Lecture 11
Kondo problem: derivation of the Bethe Ansatz

Michael Lashkevich

Kondo effect

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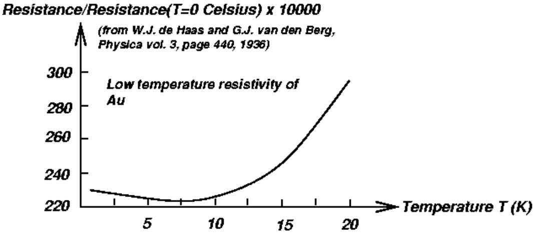

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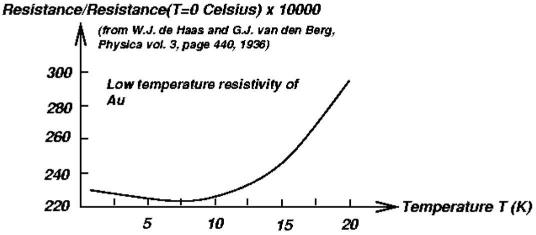

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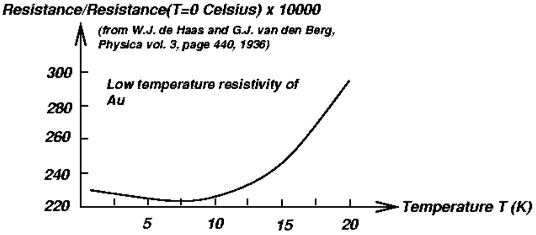

Later it turned out that the anomaly is caused by the presence of a low concentration of impurity atoms of transition metals (Mn, Fe, Cr, Co, Ce, Y). Jun Kondō (1964) explained this phenomenon by electron scattering on impurities described by the interaction (*sd model*)

$$V = J \sum_i \boldsymbol{\sigma} \boldsymbol{S}_i \delta(\boldsymbol{r} - \boldsymbol{R}_i) \tag{1}$$

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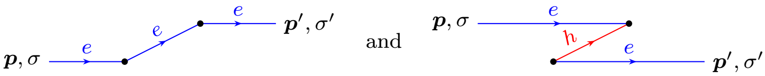

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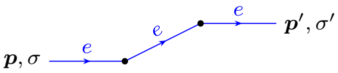

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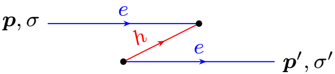

The corresponding amplitude is $f^{(2)}_{\sigma'\sigma} \sim$

$$J^2\sum_{\sigma''}\int\frac{d^3p''}{(2\pi)^3}\,\frac{(\boldsymbol{\sigma}\boldsymbol{S})_{\sigma'\sigma''}(\boldsymbol{\sigma}\boldsymbol{S})_{\sigma''\sigma}(1-f(\boldsymbol{p}''))}{\epsilon_{\boldsymbol{p}}-\epsilon_{\boldsymbol{p}''}} - J^2\sum_{\sigma''}\int\frac{d^3p''}{(2\pi)^3}\,\frac{(\boldsymbol{\sigma}\boldsymbol{S})_{\sigma''\sigma}(\boldsymbol{\sigma}\boldsymbol{S})_{\sigma'\sigma''}f(\boldsymbol{p}'')}{\epsilon_{\boldsymbol{p}''}-\epsilon_{\boldsymbol{p}'}}.$$

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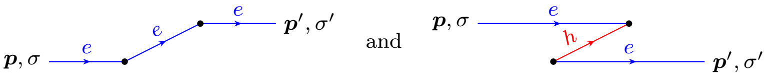

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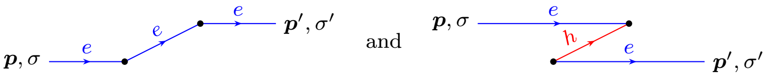

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we obtain the integral

$$f^{(2)}_{\sigma'\sigma} \sim J^2\int\frac{d^3p''}{(2\pi)^3}\left(\frac{S(S+1)\delta_{\sigma'\sigma}}{\epsilon_{\boldsymbol{p}}-\epsilon_{\boldsymbol{p}''}}+\frac{2f(\boldsymbol{p}'')-1}{\epsilon_{\boldsymbol{p}}-\epsilon_{\boldsymbol{p}''}}(\boldsymbol{\sigma}\boldsymbol{S})_{\sigma'\sigma}\right),$$

The second term diverges on the Fermi surface at $T=0$.

Integration gives the amplitude proportional to

$$f_{\sigma'\sigma} \sim J(\boldsymbol{\sigma}\boldsymbol{S})_{\sigma'\sigma}\left(1 + J\rho(\epsilon_F)\log\frac{\epsilon_F}{\max(|\epsilon_{\boldsymbol{p}} - \epsilon_F|, T)}\right). \qquad (2)$$

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This formula makes it possible to approach T_K closer, but it has a singularity at $T = T_K$. We need a nonperturbative approach at $T \lesssim T_K$.

Kondo effect. Characteristic features

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For $T \ll T_K$ it reads

$$\rho_{\text{imp}}(T) = \rho_{\text{imp}}(0)\left(1 - \kappa_R \left(\frac{T}{T_K}\right)^2 + \dots\right),$$

$$C_{\text{imp}}(T) = \gamma \frac{T}{T_K}\left(1 - \kappa_C \left(\frac{T}{T_K}\right)^2 + \dots\right),$$

$$\chi_{\text{imp}}(T) = \chi_0\left(1 - \kappa_\chi \left(\frac{T}{T_K}\right)^2 + \dots\right),$$

where κ_R, κ_C, κ_χ are quantities of order one.

sd Model. Simplifications

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We will also assume that

- the spectrum is nearly linear: $\epsilon_{\boldsymbol{p}} = \epsilon_F + v_F(p - p_F)$.

Reduction to a one-dimensional model

Decompose the creation-annihilation operators into spherical functions:

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$$c(x) = \begin{pmatrix} c_+(x) \\ c_-(x) \end{pmatrix} = \sum_p e^{ipx} \begin{pmatrix} c_{p+} \\ c_{p-} \end{pmatrix}. \tag{9}$$

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Many-particle states

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The action of the Hamiltonian on the wave function is

$$\hat{H}_N\Psi^{\sigma_1\ldots\sigma_N,s} = -i\sum_{j=1}^{N}\partial_{x_j}\Psi^{\sigma_1\ldots\sigma_N,s} + J\sum_{j=1}^{N}\sum_{\sigma'_j,s'}\delta(x_j)\boldsymbol{\sigma}_{\sigma_j\sigma'_j}\boldsymbol{S}_{ss'}\Psi^{\sigma_1\ldots\sigma'_j\ldots\sigma_N,s'} \tag{13}$$

Many-particle states

Since the Hamiltonian preserves the number of particles N (now it will be s electrons only), define the N-particle states. First define the vacuum

$$c_\sigma(x)|\Omega\rangle = S^+|\Omega\rangle = 0, \quad S^\pm = S^x \pm iS^y. \tag{11}$$

Then define the states

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Consider the case $N=1$. Look for the wave function in the form

$$\Psi^{\sigma,s}_p(x) = \begin{cases} A^{\sigma,s}_p e^{ipx}, & x<0,\\ B^{\sigma,s}_p e^{ipx}, & x>0. \end{cases} \tag{14}$$

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Substituting it to the Schrödinger equation, we obtain

$$A_p^{\sigma,s} = \sum_{\sigma',s'} R^{\sigma s}_{\sigma' s'} B^{\sigma',s'}, \qquad R = e^{iJ\boldsymbol{\sigma}\boldsymbol{S}}. \tag{15}$$

$N = 2$ states

Consider now the case $N = 2$.
Let $\Psi^{\sigma_1\sigma_2,s}(x_1, x_2) = A^{\sigma_1\sigma_2,s}e^{ip_1x_1+ip_2x_2} - A^{\sigma_2\sigma_1,s}e^{ip_1x_2+ip_2x_1}$, $x_1, x_2 < 0$.

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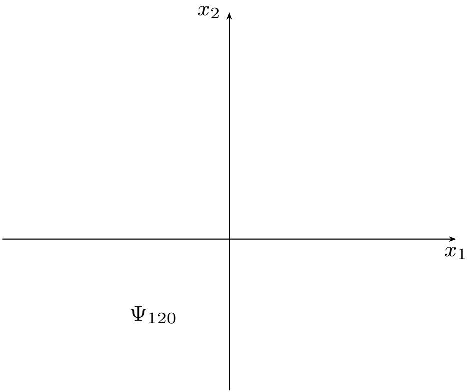

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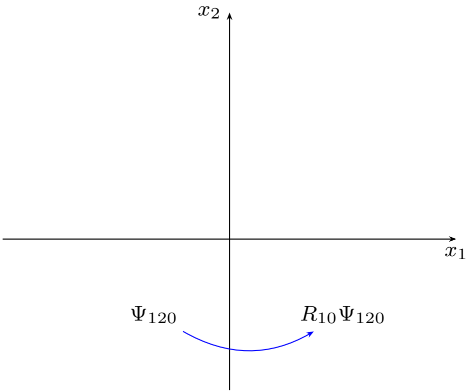

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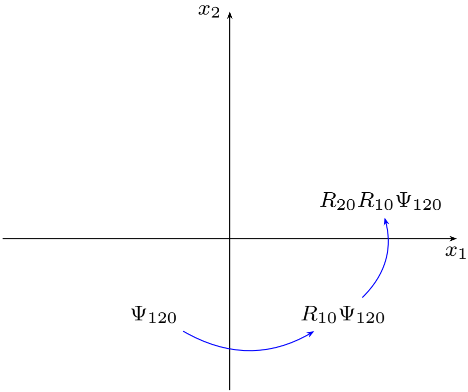

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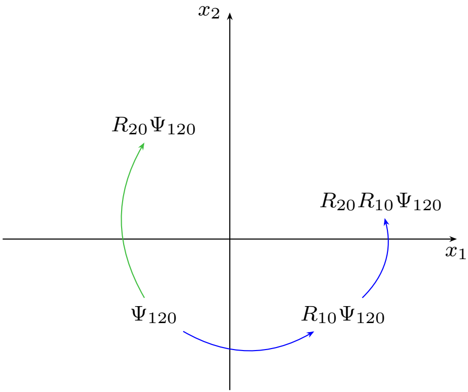

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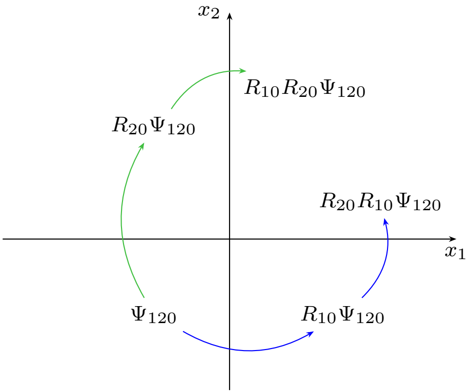

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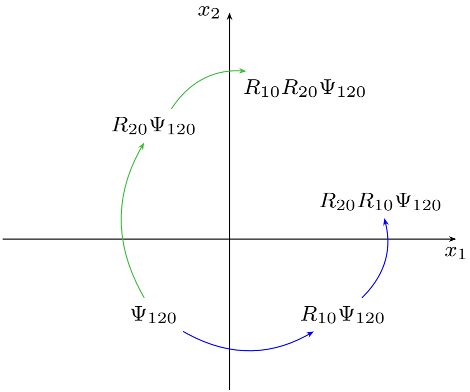

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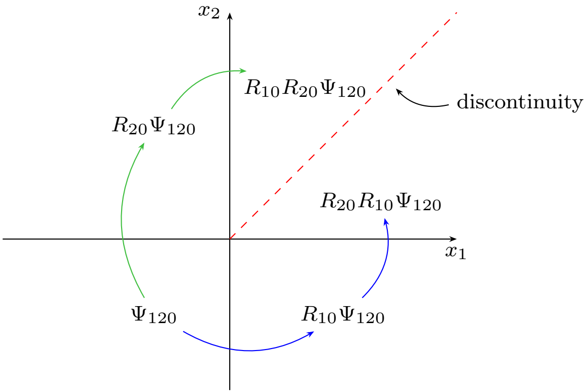

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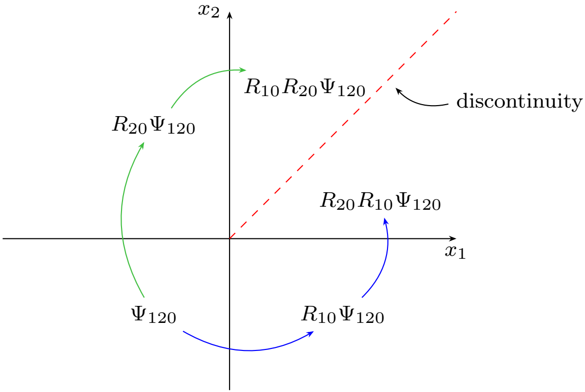

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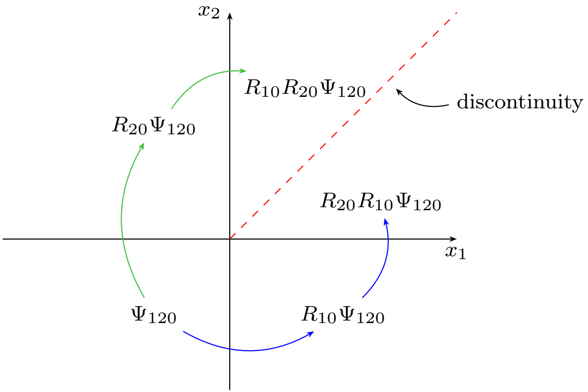

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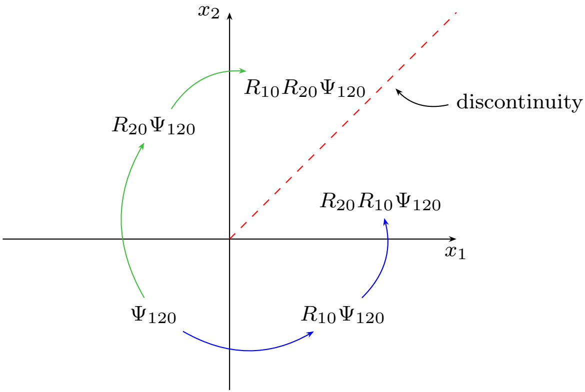

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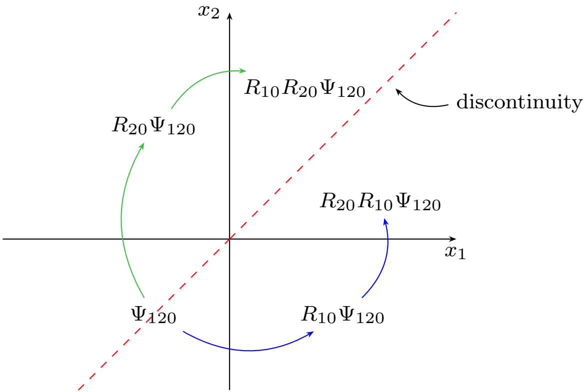

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Formal Yang–Baxter equation

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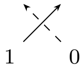

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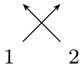

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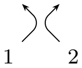

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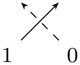
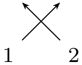
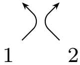
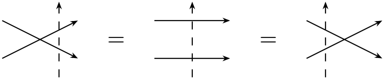
$R_{10} =$ 

$P_{12} =$ 

$=$ 

then eq. (16) reads



Impose the periodic boundary condition

$$\Psi(x_1, \ldots, x_j, \ldots, x_N) = \Psi(x_1, \ldots, x_j + L, \ldots, x_N). \tag{17}$$

In the physical 3D space it means that we have a spherical mirror of the radius $L/2$ with an appropriate boundary condition.

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$$T_j = P_{jj-1} \dots P_{j1} R_{j0} P_{jN} \dots P_{jj+1}. \tag{18}$$

Periodic boundary condition

Impose the periodic boundary condition

$$\Psi(x_1, \dots, x_j, \dots, x_N) = \Psi(x_1, \dots, x_j + L, \dots, x_N). \tag{17}$$

In the physical 3D space it means that we have a spherical mirror of the radius $L/2$ with an appropriate boundary condition.
Introduce the operator

$$T_j = P_{jj-1} \dots P_{j1} R_{j0} P_{jN} \dots P_{jj+1}. \tag{18}$$

Then the periodic boundary condition reads

$$e^{ip_j L} \Psi = T_j \Psi. \tag{19}$$

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Example: $N=2$. Let $x_1<0<x_2<x_1+L$. Then

$$\Psi^{\sigma_1\sigma_2,s}(x_1,x_2)=A^{\sigma_1\sigma_2,s}_{12,-+}e^{ip_1x_1+ip_2x_2}-A^{\sigma_2\sigma_1,s}_{21,+-}e^{ip_2x_1+ip_1x_2},$$

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By comparing the first terms we obtain

$$e^{ip_1L}A^{\sigma_1\sigma_2,s}_{21,++}=A^{\sigma_1\sigma_2,s}_{12,-+}=R^{\sigma_1,s}_{\sigma_1',s'}A^{\sigma_1'\sigma_2,s}_{12,++}=R^{\sigma_1,s}_{\sigma_1',s'}A^{\sigma_2\sigma_1',s}_{21,++}=(R_{10}P_{12}A_{21,++})^{\sigma_1\sigma_2,s}.$$

Comparing the second terms give the same result up to the permutation $1\leftrightarrow 2$.

Now rewrite the operators T_j. We have

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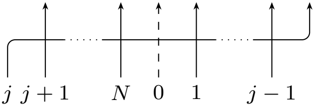

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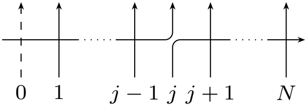

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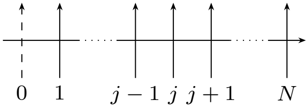

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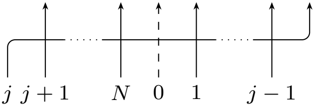
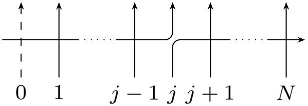
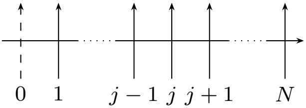
$T_j =$ 

$=$ 

$=$ 

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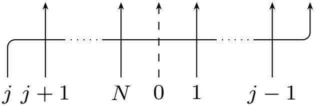

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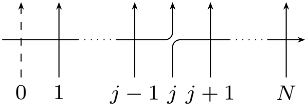

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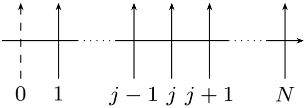

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How to diagonalize the matrix T? We want to immerse it into a set of commuting transfer matrices $T(u)$, so that $T = T(0)$.
To do it let us recall the trivial identity

$$P_{12} R_{10} R_{20} = R_{20} R_{10} P_{12} \qquad (16)$$

and try to deform it.

Matrices $R(u)$ and $S(u)$: requirements

Let us find the matrices $R(u)$ and $S(u)$, so that they satisfy the following requirements:

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If we obtain such matrices, we will have a family of transfer matrices

$$T(u)=\operatorname{tr}_{\tilde{1}} L_{\tilde{1}}(u), \qquad L_{\tilde{1}}(u)=S_{\tilde{1}N}(u)\dots S_{\tilde{1}1}(u)R_{\tilde{1}0}(u+1), \tag{24}$$

such that

$$T(0)=T, \qquad [T(u),T(v)]=0. \tag{25}$$

Matrices $R(u)$ and $S(u)$: solution

The solution can be represented as

$$\begin{aligned} S_{12}(u) &= w_0(u) + w(u)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2, \\ R_{10} &= w_0'(u) + 2w'(u)\boldsymbol{\sigma}_1\boldsymbol{S}_0. \end{aligned} \tag{26}$$

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It is convenient to introduce the notation

$$a = w_0 + w, \qquad b = w_0 - w, \qquad c = 2w,$$
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In this case, the matrix $S(u)$ has the same form as the R-matrix of the XXZ model:

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By solving the Young–Baxter equation, we find

$$
\begin{aligned}
\frac{b(u)}{a(u)} &= \frac{b'(u)}{a'(u)} = \frac{u}{u+ig}, \\
\frac{c(u)}{a(u)} &= \frac{c'(u)}{a'(u)} = \frac{ig}{u+ig},
\end{aligned}
\tag{28}
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i.e. $S(u)$ is nothing but the R-matrix of the XXX model.

Impose the unitarity condition

$$a(u)a(-u)=1, \qquad a'(u)a'(-u)=\frac{g^2+u^2}{g^2(S+1/2)^2+u^2}. \tag{29}$$

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Finally, the condition (22) gives

$$a(0) = 1, \qquad a'(1) = \frac{1+ig}{2}(e^{iJS} + e^{-iJ(S+1)}) \tag{30}$$

and

$$g = \frac{1}{S+1/2}\,\mathrm{tg}\,J(S+1/2). \tag{31}$$

Otherwise, $a(u)$, $a'(u)$ are arbitrary functions.

Return to the definitions

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$$\begin{aligned} \Lambda_A(u) &= ((S+1/2)a'(u+1) - (S-1/2)b'(u+1))a^N(u), \\ \Lambda_D(u) &= ((S+1/2)b'(u+1) - (S-1/2)a'(u+1))b^N(u). \end{aligned} \tag{35}$$

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The Bethe Ansatz has the form

$$|u_1, \ldots, u_n\rangle = B(u_1) \ldots B(u_n)|\Omega_N\rangle, \qquad S^z = N/2 + S - n. \tag{36}$$

Bethe equations

The Bethe equations are written in the standard form

$$\frac{\Lambda_D(u_i)}{\Lambda_A(u_i)} = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{a(u_j - u_i)b(u_i - u_j)}{b(u_j - u_i)a(u_i - u_j)}. \tag{37}$$

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The eigenvalues of $T(u)$ are given by

$$\Lambda(u; u_1, \ldots, u_N) = \Lambda_A(u) \prod_{i=1}^{n} \frac{a(u_i - u)}{b(u_i - u)} + \Lambda_D(u) \prod_{i=1}^{n} \frac{a(u - u_i)}{b(u - u_i)}. \tag{38}$$

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Explicitly, the system of Bethe equations have the form

$$\left(\frac{v_i+i/2}{v_i-i/2}\right)^N \frac{v_i+iS+1/g}{v_i-iS+1/g} = -\prod_{j=1}^{n}\frac{v_i-v_j+i}{v_i-v_j-i}, \tag{40}$$

$$e^{ip_jL} = e^{iJS}\prod_{i=1}^{n}\frac{v_i+i/2}{v_i-i/2}. \tag{41}$$

This reduces the solution of the Kondo problem to the joint solution of the equations (40) and (41).

Seminar