Lecture 3.
Renormalization group for the Berezinskii–Kosterlitz–Thouless transition

Michael Lashkevich

Suppose we consider a field theory system with the correlation length r_c. It is described by a bare action defined the at the UV cutoff r_0, which depends on the set of parameters λ_0. We are interested in correlations functions on a scale r, $r_0 \ll r \ll r_c$. Let $G_{\text{exact}}(\lambda_0, r_0; \cdots)$ be exact correlation functions calculated in all orders of the perturbation theory.

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$$G_{\text{exact}}(\lambda_0, r_0; x_1, \ldots, x_n) = G_{\text{tree}}\left(\lambda, r; \frac{x_1}{r}, \ldots, \frac{x_n}{r}\right).$$

if $x_i \sim r$.

Renormalization group approach

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$$\delta = \beta^2 - 2 \ll 1.$$

Bare and dressed sine-Gordon action

The bare action of the sine-Gordon model on the Euclidean plane:

$$S_{\rm SG}[\phi] = \int d^2x \left(\frac{(\partial_\mu\phi)^2}{8\pi} - \alpha_0 r_0^{\beta_0^2 - 2} \cos\beta_0\phi \right), \tag{1}$$

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For ultraviolet regularization, we will replace x^2 by $x^2 + r_0^2$. Then for $m_0^2 x^2 \ll 1$ the free field propagator (with $\alpha_0 = 0$) is equal to

$$G_0(x - x') = \log \frac{R_0^2}{(x - x')^2 + r_0^2}, \qquad R_0 = (c m_0)^{-1}, \qquad c = e^{\gamma_E}/2. \tag{3}$$

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Now write down the renormalized (dressed) action

$$S^R_{\rm SG}[\phi]=\int d^2x\left(\frac{(\partial_\mu\phi)^2}{8\pi}+\frac{m^2\phi^2}{8\pi}-\frac{\alpha}{R^2}\cos\beta\phi\right),\qquad R=(cm)^{-1}, \qquad (4)$$

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such that $S_{\mathrm{SG}}[\phi] = S_{\mathrm{SG}}^R[Z_\phi^{-1/2}\phi] + S^{\mathrm{ct}}[Z_\phi^{-1/2}\phi]$. Assume that the counterterms

$$S^{\mathrm{ct}}[\phi] = \int d^2x \, (\#(\partial_\mu \phi)^2 + \# \cos \beta\phi).$$

do not contain a counterterm for the auxiliary mass term.

The renormalization procedure

Hence we have two renormalization constants Z_ϕ and Z_α:

$$\begin{aligned} \phi &= Z_\phi^{1/2}\phi_R, \qquad & \beta_0 &= Z_\phi^{-1/2}\beta, \\ m_0 &= Z_\phi^{-1/2}m, \qquad & \alpha_0 &= Z_\alpha \alpha. \end{aligned} \tag{5}$$

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with a constant

$$M^2 = m^2 + \frac{4\pi\alpha\beta^2}{R^2} = m^2(1+4\pi c^2\alpha\beta^2). \tag{8}$$

This defines the renormalized coupling constant α for a given scale R.

The renormalization procedure

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Instead of calculating Σ it is more convenient to calculate the correlation function

$$\begin{aligned} G(x-x') = \langle \phi(x)\phi(x') \rangle &= \frac{\left\langle \phi(x)\phi(x') e^{-S_1[\phi]} \right\rangle_0}{\left\langle e^{-S_1[\phi]} \right\rangle_0} \\ &= \left\langle \phi(x)\phi(x') \right\rangle_0 - \langle \phi(x)\phi(x') S_1[\phi] \rangle_{0,\mathrm{c}} + \frac{1}{2} \langle \phi(x)\phi(x') S_1^2[\phi] \rangle_{0,\mathrm{c}} \\ &\qquad - \frac{1}{6} \langle \phi(x)\phi(x') S_1^3[\phi] \rangle_{0,\mathrm{c}} + O(\alpha_0^4). \end{aligned}$$

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$$- \frac{1}{6}\langle\phi(x)\phi(x')S_1^3[\phi]\rangle_{0,\mathrm{c}} + O(\alpha_0^4).$$

The connected averages $\langle\cdots\rangle_{0,\mathrm{c}}$ will be extracted on the fly.

The renormalization procedure

The renormalization condition can be rewritten as $\Sigma(p^2) = \Sigma_0 + \Sigma_1 p^2 + O(p^4)$. Indeed,

$$\begin{aligned} 4\pi G^{-1}(p^2) &= p^2 + m_0^2 + \Sigma(p^2) = p^2 + m_0^2 + \Sigma_0 + \Sigma_1 p^2 + O(p^4)) \\ &= (1+\Sigma_1)\Big(p^2 + m^2 + \Sigma_0(1+\Sigma_1)^{-1}\Big) + O(p^4) = 4\pi(1+\Sigma_1)G_R^{-1}(p^2). \end{aligned} \quad (9)$$

We obtain

$$Z_\phi = \frac{1}{1+\Sigma_1}, \qquad M^2 = m^2 + \frac{\Sigma_0}{1+\Sigma_1}, \qquad m^2 = \frac{m_0^2}{1+\Sigma_1}. \quad (10)$$

Instead of calculating Σ it is more convenient to calculate the correlation function

$$\begin{aligned} G(x-x') = \langle\phi(x)\phi(x')\rangle &= \frac{\left\langle \phi(x)\phi(x')e^{-S_1[\phi]}\right\rangle_0}{\left\langle e^{-S_1[\phi]}\right\rangle_0} \\ &= \langle\phi(x)\phi(x')\rangle_0 - \langle\phi(x)\phi(x')S_1[\phi]\rangle_{0,\mathrm{c}} + \frac{1}{2}\langle\phi(x)\phi(x')S_1^2[\phi]\rangle_{0,\mathrm{c}} \\ &\qquad - \frac{1}{6}\langle\phi(x)\phi(x')S_1^3[\phi]\rangle_{0,\mathrm{c}} + O(\alpha_0^4). \end{aligned}$$

The connected averages $\langle\cdots\rangle_{0,\mathrm{c}}$ will be extracted on the fly. Then the mass operator will be extracted by removing 'legs' from the diagrams.

Let us calculate

$$-\langle\phi(x)\phi(x')S_1[\phi]\rangle = \alpha_0 r_0^{\delta_0}\int d^2y\,\langle\phi(x)\phi(x')\,{:}\cos\beta_0\phi(y){:}\,\rangle_0.$$

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We have

$$\begin{aligned}\langle\phi(x)\phi(x')\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 &= \langle\phi(x)\phi(x')\rangle_0\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0\\ &\quad-\beta_0^2\langle\phi(x)\phi(y)\rangle_0\langle\phi(x')\phi(y)\rangle_0\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0.\end{aligned}$$

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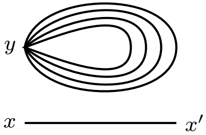
We have

$$\langle\phi(x)\phi(x')\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 = \langle\phi(x)\phi(x')\rangle_0\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 \\ - \beta_0^2\langle\phi(x)\phi(y)\rangle_0\langle\phi(x')\phi(y)\rangle_0\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0.$$

The first term is disconnected,



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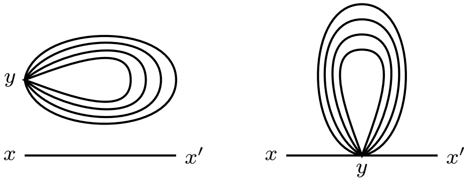
The first term is disconnected, the second one contains two external lines:



RG: first order

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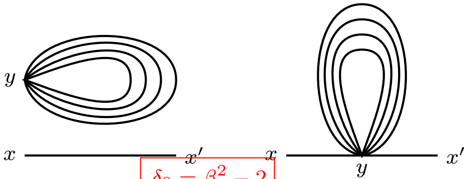

$\delta_0=\beta_0^2-2$

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Hence,

$$-\frac{1}{4\pi}\Sigma^{(1)}(y-y')=-\alpha_0\beta_0^2 r_0^{\delta_0}\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0\delta(y-y')$$

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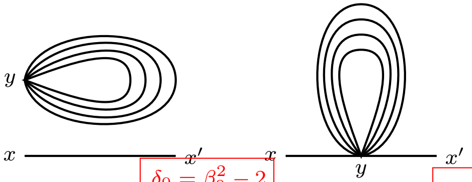

$$\delta_0 = \beta_0^2 - 2$$
$$\delta = \beta^2 - 2$$

$$\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 = R_0^{-\beta_0^2}$$

Hence,

$$-\frac{1}{4\pi}\Sigma^{(1)}(y-y') = -\alpha_0\beta_0^2 r_0^{\delta_0}\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0\delta(y-y') = -\frac{\alpha_0\beta_0^2}{R_0^2}\left(\frac{r_0}{R_0}\right)^{\delta_0}\delta(y-y').$$

Let us calculate

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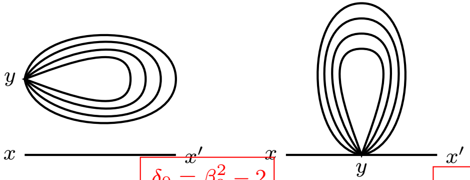

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In the momentum space:

$$\Sigma^{(1)}(p^2) = \Sigma_0^{(1)} = \frac{4\pi\alpha_0\beta_0^2}{R_0^2}\left(\frac{r_0}{R_0}\right)^{\delta_0}, \qquad \Sigma_1^{(1)} = 0. \tag{11}$$

By comparing this with the formulas

$$M^2 = m^2 + \frac{4\pi\alpha\beta^2}{R^2} = m^2(1 + 4\pi c^2\alpha\beta^2). \tag{8}$$

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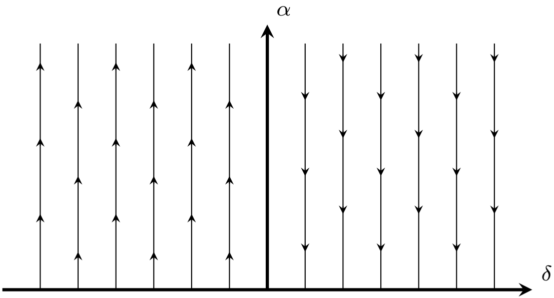
The solution is $\alpha \sim R^{-\delta}$ in consistency with (12).

RG: first order

The RG trajectories look like



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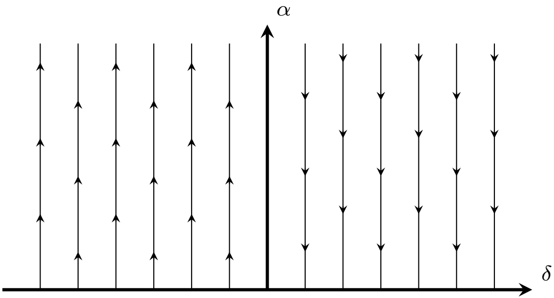

The transition point $\delta = 0$ here is a line of fixed points for any value of α. Is it really the case?

Consider the second order contribution. The connected contribution to the pair correlation function is

$$\frac{1}{2}\langle\phi(x)\phi(x')S_1^2[\phi]\rangle_{0,\mathrm{c}} = \frac{\alpha_0^2 r_0^{2\delta_0}}{2}\int d^2y_1\,d^2y_2\,\langle\phi(x)\phi(x')\,{:}\cos\beta_0\phi(y_1){:}\;{:}\cos\beta_0\phi(y_2){:}\,\rangle_{0,\mathrm{c}}$$

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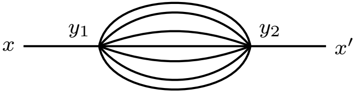
$$= \alpha_0^2\beta_0^2 r_0^{2\delta_0}\int d^2y_1\, d^2y_2\,\Big(\langle\phi(x)\phi(y_1)\rangle_0\langle\phi(x')\phi(y_2)\rangle_0\langle\,{:}\sin\beta_0\phi(y_1){:}\;{:}\sin\beta_0\phi(y_2){:}\,\rangle_0$$

These terms correspond to the diagrams



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$$-\langle\phi(x)\phi(y_1)\rangle_0\langle\phi(x')\phi(y_1)\rangle_0\Big(\langle\,{:}\cos\beta_0\phi(y_1){:}\;{:}\cos\beta_0\phi(y_2){:}\,\rangle_0 - R_0^{-2\beta_0^2}\Big)\Big).$$

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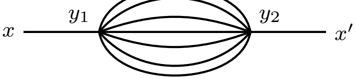

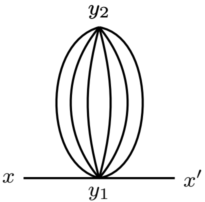

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$$= \alpha_0^2\beta_0^2 r_0^{2\delta_0}\int d^2y_1\, d^2y_2 \Big(\langle\phi(x)\phi(y_1)\rangle_0\langle\phi(x')\phi(y_2)\rangle_0\langle :\sin\beta_0\phi(y_1): :\sin\beta_0\phi(y_2):\rangle_0$$

$$- \langle\phi(x)\phi(y_1)\rangle_0\langle\phi(x')\phi(y_1)\rangle_0\Big(\langle :\cos\beta_0\phi(y_1): :\cos\beta_0\phi(y_2):\rangle_0 - R_0^{-2\beta_0^2}\Big)\Big).$$

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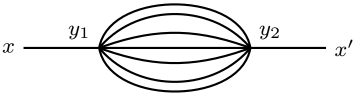

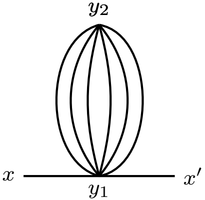

For calculation of $\Sigma^{(2)}$ we have to remove 'legs' and to subtract the contribution of one line in the first diagram:

$$-\frac{1}{4\pi}\Sigma^{(2)}(x) = \alpha_0^2\beta_0^2 r_0^{2\delta_0}\Big(\langle :\sin\beta_0\phi(x): :\sin\beta_0\phi(0):\rangle_0 - \beta_0^2 R_0^{-2\beta_0^2}\langle\phi(x)\phi(0)\rangle_0$$

$$- \delta(x)\int d^2y\, \Big(\langle :\cos\beta_0\phi(0): :\cos\beta_0\phi(y):\rangle_0 - R_0^{-2\beta_0^2}\Big)\Big)$$

Explicitly,

$$-\frac{1}{4\pi}\Sigma^{(2)}(x)=\frac{\alpha_0^2\beta_0^2 r_0^{2\delta_0}}{2R_0^{2\beta_0^2}}\left(\left(\frac{R_0}{x}\right)^{2\beta_0^2}-\left(\frac{x}{R_0}\right)^{2\beta_0^2}-2\beta_0^2\log\frac{R_0^2}{x^2}\right.$$
$$\left.-\delta(x)\int d^2y\left(\left(\frac{R_0}{y}\right)^{2\beta_0^2}+\left(\frac{y}{R_0}\right)^{2\beta_0^2}-2\right)\right).$$

Explicitly,

$$-\frac{1}{4\pi}\Sigma^{(2)}(x) = \frac{\alpha_0^2\beta_0^2 r_0^{2\delta_0}}{2R_0^{2\beta_0^2}}\left(\left(\frac{R_0}{x}\right)^{2\beta_0^2} - \left(\frac{x}{R_0}\right)^{2\beta_0^2} - 2\beta_0^2\log\frac{R_0^2}{x^2}\right.$$
$$\left. - \delta(x)\int d^2y\left(\left(\frac{R_0}{y}\right)^{2\beta_0^2} + \left(\frac{y}{R_0}\right)^{2\beta_0^2} - 2\right)\right).$$

In the momentum space we have

$$\Sigma^{(2)}(p^2) = -2\pi\alpha_0^2\beta_0^2 r_0^{2\delta_0}\left(\int d^2x\,(e^{ipx}-1)x^{-2\beta_0^2}\right.$$
$$\left. - R_0^{-4\beta_0^2}\int d^2x\,(e^{ipx}+1)x^{2\beta_0^2} - 2\beta_0^2 R_0^{-2\beta_0^2} G_0(p^2) + 2R_0^{2-2\beta_0^2}\right). \quad (13)$$

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In the momentum space we have

$$\Sigma^{(2)}(p^2)=-2\pi\alpha_0^2\beta_0^2 r_0^{2\delta_0}\left(\int d^2x\,(e^{\mathrm{i}px}-1)x^{-2\beta_0^2}\right.$$
$$\left.-R_0^{-4\beta_0^2}\int d^2x\,(e^{\mathrm{i}px}+1)x^{2\beta_0^2}-2\beta_0^2 R_0^{-2\beta_0^2}G_0(p^2)+2R_0^{2-2\beta_0^2}\right). \qquad (13)$$

The second line vanishes as $R_0\to\infty$ for $\delta_0\ll 1$.

Explicitly,

$$-\frac{1}{4\pi}\Sigma^{(2)}(x)=\frac{\alpha_0^2\beta_0^2r_0^{2\delta_0}}{2R_0^{2\beta_0^2}}\left(\left(\frac{R_0}{x}\right)^{2\beta_0^2}-\left(\frac{x}{R_0}\right)^{2\beta_0^2}-2\beta_0^2\log\frac{R_0^2}{x^2}\right.$$
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In the momentum space we have

$$\Sigma^{(2)}(p^2)=-2\pi\alpha_0^2\beta_0^2r_0^{2\delta_0}\left(\int d^2x\,(e^{\mathrm{i}px}-1)x^{-2\beta_0^2}\right.$$
$$\left.-\,R_0^{-4\beta_0^2}\int d^2x\,(e^{\mathrm{i}px}+1)x^{2\beta_0^2}-2\beta_0^2R_0^{-2\beta_0^2}G_0(p^2)+2R_0^{2-2\beta_0^2}\right). \qquad (13)$$

The second line vanishes as $R_0\to\infty$ for $\delta_0\ll 1$. The integral in the first line must be expanded in p:

$$\Sigma^{(2)}(p^2)=\pi\alpha_0^2\beta_0^2r_0^{2\delta_0}\int d^2x\,(px)^2x^{-2\beta_0^2}+O(p^4)\simeq\pi^2\alpha_0^2\beta_0^2p^2\log\frac{R_0}{r_0}+O(p^4). \qquad (14)$$

Explicitly,

$$-\frac{1}{4\pi}\Sigma^{(2)}(x)=\frac{\alpha_0^2\beta_0^2 r_0^{2\delta_0}}{2R_0^{2\beta_0^2}}\left(\left(\frac{R_0}{x}\right)^{2\beta_0^2}-\left(\frac{x}{R_0}\right)^{2\beta_0^2}-2\beta_0^2\log\frac{R_0^2}{x^2}\right.$$
$$\left.-\delta(x)\int d^2y\left(\left(\frac{R_0}{y}\right)^{2\beta_0^2}+\left(\frac{y}{R_0}\right)^{2\beta_0^2}-2\right)\right).$$

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It only contributes to Σ_1. We have

$$Z_\phi=1-\pi^2\alpha_0^2\beta_0^2\log\frac{R}{r_0},\qquad Z_\alpha=1+\delta_0\log\frac{R}{r_0}. \quad (15)$$

Substituting it to $\alpha = Z_\alpha^{-1}\alpha_0$ and $1 + \delta/2 = Z_\phi(1 + \delta_0/2)$, taking the derivation and expressing α_0, δ_0 in terms of α, δ in the r.h.s., we obtain

$$\frac{d\alpha}{dt} = -\delta\alpha, \qquad \frac{d\delta}{dt} = -4\pi^2\alpha^2, \qquad t = \log R. \tag{16}$$

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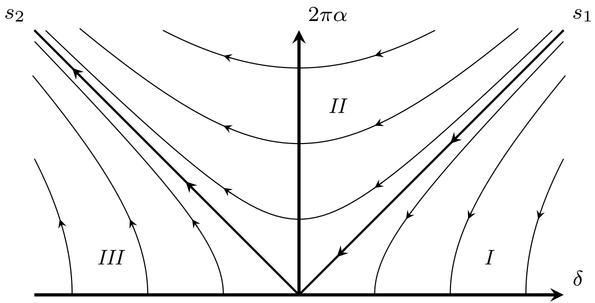
This means that the straight lines $2\pi\alpha = \pm\delta$ are RG trajectories. They divide the half-plane $\alpha > 0$ into three regions:



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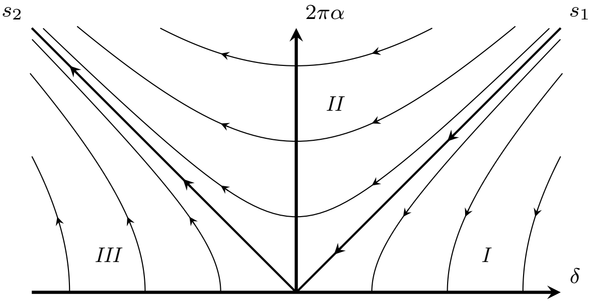
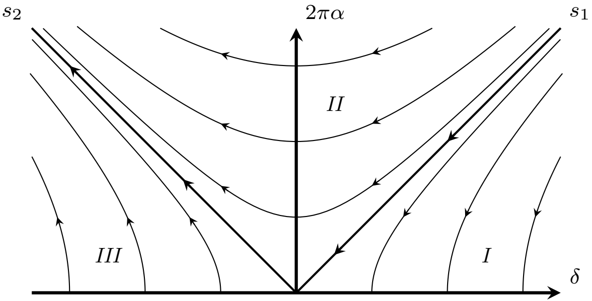


RG: regions

There are three regions:

- Region I. $\alpha \to 0$ as $R \to \infty$, so that the system looks like a free massless boson at large distances.



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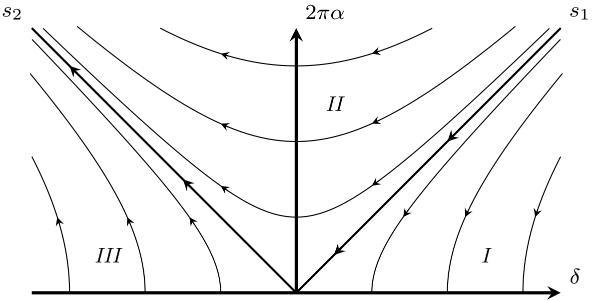

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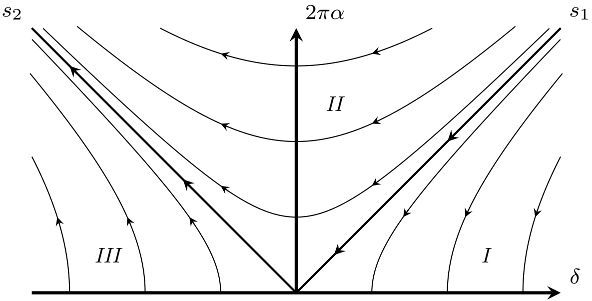
- Region II. α grows for both large and small R. The system has no conformal behavior in both IR and UV regions. Since it approaches the line s_2 at large R, it must be a massive theory.



Seminar