Lecture 2
Bosonization of the Thirring model

Consider the *massive Thirring model* in the Minkowski space:

$$S^{MT}[\psi,\bar\psi]=\int d^2x\,\left(\bar\psi(i\hat\partial-m)\psi-\frac{g}{2}(\bar\psi\gamma^\mu\psi)^2\right). \tag{1}$$

Here $\psi(x)$, $\bar\psi(x)$ are the fermion field and its Dirac conjugate, γ^μ are the Dirac matrices, and $\hat\partial=\gamma^\mu\partial_\mu$. The Dirac matrices satisfy the standard relations

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu},\qquad \gamma^{\mu+}=\gamma^0\gamma^\mu\gamma^0.$$

In the two-dimensional case the gamma-matrices can be written as:

$$\gamma^0=\begin{pmatrix} & -i\\ i & \end{pmatrix},\qquad \gamma^1=\begin{pmatrix} & i\\ i & \end{pmatrix},\qquad \gamma^3=\gamma^0\gamma^1=\begin{pmatrix}1 & \\ & -1\end{pmatrix}. \tag{2}$$

The model has a conserved current

$$j^\mu=\bar\psi\gamma^\mu\psi. \tag{3}$$

When $m=0$ there is another conserved current

$$j_3^\mu=\bar\psi\gamma^3\gamma^\mu\psi=-\epsilon^{\mu\nu}j_\nu. \tag{4}$$

In the previous lecture we considered the sine-Gordon model:

$$S^{SG}[\phi]=\int d^2x\,\left(\frac{(\partial_\mu\phi)^2}{8\pi}+\mu\cos\beta\phi\right). \tag{5}$$

This model has a topological number

$$q=\frac{\beta}{2\pi}(\phi(t,+\infty)-\phi(t,-\infty)), \tag{6}$$

which takes integer values. It can be written as

$$q=\frac{\beta}{2\pi}\int_{-\infty}^{\infty}dx\,\partial_1\phi(t,x). \tag{7}$$

This allows us to define a current responsible for the topological charge:

$$j^\mu_{\rm top}=-\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi. \tag{8}$$

This current satisfies the continuity equation $\partial_\mu j^\mu_{\rm top}=0$ due to the antisymmetry of the symbol $\epsilon^{\mu\nu}$ and the commutativity of derivatives.

In the present lecture we will make sure that the massive Thirring model and the sine-Gordon model are equivalent[2, 3], while the parameters of the two models are related according to

$$g=\pi(\beta^{-2}-1), \tag{9}$$
$$\mu\sim mr_0^{\beta^2-1}, \tag{10}$$

and the Thirring current coincides with the topological one:

$$j^\mu=j^\mu_{\rm top}. \tag{11}$$

This is an extremely important correspondence called *bosonization.* The equation (11) plays a key role in the bosonization.

Rewrite the action of the Thirring model by using the explicit form of the gamma-matrices:

$$S^{MT}[\psi,\bar\psi] = \int d^2x\,(i\psi_1^+(\partial_0+\partial_1)\psi_1 + i\psi_2^+(\partial_0-\partial_1)\psi_2 + im(\psi_1^+\psi_2 - \psi_2^+\psi_1) - 2g\psi_1^+\psi_2^+\psi_2\psi_1).$$

Substituting $z = x^1 - x^0$, $\bar z = x^1 + x^0$, we obtain

$$S^{MT}[\psi,\bar\psi] = \int d^2x\,(2i\psi_1^+\bar\partial\psi_1 - 2i\psi_2^+\partial\psi_2 + im(\psi_1^+\psi_2 - \psi_2^+\psi_1) - 2g\psi_1^+\psi_2^+\psi_2\psi_1).$$

In these components the Thirring current has the form:

$$j_z = -\psi_1^+\psi_1, \qquad j_{\bar z} = \psi_2^+\psi_2. \tag{12}$$

Consider the case $m = 0$, which admits an exact solution [1]. Start with solving classical equations of motion

$$\begin{aligned} \bar\partial\psi_1 &= -ig\psi_2^+\psi_2\psi_1 \equiv -igj_{\bar z}\psi_1, \\ \partial\psi_2 &= ig\psi_1^+\psi_1\psi_2 \equiv -igj_z\psi_2. \end{aligned} \tag{13}$$

Since $\epsilon^{\mu\nu}\partial_\mu j_\nu = \partial_\mu j_3^\mu = 0$ the current j_μ is a gradient of a free field:

$$j_\mu = -\frac{\beta}{2\pi}\partial_\mu\tilde\phi. \tag{14}$$

It is convenient to consider the field $\tilde\phi$ as a dual to another field ϕ, as it was described in the last lecture. Both satisfy the d'Alembert equation:

$$\partial_\mu\partial^\mu\phi = \partial_\mu\partial^\mu\tilde\phi = 0.$$

The general solution to these equations can be written as

$$\begin{aligned} \phi(x) &= \varphi(z) + \bar\varphi(\bar z), \\ \tilde\phi(x) &= \varphi(z) - \bar\varphi(\bar z), \end{aligned} \tag{15}$$

where $\varphi(z)$ and $\bar\varphi(\bar z)$ are arbitrary functions of the only z and $\bar z$ variables respectively. The coefficient in (14) is arbitrary. We choose it in such a way that the relation (11) is formally satisfied, if we identify the field ϕ with that in the sine-Gordon model.

We see that the massless Thirring model is equivalent to the free massless boson model. From relation (14) we have

$$\frac{\beta}{2\pi}\partial\varphi = \psi_1^+\psi_1, \qquad \frac{\beta}{2\pi}\bar\partial\bar\varphi = \psi_2^+\psi_2. \tag{16}$$

To continue searching for a classical solution, we have to substitute these functions into the equations (13). By solving the last one we obtain

$$\psi_1(z,\bar z) = F_1(z)e^{-i\frac{g\beta}{2\pi}\bar\varphi(\bar z)}, \qquad \psi_2(z,\bar z) = F_2(\bar z)e^{i\frac{g\beta}{2\pi}\varphi(z)} \tag{17}$$

with arbitrary functions F_i. By substituting them back into (16), we have

$$\frac{\beta}{2\pi}\partial\varphi(z) = F_1(z)F_1^*(z), \qquad \frac{\beta}{2\pi}\bar\partial\bar\varphi(\bar z) = F_2(\bar z)F_2^*(\bar z), \tag{18}$$

where the star denotes complex conjugation under the assumption of real arguments. It is left to integrate these equations and substitute the result into (17). As a result the fields ψ_i are expressed in terms of two functions F_i and two integration constants.

Let us turn to the quantum case. The situation is simpler just in the quantum case. Let us look for a solution to the equations (16) in the form

$$\psi_i(x) = \eta_i\sqrt{\frac{N_i}{2\pi}}e^{i\alpha_i\varphi(z)+i\beta_i\bar\varphi(\bar z)}, \qquad \psi_i^+(x) = \eta_i^{-1}\sqrt{\frac{N_i}{2\pi}}e^{-i\alpha_i\varphi(z)-i\beta_i\bar\varphi(\bar z)}, \tag{19}$$

where η_i are the algebraic factors necessary to ensure the fermionic behavior of the fields ψ_i. It turns out that solutions can be found, if we assume

$$\eta_1\eta_2 = -\eta_2\eta_1. \tag{20}$$

First of all, demand that the fields $\psi_i(x)$ behave like fermions. Consider the product

$$\psi_i(x')\psi_j(x) = \eta_i\eta_j \frac{\sqrt{N_i N_j}}{2\pi}(z'-z)^{\alpha_i\alpha_j}(\bar{z}'-\bar{z})^{\beta_i\beta_j} e^{i\alpha_i\varphi(z')+i\beta_i\bar{\varphi}(\bar{z}')+i\alpha_j\varphi(z)+i\beta_j\bar{\varphi}(\bar{z})}. \tag{21}$$

This expression continues well into the Euclidean region. From the anticommutativity requirement it is easy to obtain that

$$\alpha_i^2 - \beta_i^2 \in 2\mathbb{Z}+1, \qquad \alpha_1\alpha_2 - \beta_1\beta_2 \in 2\mathbb{Z}. \tag{22}$$

From (21) it can be seen that products like $\psi_1^+\psi_1$ are poorly defined. Let us *define* these products as follows. Consider another product:

$$\psi_1^+(x')\psi_1(x) = \frac{N_1}{2\pi}(z'-z)^{-\alpha_1^2}\left(\bar{z}'-\bar{z}\right)^{-\beta_1^2}\left(1 - i\alpha_1(z'-z)\,\partial\phi(x) - i\beta_1(\bar{z}'-\bar{z})\,\bar{\partial}\phi(x) + \cdots\right). \tag{23}$$

Take an average of this product over the circle $|z'-z|^2 = r_0^2$ and assume that r_0 is small. The leading term in the expansion in r_0 will be assumed for $\psi_1^+(x)\psi_1(x)$. Suppose that

$$\alpha_1^2 - \beta_1^2 = 1. \tag{24}$$

Then the first and third terms in the expansion (23) vanish after averaging. The leading nonzero term is the second:

$$N_1 r_0^{-2\beta_1^2}\left(\frac{-i\alpha_1\partial\varphi}{2\pi}\right).$$

It is just what we will identify with $\psi_1^+\psi_1$. The coefficient N_1 must be imaginary for consistency with the Hermicity of $\psi^+\psi$. Comparing with (16), we obtain

$$\beta = -ir_0^{-2\beta_1^2}N_1\alpha_1. \tag{25}$$

Similarly, assuming

$$\alpha_2^2 - \beta_2^2 = -1, \tag{26}$$

we obtain

$$\beta = -ir_0^{-2\alpha_2^2}N_2\beta_2. \tag{27}$$

Now consider the equations of motion (13). Substituting (16) and (19), we obtain

$$\begin{aligned} i\beta_1\,\bar{\partial}\bar{\varphi}\,e^{i\alpha_1\varphi+i\beta_1\bar{\varphi}} &= -ig\frac{\beta}{2\pi}\bar{\partial}\bar{\varphi}\,e^{i\alpha_1\varphi+i\beta_1\bar{\varphi}}, \\ i\alpha_2\,\partial\varphi\,e^{i\alpha_2\varphi+i\beta_2\bar{\varphi}} &= ig\frac{\beta}{2\pi}\partial\varphi\,e^{i\alpha_2\varphi+i\beta_2\bar{\varphi}}. \end{aligned}$$

From this we have

$$\alpha_2 = -\beta_1 = \frac{g\beta}{2\pi}, \tag{28}$$

which is surely consistent with the classical solution.

To fix the coefficients α_i, β_i, we need to define the mass term consistently in such a way that it commute with the fermion charge

$$Q = \int df_\mu\, j^\mu, \tag{29}$$

where $df_\mu = \epsilon_{\mu\nu}\,dx^\nu$ is the one-dimensional surface element. Consider the expansion

$$\psi_2^+(x')\psi_1(x) = -\eta_1\eta_2^{-1}\frac{\sqrt{N_1N_2}}{2\pi}(z'-z)^{-\alpha_1\alpha_2}(\bar{z}'-\bar{z})^{-\beta_1\beta_2}\left(e^{i(\alpha_1-\alpha_2)\varphi(z)+i(\beta_1-\beta_2)\bar{\varphi}(\bar{z})} + \cdots\right).$$

The first term survives under averaging over the corners if

$$\alpha_1\alpha_2 = \beta_1\beta_2, \tag{30}$$

which is consistent with (24–28) and yields

$$\alpha_1 = -\beta_2. \tag{31}$$

By taking the angular average we obtain the definition of the products

$$\begin{aligned} \psi_2^+\psi_1 &= -\eta_1\eta_2^{-1}\frac{\sqrt{N_1N_2}}{2\pi}r_0^{-2\alpha_1\alpha_2}e^{i(\alpha_1-\alpha_2)\phi}, \\ \psi_1^+\psi_1 &= -\eta_2\eta_1^{-1}\frac{\sqrt{N_1N_2}}{2\pi}r_0^{-2\alpha_1\alpha_2}e^{-i(\alpha_1-\alpha_2)\phi}. \end{aligned} \tag{32}$$

Check now that the operators defined in such a way commute with Q. Let $O(x)$ be a local operator. Calculate the commutator

$$\begin{aligned} [O(0), Q] &= \oint df_\mu\, j^\mu(x)\, O(0) = \oint dx^\nu\, \epsilon_{\mu\nu} j^\mu(x) O(0) = -\frac{\beta}{2\pi}\oint dx^\nu\, \epsilon_{\mu\nu}\,\partial^\mu\tilde{\phi}(x)\, O(0) \\ &= -\frac{\beta}{2\pi}\oint dx^\nu\, \epsilon_{\mu\nu}\epsilon^{\mu\lambda}\,\partial_\lambda\phi(x)\, O(0) = \frac{\beta}{2\pi}\oint dx^\lambda\,\partial_\lambda\phi(x)\, O(0) = \frac{\beta}{2\pi}\Delta\phi(x)\, O(0), \end{aligned} \tag{33}$$

Here $\Delta\phi(x)$ is the increment of the field $\phi(x)$ while x goes around zero counterclockwise. Let us apply this formula to the operator $O(x) = e^{i\alpha\varphi(z)+i\alpha'\bar{\varphi}(\bar{z})}$:

$$\begin{aligned} [e^{i\alpha\varphi(0)+i\alpha'\bar{\varphi}(0)}, Q] &= \frac{\beta}{2\pi}\Delta(\varphi(z)+\bar{\varphi}(\bar{z}))e^{i\alpha\varphi(0)+i\alpha'\bar{\varphi}(0)} \\ &= \frac{i\beta}{2\pi}\Delta\left(\alpha\log\frac{1}{z}+\alpha'\log\frac{1}{\bar{z}}\right)e^{i\alpha\varphi(0)+i\alpha'\bar{\varphi}(0)} = \beta(\alpha-\alpha')e^{i\alpha\varphi(0)+i\alpha'\bar{\varphi}(0)}. \end{aligned} \tag{34}$$

The commutator $[Q, O(x)] = 0$, if $\alpha = \alpha'$ a, therefore, $O(x) = e^{i\alpha\phi(x)}$. For $\alpha = \alpha_1 - \alpha_2$, $\alpha' = \beta_1 - \beta_2$, this condition is satisfied due to (28), (31).

Now fix the β parameter. To do it we set $O(x) = \psi_i(x)$ in (34). Since the operators ψ_i have the fermion charge equal to -1, we have

$$\psi_i(0) = [\psi_i(0), Q] = \beta(\alpha_i - \beta_i)\psi_i(0) = \beta(\alpha_1+\alpha_2)\psi_i(0)$$

Hence,

$$\alpha_1 + \alpha_2 = \beta^{-1}, \tag{35}$$

and we immediately obtain

$$\alpha_1 - \alpha_2 = \beta \tag{36}$$

and

$$\begin{aligned} \alpha_1 &= -\beta_2 = \frac{1}{2}\left(\frac{1}{\beta}+\beta\right), \\ \alpha_2 &= -\beta_1 = \frac{1}{2}\left(\frac{1}{\beta}-\beta\right). \end{aligned} \tag{37}$$

Substituting the answer into (28), we get (9).

From (25), (27) we find

$$N_1 = -N_2 = ir_0^{\frac{\beta^2}{2}+\frac{1}{2\beta^2}-1}\frac{2\beta^2}{\beta^2+1}, \tag{38}$$

From this we obtain

$$\begin{aligned} -i\psi_2^+\psi_1 &= \frac{1}{\pi}\frac{\beta^2}{\beta^2+1}r_0^{\beta^2-1}\left(i\eta_1\eta_2^{-1}\right)e^{i\beta\phi}, \\ i\psi_1^+\psi_2 &= \frac{1}{\pi}\frac{\beta^2}{\beta^2+1}r_0^{\beta^2-1}\left(i\eta_1\eta_2^{-1}\right)^{-1}e^{-i\beta\phi}. \end{aligned}$$

Since on the infinite plane the total "charge" must be equal to zero, the operators $e^{i\beta\phi}$ and $e^{-i\beta\phi}$ must occur in equal numbers for correlation functions polynomial in φ, $\bar{\varphi}$. Therefore, the factors $\left(i\eta_1\eta_2^{-1}\right)^{\pm1}$ will also cancel each other. In a more general case, they can be omitted by redefining the operators:

$$\left(i\eta_1\eta_2^{-1}\right)^{\alpha/\beta} e^{i\alpha\phi} \to e^{i\alpha\phi}.$$

Then we have

$$i(\psi_1^+\psi_2 - \psi_2^+\psi_1) = \frac{2}{\pi}\frac{\beta^2}{\beta^2+1} r_0^{\beta^2-1} \cos\beta\phi, \tag{39}$$

from which we find (10).

Rigorously speaking, so far we have found the exact solution for the massless Thirring model only. However, it follows from our reasoning that the perturbation theory in the term $m\bar{\psi}\psi$ for the Thirring model and the perturbation theory in $\cos\beta\phi$ for the sine-Gordon model coincide, which gives a strong foundation in favor of the coincidence of the theories[2, 3]. Note that the coupling constant g in the Thirring model does not renormalize, while the "mass" m is not a physical quantity and substantially renormalizes. This is because the mass term $\psi_1^+\psi_2 - \psi_2^+\psi_1$ has a scale dimension β^2 due to redefinition of the product of fields. The constant μ in the sine-Gordon model is measurable, and

$$\mu \sim m_{\text{phys}}^{2-\beta^2}, \qquad m \sim m_{\text{phys}}(m_{\text{phys}}r_0)^{1-\beta^2} = m_{\text{phys}}(m_{\text{phys}}r_0)^{\frac{g/\pi}{1+g/\pi}}, \tag{40}$$

where m_{phys} is the mass of physical excitations (for example, the Thirring fermions) in theory. The proportionality coefficient between the parameter μ and $m_{\text{phys}}^{2-\beta^2}$ is known exactly [4].

One more question remains: what does the Thirring fermions in the sine-Gordon model correspond to? From the equality between the topological and fermion currents, we can conclude that they correspond to kinks, which are nontrivial excitations with topological numbers $q = \pm1$. Simultaneously the kink excitation can be generated not only by fermion operators, but also by boson operators. Consider the operators

$$e^{iJ\varphi} = e^{\frac{iJ}{2\beta}\tilde{\phi}}, \qquad J \in \mathbb{Z}, \tag{41}$$

which entered the correlation functions of the last lecture. These operator acting on a state change the topological number: $q \to q + J$. For $J = \pm1$ they can be considered as boson creation-annihilation operators of the kinks.

References

[1] W. E. Thirring, *Annals Phys.* **3** (1958) 91

[2] S. Coleman, *Phys. Rev.* **D11** (1975) 2088

[3] S. Mandelstam, *Phys. Rev.* **D11** (1975) 3026

[4] Al. B. Zamolodchikov, *Int. J. Mod. Phys.* **A10** (1995) 1125

Problems

1. Prove that the current (4) is conserved in the massless Thirring model. Find the divergence of the current for nonzero mass.

2. In the model of free massless Dirac fermions ($m = 0$, $g = 0$) find the pair correlation functions of the fermion fields $\langle\psi_i^+(x')\psi_j(x)\rangle$.

3. Obtain all classical solutions $\phi(t, x)$ of the sine-Gordon equation with finite energy that only depend on the combination $x - vt$ with some constant v, $|v| < 1$. Find topological charges of these solutions.

4. Repeat the reasoning of the lecture in the special case of a free fermion ($g = 0$). Check that in this case $m_{\text{phys}} = m = \pi\mu$. Show that the bosonization reproduces the correct commutation relations for massless fermions.

5*. Show that in the Thirring model, in consistency with (40), in the one-loop approximation, the mass renormalizes as follows

$$m_{\text{phys}} = m\left(1 + \frac{g}{\pi}\log\frac{\Lambda}{m}\right),$$

where Λ is the momentum cutoff parameter.

While deriving the diagrammatic technique, it is convenient to use the representation for the action of the Thirring model with an auxiliary field:

$$S[\psi, \bar{\psi}, A^\mu] = \int d^D x \left(\bar{\psi}(i\hat{\partial} - \hat{A} - m)\psi + \frac{1}{2g}A^\mu A_\mu\right).$$