Lecture 5.
$O(N)$-model: $1/N$-expansion

Michael Lashkevich

Consider the general $O(N)$-model in the Minkowski space:

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$$S[\boldsymbol{n},\omega] = \frac{1}{2g}\int d^2x\,((\partial_\mu \boldsymbol{n})^2 - \omega(\boldsymbol{n}^2 - 1)). \tag{2}$$

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The integral over $\boldsymbol{n}$ is Gaussian. Indeed, the exponent looks like

$$iS[\boldsymbol{n},\omega]+ig^{-1/2}\int d^2x\,\boldsymbol{J}\boldsymbol{n} = -\frac{1}{2}\left(\frac{n_i}{g^{1/2}}, K(\omega)\delta_{ij}\frac{n_j}{g^{1/2}}\right)+\left(iJ_i, \frac{n_i}{g^{1/2}}\right)+i\int d^2x\,\frac{\omega}{2g},$$

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Thus we obtain

$$Z[J] = \int D\omega\,(\det(\partial_\mu^2+\omega))^{-N/2}\exp\left(i\int d^2x\,\frac{\omega}{2g} - \frac{1}{2}\int d^2x\,d^2x'\,J_i(x)G(x,x'|\omega)J_i(x')\right),$$

where $G(x,x'|\omega)$ is the solution of the equation

$$i(\partial_\mu^2 + \omega(x))G(x,x'|\omega) = \delta(x-x'). \tag{4}$$

Stationary point approximation as $N \to \infty$

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$$Z[J] = \int D\omega \, \exp\left(iS_{\text{eff}}[\omega] - \frac{1}{2} \int d^2x \, d^2x' \, J_i(x) G(x, x'|\omega) J_i(x') \right), \tag{5}$$

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This phenomenon is called the dynamic mass generation or dimensional transmutation.

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$$\begin{aligned} S_{\text{eff}}[\omega] &= \text{const} + i\frac{N}{2}\,\text{tr}\log\left(1 + (2/N)^{1/2}\rho(\partial_\mu^2 + m^2)^{-1}\right) + \frac{1}{(2N)^{1/2}g}\,\text{tr}\,\rho \\ &= \text{const} + i\frac{N}{2}\,\text{tr}\log(1 + i(2/N)^{1/2}\rho G) + \frac{1}{(2N)^{1/2}g}\,\text{tr}\,\rho \\ &= \text{const} + \left(\frac{1}{(2N)^{1/2}g}\,\text{tr}\,\rho - \left(\frac{N}{2}\right)^{1/2}\,\text{tr}\,\rho G\right) - i\frac{N}{2}\sum_{n=2}^{\infty}\frac{(-i)^n(2/N)^{n/2}}{n}\,\text{tr}(\rho G)^n. \end{aligned}$$

The first parenthesis vanishes. Indeed,

$$\begin{aligned} \text{tr}\,\rho &= \int d^2x\,\rho(x), \\ \text{tr}\,\rho G &= \int d^2x\,\rho(x)G(x,x) = G(0,0)\int d^2x\,\rho(x) = G(0,0)\,\text{tr}\,\rho \\ &= V\int\frac{d^2k}{(2\pi)^2}\frac{i}{k^2 - m^2 + i0}\,\text{tr}\,\rho = \frac{V}{4\pi}\log\frac{\Lambda^2}{m^2}\,\text{tr}\,\rho \end{aligned}$$

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Finally we have

$$S_{\text{eff}}[\omega] = \text{const} - i\frac{N}{2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\left(\frac{2}{N}\right)^{n/2}\int d^{2n}x\,\rho(x_1)G(x_1,x_2)\ldots\rho(x_n)G(x_n,x_1). \tag{11}$$

$1/N$ expansion: the D propagator

The term with $n = 2$ is a quadratic form in ρ:

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$$D^{-1}(x_1, x_2) = G(x_1, x_2)G(x_2, x_1)$$

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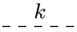
$D(k) =$ 

$$= -\left(\int \frac{d^2q}{(2\pi)^2}\, \frac{1}{(q^2 - m^2 + i0)((q+k)^2 - m^2 + i0)}\right)^{-1}. \quad (12)$$

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Explicitly, we have

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This cumbersome formula becomes quite elementary in the appropriate parameterization:

$$D(k) = 4\pi i m^2\,\frac{\operatorname{sh}\theta}{\theta},\quad k^2 = -4m^2\operatorname{sh}^2\frac{\theta}{2}. \quad (14)$$

Now expand the function $G(x, x'|\omega)$:

$$G[\omega] = \frac{1}{G^{-1} + i(2/N)^{1/2}\rho} = \sum_{n=0}^{\infty} (-i)^n \left(\tfrac{2}{N}\right)^{n/2} G(\rho G)^n,$$

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Thus the expression for S_{eff} and for $G[\omega]$ can be written in terms of the propagator

$$G_{ij}(p) = \quad i \overset{p}{\text{———————}} j \quad = G(p)\delta_{ij} = \frac{i\delta_{ij}}{p^2 - m^2 + i0} \qquad (15)$$

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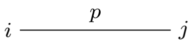
$G_{ij}(p)=$ 

$=G(p)\delta_{ij}=\dfrac{i\delta_{ij}}{p^2-m^2+i0}$ (15)

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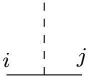

$=-i\left(\dfrac{2}{N}\right)^{1/2}\delta_{ij}.$ (16)

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$$m^2 = \Lambda^2 \exp\left(-\frac{4\pi}{(N-2)g'}\right), \qquad \frac{1}{g'} = \frac{1}{g} + \frac{\Lambda^2}{4\pi m^2 \log(\Lambda^2/m^2)}. \tag{17}$$

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The quadratic divergence originates in the rigid constraint $\boldsymbol{n}^2 = 1$. Adding to the action a term $\alpha \int d^2x \, \omega^2$ mitigates it to a logarithmic one.

We have particles φ_i, $i=1,\dots,N$ of the mass m. Consider the scattering process $\varphi_i+\varphi_j\to\varphi_{i'}+\varphi_{j'}$. Let $p_1=m\operatorname{sh}\theta_1$, $p_2=m\operatorname{sh}\theta_2$ be the momenta of the incoming particles, and $p'_1=m\operatorname{sh}\theta'_1$, $p'_2=m\operatorname{sh}\theta'_2$ be the momenta of the outgoing particles. The θ variables are called rapidities of particles.

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$$\begin{aligned} S^{i'j'}_{ij}(\theta_1,\theta_2;\theta'_1,\theta'_2)=(2\pi)^2\delta(p'_1-p_1)\delta(p'_2-p_2)S^{i'j'}_{ij}(\theta_1-\theta_2)\\ +(2\pi)^2\delta(p'_2-p_1)\delta(p'_1-p_2)S^{j'i'}_{ij}(\theta_1-\theta_2).\end{aligned}$$

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+ (2\pi)^2 \delta(p'_2 - p_1)\delta(p'_1 - p_2) S_{ij}^{j'i'}(\theta_1 - \theta_2).
\end{aligned}
$$

We can rewrite the delta-function in terms of the delta-function over space-time momenta:

$$
\begin{aligned}
S_{ij}^{i'j'}(\theta_1, \theta_2; \theta'_1, \theta'_2) &= (2\pi)^2 \delta^{(2)}(P' - P) \frac{\mathrm{sh}(\theta_1 - \theta_2)}{\mathrm{ch}\,\theta_1 \,\mathrm{ch}\,\theta_2} S_{ij}^{i'j'}(\theta_1 - \theta_2) \\
&= (2\pi)^2 \delta^{(2)}(P' - P) \frac{4m^2 \,\mathrm{sh}(\theta_1 - \theta_2)}{4\varepsilon_1 \varepsilon_2} S_{ij}^{i'j'}(\theta_1 - \theta_2),
\end{aligned}
$$

where $P^\mu = p_1^\mu + p_2^\mu$, $P'^\mu = p_1'^\mu + p_2'^\mu$.

Hence

$$S_{ij}^{i'j'}(\theta_1-\theta_2)=\delta_i^{i'}\delta_j^{j'}+\frac{M_{ij}^{i'j'}(\theta_1-\theta_2)}{4m^2\,\mathrm{sh}(\theta_1-\theta_2)}.$$

Hence

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The compatibility condition with the $O(N)$-symmetry gives

$$S_{ij}^{i'j'}(\theta)=\delta_{i'j'}\delta_{ij}S_1(\theta)+\delta_{i'i}\delta_{j'j}S_2(\theta)+\delta_{j'i}\delta_{i'j}S_3(\theta). \tag{18}$$

Calculate the S matrix in the order $1/N$. We will use the formula With the formula

$$D(k) = 4\pi i m^2 \, \frac{\operatorname{sh}\vartheta}{\vartheta}, \quad k^2 = -4m^2 \operatorname{sh}^2 \frac{\vartheta}{2}. \tag{14}$$

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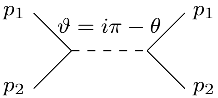

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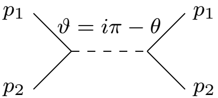

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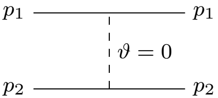

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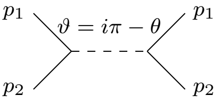

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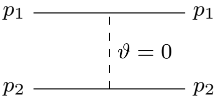

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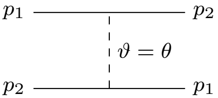

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