Levitov

S.-Y. Lee, R. Teodoerescu, P. W.

E. Bettelheim, I. Krichever, A. Zabrodin, O. Agam

- Random matrix theory; ✓
- Topological Field Theory;
- Quantum Gravity;
- Non-linear waves and soliton theory;
- Whitham universal hierarchies;
- Integrable hierarchies and Painleve transcendants
- Isomonodromic deformation theory;
- Asymptotes of orthogonal polynomials ✓
- Non-Abelian Riemann Hilbert problem;
- Stochastic Loewner Evolution (anticipated)



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Hypothesis:

The pattern is related to asymptotes of distribution of
zeros of real Orthogonal Polynomials.

Orthogonal polynomials

$$\psi_n(z) = P_n(z)e^{-\frac{1}{2\hbar}V(z)}$$

Szego theorem:

$$\int \psi_n(z)\psi_m(z)dz = \delta_{mn}$$

If V(x) real (real orthogonal polynomials)

➠

1) zeros of are distributed along a real axis

Asymptotes: [redacted]

2) Zeros form dense segments of the real axis,

➠

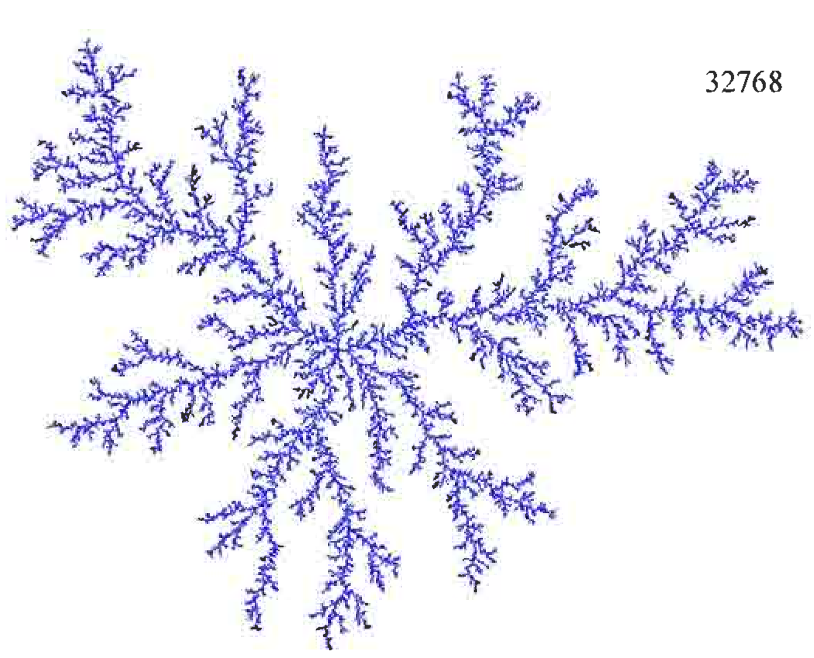
3) Asymptotes at the edges is of Airy type

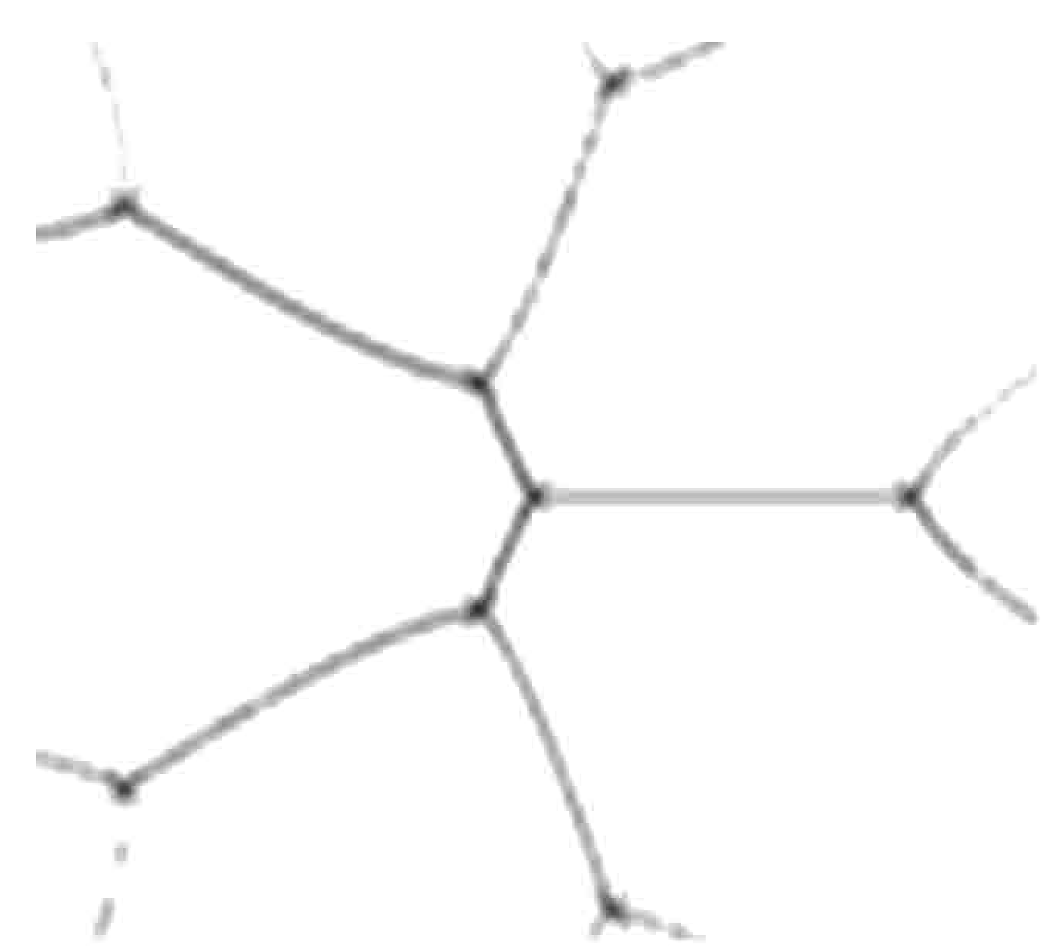
Bi-Orthogonal polynomials

$$\psi_n(z) = P_n(z) e^{-\frac{1}{2\hbar} V(z)}$$

$$\int \psi_n \overline{\psi}_m e^{-\frac{1}{2\hbar}|z|^2} dz d\bar{z} = \delta_{mn}$$

- Asymptotes: $\hbar \to 0,\ N \to \infty,\ \hbar N = \text{time} = fixed$
- Zeros are distributed along a branching graph
- Asymptotes at the edges are Painleve transcendants





Asymptotes of Orthogonal Polynomials:

Inspired by physics literature on Random Matrix Theory

Harnad-Bertola-Eynard,

Deift-Zhou-McLaughlin-Kriecherbauer-Venakides-Its, and many others

its relation with Laplacian growth:

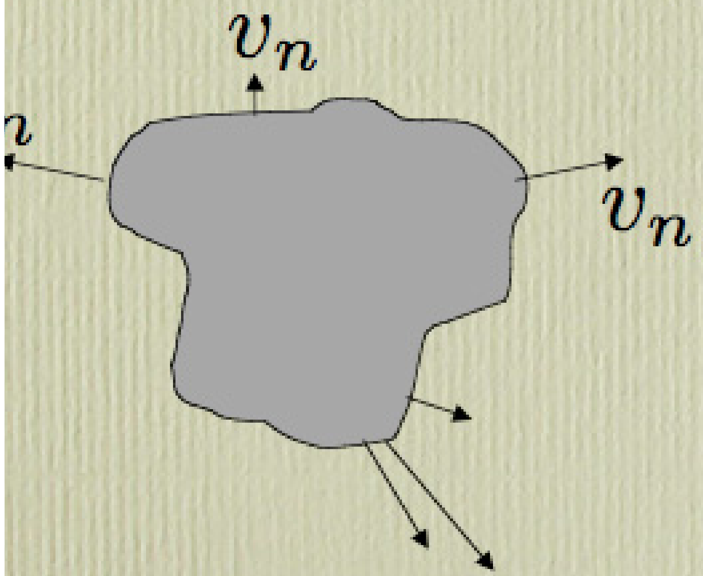
Lee-Teodorescu-Zabrodin-Bettelheim-Agam-P.W.

Continuous problem: a size of particles

Continuum version

Laplacian growth is an evolution of a plane interface between two immiscible phases, driven by a gradient of a harmonic field

continuous problem



$$\vec{v}_n = \vec{\nabla} p$$

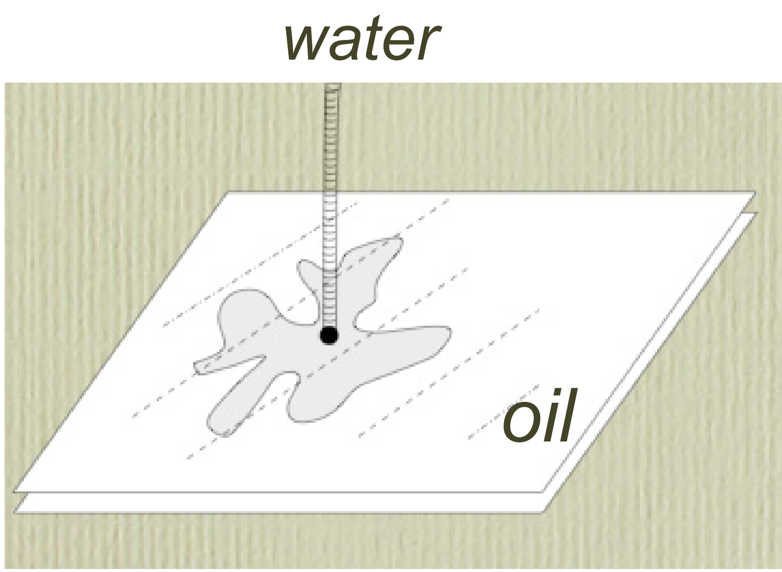
$$\Delta p = 0$$

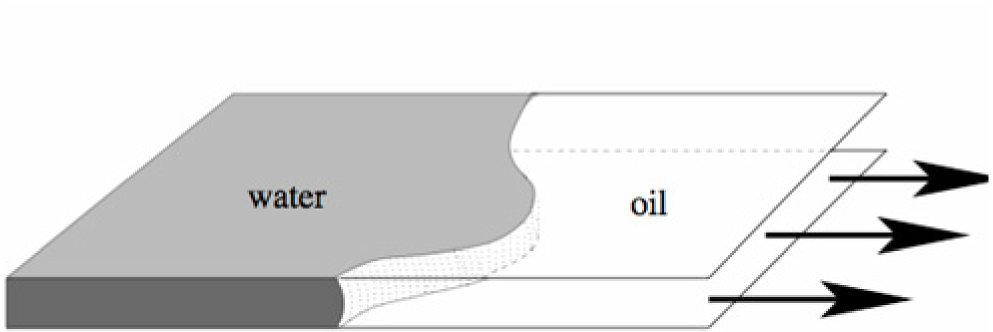
$$p \to \log|z|, \quad |z| \to \infty$$

$$p = 0 \quad \text{on the boundary}$$

normal velocity = gradient of harmonic field

Hele-Shaw cell (1894)

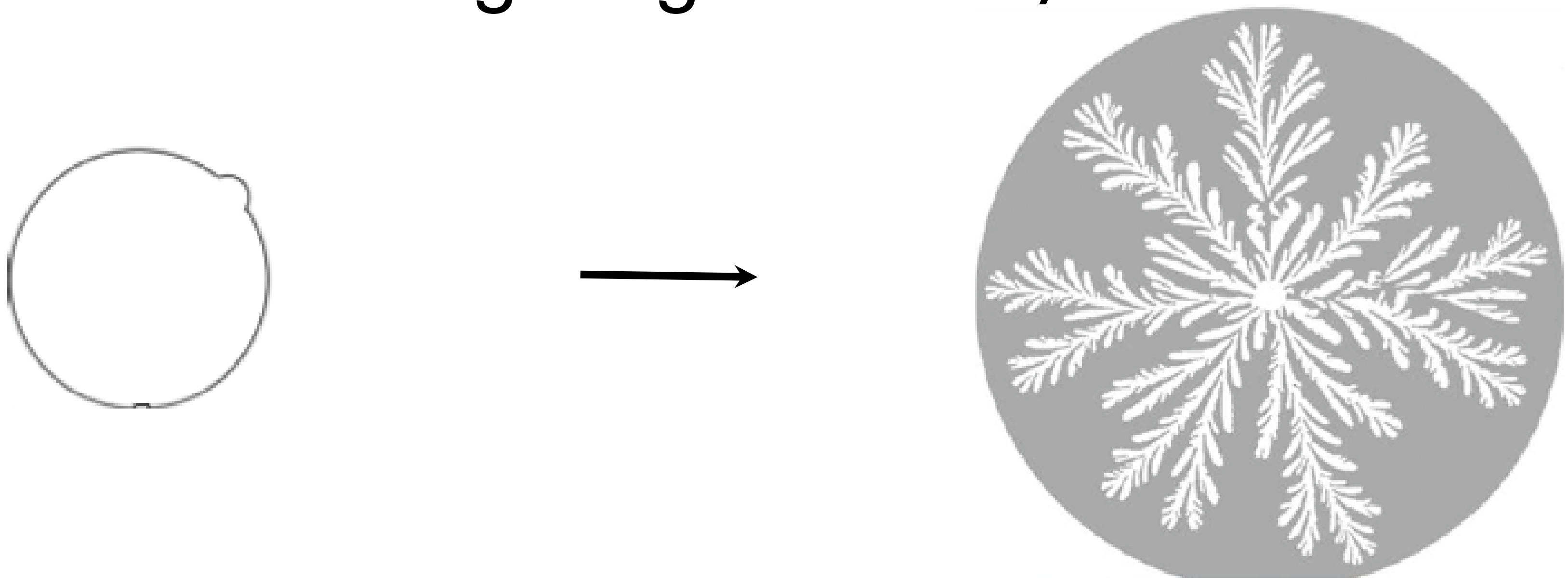




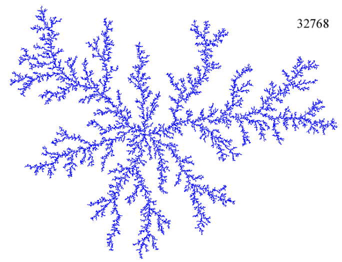
Oil (exterior)-incompressible liquid with high viscosity

Water (interior) - incompressible liquid with low viscosity

Fingering Instability



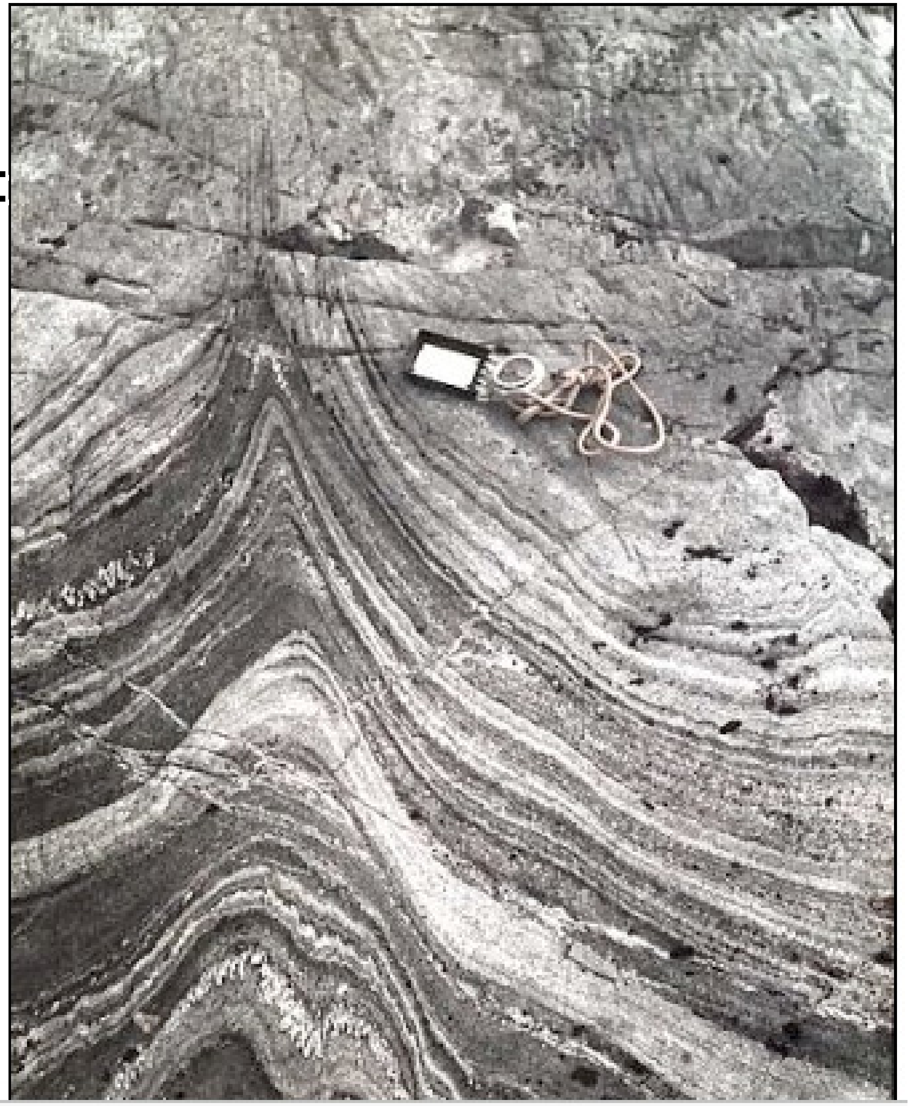
Any almost all fronts are unstable - an arbitrary small deviation from a plane front causes a complex set of fingers growing out of control



Finite time singularities:

any but plain algebraic domain lead to cusp like
singularities which occur at a finite time (the area of the domain)

Universal character of of singularities:



Cusp-like singularities:

[redacted]

Evolution of elliptic curves

Hydrodynamic problem is ill defined

$$v = -\nabla p, \in C/D, \quad p_{|\partial D} = 0, \quad p_{|z_0} \sim \log|z - z_0|$$

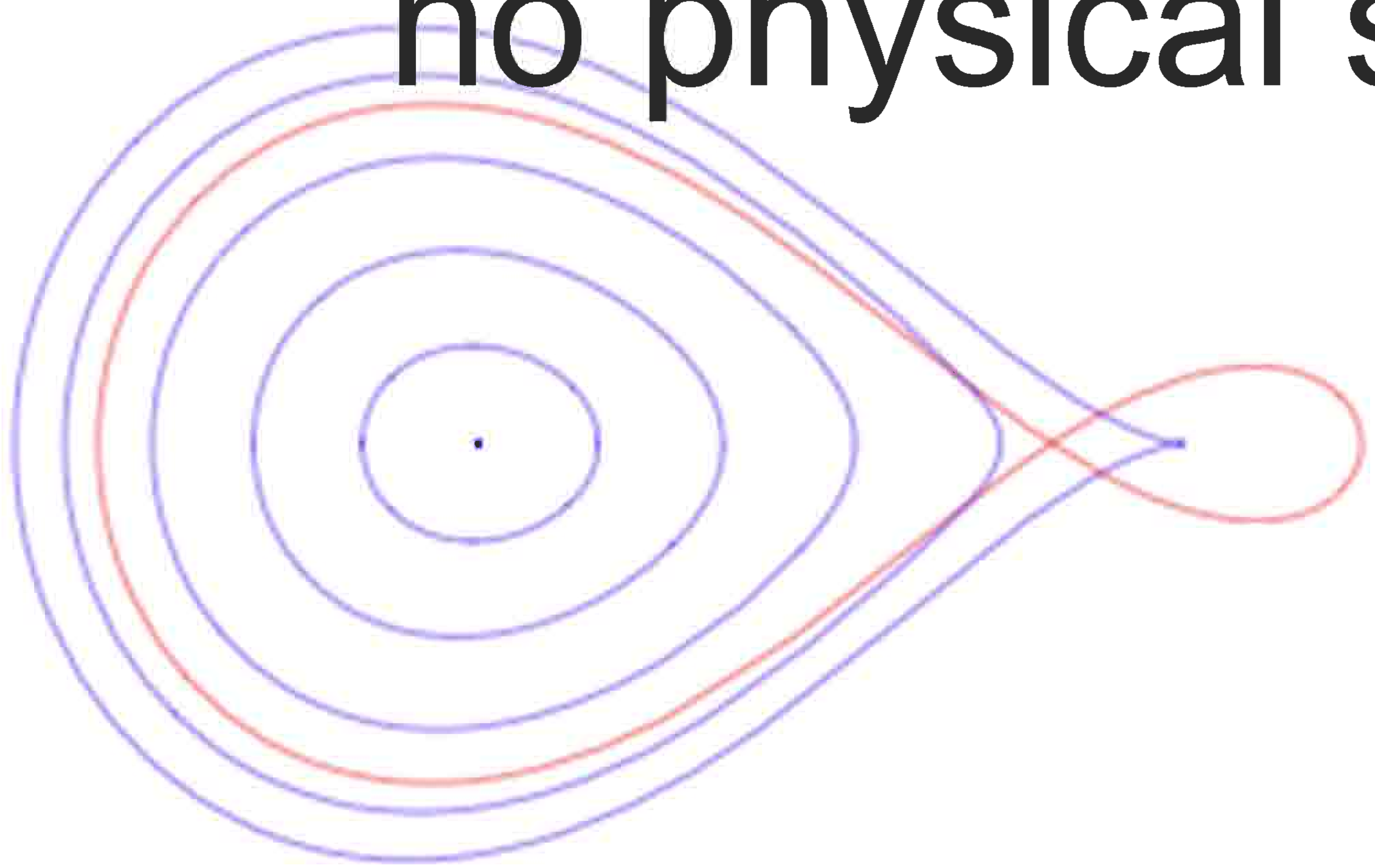
$$\Delta p = 0$$

Problem of regularization

hydrodynamic singularities

Gradient Catastrophe:

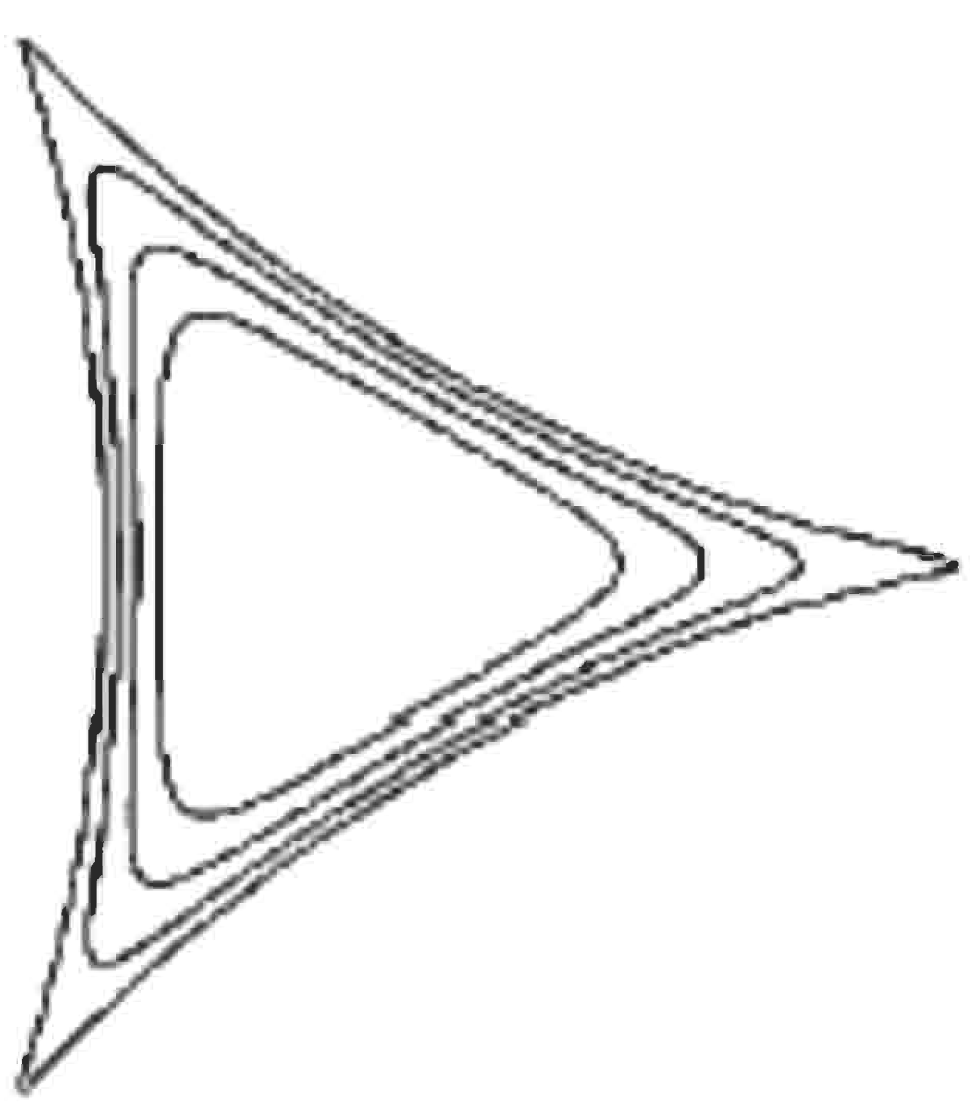
no physical solution beyond the cusp



[redacted]

[redacted]

Hypotrocoid: $$z(w) = rw + \frac{u}{w^2}$$



Hypotrochoid grows until it reaches a critical point.

Hydrodynamic problem is ill defined

Singular limit of non-linear waves

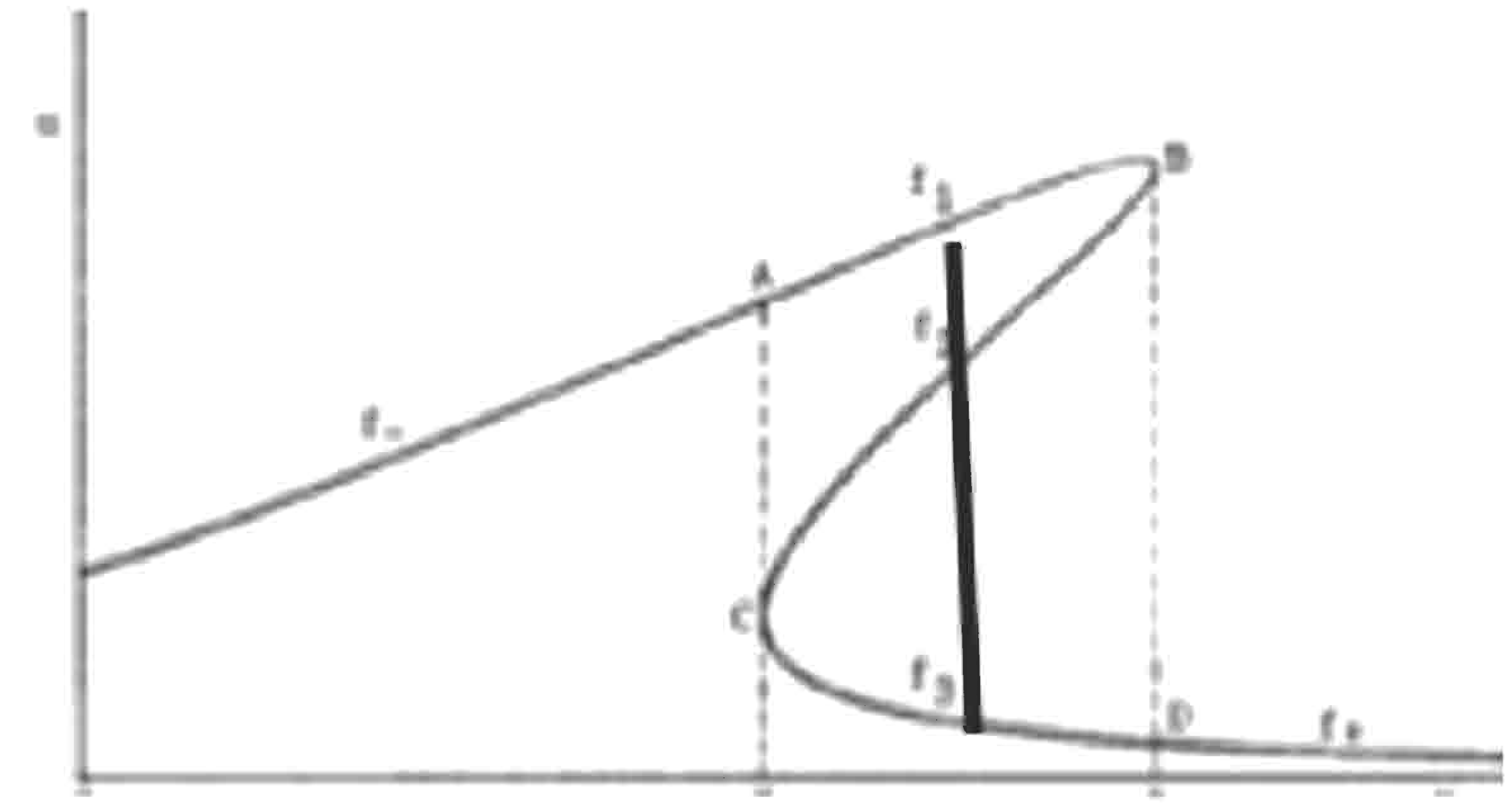
$$\dot{u} + \partial_x F(u) = \epsilon f(u_{xx}, u_{xxx})$$

Riemann Equation $\quad \epsilon \to 0$

Weak solutions: discontinuities - shocks

Diffusive regularization:

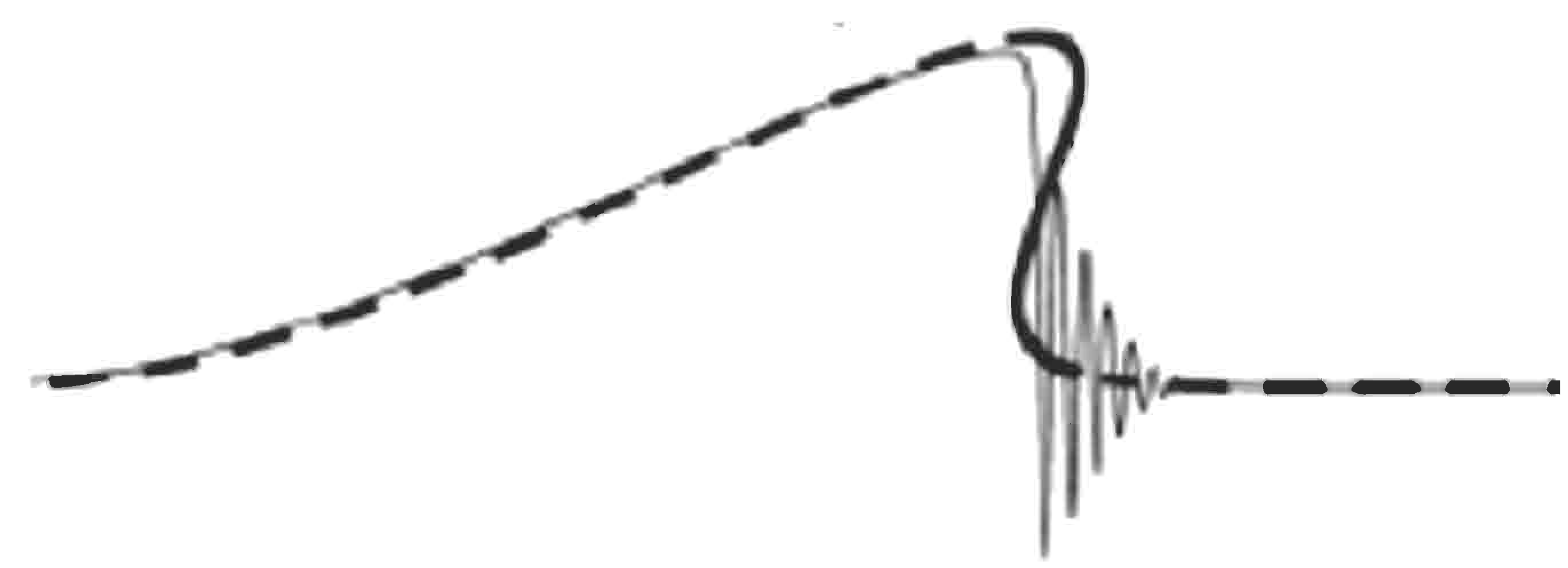
$$\dot{u} + u \cdot u_x + \epsilon u_{xx} = 0$$



Burgers equation

Hamiltonian or dispersive regularization

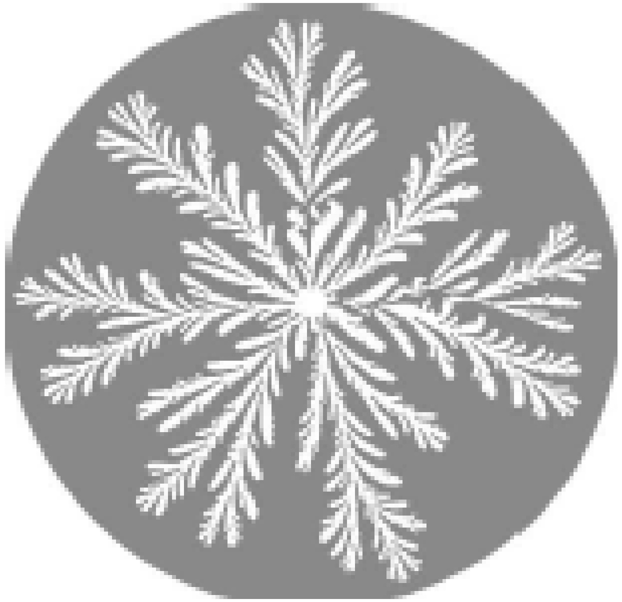
$$\dot{u} + u \cdot u_x + \epsilon u_{xxx} = 0$$



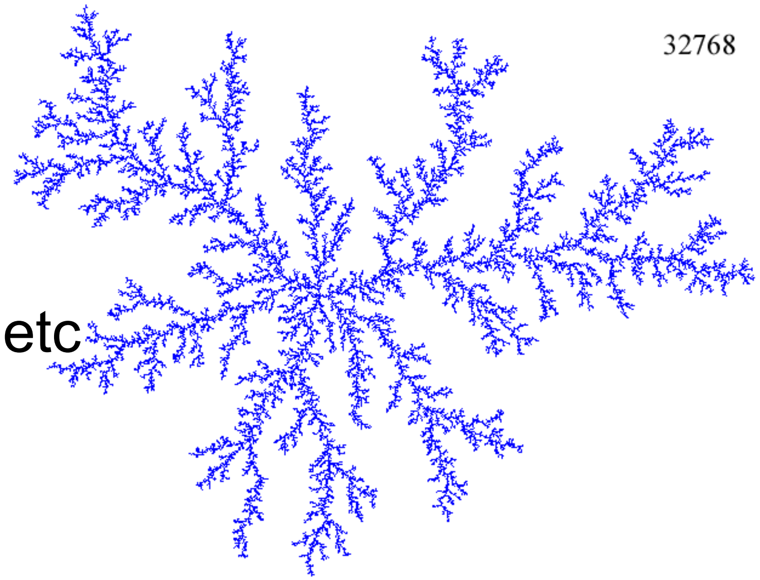
Korteweg de Vries equation

Idealized Hydrodynamics fails after a singularity is reached

Evolution continuos beyond singularities



Fluid dynamics:
surface tension, etc.



Diffusion limited aggregation:
geometrical growth, no surface tension, etc

Integrability of continuum problem (fluid mechanics)

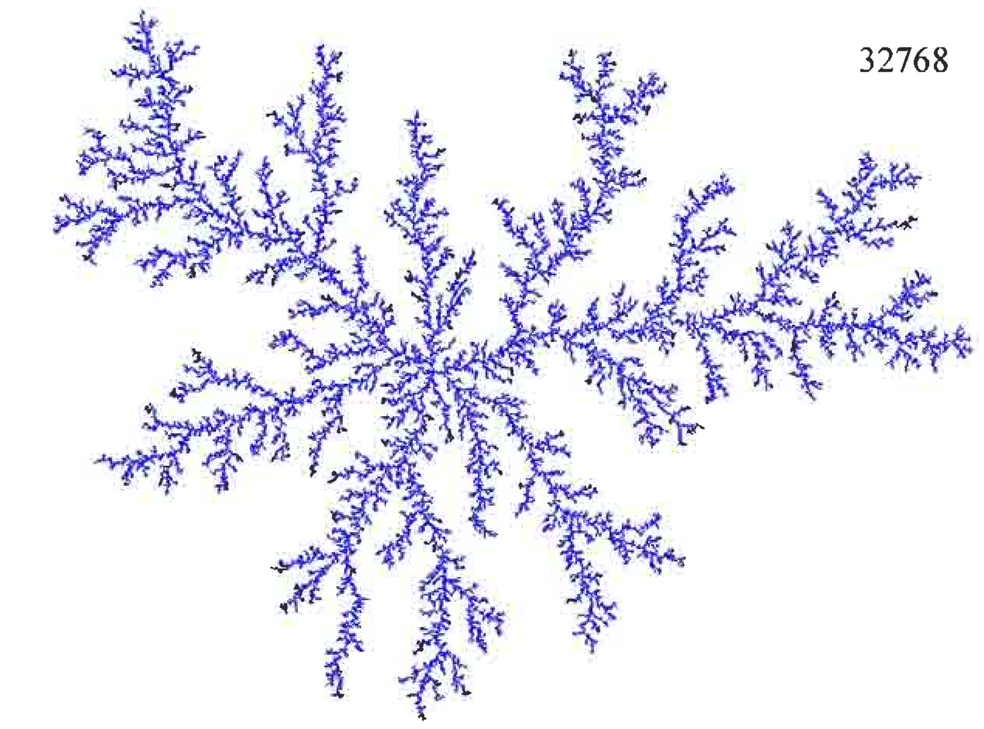
A. Zabrodin and P.W.
(2001)

Search for a regularization which <u>preserves the algebraic structure</u> (i.e. integrable structure) of the of the curve.

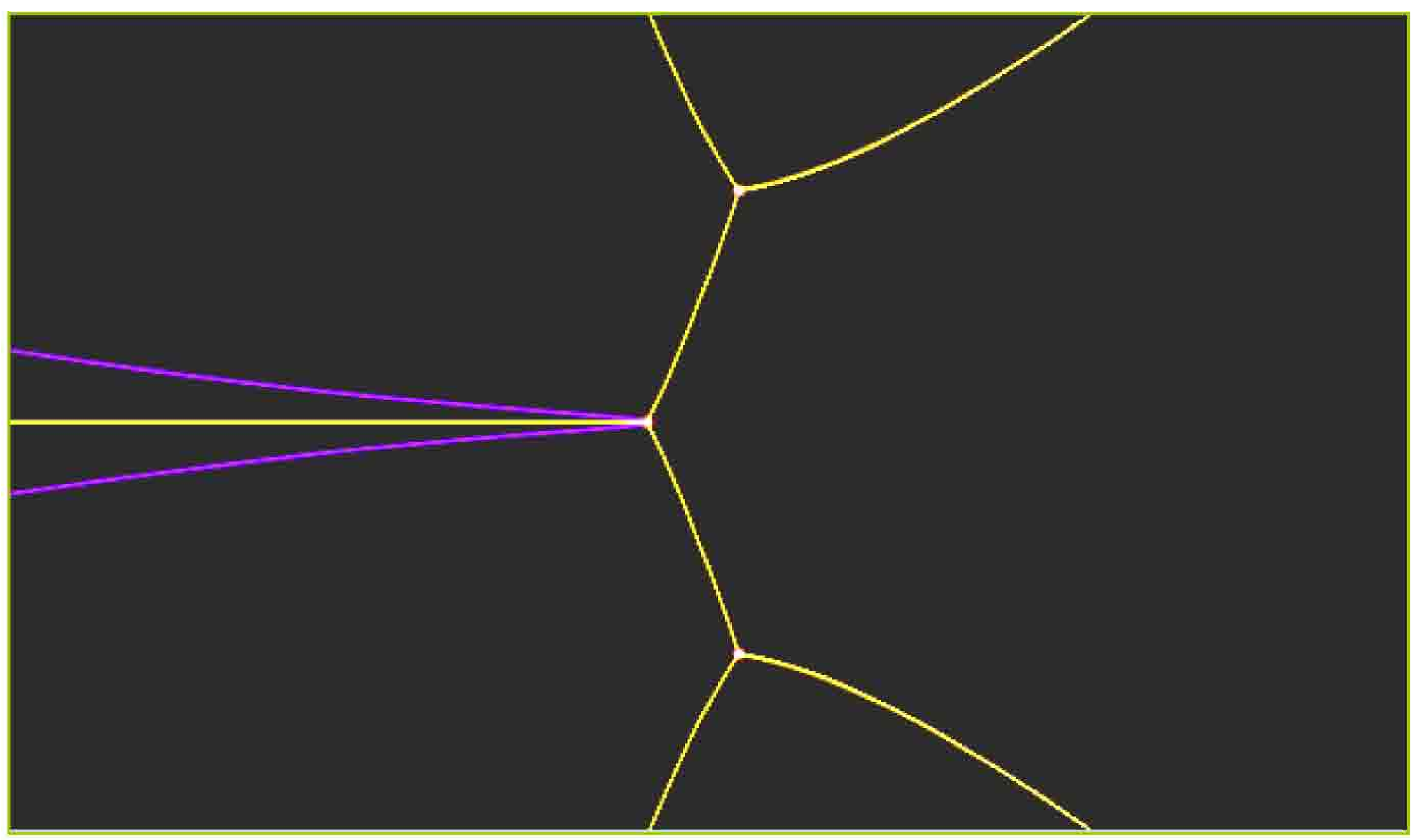
Weak solution of hydrodynamics:

<u>preserving the algebraic structure</u> of

the curve (i.e. integrable structure)



$$v = -\nabla p, \in C/D, \quad p_{|\partial D} = 0, \quad p_{|z_0} \sim \log|z - z_0|$$



$$\Delta p = 0$$

Pressure is harmonic everywhere except moving lines of discontinuities - shocks

Shocks are <u>uniquely</u> determined by integrability

Physical conditions of the weak solution:

1) Fluid is compressible on shock lines;

2) Fluids remains irrotational;

Mathematical conditions of the weak solution:

Boundary is an admissible Boutroux curve;

Schwarz function:

$$f(z, \bar{z}) = 0 \qquad \bar{z} = S(z)$$

$$d\Omega(z) = S(z)dz$$

$$d\Omega_{-1}(z) = t\frac{dz}{z}$$

Harmonic moments

[redacted]

$$d\Omega_{+}(z) = \sum_{k>0} k t_k z^{k-1} dz$$

⇨ conserved

$$\text{Boutroux curve : all cycles} \oint d\Omega = \text{imaginary}$$

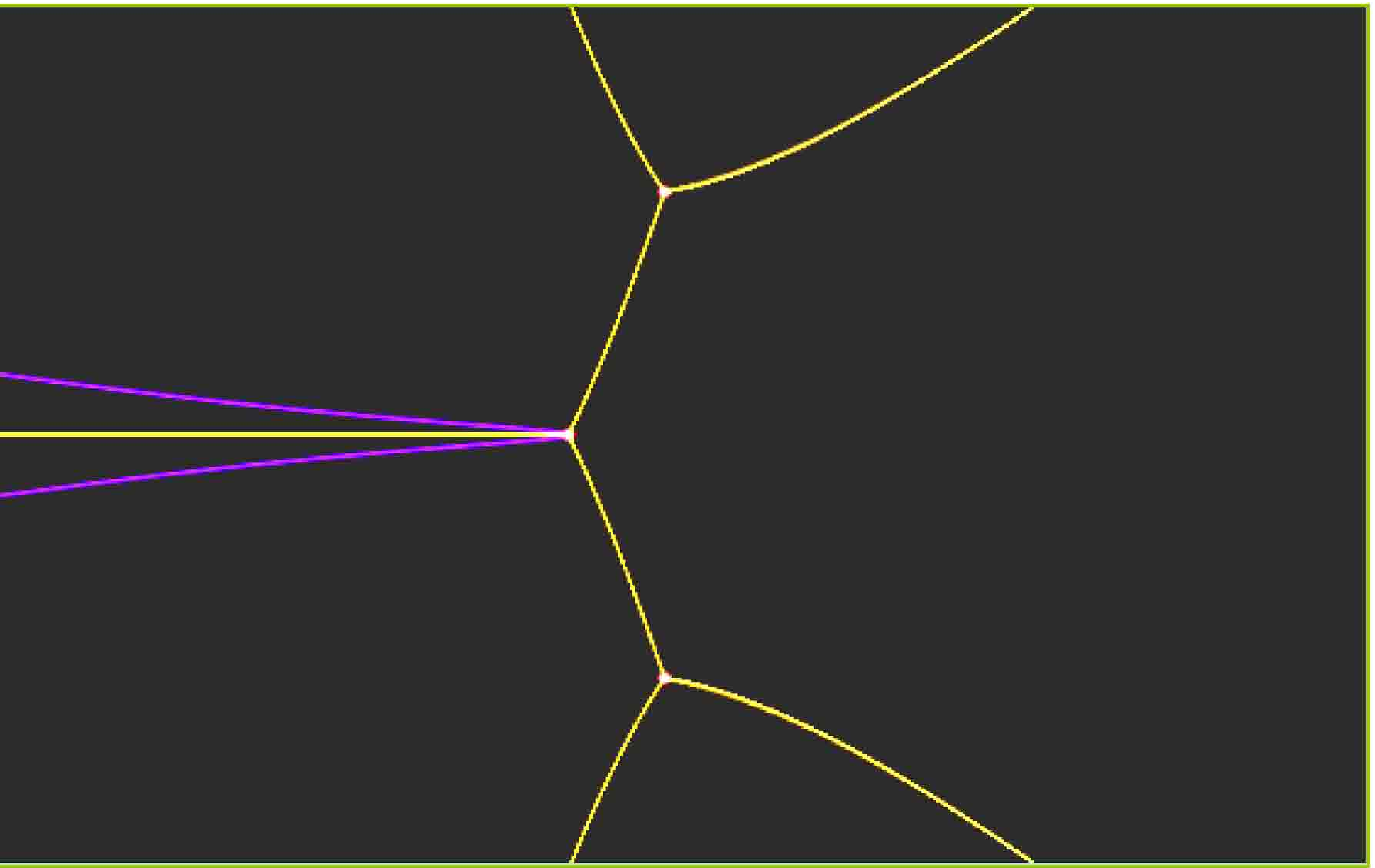
Global formulation of the Laplacian Growth

- $d\Omega_+(z)$ a given time independent differential
- $d\Omega_{-1}(z) = t\frac{dz}{z}$

Boutroux curve : all cycles $\oint \mathrm{d}\Omega = \text{imaginary}$

Find an evolution of Boutroux curve under with respect to its residue under condition that the principal part is constant.

weak solution of Hele-Shaw flow



Elliptic curve Boutroux self-similar curve - an elementary fork

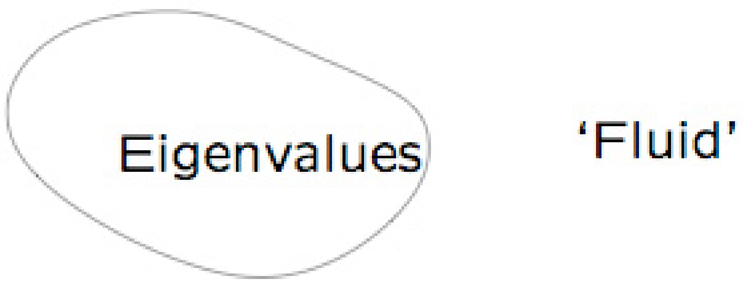
$$(e_1, e_2, e_3) = \sqrt{\frac{3}{h^2 - \frac{3}{4}}}\left(-1, \tfrac{1}{2} + \mathrm{i}h, \tfrac{1}{2} - \mathrm{i}h\right)\sqrt{t} \qquad h \approx 3.2463822537442788756 76.$$

$$m = \frac{1}{2} + \frac{3}{4}\frac{1}{\sqrt{\frac{9}{4} + h^2}} \qquad (16m^2 - 16m + 1)E(m) = (8m^2 - 9m + 1)K(m).$$

Random matrices

• *Normal* matrices satisfy $\mathbf{M}^*\mathbf{M} = \mathbf{M}\mathbf{M}^*$. They have complex eigenvalues: how are they distributed if $\mathbf{M}$ is random?

• Answer: in a domain D which, after scaling with $\sqrt{N}$, to leading order grows like a Hele-Shaw bubble (Wiegmann & Zabrodin 2002).



Eigenvalues distribution of Norman Random Matrix ensemble

$$e^{-\frac{1}{\hbar}\mathrm{tr}\left(MM^\dagger+V(M)+\bar{V}(M^\dagger)\right)}dMdM^\dagger$$

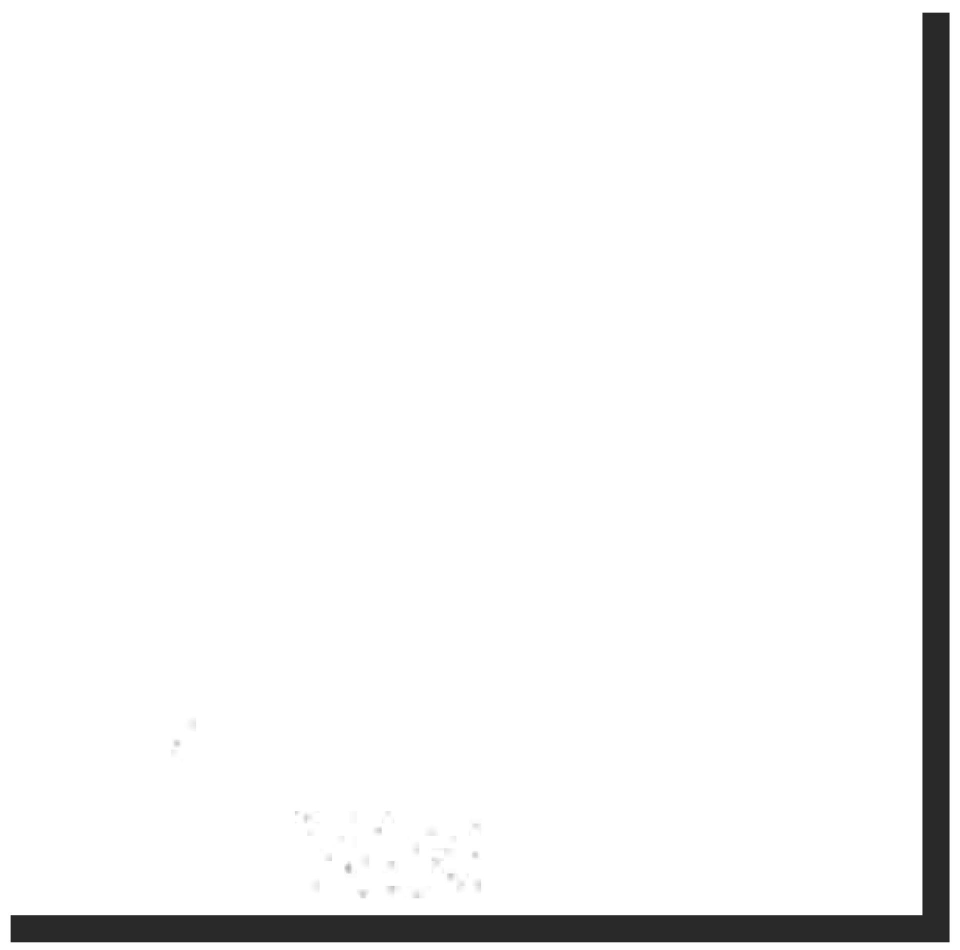
$$M=U^{-1}\mathrm{diag}(z_1,\ldots,z_N)U$$

$$\mathcal{P}(z_1,\ldots,z_N)d^2z_1\ldots d^2z_N$$

[redacted] $$\prod_{m<n}|z_n-z_m|^2e^{-\frac{1}{\hbar}\sum_n|z_n|^2+V(z_n)+\overline{V(z_n)}}$$

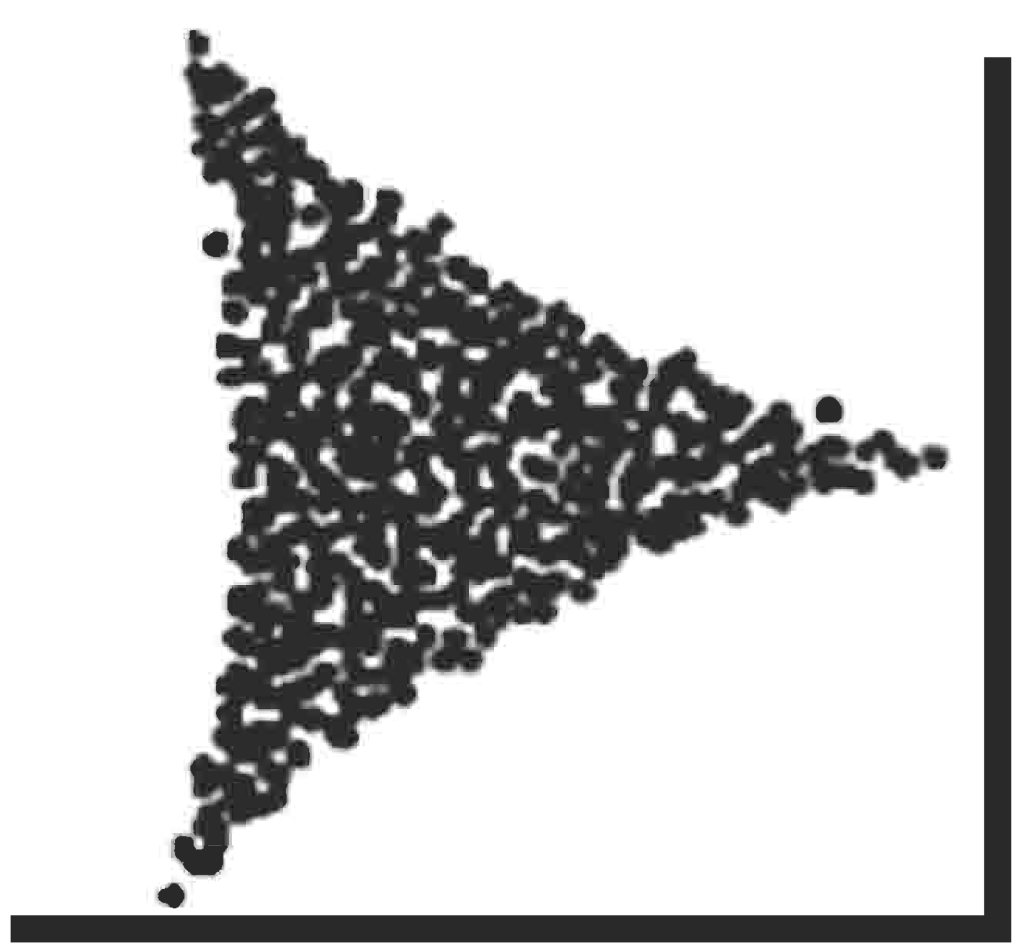
equilibrium measure

$$e^{-\frac{1}{\hbar}\mathrm{tr}\left(MM^{\dagger}+V(M)+\bar{V}(M^{\dagger})\right)}dM\,dM^{\dagger}$$

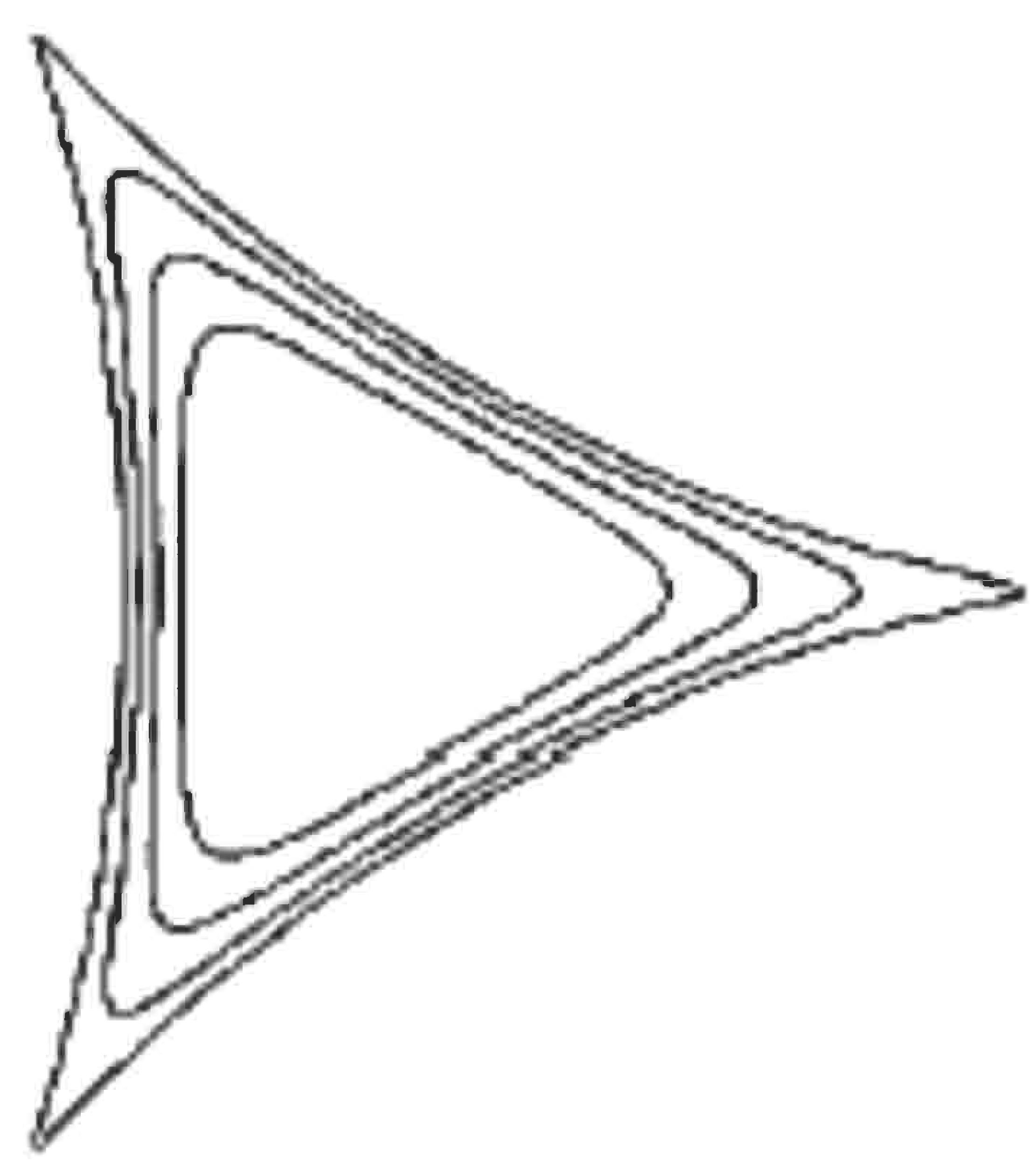


$$V(z)=0$$ Gaussian ensemble

$$\prod_{m<n}|z_n-z_m|^2\,e^{-\frac{1}{\hbar}\sum_n |z_n|^2+V(z_n)+\overline{V(z_n)}}$$



$$V(z)\sim z^3$$ Non-Gaussian ensemble



Bi-Orthogonal polinomials and Random Matrices

[redacted]

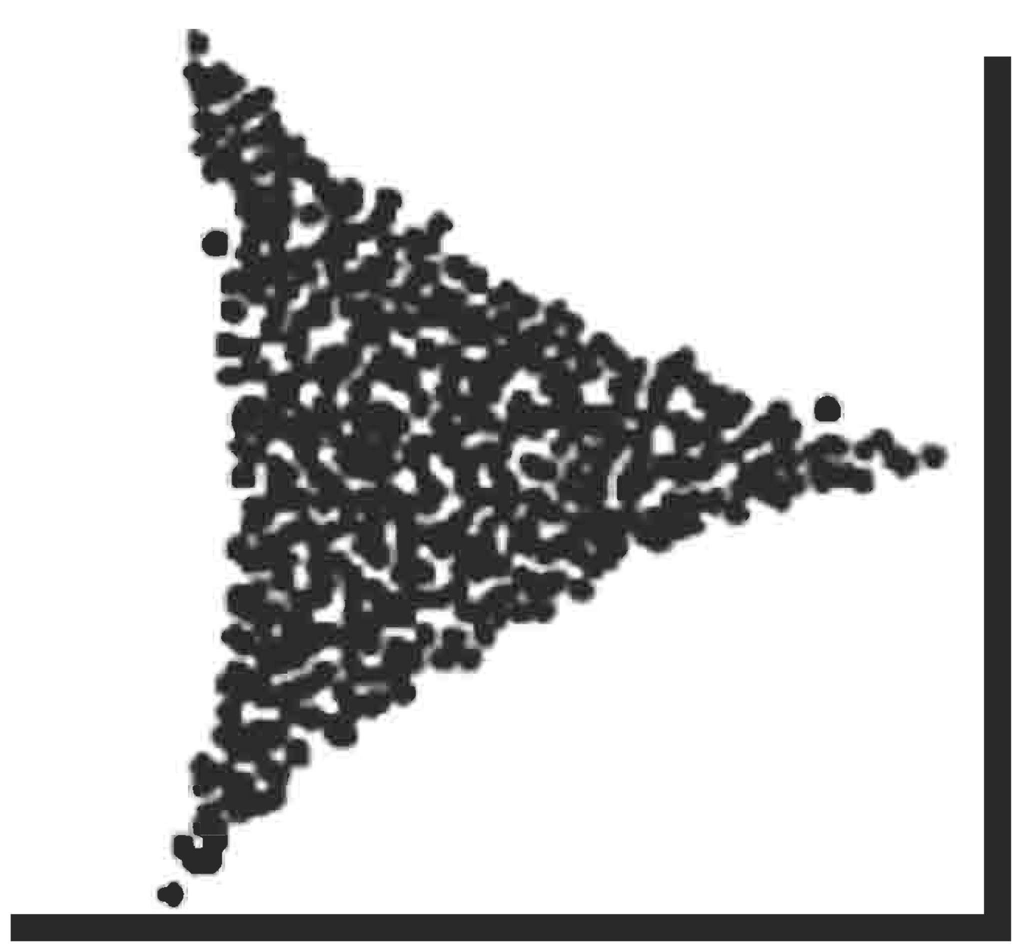
$$\psi_n(z) = P_n(z)e^{-\frac{1}{2\hbar}V(z)}$$

$$\int \psi_n \overline{\psi}_m e^{-\frac{1}{2\hbar}|z|^2} dz d\bar{z} = \delta_{mn}$$

Semiclassical limit of Matrix Growth: N□ N+1

is <u>equivalent</u> to the Hele-Shaw flow.

$$\rho_N(z) = \sum_{n=1}^{N-1} |\psi_n(z)|^2 \approx 1, \quad z \in D, \quad \text{or } 0, \quad z \in C/D$$



Proved by
Haakan Hedenmalm and Nikolai Makarov

Bi-Orthogonal Polynomials

[redacted]

[redacted]

Semiclassical Limit: back to hydrodynamics

$$\hbar \to 0, \; n \to \infty, \; n\hbar = \pi t = \text{fixed}$$

$$\boxed{v(z) = \lim_{\hbar\to 0} \hbar \partial_t \log \Psi(t,z), \quad p(z) = \lim_{\hbar\to 0} \hbar \partial_z \log \Psi(t,z)}$$

Asymptotes of Orthogonal Polynomials solve Hele-Shaw flow

Classical limit:

$$v(z) = \lim_{\hbar \to 0} \hbar \partial_t \log \Psi(t, z), \quad p(z) = \lim_{\hbar \to 0} \hbar \partial_z \log \Psi(t, z)$$

[redacted]

does not exists at the anti-Stokes lines,

where

polynomials accumulate zeros:

anti-Stokes lines - lines of discontinuities pressure and velocity -

\- *shock fronts of the flow*

Hydrodynamics point of view:

anti-Stokes lines are shock fronts

How to determine lines of zeros of bi-Orthogonal polynomials?

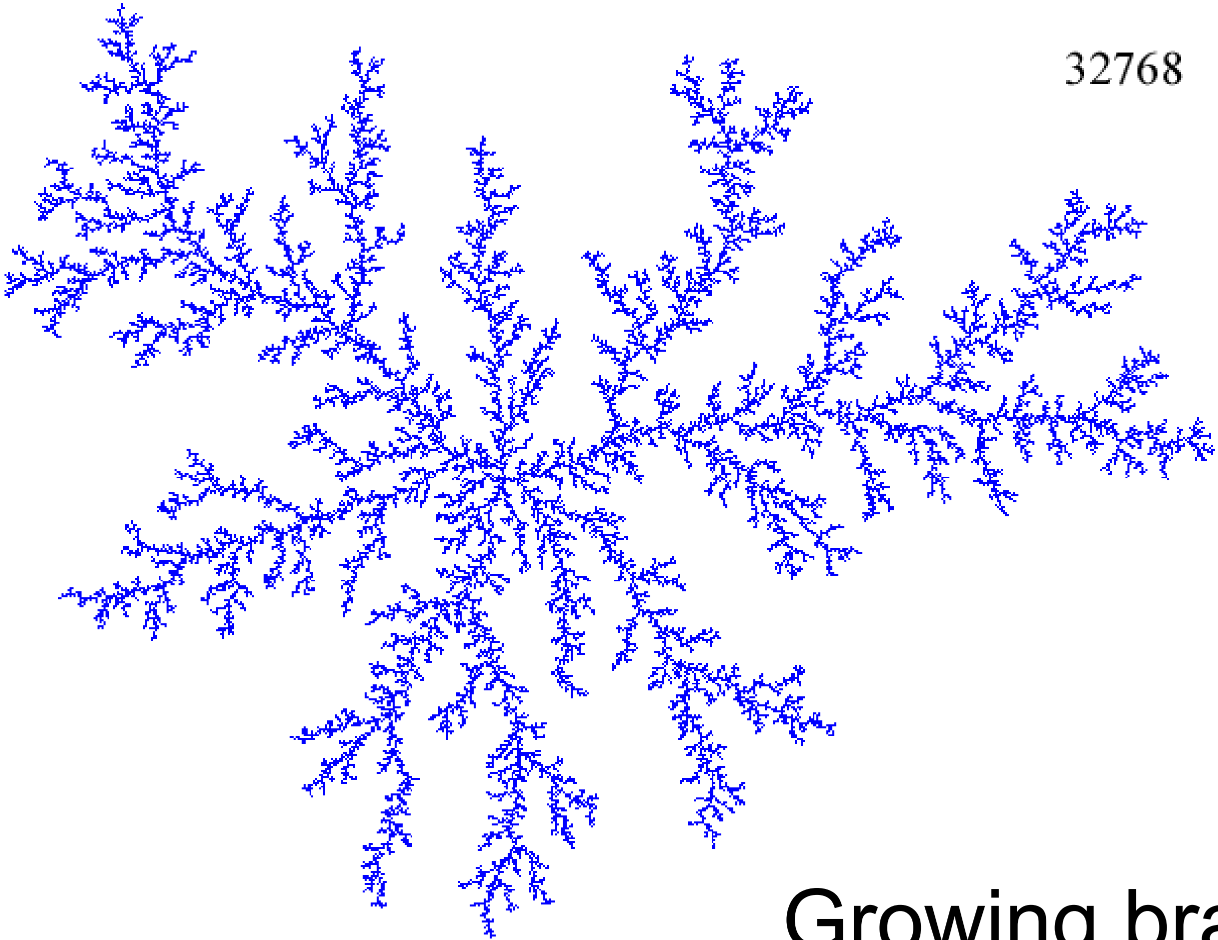
- $d\Omega_+(z)$ a measure of the polynomials
- $d\Omega_{-1}(z) = t\frac{dz}{z}$
- $\boxed{\text{Boutroux curve : all cycles} \oint \mathrm{d}\Omega = \text{imaginary}}$

- weak solution of Hele-Shaw flow

[redacted]

$$y^2(x) = \prod_{i=1}^{8}(x - e_i)$$

QuickTime™ and a
Motion JPEG OpenDML decompressor
are needed to see this picture.



Growing branches as Stokes lines

<u>Stochastic geometry</u>:
measure on a space of
Riemann surfaces

Черноголовка 2007