Lecture 3.
Renormalization group for the Berezinskii–Kosterlitz–Thouless transition

Michael Lashkevich

Renormalization group approach

Suppose we consider a field theory system with the correlation length r_c. It is described by a bare action defined the at the UV cutoff r_0, which depends on the set of parameters λ_0. We are interested in correlations functions on a scale r, $r_0 \ll r \ll r_c$. Let $G_{\mathrm{exact}}(\lambda_0, r_0; \cdots)$ be exact correlation functions calculated in all orders of the perturbation theory.

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$$\delta = \beta^2 - 2 \ll 1.$$

Bare and dressed sine-Gordon action

The bare action of the sine-Gordon model on the Euclidean plane:

$$S_{\rm SG}[\phi] = \int d^2x \left(\frac{(\partial_\mu \phi)^2}{8\pi} - \alpha_0 r_0^{\beta_0^2 - 2} \cos \beta_0 \phi \right), \tag{1}$$

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For ultraviolet regularization, we will replace x^2 by $x^2 + r_0^2$. Then for $m_0^2 x^2 \ll 1$ the free field propagator (with $\alpha_0 = 0$) is equal to

$$G_0(x - x') = \log \frac{R_0^2}{(x - x')^2 + r_0^2}, \qquad R_0 = (cm_0)^{-1}, \qquad c = e^{\gamma_E}/2. \tag{3}$$

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such that $S_{\text{SG}}[\phi] = S^R_{\text{SG}}[Z_\phi^{-1/2}\phi] + S^{\text{ct}}[Z_\phi^{-1/2}\phi]$. Assume that the counterterms

$$S^{\text{ct}}[\phi] = \int d^2x \, (\#(\partial_\mu\phi)^2 + \# \cos \beta\phi).$$

do not contain a counterterm for the auxiliary mass term.

The renormalization procedure

Hence we have two renormalization constants Z_ϕ and Z_α:

$$\begin{aligned} \phi &= Z_\phi^{1/2}\phi_R, & \beta_0 &= Z_\phi^{-1/2}\beta, \\ m_0 &= Z_\phi^{-1/2}m, & \alpha_0 &= Z_\alpha\alpha. \end{aligned} \tag{5}$$

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Suppose that in the momentum space

$$G_R(p^2) = \frac{4\pi}{p^2+M^2} + O(p^4) \quad \text{as } p^2 \to 0, \tag{7}$$

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$G \simeq \left(G_0^{-1} + \frac{\Sigma}{4\pi}\right)^{-1} \simeq G_0 - G_0 \Sigma G_0$

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with a constant

$$M^2 = m^2 + \frac{4\pi\alpha\beta^2}{R^2} = m^2(1 + 4\pi c^2 \alpha\beta^2). \tag{8}$$

This defines the renormalized coupling constant α for a given scale R.

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Instead of calculating Σ it is more convenient to calculate the correlation function

$$\begin{aligned}
G(x-x') &= \langle \phi(x)\phi(x') \rangle = \frac{\left\langle \phi(x)\phi(x') e^{-S_1[\phi]} \right\rangle_0}{\left\langle e^{-S_1[\phi]} \right\rangle_0} \\
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$$= \langle\phi(x)\phi(x')\rangle_0 - \langle\phi(x)\phi(x')S_1[\phi]\rangle_{0,\mathrm{c}} + \frac{1}{2}\langle\phi(x)\phi(x')S_1^2[\phi]\rangle_{0,\mathrm{c}}$$
$$- \frac{1}{6}\langle\phi(x)\phi(x')S_1^3[\phi]\rangle_{0,\mathrm{c}} + O(\alpha_0^4).$$

The connected averages $\langle\cdots\rangle_{0,\mathrm{c}}$ will be extracted on the fly.

The renormalization condition can be rewritten as $\Sigma(p^2) = \Sigma_0 + \Sigma_1 p^2 + O(p^4)$. Indeed,

$$\begin{aligned} 4\pi G^{-1}(p^2) &= p^2 + m_0^2 + \Sigma(p^2) = p^2 + m_0^2 + \Sigma_0 + \Sigma_1 p^2 + O(p^4)) \\ &= (1+\Sigma_1)\Big(p^2 + m^2 + \Sigma_0(1+\Sigma_1)^{-1}\Big) + O(p^4) = 4\pi(1+\Sigma_1)G_R^{-1}(p^2). \end{aligned} \tag{9}$$

We obtain

$$Z_\phi = \frac{1}{1+\Sigma_1}, \qquad M^2 = m^2 + \frac{\Sigma_0}{1+\Sigma_1}, \qquad m^2 = \frac{m_0^2}{1+\Sigma_1}. \tag{10}$$

Instead of calculating Σ it is more convenient to calculate the correlation function

$$\begin{aligned} G(x-x') = \langle \phi(x)\phi(x') \rangle &= \frac{\langle \phi(x)\phi(x') e^{-S_1[\phi]} \rangle_0}{\langle e^{-S_1[\phi]} \rangle_0} \\ &= \langle \phi(x)\phi(x') \rangle_0 - \langle \phi(x)\phi(x') S_1[\phi] \rangle_{0,\mathrm{c}} + \frac{1}{2} \langle \phi(x)\phi(x') S_1^2[\phi] \rangle_{0,\mathrm{c}} \\ &\quad - \frac{1}{6} \langle \phi(x)\phi(x') S_1^3[\phi] \rangle_{0,\mathrm{c}} + O(\alpha_0^4). \end{aligned}$$

The connected averages $\langle \cdots \rangle_{0,\mathrm{c}}$ will be extracted on the fly. Then the mass operator will be extracted by removing 'legs' from the diagrams.

Let us calculate

$$-\langle\phi(x)\phi(x')S_1[\phi]\rangle = \alpha_0 r_0^{\delta_0}\int d^2y\,\langle\phi(x)\phi(x')\,{:}\cos\beta_0\phi(y){:}\,\rangle_0.$$

RG: first order

Let us calculate

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We have

$$\langle\phi(x)\phi(x')\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 = \langle\phi(x)\phi(x')\rangle_0\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 - \beta_0^2\langle\phi(x)\phi(y)\rangle_0\langle\phi(x')\phi(y)\rangle_0\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0.$$

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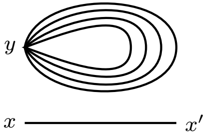
The first term is disconnected,



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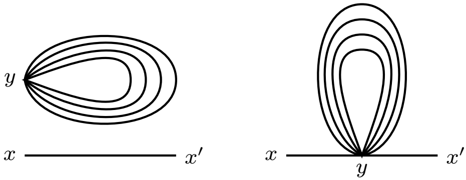
The first term is disconnected, the second one contains two external lines:



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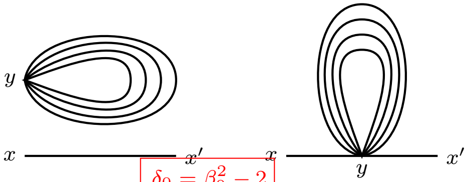

$\delta_0 = \beta_0^2 - 2$

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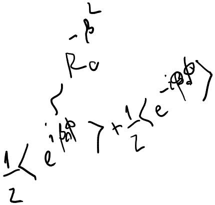

Hence,

$$-\frac{1}{4\pi}\Sigma^{(1)}(y-y') = -\alpha_0\beta_0^2 r_0^{\delta_0}\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0\delta(y-y')$$

RG: first order

Let us calculate

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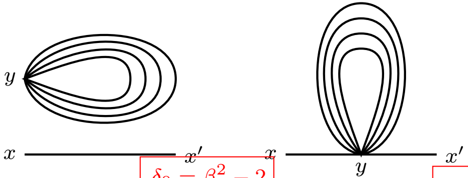

$\delta_0 = \beta_0^2 - 2$

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$\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 = R_0^{-\not{2}\beta_0^2}$

Hence,

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RG: first order

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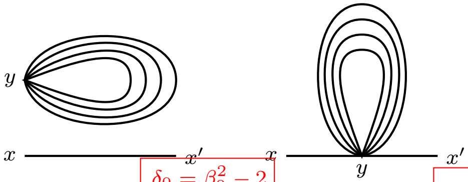

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In the momentum space:

$$\Sigma^{(1)}(p^2) = \Sigma_0^{(1)} = \frac{4\pi\alpha_0\beta_0^2}{R_0^2}\left(\frac{r_0}{R_0}\right)^{\delta_0}, \qquad \Sigma_1^{(1)} = 0. \tag{11}$$

RG: first order

By comparing this with the formulas

$$M^2 = m^2 + \frac{4\pi\alpha\beta^2}{R^2} = m^2(1 + 4\pi c^2\alpha\beta^2). \tag{8}$$

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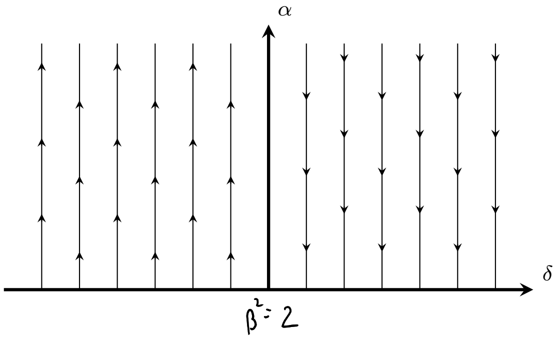
The solution is $\alpha \sim R^{-\delta}$ in consistency with (12).

The RG trajectories look like



RG: first order

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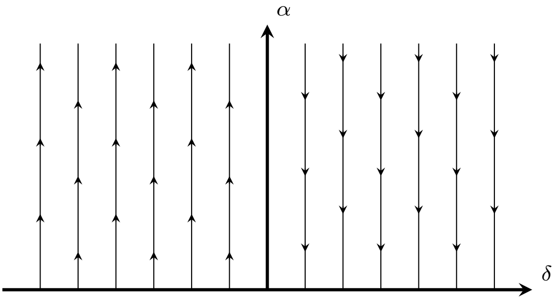

The transition point $\delta = 0$ here is a line of fixed points for any value of α. Is it really the case?

Consider the second order contribution. The connected contribution to the pair correlation function is

$$\frac{1}{2}\langle\phi(x)\phi(x')S_1^2[\phi]\rangle_{0,\mathrm{c}} = \frac{\alpha_0^2 r_0^{2\delta_0}}{2}\int d^2y_1\, d^2y_2\, \langle\phi(x)\phi(x')\, {:}\cos\beta_0\phi(y_1){:}\; {:}\cos\beta_0\phi(y_2){:}\,\rangle_{0,\mathrm{c}}$$

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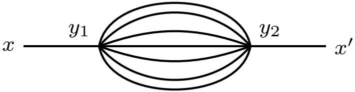
These terms correspond to the diagrams



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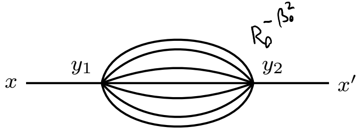
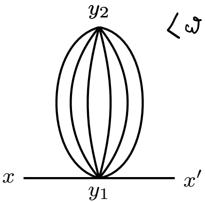
$$- \langle\phi(x)\phi(y_1)\rangle_0\langle\phi(x')\phi(y_1)\rangle_0\Big(\langle\, {:}\cos\beta_0\phi(y_1){:}\; {:}\cos\beta_0\phi(y_2){:}\,\rangle_0 - R_0^{-2\beta_0^2}\Big)\Big).$$

$\uparrow$ $\langle\cos\beta\phi(y_1)\rangle\,\langle\cos\beta\phi(y_2)\rangle$

These terms correspond to the diagrams





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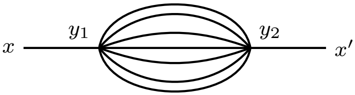

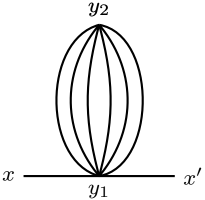

For calculation of $\Sigma^{(2)}$ we have to remove 'legs' and to subtract the contribution of one line in the first diagram:

$$-\frac{1}{4\pi}\Sigma^{(2)}(x) = \alpha_0^2\beta_0^2 r_0^{2\delta_0}\Big(\langle\, {:}\sin\beta_0\phi(x){:}\ {:}\sin\beta_0\phi(0){:}\,\rangle_0 - \beta_0^2 R_0^{-2\beta_0^2}\langle\phi(x)\phi(0)\rangle_0$$

$$-\delta(x)\int d^2y\ \Big(\langle\, {:}\cos\beta_0\phi(0){:}\ {:}\cos\beta_0\phi(y){:}\,\rangle_0 - R_0^{-2\beta_0^2}\Big)\Big)$$

RG: second order

Explicitly,

$\langle e^{\pm i\beta_0\phi(x)} e^{\mp i\beta_0\phi(0)} \rangle$ $\langle e^{\pm i\beta_0\phi(x)} e^{\pm i\beta_0\phi(0)} \rangle$

$$-\frac{1}{4\pi}\Sigma^{(2)}(x) = \frac{\alpha_0^2\beta_0^2 r_0^{2\delta_0}}{2R_0^{2\beta_0^2}}\left(\left(\frac{R_0}{x}\right)^{2\beta_0^2} - \left(\frac{x}{R_0}\right)^{2\beta_0^2} - 2\beta_0^2\log\frac{R_0^2}{x^2} - \delta(x)\int d^2y\left(\left(\frac{R_0}{y}\right)^{2\beta_0^2} + \left(\frac{y}{R_0}\right)^{2\beta_0^2} - 2\right)\right).$$

$y = |y_2 - y_1|$

Explicitly,

$$-\frac{1}{4\pi}\Sigma^{(2)}(x)=\frac{\alpha_0^2\beta_0^2 r_0^{2\delta_0}}{2R_0^{2\beta_0^2}}\left(\left(\frac{R_0}{x}\right)^{2\beta_0^2}-\left(\frac{x}{R_0}\right)^{2\beta_0^2}-2\beta_0^2\log\frac{R_0^2}{x^2}\right.$$
$$\left.-\delta(x)\int d^2y\left(\left(\frac{R_0}{y}\right)^{2\beta_0^2}+\left(\frac{y}{R_0}\right)^{2\beta_0^2}-2\right)\right).$$

In the momentum space we have

$$\Sigma^{(2)}(p^2)=-2\pi\alpha_0^2\beta_0^2 r_0^{2\delta_0}\left(\int d^2x\,(e^{\mathrm{i}px}-1)x^{-2\beta_0^2}\right.$$
$$\left.-R_0^{-4\beta_0^2}\int d^2x\,(e^{\mathrm{i}px}+1)x^{2\beta_0^2}-2\beta_0^2 R_0^{-2\beta_0^2}G_0(p^2)+2R_0^{2-2\beta_0^2}\right). \quad (13)$$

Explicitly,

$$-\frac{1}{4\pi}\Sigma^{(2)}(x)=\frac{\alpha_0^2\beta_0^2r_0^{2\delta_0}}{2R_0^{2\beta_0^2}}\left(\left(\frac{R_0}{x}\right)^{2\beta_0^2}-\left(\frac{x}{R_0}\right)^{2\beta_0^2}-2\beta_0^2\log\frac{R_0^2}{x^2}\right.$$
$$\left.-\delta(x)\int d^2y\left(\left(\frac{R_0}{y}\right)^{2\beta_0^2}+\left(\frac{y}{R_0}\right)^{2\beta_0^2}-2\right)\right).$$

In the momentum space we have

$$\Sigma^{(2)}(p^2)=-2\pi\alpha_0^2\beta_0^2r_0^{2\delta_0}\left(\int d^2x\,(e^{ipx}-1)x^{-2\beta_0^2}\right.$$
$$\left.-R_0^{-4\beta_0^2}\int d^2x\,(e^{ipx}+1)x^{2\beta_0^2}-2\beta_0^2R_0^{-2\beta_0^2}G_0(p^2)+2R_0^{2-2\beta_0^2}\right).\quad(13)$$

The second line vanishes as $R_0\to\infty$ for $\delta_0\ll1$.

RG: second order

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It only contributes to Σ_1. We have

$$Z_\phi=1-\pi^2\alpha_0^2\beta_0^2\log\frac{R}{r_0},\qquad Z_\alpha=1+\delta_0\log\frac{R}{r_0}. \quad (15)$$

RG: second order

Substituting it to $\alpha = Z_\alpha^{-1}\alpha_0$ and $1 + \delta/2 = Z_\phi(1 + \delta_0/2)$, taking the derivation and expressing α_0, δ_0 in terms of α, δ in the r.h.s., we obtain

$$\frac{d\alpha}{dt} = -\delta\alpha, \qquad \frac{d\delta}{dt} = -4\pi^2\alpha^2, \qquad t = \log R. \tag{16}$$

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$$\frac{d(2\pi\alpha \mp \delta)}{dt} = \pm 2\pi\alpha(2\pi\alpha \mp \delta). \tag{16a}$$

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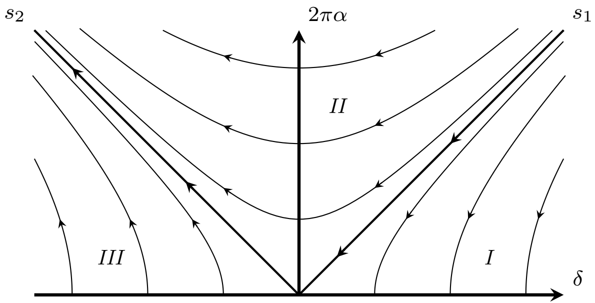
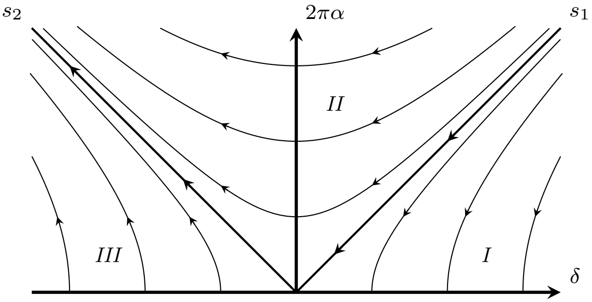
This means that the straight lines $2\pi\alpha = \pm\delta$ are RG trajectories. They divide the half-plane $\alpha > 0$ into three regions:

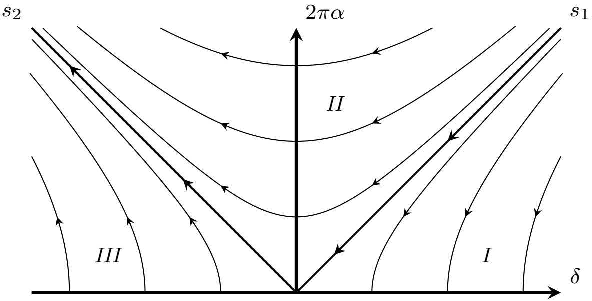


There are three regions:



There are three regions:

- Region I. $\alpha \to 0$ as $R \to \infty$, so that the system looks like a free massless boson at large distances.



RG: regions

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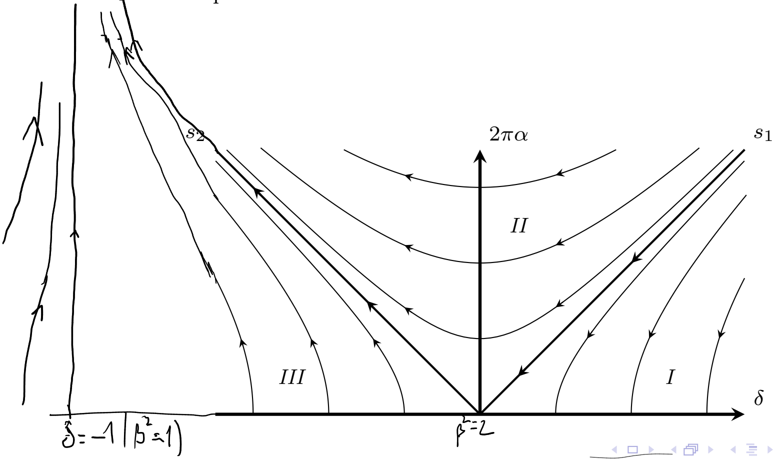

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- Region II. α grows for both large and small R. The system has no conformal behavior in both IR and UV regions. Since it approaches the line s_2 at large R, it must be a massive theory.

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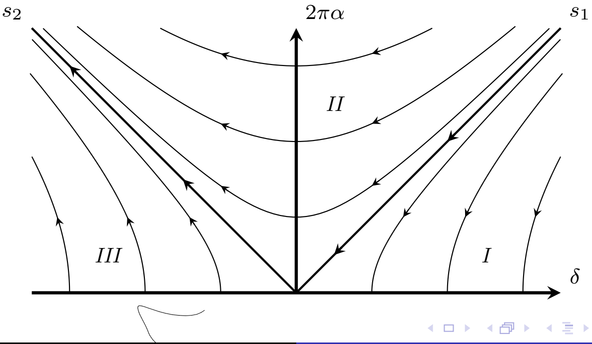
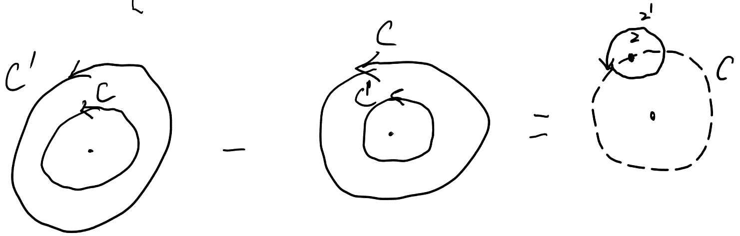


$$T(z')\,T(z) = \frac{c/2}{(z'-z)^4} + \frac{2T(z)}{(z'-z)^2} + \frac{\partial T(z)}{z'-z} + O(1)$$

$$L_n = \oint \frac{dz}{2\pi i}\, z^{n+1}\, T(z)$$

$$[L_m, L_n] = \left[\oint \frac{dz'}{2\pi i}, \oint \frac{dz}{2\pi i}\right] z'^{\,m+1} z^{n+1}\, T(z')\,T(z) =$$



$$\oint \frac{dz_2}{2\pi i} \frac{c}{2} z_2^{m+n-1} \left(\frac{\Gamma(m+2)}{\Gamma(m-1)} - \frac{\Gamma(n+2)}{\Gamma(n-1)} \right) +$$

$$+ 2(m-n) \oint \frac{dz_2}{2\pi i} T(z_2)\, z_2^{m+n+1} \quad \int \frac{dz_2}{2\pi i}$$

$$\oint \frac{dz_1}{2\pi i} z_1^{m+1} z_2^{n+1} \times$$

$$C(z_1)$$

$$C$$

$$+ \oint \frac{dz_2}{2\pi i} \partial T(z_2) \left(z_2^{m+n+1} - z_2^{m+n+1} \right) \quad = \times \; \frac{\partial T(z_2)}{z_1 - z_2} =$$

$$= \frac{c}{2} \delta_{m,-n} \left((m+1)m(m-1) - (-m+1)(-m)(-m-1) \right) +$$

$$= \int \frac{dz_2}{2\pi i} z_2^{m+n+2} \partial T(z_2)$$

$$+ 2(m-n) L_{m+n} = c(m+1)m(m-1)\delta_{m,-n} + 2(m-n)L_{m+n}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\,\delta_{m+n,0}$$

Virasoro algebra $\quad L_n \sim T_{zz} \quad \bar{L}_n \sim T_{\bar{z}\bar{z}}$

1) $m, n = 0, \pm 1$ $\quad [L_1, L_{-1}] = 2L_0, \quad [L_0, L_{\pm 1}] = \mp L_{\pm 1}$

$sl(2)$

$$L_{-1} = \oint \frac{dz}{2\pi i} T(z) \to \partial$$

$$[L_{-1}, O(z,\bar{z})] = \partial O(z,\bar{z}) \qquad J^{\mu\nu} = \int dx^1 (x^\mu T^{\nu 0} - x^\nu T^{\mu 0})$$

$$L_0 = \oint \frac{dz}{2\pi i} z\, T(z) \to \left(z \to \lambda z\right), \quad \lambda = (1+\varepsilon) \qquad L_0 - \bar{L}_0 = S, \quad L_0 + \bar{L}_0 = D$$

$$L_1 \quad z^2 \quad \left(z \to \frac{1}{z^{-1}+\varepsilon}\right)$$

~~$\oint \frac{dz}{2\pi i} z^2 T(z) = \oint \frac{d\zeta}{2\pi i} \zeta^{-4} T(\zeta^{-1}) = T(\zeta)$~~

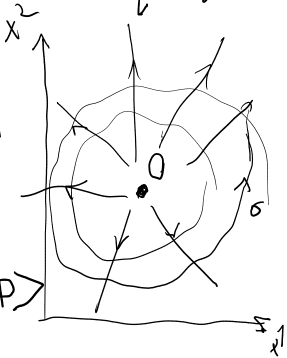
$$z \to z + \varepsilon(z)$$

Free boson $c = 1$

Free real MF $c = \frac{1}{2}$

Free DF $c = 1$

$z = e^{\tau + i\sigma}$



Highest weight reps

$$\begin{cases} L_n|\Delta\rangle = 0, & n > 0 \\ L_0|\Delta\rangle = \Delta|\Delta\rangle \end{cases}$$

Δ conf. dim

$$L_{-n_1} \cdots L_{-n_k}|\Delta\rangle$$

$$\Phi(z,\bar z) \longleftrightarrow |\Phi\rangle$$

$$[L_0, \Phi(z,\bar z)] = \partial\Phi(z,\bar z), \quad [\bar L_0, \Phi] = \bar\partial\Phi$$

$\Phi_\Delta(0,0)|vac\rangle = |\Delta\rangle \to \Phi_\Delta(z,\bar{z})$

$L_n|\Delta\rangle = 0, \quad n>0$

$[L_n, \Phi_\Delta(z)] = ?$

$L_0|\Delta\rangle = \Delta|\Delta\rangle$

$[L_{-1}, \Phi_\Delta] = \partial\Phi_\Delta$

$T(z)\Phi_\Delta(0) = ?$

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$T(z)|\Delta\rangle = \frac{\Delta}{z^2}|\Delta\rangle + \frac{\partial\Phi_\Delta(0)|vac\rangle}{z} + O(1)$$

$$T(z')\Phi_\Delta(z) = \frac{\Delta\Phi_\Delta(z)}{(z'-z)^2} + \frac{\partial\Phi_\Delta(z)}{z'-z} + O(1)$$

primary conformal operators

$\mathcal{D}^d$

$$\langle vac|\Phi_{\Delta_N}(z_N)\cdots\Phi_{\Delta_1}(z_1)L_{-n}|\Delta\rangle = \partial\langle vac|\cdots|\Delta\rangle$$

$\Delta + n$

$$\langle vac|L_0 = 0,\quad \langle vac|L_{-n} = 0,\ n>0$$

$$L_0 L_{-n_1}\cdots L_{-n_k}|\Delta\rangle = \left(\Delta + \sum n_i\right) L_{-n_1}\cdots L_{-n_k}|\Delta\rangle$$