Lecture 3
Algebraic Bethe Ansatz

A mini-course "Solvable lattice models and Bethe Ansatz"
(Ariel University, spring 2021)

Michael Lashkevich

Landau Institute for Theoretical Physics,
Kharkevich Institute for Information Transmission Problems

Recall the definition of the L-operator:

$$L(u) = R_{0N}(u) \dots R_{02}(u) R_{01}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \quad (1)$$

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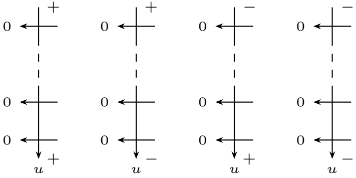
These four operators can be represented graphically as



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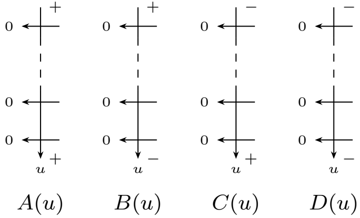

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$$[S^z, A(u)] = [S^z, D(u)] = 0, \quad [S^z, B(u)] = -B(u), \quad [S^z, C(u)] = C(u).$$

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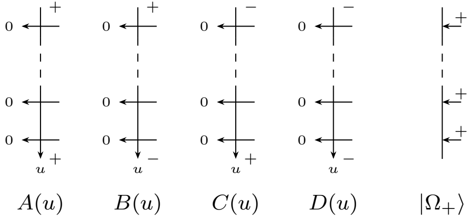

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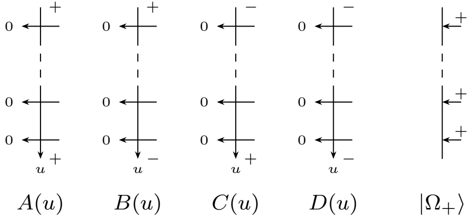

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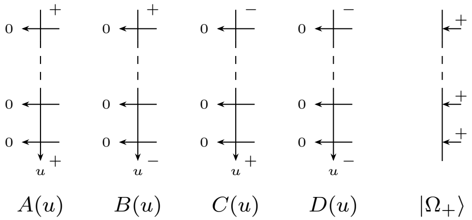

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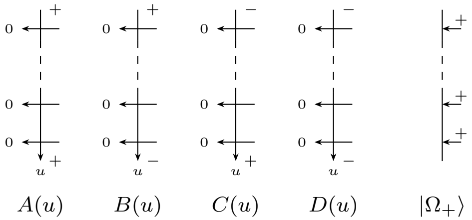

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How to flip spins? By means of $B(u)$ operators.

$n = 1$ case

Indeed, let

$$|u_1, u_2, \dots, u_k\rangle = B(u_1)B(u_2)\dots B(u_k)|\Omega_+\rangle. \tag{2}$$

Then

$$S^z|u_1, \dots, u_k\rangle = \left(\frac{N}{2} - k\right)|u_1, \dots, u_k\rangle.$$

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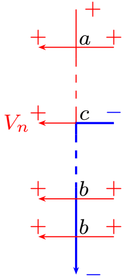

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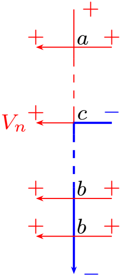

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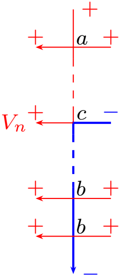

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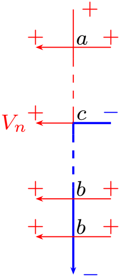

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We see that

$$B(u)|\Omega_+\rangle \sim \sum_{n=1}^{N} z^n(u)|n\rangle, \qquad z(u) = \frac{b(u)}{a(u)}.$$

It is a Bethe wave function, if $z^N(u) = 1$.

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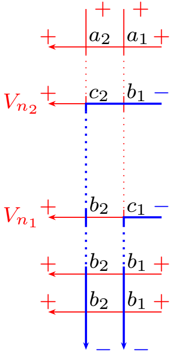

where $a_i = a(u_i)$, $a_{ij} = a(u_i - u_j)$ and so on, $z_i = b_i/a_i$.

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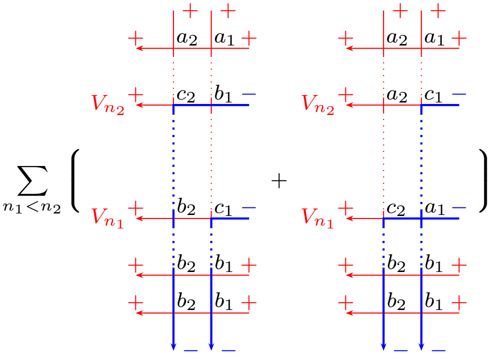

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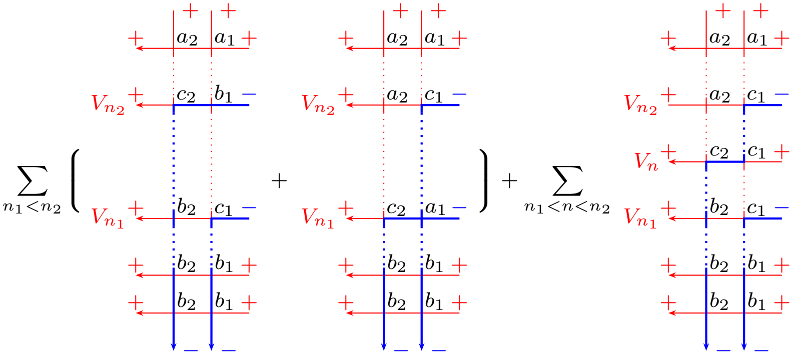

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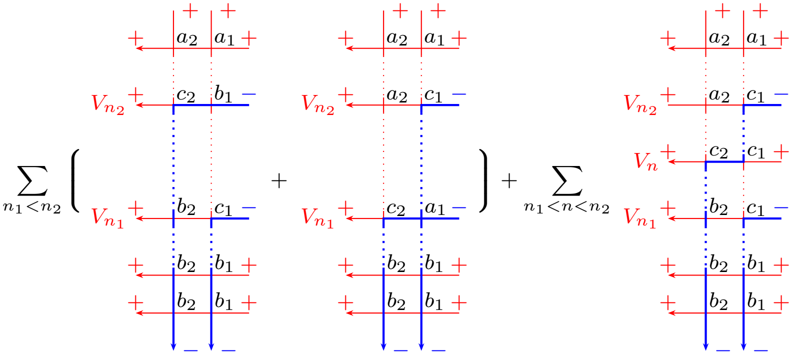

$$= \frac{a_1^N a_2^N c_1 c_2}{b_1 b_2} \sum_{n_1<n_2} \left(\frac{a_{21}}{b_{21}} z_1^{n_1} z_2^{n_2} + \frac{a_{12}}{b_{12}} z_1^{n_2} z_2^{n_1} \right) |n_1, n_2\rangle,$$

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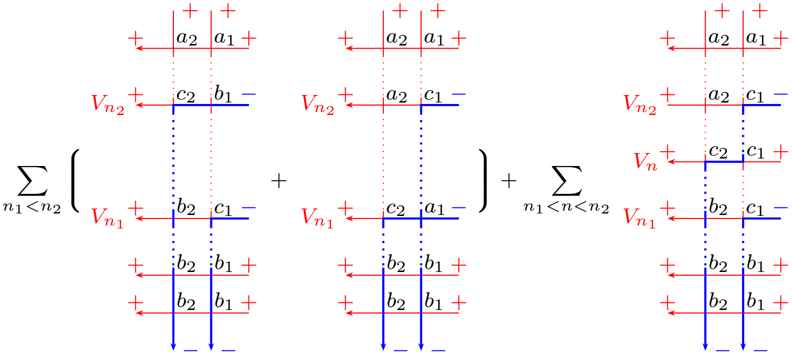

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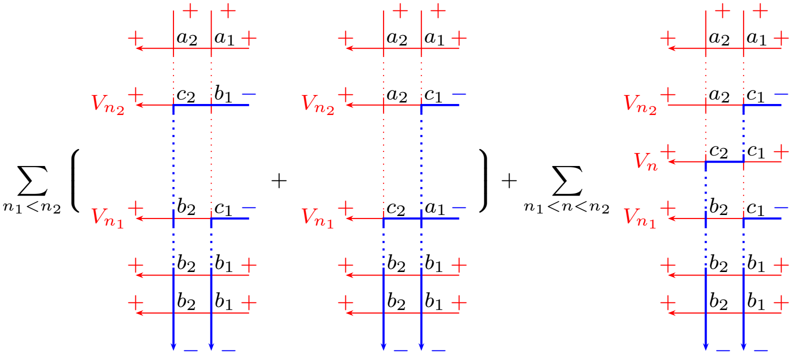

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$$S(z_1, z_2) = \frac{a_{12} b_{21}}{b_{12} a_{21}}.$$

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First, find the commutation relations of the operators $A(u), \ldots, D(u)$. We have

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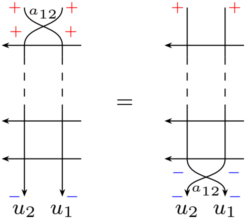

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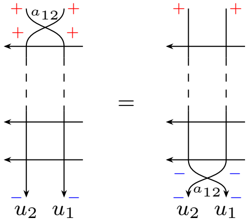

First, the $^{++}_{--}$-component of this relation gives

$$B(u_1)B(u_2) = B(u_2)B(u_1). \tag{3}$$

It means that the states (2) are symmetric in u_i.

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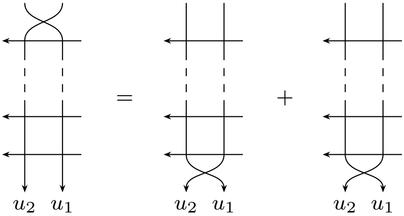

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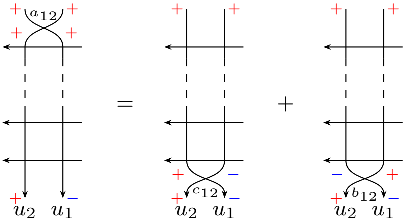

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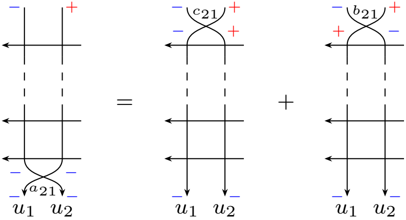

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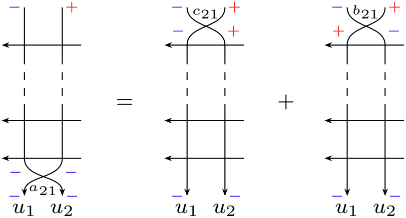

$$= c(u_2 - u_1)B(u_2)D(u_1) + b(u_2 - u_1)D(u_2)B(u_1)$$

Finally we have

$$a(u_1 - u_2)B(u_1)A(u_2) = c(u_1 - u_2)B(u_2)A(u_1) + b(u_1 - u_2)A(u_2)B(u_1), \quad (4)$$
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$$\begin{aligned}
T(u)|u_1\rangle &= (A(u)+D(u))B(u_1)|\Omega_+\rangle \\
&= \left(\frac{a(u_1-u)}{b(u_1-u)}B(u_1)A(u) + \frac{a(u-u_1)}{b(u-u_1)}B(u_1)D(u)\right. \\
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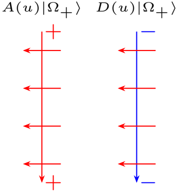

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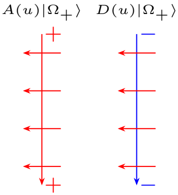

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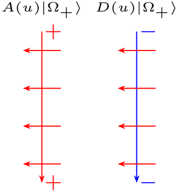
Taking into account that $A(u)|\Omega_+\rangle = a^N(u)|\Omega_+\rangle$, $D(u)|\Omega_+\rangle = b^N(u)|\Omega_+\rangle$. We conclude that



$$\begin{aligned} T(u)|u_1\rangle &= \left(\frac{a(u_1 - u)}{b(u_1 - u)} a^N(u) + \frac{a(u - u_1)}{b(u - u_1)} b^N(u) \right) |u_1\rangle \\ &\quad - \left(\frac{c(u_1 - u)}{b(u_1 - u)} a^N(u_1) + \frac{c(u - u_1)}{b(u - u_1)} b^N(u_1) \right) |u\rangle. \end{aligned}$$

The vector $|u_1\rangle$ is an eigenvector of $T(u)$ it the second term vanishes:

$$\left(\frac{b(u_1)}{a(u_1)} \right)^N = - \left(\frac{c(u_1 - u)b(u - u_1)}{c(u - u_1)b(u_1 - u)} \right) = 1 \quad \Leftrightarrow \quad z_1^N = 1.$$

Action of the transfer matrix: $n = 1$

Apply these formulas to the action of the transfer matrix. Let us start from $n = 1$:

$$\begin{aligned} T(u)|u_1\rangle &= (A(u) + D(u))B(u_1)|\Omega_+\rangle \\ &= \left(\frac{a(u_1 - u)}{b(u_1 - u)} B(u_1)A(u) + \frac{a(u - u_1)}{b(u - u_1)} B(u_1)D(u) \right. \\ &\qquad \left. - \frac{c(u_1 - u)}{b(u_1 - u)} B(u)A(u_1) - \frac{c(u - u_1)}{b(u - u_1)} B(u)D(u_1) \right) |\Omega_+\rangle. \end{aligned}$$

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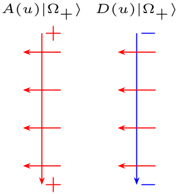

$$\begin{aligned} T(u)|u_1\rangle = &\left(\frac{a(u_1 - u)}{b(u_1 - u)} a^N(u) + \frac{a(u - u_1)}{b(u - u_1)} b^N(u) \right) |u_1\rangle \\ &- \left(\frac{c(u_1 - u)}{b(u_1 - u)} a^N(u_1) + \frac{c(u - u_1)}{b(u - u_1)} b^N(u_1) \right) |u\rangle. \end{aligned}$$

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The corresponding eigenvector is

$$\Lambda(u; u_1) = a^N(u) \frac{a(u_1 - u)}{b(u_1 - u)} + b^N(u) \frac{a(u - u_1)}{b(u - u_1)}.$$

Action of the transfer matrix: general case

The Takhtajan–Faddeev formulas:

$$
\begin{aligned}
A(u)|u_1,\ldots,u_n\rangle &= \alpha(u;u_1,\ldots,u_n)|u_1,\ldots,u_n\rangle \\
&\quad - \sum_{i=1}^{n} \frac{c(u_i-u)}{b(u_i-u)}\alpha(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n)|u,u_1,\ldots,\widehat{u_i},\ldots,u_n\rangle, \\
D(u)|u_1,\ldots,u_n\rangle &= \delta(u;u_1,\ldots,u_n)|u_1,\ldots,u_n\rangle \\
&\quad - \sum_{i=1}^{n} \frac{c(u-u_i)}{b(u-u_i)}\delta(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n)|u,u_1,\ldots,\widehat{u_i},\ldots,u_n\rangle,
\end{aligned}
\tag{6}
$$

where

$$
\alpha(u;u_1,\ldots,u_n) = a^N(u)\prod_{i=1}^{n}\frac{a(u_i-u)}{b(u_i-u)}, \qquad \delta(u;u_1,\ldots,u_n) = b^N(u)\prod_{i=1}^{n}\frac{a(u-u_i)}{b(u-u_i)}.
\tag{7}
$$

Action of the transfer matrix: general case

The Takhtajan–Faddeev formulas:

$$\begin{aligned} A(u)|u_1,\ldots,u_n\rangle &= \alpha(u;u_1,\ldots,u_n)|u_1,\ldots,u_n\rangle \\ &\quad - \sum_{i=1}^n \frac{c(u_i-u)}{b(u_i-u)}\alpha(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n)|u,u_1,\ldots,\widehat{u_i},\ldots,u_n\rangle, \\ D(u)|u_1,\ldots,u_n\rangle &= \delta(u;u_1,\ldots,u_n)|u_1,\ldots,u_n\rangle \\ &\quad - \sum_{i=1}^n \frac{c(u-u_i)}{b(u-u_i)}\delta(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n)|u,u_1,\ldots,\widehat{u_i},\ldots,u_n\rangle, \end{aligned} \tag{6}$$

where

$$\alpha(u;u_1,\ldots,u_n) = a^N(u)\prod_{i=1}^n \frac{a(u_i-u)}{b(u_i-u)}, \qquad \delta(u;u_1,\ldots,u_n) = b^N(u)\prod_{i=1}^n \frac{a(u-u_i)}{b(u-u_i)}. \tag{7}$$

Thus the action of the transfer matrix is

$$T(u)|u_1,\ldots,u_n\rangle = \Lambda(u;u_1,\ldots,u_n)|u_1,\ldots,u_n\rangle + \text{unwanted terms},$$

where

$$\Lambda(u;u_1,\ldots,u_n) = \alpha(u;u_1,\ldots,u_n) + \delta(u;u_1,\ldots,u_n)$$

Bethe equations and transfer matrix eigenvalue

Since $\frac{c(u)}{b(u)} = -\frac{c(-u)}{b(-u)}$, the unwanted terms in the r.h.s. have the form

$$\frac{c(u_i - u)}{b(u_i - u)}(\alpha(u_i; u_1, \ldots, \widehat{u_i}, \ldots, u_n) - \delta(u_i; u_1, \ldots, \widehat{u_i}, \ldots, u_n))|u, u_1, \ldots, \widehat{u_i}, \ldots, u_n\rangle.$$

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They vanish, if the Bethe equations are satisfied:

$$\alpha(u_i; u_1, \dots, \widehat{u_i}, \dots, u_n) = \delta(u_i; u_1, \dots, \widehat{u_i}, \dots, u_n)$$

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$$\frac{c(u_i-u)}{b(u_i-u)}(\alpha(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n)-\delta(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n))|u,u_1,\ldots,\widehat{u_i},\ldots,u_n\rangle.$$

They vanish, if the Bethe equations are satisfied:

$$\alpha(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n)=\delta(u_i;u_1,\ldots,\widehat{u_i},\ldots,u_n)$$

or

$$\left(\frac{b(u_i)}{a(u_i)}\right)^N=\prod_{\substack{j=1\\(j\neq i)}}^{n}\frac{a(u_j-u_i)b(u_i-u_j)}{b(u_j-u_i)a(u_i-u_j)}. \tag{8}$$

Since $\frac{c(u)}{b(u)} = -\frac{c(-u)}{b(-u)}$, the unwanted terms in the r.h.s. have the form

$$\frac{c(u_i - u)}{b(u_i - u)}(\alpha(u_i; u_1, \ldots, \widehat{u_i}, \ldots, u_n) - \delta(u_i; u_1, \ldots, \widehat{u_i}, \ldots, u_n))|u, u_1, \ldots, \widehat{u_i}, \ldots, u_n\rangle.$$

They vanish, if the Bethe equations are satisfied:

$$\alpha(u_i; u_1, \ldots, \widehat{u_i}, \ldots, u_n) = \delta(u_i; u_1, \ldots, \widehat{u_i}, \ldots, u_n)$$

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$$\left(\frac{b(u_i)}{a(u_i)}\right)^N = \prod_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{a(u_j - u_i) b(u_i - u_j)}{b(u_j - u_i) a(u_i - u_j)}. \tag{8}$$

Subject to these equations the vectors $|u_1, \ldots, u_n\rangle$ are eigenvectors with the eigenvalues

$$\Lambda(u; u_1, \ldots, u_n) = a^N(u) \prod_{i=1}^{n} \frac{a(u_i - u)}{b(u_i - u)} + b^N(u) \prod_{i=1}^{n} \frac{a(u - u_i)}{b(u - u_i)}. \tag{9}$$

Explicit form of the Bethe equations

Let

$$s(u)=\begin{cases}\sin u & \text{for } |\Delta|<1;\\ \operatorname{sh} u & \text{for } \Delta<-1;\end{cases}\qquad c(u)=\begin{cases}\cos u & \text{for } |\Delta|<1;\\ \operatorname{ch} u & \text{for } \Delta<-1.\end{cases}$$

The explicit form of the Bethe equations:

$$\left(\frac{s(u_i)}{s(\lambda-u_i)}\right)^N=\prod_{\substack{j=1\\(j\neq i)}}^{n}\frac{s(u_i-u_j+\lambda)}{s(u_i-u_j-\lambda)}. \tag{10}$$

Explicit form of the Bethe equations

Let

$$s(u) = \begin{cases} \sin u & \text{for } |\Delta| < 1; \\ \operatorname{sh} u & \text{for } \Delta < -1; \end{cases} \qquad c(u) = \begin{cases} \cos u & \text{for } |\Delta| < 1; \\ \operatorname{ch} u & \text{for } \Delta < -1. \end{cases}$$

The explicit form of the Bethe equations:

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Let

$$u_i = \frac{\lambda}{2} + \mathrm{i} v_i, \quad e^{\mathrm{i}p(v)} = \frac{s(\lambda/2 + \mathrm{i}v)}{s(\lambda/2 - \mathrm{i}v)}, \quad e^{\mathrm{i}\theta(v)} = \frac{s(\lambda + \mathrm{i}v)}{s(\lambda - \mathrm{i}v)}.$$

The variables v_i are defined in such a way that $|z_i| = 1$ for real values of v_i.

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$$u_i=\frac{\lambda}{2}+\mathrm{i}v_i,\quad e^{\mathrm{i}p(v)}=\frac{s(\lambda/2+\mathrm{i}v)}{s(\lambda/2-\mathrm{i}v)},\quad e^{\mathrm{i}\theta(v)}=\frac{s(\lambda+\mathrm{i}v)}{s(\lambda-\mathrm{i}v)}.$$

The variables v_i are defined in such a way that $|z_i|=1$ for real values of v_i. Take logarithm of the Bethe equations:

$$Np(v_i)=2\pi I_i+\sum_{j=1}^{n}\theta(v_i-v_j),$$

where $I_i\in\mathbb{Z}+\frac{1}{2}$ if $n\in 2\mathbb{Z}$ and $I_i\in\mathbb{Z}$ if $n\in 2\mathbb{Z}+1$.