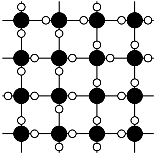
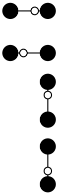
Lecture 9
Ice model and commuting transfer matrices

Ice model: configurations

The 'ice model' (● is Oxygen, ○ is Hydrogen):





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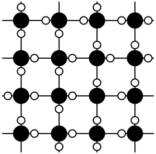
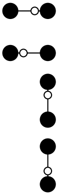


Each oxygen atom has two hydrogen atom next to it.



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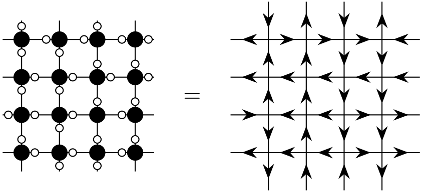

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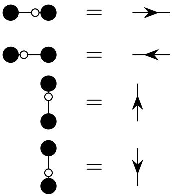

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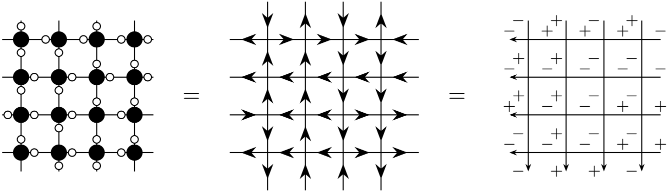
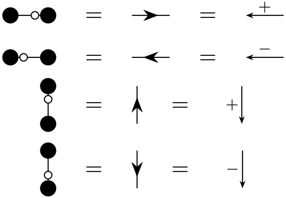


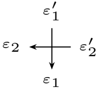
Each oxygen atom has two hydrogen atom next to it. Small arrows on the right figure define the orientation of the lattice lines and vertices, which will be important later.



Ice model: Boltzmann weights

Six-vertex model: the Boltzmann weights are associated with vertices:

$$Z = \sum_{\substack{\text{configu-}\\\text{rations}}} \prod_{\text{vertices}} R^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2}, \qquad R^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2} =$$



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Ice condition

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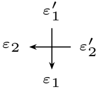

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Ice condition

We have six vertices

$R^{++}_{++} = a =$, $R^{--}_{--} = a' =$

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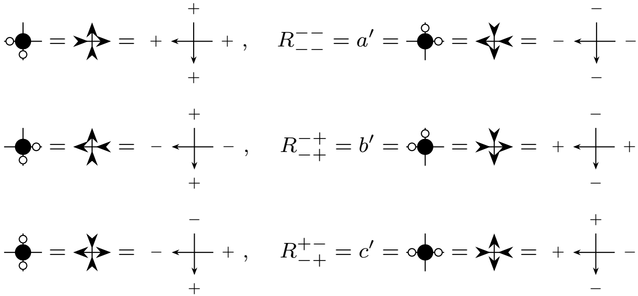

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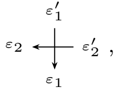

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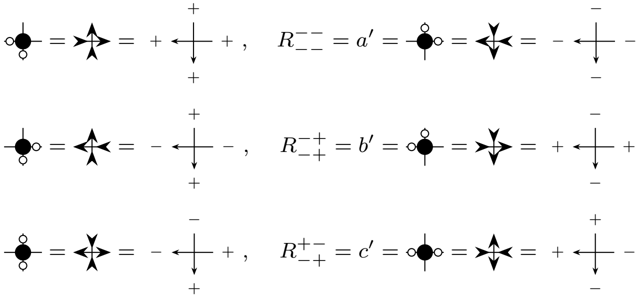

$$R = \begin{pmatrix} a & & & \\ & b & c & \\ & c' & b' & \\ & & & a' \end{pmatrix}$$ in the basis $(++), (+-), (-+), (--)$.

Six-vertex models: solvable case

The six-vertex model is solvable, if

$$R_{-\varepsilon_1\,-\varepsilon_2}^{-\varepsilon_1'\,-\varepsilon_2'} = R_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}$$

or

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The transfer matrix

$$T^{\varepsilon'_1\dots\varepsilon'_N}_{\varepsilon_1\dots\varepsilon_N} = \sum_{\mu_1\dots\mu_N} R^{\mu_2\varepsilon'_1}_{\mu_1\varepsilon_1} R^{\mu_3\varepsilon'_2}_{\mu_2\varepsilon_2} \dots R^{\mu_1\varepsilon'_N}_{\mu_N\varepsilon_N}. \quad (1)$$

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Let us consider the matrix R as an operator in the tensor product of two two-dimensional spaces:

$$R : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2, \qquad v_{\varepsilon_1} \otimes v_{\varepsilon_2} \mapsto R^{\varepsilon_1 \varepsilon_2}_{\varepsilon_1' \varepsilon_2'} v_{\varepsilon_1'} \otimes v_{\varepsilon_2'}.$$

Here v_ε is the natural basis in $V = \mathbb{C}^2$.

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Here v_ε is the natural basis in $V = \mathbb{C}^2$. Consider the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_k$ of identical spaces $V_i \simeq V$. Let R_{ij} is the R matrix acting on $V_i \otimes V_j$.

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Then the transfer matrix can be written as

$$T = \operatorname{tr}_{V_0}(R_{0N} \ldots R_{02} R_{01}) : \; V_1 \otimes V_2 \otimes \cdots \otimes V_N \to V_1 \otimes V_2 \otimes \cdots \otimes V_N. \tag{2}$$

The space $V_1 \otimes \cdots \otimes V_N$ is called quantum space, while the space V_0 is called auxiliary space.

The operator under the trace is

$$L = R_{0N} \dots R_{02} R_{01} : \; V_0 \otimes V_1 \otimes \cdots \otimes V_N \to V_0 \otimes V_1 \otimes \cdots \otimes V_N . \tag{3}$$

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We will consider it as an operator in the quantum space and a matrix in the auxiliary space

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D : V_1 \otimes V_2 \otimes \dots \otimes V_N \to V_1 \otimes V_2 \otimes \dots \otimes V_N .$$

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Then

$$T = \mathrm{tr}_{V_0} L = A + D. \qquad (4)$$

Commuting transfer matrices and Yang–Baxter equation

Integrability demands the existence of extra commuting integrals of motion I_n:

$$[T, I_n] = 0, \quad [I_m, I_n] = 0.$$

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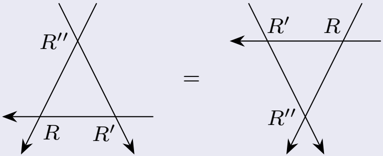
Theorem

If there exist nondegenerate matrices R', R'' such that

$$R''_{12} R'_{13} R_{23} = R_{23} R'_{13} R''_{12}, \tag{5}$$

or, graphically



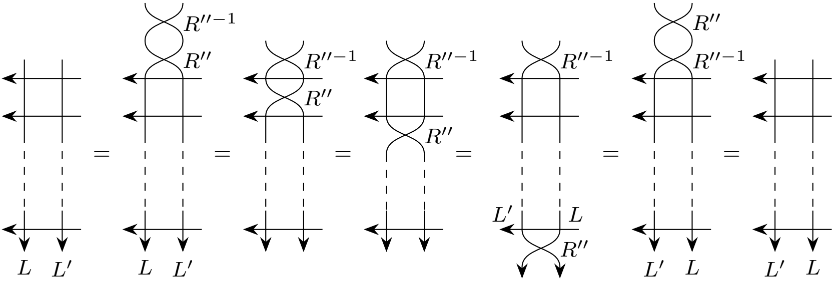
(5′)

then

$$[T, T'] = 0 \tag{6}$$

A graphical proof:

$T'T =$



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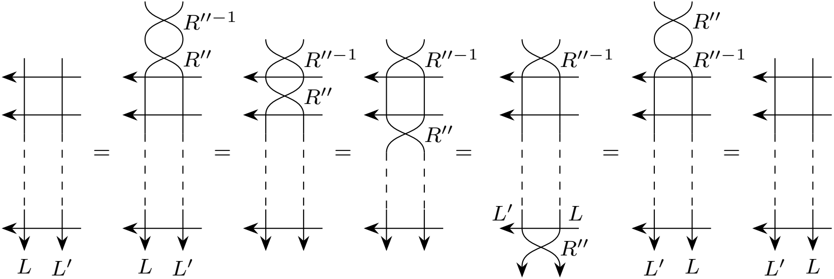

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A more conventional proof is based on the relation

$$R''_{12} L'_1 L_2 = L_2 L'_1 R''_{12},$$

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Commuting transfer matrices: a proof

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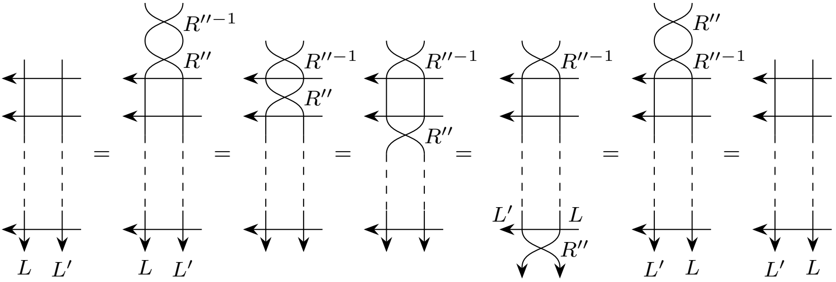

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$$\begin{aligned} T'T &= \mathrm{tr}_{V_1 \otimes V_2}(L'_1 L_2) = \mathrm{tr}_{V_1 \otimes V_2}((R''_{12})^{-1} R''_{12} L'_1 L_2) = \mathrm{tr}_{V_1 \otimes V_2}((R''_{12})^{-1} L_2 L'_1 R''_{12}) \\ &= \mathrm{tr}_{V_1 \otimes V_2}(R''_{12} (R''_{12})^{-1} L_2 L'_1) = \mathrm{tr}_{V_1 \otimes V_2}(L_2 L'_1) = TT'. \end{aligned}$$

Yang–Baxter equation: solution for the six-vertex model

The solution can be found in the form

$$\begin{aligned} R &= R(\lambda, u_2 - u_3), \\ R' &= R(\lambda, u_1 - u_3), \\ R'' &= R(\lambda, u_1 - u_2) \end{aligned} \tag{8}$$

with a given matrix-valued function $R(\lambda, u)$.

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with a given matrix-valued function $R(\lambda, u)$. Since the common factor of a, b, c is arbitrary, assume $a = 1$. Trigonometric solution(s):

$$\begin{aligned} b(u) &= \frac{\sin u}{\sin(\lambda - u)}, & b(u) &= \frac{\operatorname{sh} u}{\operatorname{sh}(\lambda - u)}, \\ c(u) &= \frac{\sin \lambda}{\sin(\lambda - u)} & c(u) &= \frac{\operatorname{sh} \lambda}{\operatorname{sh}(\lambda - u)} \\ &(a < b + c,\ b < a + c,\ c < a + b); & &(c > a + b). \end{aligned}$$

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The cases $a > b + c$ and $b > a + c$ and not interesting from the thermodynamic point of view and will be discussed later.

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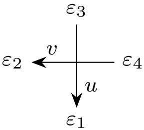
Thus we will omit the parameter λ from now on:

$$R(u) \equiv R(\lambda, u),\ a(u) \equiv a(\lambda, u) \text{ etc.}$$

Yang–Baxter equation: spectral parameter

The spectral parameters can be associated to lines:

$$R(\lambda, u - v)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_3 \varepsilon_4} =$$



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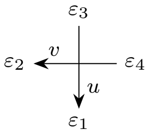
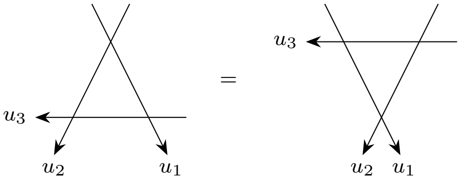


This R matrix is the solution to the Yang–Baxter equation in the form

$$\begin{aligned} &R_{12}(\lambda, u_1-u_2)R_{13}(\lambda, u_1-u_3)R_{23}(\lambda, u_2-u_3) \\ &\qquad = R_{23}(\lambda, u_2-u_3)R_{13}(\lambda, u_1-u_3)R_{12}(\lambda, u_1-u_2). \end{aligned} \tag{10}$$

Graphically:



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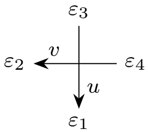
$$R(\lambda, u-v)^{\varepsilon_3\varepsilon_4}_{\varepsilon_1\varepsilon_2} =$$ 

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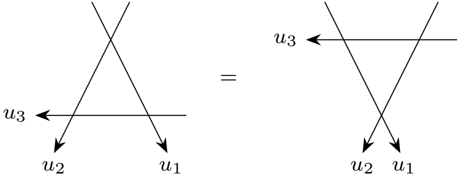

(10′)

Besides, the R matrix satisfy the relations

$$b(u)R(\lambda-u)^{\varepsilon_3\varepsilon_4}_{\varepsilon_1\varepsilon_2} = R(u)^{\varepsilon_2\ -\varepsilon_3}_{\varepsilon_4\ -\varepsilon_1}, \qquad R_{12}(u)R_{21}(-u) = 1, \qquad R(0) = P =$$  .

(11)

Integrals of motion

We have

$$[T(u), T(u')] = 0 \quad \forall u, u'. \tag{12}$$

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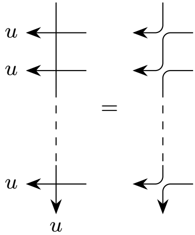

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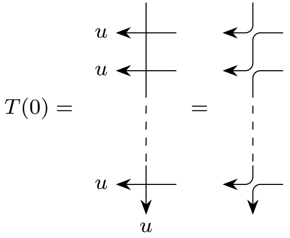

(13)

Then decompose the product $T^{-1}(0)T(u)$ in u:

$$T^{-1}(0)T(u) = 1 - \sum_{n=1}^{\infty} \frac{H_n u^n}{n!}. \tag{14}$$

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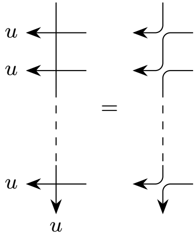
$T(0) =$  (13)

Then decompose the product $T^{-1}(0)T(u)$ in u:

$$T^{-1}(0)T(u) = 1 - \sum_{n=1}^{\infty} \frac{H_n u^n}{n!}. \tag{14}$$

Hamiltonians H_n commute with $T(u)$ and mutually commute:

$$[T(0), H_n] = [H_m, H_n] = 0 \quad \forall m, n. \tag{15}$$

The set $T(0)$, $H_1, \ldots, H_{N-1}$ form a set of independent integrals of motion.

We have

$$[T(u), T(u')] = 0 \quad \forall u, u'. \tag{12}$$

But not all the integrals of motion $T(u)$ are independent.
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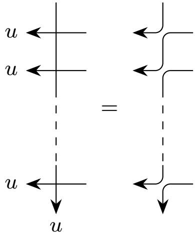
$T(0) =$  (13)

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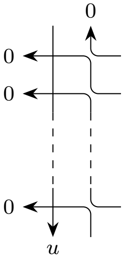
The set $T(0)$, $H_1, \ldots, H_{N-1}$ form a set of independent integrals of motion.
Operators H_n are local in the sense that each of them is a sum of term, which involves a finite number $(n+1)$ of neighboring nodes.

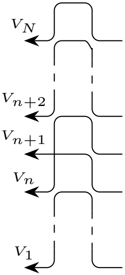
Let us find the Hamiltonian H_1 explicitly:

$$-H_1 = T^{-1}(0)T'(0) =$$

$$\left.\frac{d}{du}\right|_{u=0}$$



$$= \left.\frac{d}{du}\right|_{u=0} \sum_{n=1}^{N}$$



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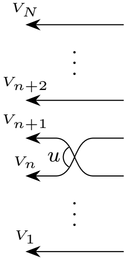

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Six-vertex model and XXZ Heisenberg chain

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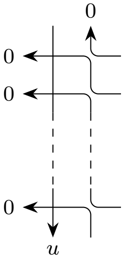

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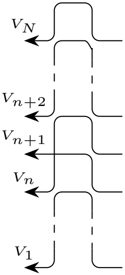

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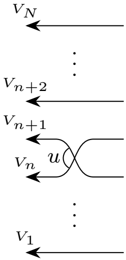

$$= \sum_{n=1}^{N} \check{R}'_{n,n+1}(0),$$

where

$$\check{R}(u) = PR(u) = \begin{pmatrix} a(u) & & & \\ & c(u) & b(u) & \\ & b(u) & c(u) & \\ & & & a(u) \end{pmatrix} = 1 + \frac{u}{\sin\lambda} \begin{pmatrix} 0 & & & \\ & \cos\lambda & 1 & \\ & 1 & \cos\lambda & \\ & & & 0 \end{pmatrix} + O(u^2)$$

$$= 1 - \frac{u}{\sin\lambda}\left(h - \frac{\cos\lambda}{2}\right) + O(u^2),$$

Here

$$h = -\frac{1}{2}(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \cos\lambda\, \sigma^z \otimes \sigma^z).$$

Hence

$$H_1 \sin\lambda = H_{\mathrm{XXZ}} + \frac{N\Delta}{2},$$

where H_{XXZ} is the Hamiltonian of the XXZ Heisenberg chain:

$$H_{\mathrm{XXZ}} = -\frac{1}{2}\sum_{n=1}^{N}(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z) \tag{16}$$

with Δ given by (9):

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} = \begin{cases} -\cos\lambda \\ -\operatorname{ch}\lambda \end{cases} .$$

XXZ Heisenberg chain: pseudovacuums

Due to the ice condition the z component of total spin

$$S^z = \frac{1}{2}\sum_{i=1}^{N}\sigma_n^z$$

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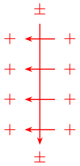
$T(u)|\Omega_+\rangle$



Evidently,

$$S^z|\Omega_\pm\rangle = \pm\frac{N}{2}|\Omega_\pm\rangle, \qquad T(u)|\Omega_\pm\rangle = (a^N(u) + b^N(u))|\Omega_\pm\rangle,$$

$$H_{XXZ}|\Omega_\pm\rangle = -\frac{N\Delta}{2}|\Omega_\pm\rangle.$$

XXZ Heisenberg chain: pseudoparticles

States of fixed spin $S^z = N/2 - k$ are linear combinations of the states

$$|n_1, \dots, n_k\rangle = \sigma^-_{n_1} \dots \sigma^-_{n_k} |\Omega_+\rangle, \qquad \sigma^\pm = \frac{\sigma^x \pm i\sigma^y}{2}. \tag{19}$$

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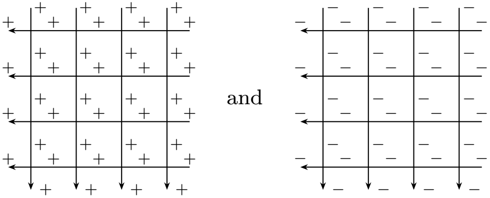

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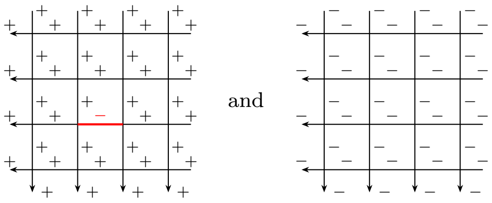

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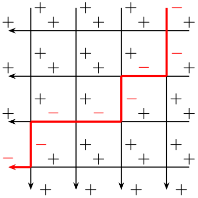

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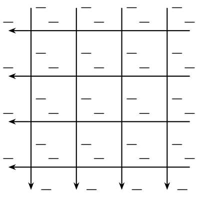

Six-vertex model: three regimes

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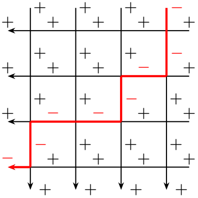

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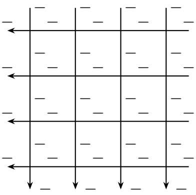

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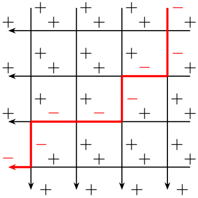

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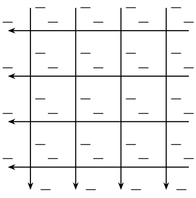

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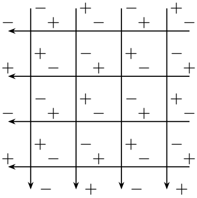

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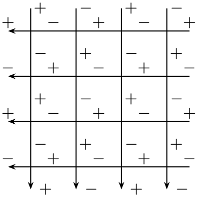

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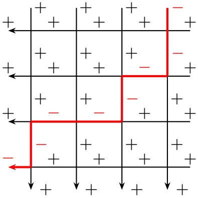

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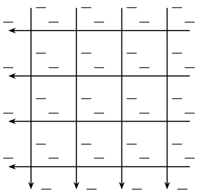

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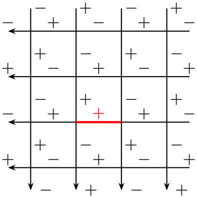

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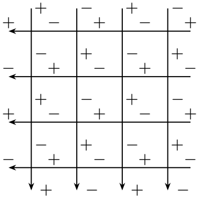

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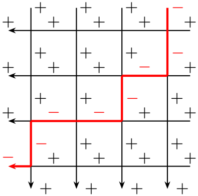

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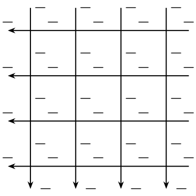

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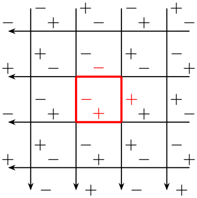

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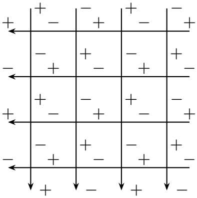

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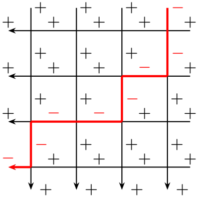

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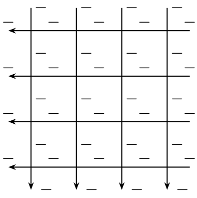

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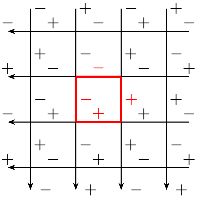

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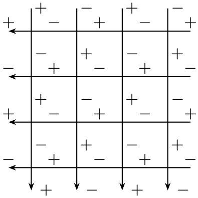

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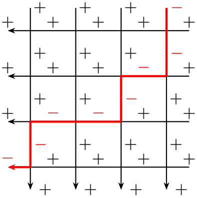

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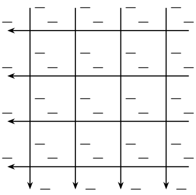

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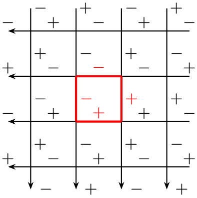

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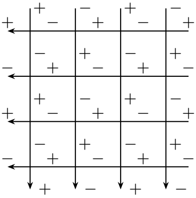

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3. Disordered regime: $|\Delta| < 1$. No ground configurations. It turns out that this regime is always critical.

Heisenberg chain: two-particle states

Consider the case $k = 2$. Let us search for an eigenstate in the form

$$|\Psi_2(z_1, z_2)\rangle = \sum_{n_1 < n_2} (A_{12} z_1^{n_1} z_2^{n_2} + A_{21} z_2^{n_1} z_1^{n_2}) |n_1, n_2\rangle. \qquad (22)$$

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$$H_{XXZ}|\Psi_2(z_1, z_2)\rangle = \left(-\frac{N\Delta}{2} + \epsilon(z_1) + \epsilon(z_2)\right)|\Psi_2(z_1, z_2)\rangle.$$

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$$H_{XXZ} |\Psi_2(z_1, z_2)\rangle = \left(-\frac{N\Delta}{2} + \epsilon(z_1) + \epsilon(z_2) \right) |\Psi_2(z_1, z_2)\rangle.$$

When is it the case? First, check the action on the terms with $n_2 - n_1 = 1$. We obtain

$$\frac{A_{21}}{A_{12}} = S(z_1, z_2) \equiv -\frac{1 + z_1 z_2 - 2\Delta z_2}{1 + z_1 z_2 - 2\Delta z_1}. \quad (23)$$

Consider the case $k = 2$. Let us search for an eigenstate in the form

$$|\Psi_2(z_1, z_2)\rangle = \sum_{n_1 < n_2} (A_{12} z_1^{n_1} z_2^{n_2} + A_{21} z_2^{n_1} z_1^{n_2}) |n_1, n_2\rangle. \tag{22}$$

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Second, we have to impose the periodicity condition:

$$z_1^N S(z_1, z_2) = 1, \quad z_2^N S(z_2, z_1) = 1. \tag{24}$$

Heisenberg chain: Bethe Ansatz

Consider general k. The Bethe Ansatz is

$$|\Psi_k(z_1,\ldots,z_k)\rangle = \sum_{n_1<\ldots<n_k} \sum_{\sigma\in S_k} A_{\sigma_1\ldots\sigma_k} \prod_{j=1}^{k} z_{\sigma_j}^{n_j} |n_1,\ldots,n_k\rangle.$$

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Next time we rederive the Bethe equations in a different way and solve them for the ground state. We also will find the corresponding eigenvalue of the transfer matrix.

Seminar