Lecture 10
Algebraic Bethe Ansatz. Solving Bethe equations

Michael Lashkevich

Recall the definition of the L-operator:

$$L(u) = R_{0N}(u) \dots R_{02}(u) R_{01}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \qquad (1)$$

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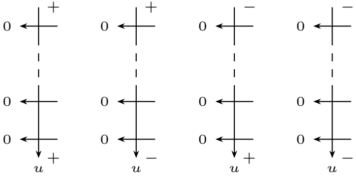
These four operators can be represented graphically as



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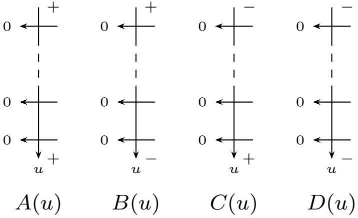

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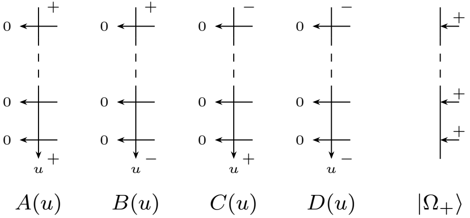

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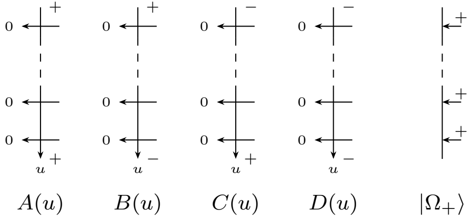

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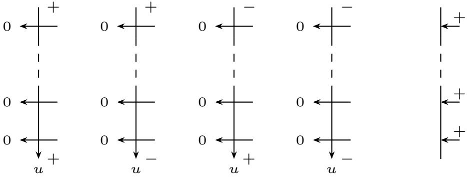

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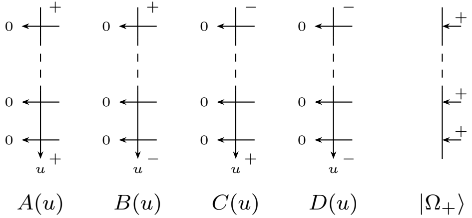

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How to flip spins? By means of $B(u)$ operators.

Indeed, let

$$|u_1, u_2, \dots, u_n\rangle = B(u_1)B(u_2)\dots B(u_n)|\Omega_+\rangle. \tag{2}$$

Then

$$S^z|u_1, \dots, u_n\rangle = \left(\frac{N}{2} - n\right)|u_1, \dots, u_n\rangle.$$

$n = 1$ case

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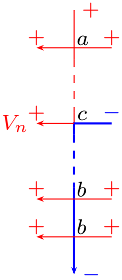

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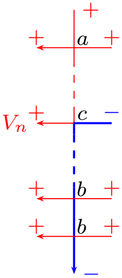

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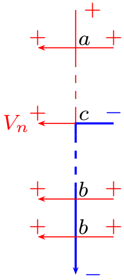

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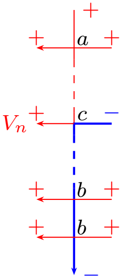

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We see that

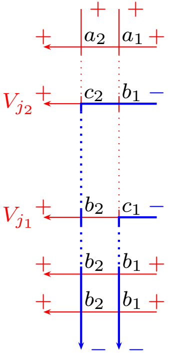
$$B(u)|\Omega_+\rangle \sim \sum_{j=1}^{N} z^j(u)|j\rangle, \qquad z(u) = \frac{b(u)}{a(u)}.$$

It is a Bethe wave function, if $z^N(u) = 1$.

$n = 2$ case

Let $n = 2$. We have $B(u_1)B(u_2)|\Omega_+\rangle =$

$$\sum_{j_1<j_2} \Bigg($$



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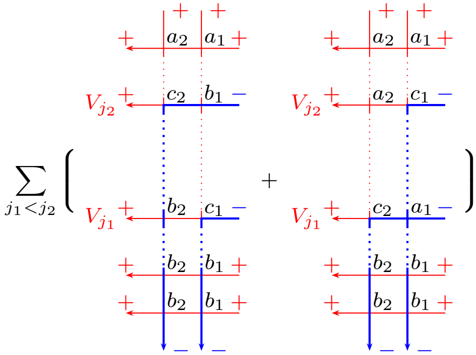

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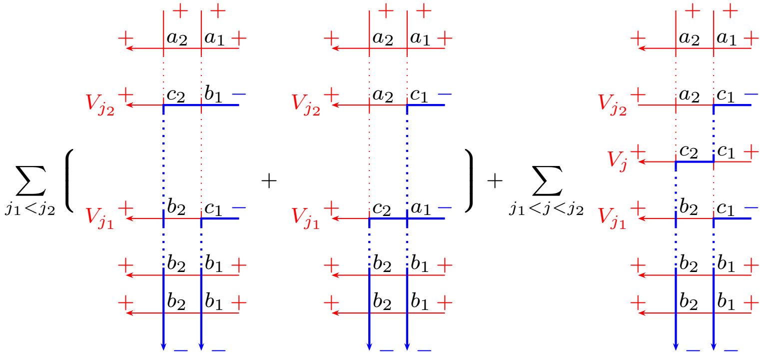

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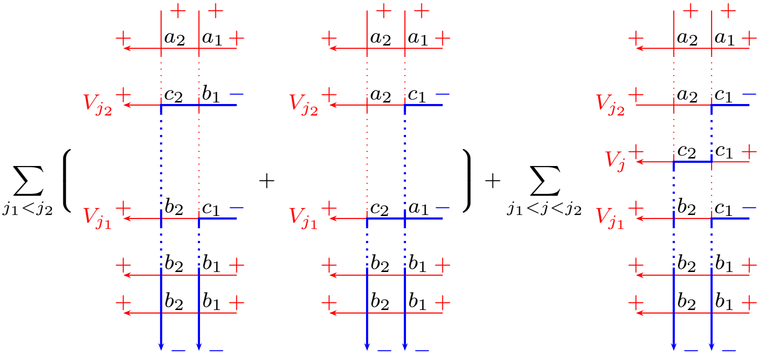

$$= \frac{a_1^N a_2^N c_1 c_2}{b_1 b_2} \sum_{j_1<j_2} \left(\frac{a_{21}}{b_{21}} z_1^{j_1} z_2^{j_2} + \frac{a_{12}}{b_{12}} z_1^{j_2} z_2^{j_1} \right) |j_1, j_2\rangle,$$

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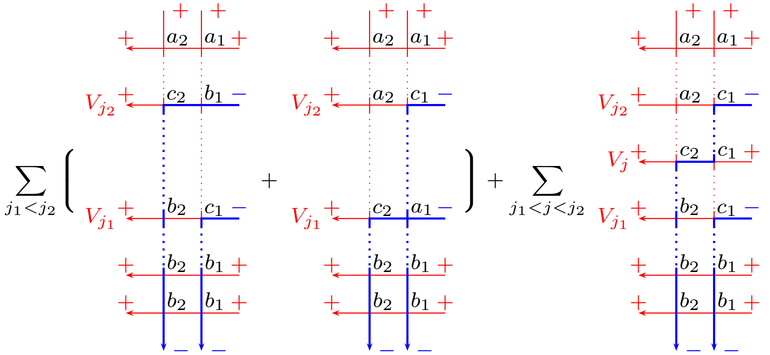

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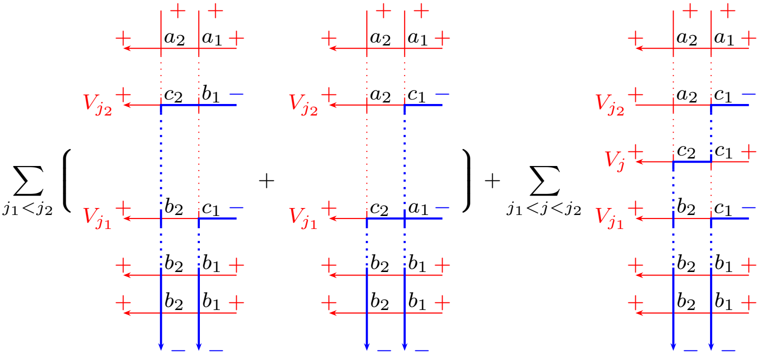

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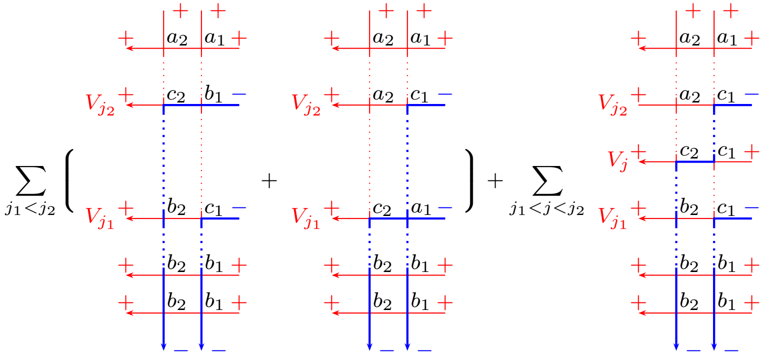

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It can be checked that

$$S(z_1, z_2) = \frac{a_{12}b_{21}}{b_{12}a_{21}}.$$

We want to derive the Bethe equations in an alternative way. Since $T(u) = A(u) + D(u)$, we have to calculate the vectors

$$T(u)|u_1, \ldots, u_k\rangle = (A(u) + D(u))B(u_1)B(u_2)\cdots B(u_k)|\Omega_+\rangle.$$

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First, find the commutation relations of the operators $A(u), \ldots, D(u)$. We have

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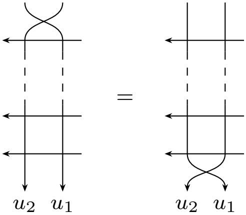

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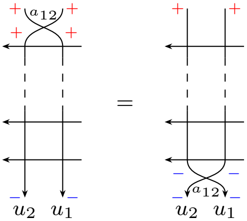

First, the $^{++}_{--}$-component of this relation gives

$$B(u_1)B(u_2) = B(u_2)B(u_1). \tag{3}$$

It means that the states (2) are symmetric in u_i.

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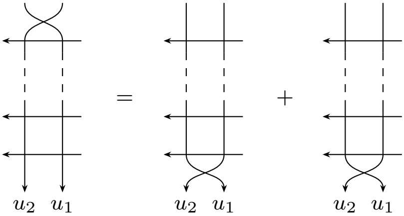

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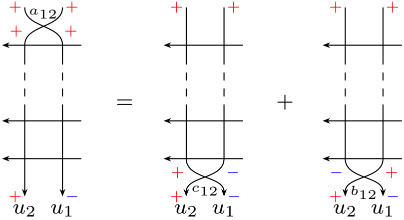

$$= c(u_1 - u_2)B(u_2)A(u_1) + b(u_1 - u_2)A(u_2)B(u_1)$$

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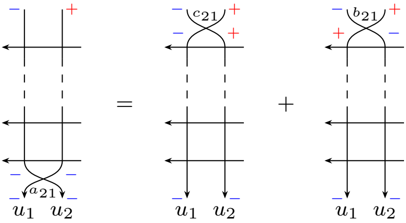

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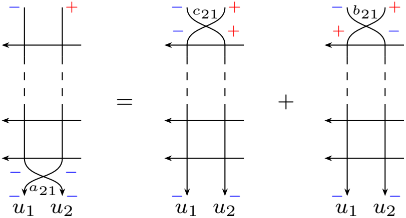

$$= c(u_2 - u_1)B(u_2)D(u_1) + b(u_2 - u_1)D(u_2)B(u_1)$$

Finally we have

$$a(u_1 - u_2)B(u_1)A(u_2) = c(u_1 - u_2)B(u_2)A(u_1) + b(u_1 - u_2)A(u_2)B(u_1), \quad (4)$$

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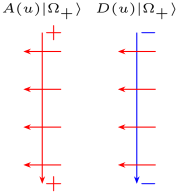

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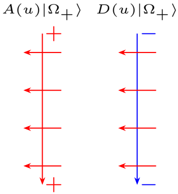

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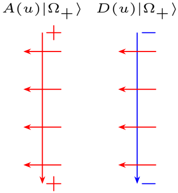

$$\begin{aligned} T(u)|u_1\rangle &= \left(\frac{a(u_1 - u)}{b(u_1 - u)} a^N(u) + \frac{a(u - u_1)}{b(u - u_1)} b^N(u)\right)|u_1\rangle \\ &\quad - \left(\frac{c(u_1 - u)}{b(u_1 - u)} a^N(u_1) + \frac{c(u - u_1)}{b(u - u_1)} b^N(u_1)\right)|u\rangle. \end{aligned}$$

The vector $|u_1\rangle$ is an eigenvector of $T(u)$ it the second term vanishes:

$$\left(\frac{b(u_1)}{a(u_1)}\right)^N = -\left(\frac{c(u_1 - u)b(u - u_1)}{c(u - u_1)b(u_1 - u)}\right) = 1 \quad \Leftrightarrow \quad z_1^N = 1.$$

Action of the transfer matrix: $n = 1$

Apply these formulas to the action of the transfer matrix. Let us start from $k = 1$:

$$\begin{aligned}
T(u)|u_1\rangle &= (A(u) + D(u))B(u_1)|\Omega_+\rangle \\
&= \left(\frac{a(u_1 - u)}{b(u_1 - u)} B(u_1)A(u) + \frac{a(u - u_1)}{b(u - u_1)} B(u_1)D(u) \right. \\
&\quad \left. - \frac{c(u_1 - u)}{b(u_1 - u)} B(u)A(u_1) - \frac{c(u - u_1)}{b(u - u_1)} B(u)D(u_1) \right) |\Omega_+\rangle.
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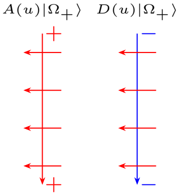

$$\begin{aligned}
T(u)|u_1\rangle = & \left(\frac{a(u_1 - u)}{b(u_1 - u)} a^N(u) + \frac{a(u - u_1)}{b(u - u_1)} b^N(u) \right) |u_1\rangle \\
& - \left(\frac{c(u_1 - u)}{b(u_1 - u)} a^N(u_1) + \frac{c(u - u_1)}{b(u - u_1)} b^N(u_1) \right) |u\rangle.
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The corresponding eigenvector is

$$\Lambda(u; u_1) = a^N(u) \frac{a(u_1 - u)}{b(u_1 - u)} + b^N(u) \frac{a(u - u_1)}{b(u - u_1)}.$$

Action of the transfer matrix: general case

The Takhtajan–Faddeev formulas:

$$\begin{aligned} A(u)|u_1,\ldots,u_n\rangle &= \alpha(u;u_1,\ldots,u_n)|u_1,\ldots,u_n\rangle \\ &\quad - \sum_{i=1}^n \frac{c(u_i-u)}{b(u_i-u)}\alpha(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n)|u,u_1,\ldots,\widehat{u_i},\ldots,u_n\rangle, \\ D(u)|u_1,\ldots,u_n\rangle &= \delta(u;u_1,\ldots,u_n)|u_1,\ldots,u_n\rangle \\ &\quad - \sum_{i=1}^n \frac{c(u-u_i)}{b(u-u_i)}\delta(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n)|u,u_1,\ldots,\widehat{u_i},\ldots,u_n\rangle, \end{aligned} \tag{6}$$

where

$$\alpha(u;u_1,\ldots,u_n) = a^N(u)\prod_{i=1}^n \frac{a(u_i-u)}{b(u_i-u)}, \quad \delta(u;u_1,\ldots,u_n) = b^N(u)\prod_{i=1}^n \frac{a(u-u_i)}{b(u-u_i)}. \tag{7}$$

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Thus the action of the transfer matrix is

$$T(u)|u_1,\ldots,u_n\rangle = \Lambda(u;u_1,\ldots,u_n)|u_1,\ldots,u_n\rangle + \text{unwanted terms},$$

where

$$\Lambda(u;u_1,\ldots,u_n) = \alpha(u;u_1,\ldots,u_n) + \delta(u;u_1,\ldots,u_n)$$

Since $\frac{c(u)}{b(u)} = -\frac{c(-u)}{b(-u)}$, the unwanted terms in the r.h.s. have the form

$$\frac{c(u_i - u)}{b(u_i - u)}(\alpha(u_i; u_1, \ldots, \widehat{u_i}, \ldots, u_n) - \delta(u_i; u_1, \ldots, \widehat{u_i}, \ldots, u_n))|u, u_1, \ldots, \widehat{u_i}, \ldots, u_n\rangle.$$

Bethe equations and transfer matrix eigenvalue

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They vanish, if the Bethe equations are satisfied:

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or

$$\left(\frac{b(u_i)}{a(u_i)}\right)^N = \prod_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{a(u_j - u_i) b(u_i - u_j)}{b(u_j - u_i) a(u_i - u_j)}. \qquad (8)$$

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Subject to these equations the vectors $|u_1,\dots,u_n\rangle$ are eigenvectors with the eigenvalues

$$\Lambda(u;u_1,\dots,u_n) = a^N(u)\prod_{i=1}^{n}\frac{a(u_i-u)}{b(u_i-u)} + b^N(u)\prod_{i=1}^{n}\frac{a(u-u_i)}{b(u-u_i)}. \tag{9}$$

Explicit form of the Bethe equations

Let

$$s(u)=\begin{cases}\sin u & \text{for } |\Delta|<1;\\ \operatorname{sh} u & \text{for } \Delta<-1;\end{cases}\qquad c(u)=\begin{cases}\cos u & \text{for } |\Delta|<1;\\ \operatorname{ch} u & \text{for } \Delta<-1.\end{cases}$$

The explicit form of the Bethe equations:

$$\left(\frac{s(u_i)}{s(\lambda-u_i)}\right)^N=\prod_{\substack{j=1\\(j\neq i)}}^{n}\frac{s(u_i-u_j+\lambda)}{s(u_i-u_j-\lambda)}. \tag{10}$$

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$$u_i=\frac{\lambda}{2}+\mathrm{i}v_i,\quad e^{\mathrm{i}p(v)}=\frac{s(\lambda/2+\mathrm{i}v)}{s(\lambda/2-\mathrm{i}v)},\quad e^{\mathrm{i}\theta(v)}=\frac{s(\lambda+\mathrm{i}v)}{s(\lambda-\mathrm{i}v)}.$$

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$$u_i = \frac{\lambda}{2} + \mathrm{i} v_i, \quad e^{\mathrm{i}p(v)} = \frac{s(\lambda/2 + \mathrm{i}v)}{s(\lambda/2 - \mathrm{i}v)}, \quad e^{\mathrm{i}\theta(v)} = \frac{s(\lambda + \mathrm{i}v)}{s(\lambda - \mathrm{i}v)}.$$

The variables v_i are defined in such a way that $|z_i| = 1$ for real values of v_i. Take logarithm of the Bethe equations:

$$Np(v_i) = 2\pi I_i + \sum_{j=1}^{n} \theta(v_i - v_j),$$

where $I_i \in \mathbb{Z} + \frac{1}{2}$ if $n \in 2\mathbb{Z}$ and $I_i \in \mathbb{Z}$ if $n \in 2\mathbb{Z} + 1$.

Solving the Bethe equations for the ground state

Let us obtain the ground state, i.e. the state with the largest eigenvalue of the transfer matrices in the thermodynamic limit.

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$$p'(v) = \rho(v) + \int_{-v_F}^{v_F} \frac{dv'}{2\pi}\,\theta'(v-v')\rho(v'), \qquad \int_{-v_F}^{v_F} \frac{dv}{2\pi}\,\rho(v) = \frac{n}{N}, \tag{11}$$

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$$\rho_k = \frac{p'_k}{1 + \theta'_k} = \frac{1}{2 \operatorname{ch} \frac{\lambda k}{2}}$$
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We have for the density
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in both cases. Then
$$\frac{n}{N} = \int_{-\bar{v}_F}^{\bar{v}_F} \frac{dv}{2\pi}\, \rho(v) = \rho_0 = \frac{1}{2} \quad \Rightarrow \quad \frac{S^z}{N} \to 0.$$

Recall the expression for the eigenvalue

$$\Lambda(u; u_1, \ldots, u_n) = a^N(u) \prod_{i=1}^{n} \frac{a(u_i - u)}{b(u_i - u)} + b^N(u) \prod_{i=1}^{n} \frac{a(u - u_i)}{b(u - u_i)}. \qquad (9)$$

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Now let us calculate the free energy per vertex of the six-vertex model:

$$f = -\lim_{N\to\infty} \frac{\log \Lambda_{\max}(u)}{N} = -\max\Bigg(\log a(u) + \int_{-v_F}^{v_F} \frac{dv}{2\pi}\, \rho(v) \log \frac{a(iv - u + \lambda/2)}{b(iv - u + \lambda/2)},$$

$$\log b(u) + \int_{-v_F}^{v_F} \frac{dv}{2\pi}\, \rho(v) \log \frac{a(u - iv - \lambda/2)}{b(u - iv - \lambda/2)}\Bigg).$$

Recall the expression for the eigenvalue

$$\Lambda(u;u_1,\dots,u_n)=a^N(u)\prod_{i=1}^n\frac{a(u_i-u)}{b(u_i-u)}+b^N(u)\prod_{i=1}^n\frac{a(u-u_i)}{b(u-u_i)}. \tag{9}$$

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For $v_F=\bar v_F$ we can use the Fourier transform. For $|\Delta|<1$ we have

$$f=\min\left(-\log a(u)-\int\frac{dk}{k}\rho_{-k}p'_k e^{ku},-\log b(u)-\int\frac{dk}{k}\rho_k p'_k e^{k(\lambda-u)}\right).$$

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By symmetrizing the we find that the two alternatives coincide, so that

$$\begin{aligned}f&=-\log a(u)-\int_0^\infty\frac{dk}{k}\,\frac{\operatorname{sh}uk\operatorname{sh}\frac{\pi-\lambda}{2}k}{\operatorname{sh}\frac{\pi}{2}k\operatorname{ch}\frac{\lambda}{2}k}\\&=-\log b(u)-\int_0^\infty\frac{dk}{k}\,\frac{\operatorname{sh}(\lambda-u)k\operatorname{sh}\frac{\pi-\lambda}{2}k}{\operatorname{sh}\frac{\pi}{2}k\operatorname{ch}\frac{\lambda}{2}k}.\end{aligned} \tag{14}$$

Free energy

In the case $\Delta < -1$ the free energy reads

$$f = \min\left(-\log a(u) - \frac{1}{\pi}\sum_{k\in 2Z}\frac{1}{k}\rho_{-k}p'_k e^{ku}, -\log b(u) - \frac{1}{\pi}\sum_{k\in 2Z}\frac{1}{k}\rho_k p'_k e^{k(\lambda-u)}\right).$$

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Why are these two cases so different?

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Why are these two cases so different? Because in the case $|\Delta| < 1$ there is a gapless spectrum, while in the case $\Delta < -1$ there is a gap between the two largest eigenvalues of $T(u)$ and all other eigenvalues.

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What if $v_F < \bar{v}_F$? This case corresponds to general homogeneous six-vertex model with arbitrary a, a', b, b', c, c'. The ratio c/c' is inessential, but nonunit rations a/a', b/b' correspond to an external field. They can be related to v_F. The integral equations do not have an analytic solution, but can be solved numerically. The two alternatives for the free energy are different.

Seminar