Lecture 4.
$O(3)$-model: mass generation by instantons

Michael Lashkevich

$O(3)$-model: topology of $\boldsymbol{n}$-field

Consider the $O(3)$-model on the Euclidean plane:

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$$q = \frac{1}{S}\int_{x'(S^2)} d^2x' \sqrt{g'} = \frac{1}{S}\int_{S^2} d^2x \frac{\partial(x')}{\partial(x)} \sqrt{g'}.$$

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Assuming the spherical coordinates on $S^{2\prime}$ with the standard metric:

$$q = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \frac{\partial(\theta', \varphi')}{\partial(\theta, \varphi)} \sin\theta' = \frac{1}{4\pi}\int d^2x \, \frac{\partial(\theta', \varphi')}{\partial(x^1, x^2)} \sin\theta'.$$

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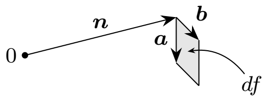

Self-duality equations

From the identity

$$\int d^2x\,(\partial_\mu \boldsymbol{n} + \epsilon_{\mu\nu}\boldsymbol{n}\times\partial^\nu \boldsymbol{n})^2 = 2\int d^2x\,(\partial_\mu \boldsymbol{n})^2 - 2\int d^2x\,\boldsymbol{n}\,(\partial_\mu \boldsymbol{n}\times\partial_\nu \boldsymbol{n})\epsilon^{\mu\nu} \quad (8)$$

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These are first-order differential equations. Every their solution is a solution to the equations of motion, but not vice versa.

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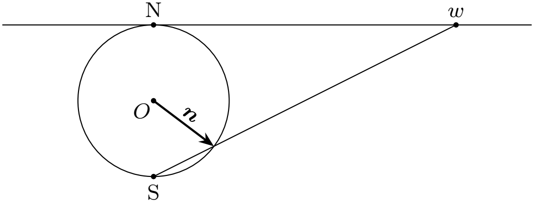

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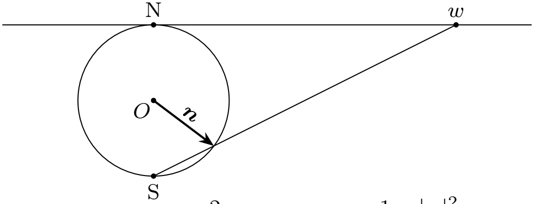

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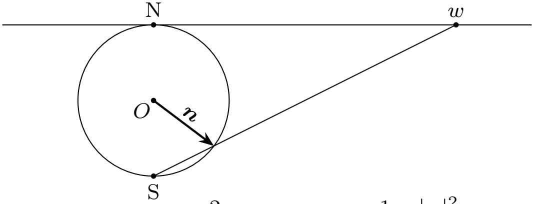

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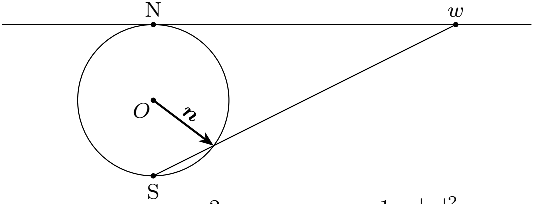

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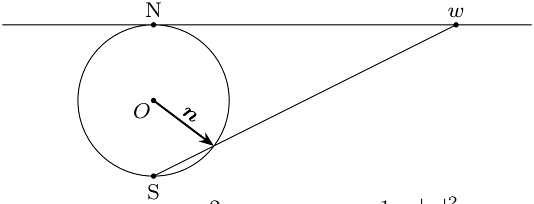

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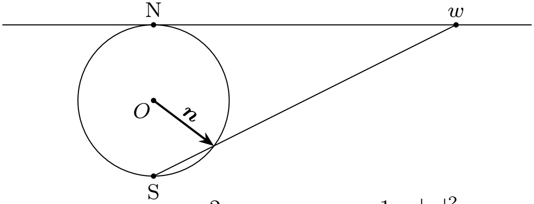

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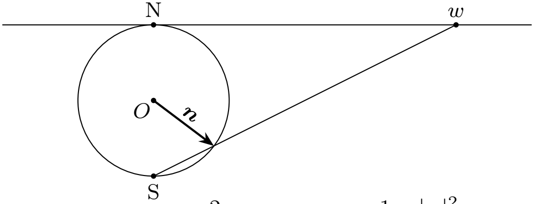

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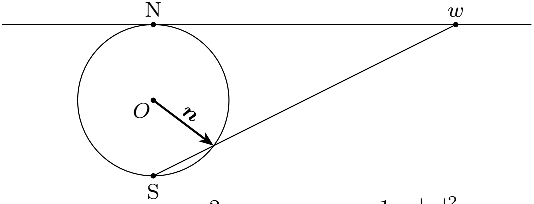

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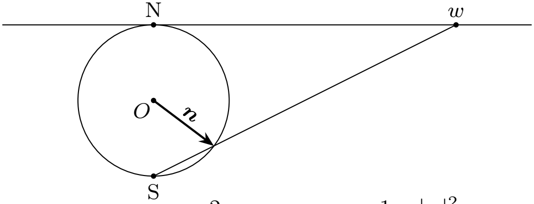

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The values $a_j, b_j \in \mathbb{C} \cup \{\infty\}$, but $a_i \neq b_j$ $(\forall i, j)$.

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$$n=q. \tag{17a}$$

Solutions to the self-duality equations

Let w_0 be large enough and generic. Then the equation $w(n,\vec{a},\vec{b},c;z)=w_0$ has exactly n solutions. $\Rightarrow$ It has exactly n solution un to multiplicity for any w_0. $\Rightarrow$

$$n=q. \tag{17a}$$

Similarly, for $q<0$ we have

$$w(q,\vec{a},\vec{b},c;\bar{z})=c\prod_{j=1}^{-q}\frac{\bar{z}-a_j}{\bar{z}-b_j}, \tag{18}$$

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$$Z_q=\frac{e^{-4\pi q/g}}{(q!)^2}\int d\mu(\vec a,\vec b,c)\,Z[w(q,\vec a,\vec b,c;z)],$$
$$Z[w]=\int D\varphi\,D\bar\varphi\,\exp\left(-\frac{8}{g}\int d^2x\,\frac{|w|^2}{(1+|w|^2)^2}\bar\partial\varphi\partial\bar\varphi\right) \tag{23}$$

with a conformal invariant measure μ.

Functional integral

The only conformal invariant measure is

$$\mu(\vec{a},\vec{b},c)=k^q\,\frac{d^2c}{|c|^2}\prod_{j=1}^q d^2a_j\,d^2b_j\,\prod_{i<j}|a_i-a_j|^4|b_i-b_j|^4\prod_{i,j}|a_i-b_j|^{-4} \tag{24}$$

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A rather complex calculation results in $\alpha=1/2$ and $f(c)=|c|^2/(1+|c|^2)^2$, so the integral over c gives just a finite factor. Thus we have

$$Z_q\sim\frac{\lambda^q}{(q!)^2}\int\prod_{j=1}^q d^2a_j\,d^2b_j\,\prod_{i<j}|a_i-a_j|^2|b_i-b_j|^2\prod_{i,j}|a_i-b_j|^{-2}, \qquad (25)$$

where $\lambda\sim e^{-4\pi/g}$.

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In particular, it is known that for $\theta = \pi$ the $O(3)$-model is massless, but not scale-invariant. For $\theta \neq 0, \pi \pmod{2\pi}$ the theory does not seem to be exactly solvable.