# Lecture 2. Bosonization of the Thirring model

Michael Lashkevich

The *massive Thirring model* in the Minkowski space:

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Excitations: fermion, antifermion, and for $g>0$ neutral boson bound states.

# Sine-Gordon model

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$$j^\mu_{\text{top}} = -\frac{\beta}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi, \quad \partial_\mu j^\mu_{\text{top}} = -\frac{\beta}{2\pi} \epsilon^{\mu\nu} \partial_\mu \partial_\nu \phi = 0. \tag{8}$$

# Digression: Stationary solutions of field equations

Consider a two-dimensional model of one scalar field with the action

$$S[\phi] = \int d^2x \left( \frac{(\partial_\mu \phi)^2}{2} - U(\phi) \right).$$

Suppose that the potential $U(\phi)$ possesses a set of degenerate absolute minima $\phi_i$. Let us order these minima so that $\phi_i < \phi_{i+1}$.

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If there is a static solution, we may define a family of solutions moving with any velocity $-1 < v < 1$:

$$\phi(x) = \varphi\left( \frac{x^1 - v x^0}{\sqrt{1 - v^2}} \right).$$

## Digression: Classical solutions of the sine-Gordon equation

There are kink ($q=1$) and antikink ($q=-1$) solutions:

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$$\phi(x) = 4\,\mathrm{arctg}\frac{\sqrt{1-\omega^2}\cos\omega\tau}{\omega\cos\sqrt{1-\omega^2}\xi}, \quad \tau = \frac{x^0 - vx^1 - x_0^0}{\sqrt{1-v^2}}, \quad \xi = \frac{x^1 - vx^0 - x_0^1}{\sqrt{1-v^2}},$$

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One may expect the quantum spectrum to consist of a kink ($q = 1$), antikink ($q = -1$) and a discrete series of breathers. The density of the mass spectrum of breathers decreases with increasing $\beta$.

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The conserved currents also coincide

$$j^\mu = j^\mu_{\mathrm{top}}, \tag{11}$$

so that the fermion number in the Thirring model coincides with the topological number in the sine-Gordon model.

Rewrite the action in terms of the light cone variables:

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The equations of motion (13) have the solution

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But the exponential form of $F_i$ seems to be strange. To understand them consider the operator product expansions of $\psi_i$.

We have

$$\psi_i(x')\psi_j(x) = \eta_i\eta_j \frac{\sqrt{N_i N_j}}{2\pi}(z'-z)^{\alpha_i\alpha_j}(\bar z'-\bar z)^{\beta_i\beta_j} \times e^{i\alpha_i\varphi(z')+i\beta_i\bar\varphi(\bar z')+i\alpha_j\varphi(z)+i\beta_j\bar\varphi(\bar z)}. \quad (21)$$

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# Massless Thirring model: conditions for $\alpha_i, \beta_i, N_i$

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We need one more equation. To find it, we have to analyze the mass term.

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Now check the consistency with the conservancy of the fermion charge $Q$ means that it should be a function of $\phi$.

Consider now the mass term $-m\bar\psi\psi = im(\psi_1^+\psi_2 - \psi_2^+\psi_1)$. Both terms must be well-defined and be consistent with the conservancy of the fermion charge. Consider the expansion

$$\psi_2^+(x')\psi_1(x) = -\eta_1\eta_2^{-1}\frac{\sqrt{N_1N_2}}{2\pi}(z'-z)^{-\alpha_1\alpha_2}(\bar z'-\bar z)^{-\beta_1\beta_2} \times\left(e^{i(\alpha_1-\alpha_2)\varphi(z)+i(\beta_1-\beta_2)\bar\varphi(\bar z)}+\cdots\right). \quad (28)$$

The first term survives in the averaged product defined similarly to $j_z$, if

$$\alpha_1\alpha_2 = \beta_1\beta_2 \quad\Rightarrow\quad \alpha_1 = -\beta_2. \quad (29)$$

With this assumption we may define

$$\psi_2^+\psi_1(x) = -\eta_1\eta_2^{-1}\frac{\sqrt{N_1N_2}}{2\pi}r_0^{-2\alpha_1\alpha_2}e^{i(\alpha_1-\alpha_2)\varphi(z)+i(\beta_1-\beta_2)\bar\varphi(\bar z)}.$$

Now check the consistency with the conservancy of the fermion charge $Q$ means that it should be a function of $\phi$. Consider the commutator

$$[O(0),Q] = \oint df_\mu\, j^\mu(x)O(0) = \oint dx^\nu\,\epsilon_{\mu\nu}j^\mu(x)O(0) = -\frac{\beta}{2\pi}\oint dx^\nu\,\epsilon_{\mu\nu}\partial^\mu\tilde\phi(x)\,O(0)$$

$$= -\frac{\beta}{2\pi}\oint dx^\nu\,\epsilon_{\mu\nu}\epsilon^{\mu\lambda}\partial_\lambda\phi(x)\,O(0) = \frac{\beta}{2\pi}\oint dx^\lambda\,\partial_\lambda\phi(x)\,O(0) = \frac{\beta}{2\pi}\Delta\phi(x)\,O(0),$$

where $\Delta$ means the increment of the field while passing the closed contour.

# Thirring model: mass term

Let $O(x) = e^{\alpha\varphi(z)+\alpha'\bar\varphi(\bar z)}$. Then

$$[O(0), Q] = \frac{\beta}{2\pi}\Delta(\varphi(z)+\bar\varphi(\bar z))e^{i\alpha\varphi(z)+i\alpha'\bar\varphi(\bar z)}$$

$$= \frac{i\beta}{2\pi}\Delta\left(\alpha\log\frac{1}{z}+\alpha'\log\frac{1}{\bar z}\right)e^{i\alpha\varphi(0)+i\alpha'\bar\varphi(0)} = \beta(\alpha-\alpha')e^{i\alpha\varphi(0)+i\alpha'\bar\varphi(0)}.$$

Let $O(x) = e^{\alpha\varphi(z)+\alpha'\bar\varphi(\bar z)}$. Then

$$[O(0), Q] = \frac{\beta}{2\pi}\Delta(\varphi(z) + \bar\varphi(\bar z))e^{i\alpha\varphi(z)+i\alpha'\bar\varphi(\bar z)}$$
$$= \frac{i\beta}{2\pi}\Delta\left(\alpha \log\frac{1}{z} + \alpha' \log\frac{1}{\bar z}\right) e^{i\alpha\varphi(0)+i\alpha'\bar\varphi(0)} = \beta(\alpha - \alpha')e^{i\alpha\varphi(0)+i\alpha'\bar\varphi(0)}.$$

Hence, the operator $Q$ is commuting with $\psi_2^+\psi_1$, since $\alpha_1 - \alpha_2 = \beta_1 - \beta_2$.

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Hence, the operator $Q$ is commuting with $\psi_2^+\psi_1$, since $\alpha_1-\alpha_2=\beta_1-\beta_2$. Now fix the $\beta$ parameter. Set $O(x)=\psi_i(x)$. Since the operators $\psi_i$ have the fermion charge $-1$, we have

$$\psi_i(0)=[\psi_i(0),Q]=\beta(\alpha_i-\beta_i)\psi_i(0)=\beta(\alpha_1+\alpha_2)\psi_i(0)$$

Let $O(x) = e^{\alpha\varphi(z)+\alpha'\bar{\varphi}(\bar{z})}$. Then

$$[O(0), Q] = \frac{\beta}{2\pi}\Delta(\varphi(z) + \bar{\varphi}(\bar{z}))e^{i\alpha\varphi(z)+i\alpha'\bar{\varphi}(\bar{z})}$$
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Hence, the operator $Q$ is commuting with $\psi_2^+\psi_1$, since $\alpha_1 - \alpha_2 = \beta_1 - \beta_2$. Now fix the $\beta$ parameter. Set $O(x) = \psi_i(x)$. Since the operators $\psi_i$ have the fermion charge $-1$, we have

$$\psi_i(0) = [\psi_i(0), Q] = \beta(\alpha_i - \beta_i)\psi_i(0) = \beta(\alpha_1 + \alpha_2)\psi_i(0)$$

Hence

$$\alpha_1 + \alpha_2 = \beta^{-1} \quad \Rightarrow \quad \alpha_1 - \alpha_2 = \beta.$$

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Hence

$$\alpha_1 = -\beta_2 = \frac{1}{2}\left(\frac{1}{\beta}+\beta\right),$$
$$\alpha_2 = -\beta_1 = \frac{1}{2}\left(\frac{1}{\beta}-\beta\right). \tag{30}$$

Let $O(x) = e^{\alpha\varphi(z)+\alpha'\bar\varphi(\bar z)}$. Then

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Using $\alpha_2 = g\beta/2\pi$ we obtain the relation between coupling constants:

$$g = \pi(\beta^{-2} - 1). \tag{9}$$

Substituting it we obtain

$$N_1 = -N_2 = ir_0^{\frac{\beta^2}{2}+\frac{1}{2\beta^2}-1}\frac{2\beta^2}{\beta^2+1}, \tag{31}$$

Substituting it we obtain

$$N_1 = -N_2 = i r_0^{\frac{\beta^2}{2} + \frac{1}{2\beta^2} - 1} \frac{2\beta^2}{\beta^2 + 1}, \tag{31}$$

Hence

$$-i\psi_2^+ \psi_1 = \frac{1}{\pi} \frac{\beta^2}{\beta^2 + 1} r_0^{\beta^2 - 1} \left( i\eta_1 \eta_2^{-1} \right) e^{i\beta\phi},$$

$$i\psi_1^+ \psi_2 = \frac{1}{\pi} \frac{\beta^2}{\beta^2 + 1} r_0^{\beta^2 - 1} \left( i\eta_1 \eta_2^{-1} \right)^{-1} e^{-i\beta\phi}.$$

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$$\begin{aligned} -i\psi_2^+\psi_1 &= \frac{1}{\pi}\frac{\beta^2}{\beta^2+1}r_0^{\beta^2-1}\left(i\eta_1\eta_2^{-1}\right)e^{i\beta\phi}, \\ i\psi_1^+\psi_2 &= \frac{1}{\pi}\frac{\beta^2}{\beta^2+1}r_0^{\beta^2-1}\left(i\eta_1\eta_2^{-1}\right)^{-1}e^{-i\beta\phi}. \end{aligned}$$

Consider the mass contribution

$$im(\psi_2^+\psi_1 - \psi_1^+\psi_2) \sim \left(i\eta_1\eta_2^{-1}\right)e^{i\beta\phi} + \left(i\eta_1\eta_2^{-1}\right)^{-1}e^{-i\beta\phi}$$

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Substituting it we obtain

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from which we find

$$\mu \sim m r_0^{\beta^2-1}, \tag{10}$$

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