Lecture 12
Kondo Problem: Solving Bethe Equations

Michael Lashkevich

Bethe equations

Recall the Bethe equations for the *sd* model:

$$e^{ip_a L} = e^{iJS} \prod_{i=1}^{n} \frac{v_i + i/2}{v_i - i/2}, \tag{1}$$

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Now we will study these equations in the thermodynamic limit $L \to \infty$, $N \to \infty$.

Take logarithm of the Bethe equations:

$$p_a L = 2\pi I_a + JS - \sum_{i=1}^{n} (\pi + p(v_i)), \tag{5}$$

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Charge energy for $J = 0$. Usual description

Let $J = 0$: the case of free electrons. We have two pictures:

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$$p_a L = 2\pi I_a \quad I_a \in \mathbb{Z},$$

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The ground state is defined by $-\frac{N}{2} \leq I_a \leq 0$, and the energy is equal to

$$E_0 = -\frac{\pi N^2}{2L} = -\frac{\epsilon_F N}{2}.$$

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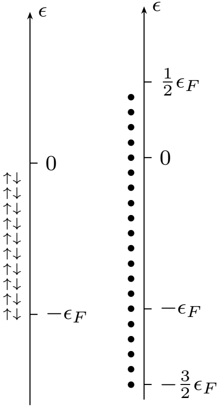
All other energies must be larger. Therefore, we obtain the admissibility condition for solutions to the Bethe Ansatz equations

$$\boxed{I_a \geq -N/2.}$$

Two descriptions: comparison



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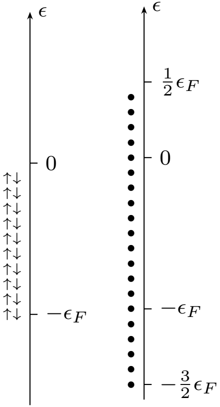

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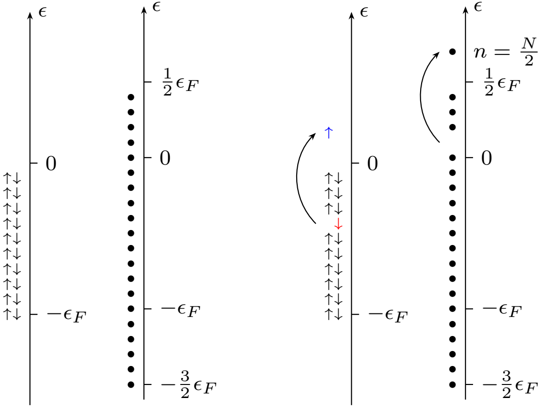

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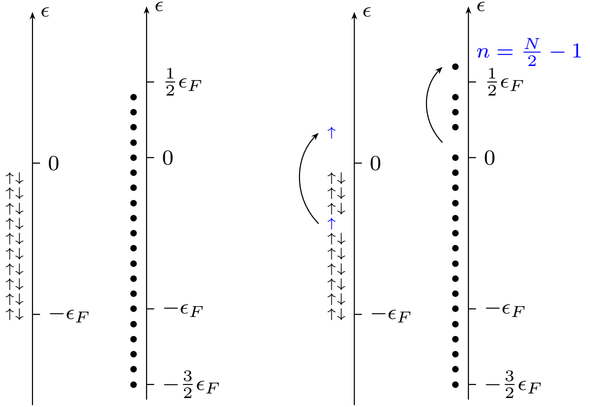

$I'_a = I_a + L\Delta E/2\pi - 1$ since $\Delta \sum(-\pi - p(v_i)) \simeq \frac{2\pi}{N}$.

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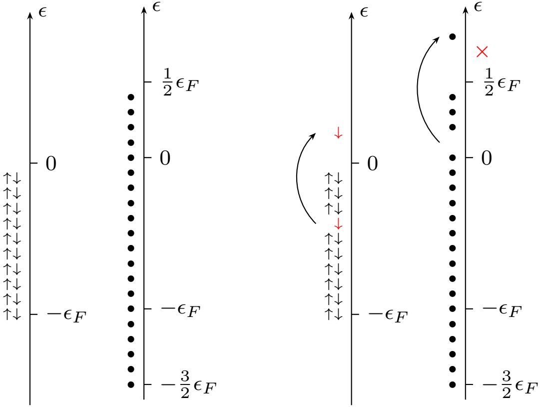

$I'_a = I_a + L\Delta E/2\pi + 1$? BUT: Calculation of state with $n > \frac{N}{2}$ ($S^z_{\rm tot} < 0$) is problematic within the Bethe Ansatz technique.

States of nonzero spin

Assume J to be arbitrary. Return to the Bethe equations for v_i:

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The minimum of energy corresponds to larger v_is and, hence, to larger J_is. Therefore, for the ground state in the spin space J_is densely fill the region

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$$-b \le v_i < \infty$$

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Now let us calculate the spin energy. Since $\delta_S(v) = \pi$, we have

$$E^{\rm el}_{\rm sp} = -\frac{2\pi}{L}\sum_{i=0}^{n-1}(J_{\min}+i) + \epsilon_F\left(\frac{N}{2} - n + \frac{n}{N}\right) = \frac{2\epsilon_F}{N}\left(\frac{N}{2} - n\right)^2 = \frac{2\epsilon_F}{N}(S^z_{\rm el})^2.$$

The superscript el means that later we will separate this contribution as the energy of the electron subsystem.

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Hence

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Now let us calculate the spin energy. Since $\delta_S(v) = \pi$, we have

$$E^{\text{el}}_{\text{sp}} = -\frac{2\pi}{L}\sum_{i=0}^{n-1}(J_{\text{min}}+i) + \epsilon_F\left(\frac{N}{2} - n + \frac{n}{N}\right) = \frac{2\epsilon_F}{N}\left(\frac{N}{2}-n\right)^2 = \frac{2\epsilon_F}{N}(S^z_{\text{el}})^2.$$

The superscript el means that later we will separate this contribution as the energy of the electron subsystem.
If we define external magnetic field H as $E^{\text{el}}_{\text{sp}}(H) = E^{\text{el}}_{\text{sp}} - S^z_{\text{el}}H$, and minimize this energy in S^z_{el}, we obtain the Pauli paramagnetism of the s electrons:

$$H = \frac{4\epsilon_F}{N}S^z = 4\epsilon_F M_{\text{el}}, \tag{15}$$

where M_{el} is the magnetization, i.e. spin per electron.

It is easy to find the energy of these states in the case $J = 0$. Indeed, evidently

$$\frac{N+2-n}{2} - J_{\text{min}} = n$$

Hence

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If we define external magnetic field H as $E_{\text{sp}}^{\text{el}}(H) = E_{\text{sp}}^{\text{el}} - S_{\text{el}}^z H$, and minimize this energy in S_{el}^z, we obtain the Pauli paramagnetism of the s electrons:

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where M_{el} is the magnetization, i.e. spin per electron. This formula will make it possible to express impurity energy and magnetization in terms of external magnetic field.

Bethe equations in the thermodynamic limit

Let us write down the Bethe equations for the ground state in the spin space in the thermodynamic limit:

$$\rho(v) = a_1(v) + \frac{1}{N} a_{2S}(v + 1/g) - \int_{-b}^{\infty} \frac{dv'}{2\pi}\, a_2(v - v')\rho(v'), \quad -b \le v < \infty, \tag{16}$$

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$$E_{\rm sp} = \frac{N\epsilon_F}{2} + \frac{\epsilon_F JS}{\pi} - \frac{N\epsilon_F}{\pi} \int_{-b}^{\infty} \frac{dv}{2\pi} \rho(v)(\pi + p(v)). \tag{19}$$

Splitting the integral equation

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The $b = \infty$ case

In the $b = \infty$ case we may use the Fourier transform. Since

$$\tilde{a}_t(k) = \int_{-\infty}^{\infty} \frac{dv}{2\pi}\, a_t(v) e^{ikv} = e^{-t|k|/2}, \tag{27}$$

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From this we obtain

$$M_{\mathrm{el}} = 1/2 - \tilde{\rho}_0(0) = 0, \tag{30a}$$
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The limit $b \to \infty$ corresponds to $H \to +0$. Therefore the total spin of the system is $S - 1/2$ and, hence, the ground state is $2S$-fold degenerate.

For finite b both the equations for $\rho_0(v)$ and $\rho_1(v)$ have the form

$$f(x)+\int_0^\infty \frac{dx'}{2\pi}\,K(x-x')f(x')=g(x),\qquad x>0. \tag{31}$$

Wiener–Hopf method

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The function $\tilde{f}_+(k)$ ($\tilde{f}_-(k)$) has no singularities in the upper (lower) half-plane. Here and below, such a property will be assumed for all functions with the $\pm$ subscripts.

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$$1 + \tilde{K}(k) = \frac{\tilde{K}_+(k)}{\tilde{K}_-(k)}. \tag{34}$$

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$$\tilde{K}_-(k) g(k) = \tilde{q}_+(k) + \tilde{q}_-(k). \tag{35}$$

Multiplying (33) by $\tilde{K}_-(k)$, we obtain

$$\tilde{K}_+(k)\tilde{f}_+(k) + \tilde{K}_-(k)\tilde{f}_-(k) = \tilde{q}_+(k) + \tilde{q}_-(k). \tag{36}$$

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$$\tilde{K}_+(k)\tilde{f}_+(k) - \tilde{q}_+(k) = \tilde{q}_-(k) - \tilde{K}_-(k)\tilde{f}_-(k).$$

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The left-hand side has no singularities in the upper half-plane, and the right-hand side in the lower one. Thus, both sides of this equation have no singularities. Under some additional restrictions on the growth of the functions (which must be checked separately in each case), it follows that

$$\tilde{K}_+(k)\tilde{f}_+(k) = \tilde{q}_+(k), \qquad \tilde{K}_-(k)\tilde{f}_-(k) = \tilde{q}_-(k). \tag{37}$$

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Finally,

$$f(x) = \int_{-\infty}^{\infty} dk\, \frac{\tilde{q}_+(k)}{\tilde{K}_+(k)} e^{-ikx}, \qquad x > 0. \tag{38}$$

Wiener–Hopf method. Application to $b < \infty$

Let

$$f_i(x) = \rho_i(x - b).$$

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We use a trick to obtain a few simple results. Rewrite equation (33) in the form

$$\tilde{f}_{i+}(k) + \frac{\tilde{f}_{i-}(k)}{1+\tilde{K}(k)} = \frac{\tilde{g}_i(k)}{1+\tilde{K}(k)}. \tag{40}$$

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Perform the inverse Fourier transform:

$$f_i(x) + \int_{-\infty}^{0} \frac{dx'}{2\pi} R(x - x') f_i(x') = h_i(x), \tag{41}$$

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$$R(x)=\int_{-\infty}^{\infty}dk\,e^{-ikx}\left(\frac{1}{1+\tilde{K}(k)}-1\right)=-\int_{-\infty}^{\infty}dk\,\frac{e^{-ikx}}{1+e^{|k|}},$$
$$h_0(x)=\frac{\pi}{\operatorname{ch}\pi(x-b)},\qquad h_1(x)=\int_{-\infty}^{\infty}dk\,e^{-ik(x-b+1/g)}\,\frac{e^{-(2S-1)|k|/2}}{2\operatorname{ch}\frac{k}{2}}. \tag{42}$$

Application to $b < \infty$

If $b \gg 1$, for small enough x we may approximate

$$h_0(x) \simeq 2\pi e^{\pi(x-b)}. \tag{43}$$

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Hence $\frac{\tilde{f}_{1-}(k)}{\tilde{f}_{0-}(k)} = e^{\pi/g}$, and we have a precise result for the susceptibility:

$$\chi_{\rm im} = \frac{M_{\rm im}}{H} = \frac{1}{4\epsilon_F}\frac{M_{\rm im}}{M_{\rm el}} = \frac{e^{\pi/g}}{4\epsilon_F}, \quad \text{if } S = 1/2. \tag{46}$$

An accurate calculation by the Wiener–Hopf method gives the formula

$$M_{\rm im}(H)=S-\frac{1}{2}+ \\ +\frac{i}{4\pi^{3/2}}\int_{-\infty}^{\infty}d\omega\left(\frac{H}{T_H}\right)^{-2i\omega}\frac{\Gamma(i\omega+1/2)}{\omega+i0}\left(\frac{-i\omega+0}{e}\right)^{-2iS\omega}\left(\frac{i\omega+0}{e}\right)^{i(2S-1)\omega}, \tag{47}$$

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and

$$M_{\rm im}(H) = (S-1/2)\left(1 + \frac{1}{\log(T_K/H)^2} - \frac{\log\log(T_K/H)^2}{\log^2(T_K/H)^2} + \cdots\right), \quad H \ll T_K, \quad S > 1/2$$

$$M_{\rm im}(H) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \frac{(-1)^n}{n!(n+\frac{1}{2})} \left(\frac{H}{T_H}\right)^{2n+1}, \quad S = 1/2.$$

String solutions

The Bethe equations admit complex roots. For large values of N these roots form the so called strings:

$$v_{j,k}^{p} = v_{j}^{p} + \frac{i}{2}(p + 1 - 2k) + O(e^{-\mathrm{const}\,N}), \qquad k = 1, 2, \ldots, p. \tag{49}$$

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$$e^{ip_a L} = e^{iJS} \prod_{p=1}^{\infty} \prod_{j=1}^{n_p} e_p(v_j^p), \tag{50}$$

$$(e_p(v_j^p))^N e_{p,S}(v_j^p + 1/g) = \prod_{p'=1}^{\infty} \prod_{j'=1}^{n_m} E_{pp'}(v_j^p - v_{j'}^{p'}), \tag{51}$$

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where

$$e_p(v) = -e^{iP_p(v)} = \frac{v+ip/2}{v-ip/2}, \qquad e_{p,S}(v) = -e^{i\Delta_{p,S}(v)} = \prod_{k=1}^{p} \frac{v+\frac{i}{2}(p+1-2k)+iS}{v+\frac{i}{2}(p+1-2k)-iS},$$

$$E_{pp'}(v) = e^{i\Phi_{pp'}(v)} = e_{|p-p'|}(v)e^2_{|p-p'|+2}(v)\ldots e^2_{p+p'-2}(v)e_{p+p'}(v).$$

Finite temperatures

Bethe equations may be applied to finite temperatures. To do it, we need to introduce two types of densities: density of states $\rho_p(v)$ (p means the type of a string) and density of particles $\rho_p^\bullet(v)$. The Bethe equations make it possible to express $\rho_p(v)$ in term of $\rho_p^\bullet(v)$. It is convenient to use also the density of holes $\rho_p^\circ(v) = \rho_p(v) - \rho_p^\bullet(v)$.

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Introduce the entropy of a set of states described by these densities:

$$\begin{aligned} \mathcal{S} &= \log \prod_{p,v} \frac{(N\rho_p(v)\frac{dv}{2\pi})!}{(N\rho_p^\bullet(v)\frac{dv}{2\pi})!(N\rho_p^\circ(v)\frac{dv}{2\pi})!} \\ &= N\sum_{p=1}^{\infty}\int\frac{dv}{2\pi}\left(\rho_p(v)\log\rho_p(v) - \rho_p^\bullet(v)\log\rho_p^\bullet(v) - \rho_p^\circ(v)\log\rho_p^\circ(v)\right). \end{aligned} \tag{52}$$

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Then we have to minimize the free energy

$$F[\rho^\bullet] = E - T\mathcal{S} - HS^z.$$

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This minimization leads to a set of nonlinear equations (the Yang–Yang equations) of the form

$$\epsilon_p(v) + \sum_{p'}\int \frac{dv'}{2\pi}\Phi_{pp'}(v-v')\log(1+e^{-\epsilon_{p'}(v')}) = \frac{1}{T}\left(P_p(v) + \frac{1}{N}\Delta_{p,S}(v) + pH\right),$$

where

$$\frac{\rho_p^\bullet(v)}{\rho_p(v)} = \frac{1}{e^{\epsilon_p(v)}+1}.$$

All thermodynamic quantities are expressed in terms of the pseudoenergies $\epsilon_p(v)$.

Seminar