Lecture 6.
$O(N)$-model: integrability and the exact S-matrix

Michael Lashkevich

Conformal invariance of the action

Recall the action of the $O(N)$-model

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and the inversion transformation

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We have two integrals of motion So, there are at least two integrals of motion of spin 1 and spin 3:

$$I_1 = \int dz \frac{1}{2}(\partial \boldsymbol{n})^2, \qquad I_3 = \int dz \left(\frac{1}{2}(\partial^2 \boldsymbol{n})^2 - \frac{3+\alpha}{2(2\beta+\alpha')}(\partial \boldsymbol{n})^4\right), \tag{10}$$

which satisfy the equations $\bar{\partial} I_1 = 0$, $\bar{\partial} I_3 = 0$.

IMs and elasticity of scattering

By taking into account both components and both chiralities, we obtain four integrals of motion (IMs): $I_{\pm 1}$, $I_{\pm 3}$, which satisfy the equations $\dot{I}_s = 0$. We have $I_1 \sim p_z$, $I_{-1} \sim p_{\bar{z}}$.

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IMs and elasticity of scattering

By taking into account both components and both chiralities, we obtain four integrals of motion (IMs): $I_{\pm 1}$, $I_{\pm 3}$, which satisfy the equations $\dot{I}_s = 0$. We have $I_1 \sim p_z$, $I_{-1} \sim p_{\bar{z}}$. Let $|\theta_1, \ldots, \theta_n\rangle$ be a common eigenstate of IMs with n particles with rapidities θ_i, i.e. $p_{iz} = -\frac{m}{2} e^{\theta_i}$, $p_{i\bar{z}} = \frac{m}{2} e^{-\theta_i}$. The eigenvalue of I_s must be a homogeneous Laurent polynomial of e^{θ_i} of the power s.

Locality. Local IMs are space integrals, so that if particles are far from each other, their eigenvalues must be sums of eigenvalues on one-particle states. Hence

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In fact, the model has integrals of motion with all $s \in 2\mathbb{Z} + 1$. In this case only $n \to n$ processes are allowed with $\theta'_i = \theta_i$. It is the ideally elastic scattering characteristic for integrable models.

Factorizable scattering

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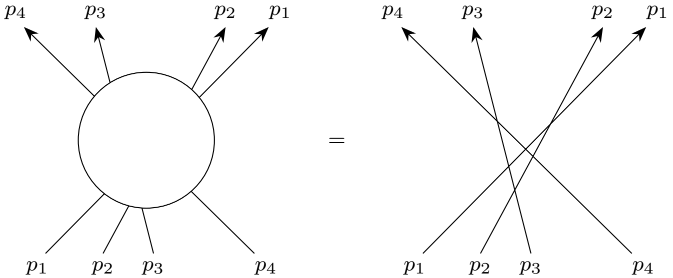
Graphically it can be depicted as



(11)

Asymptotic n-particle wave function

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$$\psi_{\beta_1 p_1,\ldots,\beta_n p_n}(\alpha_1 x_1,\ldots,\alpha_n x_n) = \sum_{\tau\in S_n} A^{\alpha_{\sigma_1}\ldots\alpha_{\sigma_n}}_{\beta_1\quad\ldots\beta_n}[\tau] e^{i\sum_{i=1}^n p_{\tau_i} x_{\sigma_i}}$$

$$\text{for}\quad x_{\sigma_1} < x_{\sigma_2} < \cdots < x_{\sigma_n}, \quad |x_i - x_j| \gg R. \qquad (12)$$

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Exchange of two neighboring particles means their scattering. Therefore

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Consistency requirements: three particle permutation

Due to the factorization the two-particle S-matrix satisfy a set of equations.

Consistency requirements: three particle permutation

Due to the factorization the two-particle S-matrix satisfy a set of equations.
Let us inverse the order of three consecutive particles, say $1, 2, 3$. We may do it in two ways:

$$\begin{array}{ccccccc} & & 132 & \to & 312 & & \\ & \nearrow & & & & \searrow & \\ 123 & & & & & & 321. \\ & \searrow & & & & \nearrow & \\ & & 213 & \to & 231 & & \end{array}$$

Consistency requirements: three particle permutation

The first way $123 \to 132 \to 312 \to 321$ leads to the relation

$$A^{\alpha_3\alpha_2\alpha_1\cdots}_{\cdots}[321\ldots]$$

$$= \sum_{\beta_1,\beta_2,\beta_3} \left(\sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}(p_1,p_2) S^{\gamma_1\alpha_3}_{\beta_1\gamma_3}(p_1,p_3) S^{\gamma_2\gamma_3}_{\beta_2\beta_3}(p_2,p_3) \right) A^{\beta_1\beta_2\beta_3\cdots}_{\cdots}[123\ldots]$$

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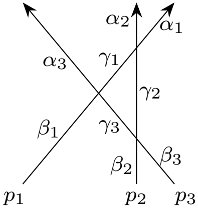
or graphically

$$\sum_{\text{internal lines}}$$



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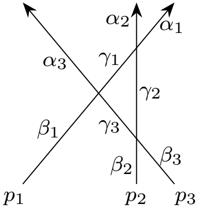

We will write it as

$$A_{321\ldots} = S_{12}(p_1,p_2)S_{13}(p_1,p_3)S_{23}(p_2,p_3)A_{123\ldots},$$

Consistency requirements: three particle permutation

The second way $123 \to 213 \to 231 \to 321$ leads to the relation

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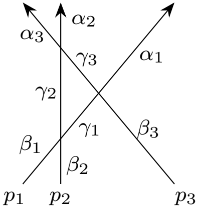

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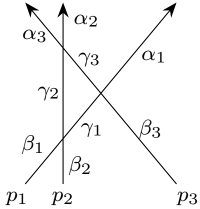

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Consistency requirements: three particle permutation

The result of these two permutation processes must be the same. Hence, we have the

Yang–Baxter equation

$$\sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}(p_1,p_2)S^{\gamma_1\alpha_3}_{\beta_1\gamma_3}(p_1,p_3)S^{\gamma_2\gamma_3}_{\beta_2\beta_3}(p_2,p_3) = \sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_2\alpha_3}_{\gamma_2\gamma_3}(p_2,p_3)S^{\alpha_1\gamma_3}_{\gamma_1\beta_3}(p_1,p_3)S^{\gamma_1\gamma_2}_{\beta_1\beta_2}(p_1,p_2), \quad (14)$$

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or, shorter,

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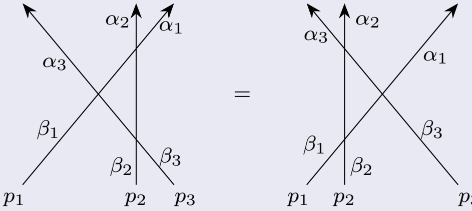
or, graphically,



(16)

Consistency requirements: inversability

Now demand that two consequent permutation of the same two particle leads to identical map: $12 \to 21 \to 12$. This implies the

Unitarity condition

$$\sum_{\gamma_1,\gamma_2} S^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}(p_1,p_2)S^{\gamma_2\gamma_1}_{\beta_2\beta_1}(p_2,p_1) = \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}, \tag{17}$$

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Now demand that two consequent permutation of the same two particle leads to identical map: $12 \to 21 \to 12$. This implies the

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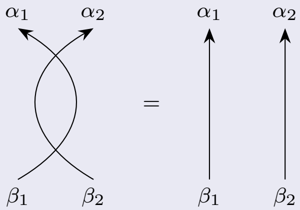
$$\sum_{\gamma_1,\gamma_2} S^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}(p_1,p_2)S^{\gamma_2\gamma_1}_{\beta_2\beta_1}(p_2,p_1)=\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}, \tag{17}$$

or

$$S_{12}(p_1,p_2)S_{21}(p_2,p_1)=1, \tag{18}$$

or



(19)

The last condition is due to the theory is relativistic.

Crossing symmetry

$$S^{\alpha_1\alpha_2}_{\beta_1\beta_2}(p_1,p_2)=\sum_{\alpha'_1\beta'_1}C_{\beta_1\beta'_1}S^{\alpha_2\beta'_1}_{\beta_2\alpha'_1}(p_2,-p_1)C_{\alpha'_1\alpha_1}. \tag{20}$$

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Relativistic condition: crossing symmetry

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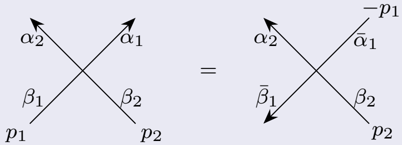

(21)

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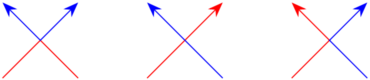
A pole $iu_0\in[0,i\pi]$ corresponds to a bound state, if $\operatorname*{Res}_{u=u_0} S^{\alpha\beta}_{\alpha\beta}(iu)>0$.

Recall that for the $O(N)$ ($N \geq 3$) model we have

$$S^{\alpha'\beta'}_{\alpha\ \beta}(\theta) = \delta^{\alpha'\beta'}\delta_{\alpha\beta}S_1(\theta) + \delta^{\alpha'}_{\alpha}\delta^{\beta'}_{\beta}S_2(\theta) + \delta^{\beta'}_{\alpha}\delta^{\alpha'}_{\beta}S_3(\theta). \tag{26}$$



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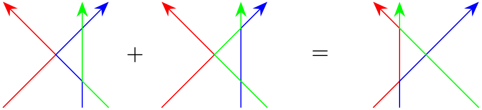

(28)

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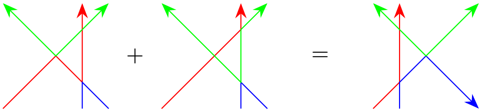

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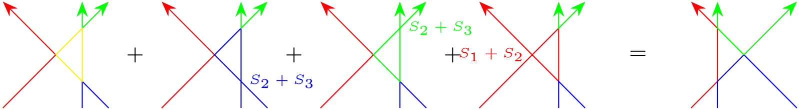
$$\begin{aligned} &NS_1(\theta)S_3(\theta+\theta')S_1(\theta') + S_1(\theta)S_3(\theta+\theta')(S_2(\theta') + S_3(\theta')) \\ &\qquad\quad + (S_2(\theta) + S_3(\theta))S_3(\theta+\theta')S_1(\theta') \\ &\qquad\quad + S_1(\theta)(S_1(\theta+\theta') + S_2(\theta+\theta'))S_1(\theta') \\ &\qquad = S_3(\theta)S_1(\theta+\theta')S_3(\theta'). \end{aligned} \tag{29}$$



Bootstrap equations for the S-matrix of the $O(N)$ model

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Let us solve these equations.

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Let $h(\theta) = S_2(\theta)/S_3(\theta)$. Then it takes the form

$$h(\theta) + h(\theta') = h(\theta+\theta').$$

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Therefore, $h(\theta) \sim \theta$ and

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Substituting it into the third equation (29), we get

$$\kappa = \frac{N-2}{2}\lambda.$$

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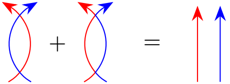
(36)



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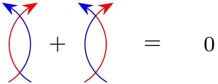
(37)



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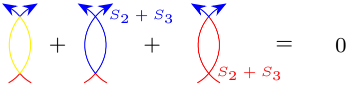

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we will again have a solution. We will search the 'minimal' solution, which has the least number of poles and zeros on the physical sheet.

Bootstrap equations for the S-matrix of the $O(N)$ model

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The function that has poles and zeros at (39) and (40) is

$$S_2^{(\pm)}(\theta) = Q^{(\pm)}(\theta)Q^{(\pm)}(i\pi - \theta), \quad Q^{(\pm)}(\theta) = \frac{\Gamma\left(\pm\frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} \pm \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(-i\frac{\theta}{2\pi}\right)}. \tag{41}$$

S-matrix: $N \to \infty$ behavior and final choice

Take the limit $N \to \infty$. We obtain

$$S_1^{(\pm)}(\theta) = -\frac{2\pi i}{N(i\pi - \theta)}, \tag{42}$$

$$S_2^{(\pm)}(\theta) = 1 \mp \frac{2\pi i}{N \operatorname{sh} \theta}, \tag{43}$$

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where $P_{12} : a \times b \mapsto b \times a$ is the permutation operator of the spaces 1 and 2. This means that for the particles in the $O(N)$-model a kind of the Pauli principle applies, although we considered the particles to be bosons. Two particles cannot have the same momentum.

Seminar