Lecture 1
Eight-vertex model and commuting transfer matrices

Since we shall speak about some models of physical importance, let us formulate first the general physical framework. Consider a classical system of interacting particles. Let C be a state of this system, which can run some (generally infinite) set of admissible states. Let $E(C)$ be the energy of the state C, defined by usual Hamiltonian mechanics. Suppose that the system weakly interacts with a thermal bath of the temperature T. The most fundamental postulate of statistical mechanics, the *Gibbs law*, says that the probability of the state C is given by

$$w(C) = \frac{1}{Z}\, e^{-E(C)/T}. \tag{1}$$

It is easy to understand, what is the proportionality coefficient $1/Z$. As the total probability for the system to be in any state is unity, we have for the *partition functions* of the system

$$Z = \sum_C e^{-E(C)/T}. \tag{2}$$

Since the space of configurations can be continuous, the sum may turn out to be an integration, but we shall not consider this general case in these lectures.

The partition function encodes the most fundamental observable thermodynamic function of the model, the *free energy*:

$$F = E - TS = -T \log Z. \tag{3}$$

Here E is the *total energy* of the system,

$$E = \sum E(C)w(C) = \frac{T^2}{Z(T)}\frac{d}{dT}Z(T) = -T^2\frac{d}{dT}\frac{F}{T} = F - T\frac{dF}{dT},$$

while

$$S = -\sum w(C)\log w(C) = -\frac{dF}{dT}$$

is the *entropy*. The first equality in (3) is the thermodynamic definition of the free energy, which, in principle, was established before Gibbs, while the second equality provides its statistical interpretation.

Let $f_i(C)$ be some functions of the state of the system. The *correlation functions* are generic expectation values

$$\langle f_1 \dots f_n \rangle = \sum_C w(C) f_1(C) \dots f_n(C).$$

Note, that the correlation functions can be expressed in terms of derivatives of a generalized partition function of the system with external fields F_i coupled to the variables f_i:

$$Z(F_1, \dots, F_N) = \sum_C e^{-E(C;F_1,\dots,F_N)/T}, \qquad E(C;F_1,\dots,F_N) = E(C) - \sum_i F_i f_i.$$

Usually, from the physical point of view the most interesting objects are local correlation functions, i.e. the correlation functions of the amounts $f_i(C)$ that can be measured at some space points. We shall discuss examples of such functions in detail later.

Naturally, evaluation of the partition function and correlation functions is a difficult task, except some trivial examples solved by classics of the science. Most these solved examples are systems of independent particles, each of which possesses a finite set or, at least, a finite-dimensional space of states and a simple function for $E(C)$ (like a quadratic function in the Boltzmann gas). But how to do in the case of interacting particles? The usual approach in physics is developing some approximate methods, based either on the perturbation theory or on some experimentally supported assumptions. It is a very effective way, but sometimes physicists need some additional support to their assumptions.

How mathematicians can help physicists? First, they can propose some rigorous estimates, which can prove the effects predicted by physicists. Many of such important estimates were proposed (see, e. g. [1,2]).

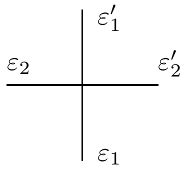
Second, they may propose some sofisticated methods to solve exactly some particular nontrivial examples. Though these methods can be not completely rigorous, they better convince physicists, because they are more in the way of physical thinking, and can be used by physicists themselves.

Let us now slightly specify the problem. Forget about motion of particles. We can do it for some problems, for example, if positions of particles are fixed by the crystalline lattice. The set of states of the system (*configurations*) is reduced to the direct product of sets of internal states of particles ('spins'). We shall assume these 'spins' to be discrete variables. Physically the discrete 'spins' may originate in quantum states of atoms (e.g. physical spin states) with diagonal interaction or in other sources, e.g. in types of atoms in the substitutional solutions or in positions of atoms in different quantum wells as in the ice-type systems.

In these lectures we shall discuss two-dimensional models of statistical mechanics. Why two-dimensional? Because in one dimension the lattice statistical models are trivially reduced to quantum mechanics of small systems without phase transitions or other interesting features. On the other hand, in three dimensions the problem is too complicated and the classes of solvable models are too narrow. We restrict our attention onto two classes of two-dimensional solvable lattice models, which are in a sense basic: eight-vertex model and solid-on-solid (SOS) solvable model.

Let us start with the ice model on the square lattice called also the six-vertex model [3]. Consider a square lattice made of oxygen ions (O^{2-}) in the vertices and hydrogen ions (H^+) on the vertical and horizontal bonds of the lattice, which we shall call edges. (Do not think of all this too seriously, because experimentalists are unable to produce any two-dimensional ice-like systems, but this picture of ice can help you both to remember the formulation of the model and to better understand the physical origin of such kind of problems.) We know that the hydrogen ions can form with the oxygen ions two types of bonds: strong and narrow polar bond and weak and long 'hydrogen' bond. It means that a hydrogen ion lying on an edge of the lattice must be positioned near one of the oxygen ion and far from another oxygen ion. There are two such positions on each edge. It forms discrete 'spin' state on each edge. We shall think that the 'spin' is equal to '+' or '+1' if the hydrogen ion is positioned near the right end of a horizontal edge or near the upper end of the vertical edge, and is equal to '−' or '−1' if it is positioned near the left or lower end of the edge.

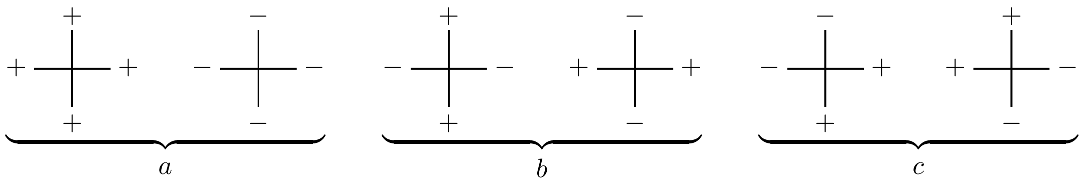
Further, from the neutrality condition we conclude, that in the vicinity of each oxygen ion there must be just two hydrogen ions. It imposes a restriction onto values of 'spins' at edges surrounding each vertex. Namely, denote these spins by ε_1, ε_2, ε_1', ε_2':



Then this restriction, called the *ice rule*, can be expressed as

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_1' + \varepsilon_2' \tag{4}$$

There are six configurations around each vertex that satisfy the ice rule:



Let us think that each of this vertex configuration may have its own energy $E_1, \ldots, E_6$. The energy of the system is equal to the sum of energies of vertices. It defines the model nearly completely. The only thing to specify is the boundary conditions. We shall consider three possibilities:

1. The model of M columns and N rows with cyclic (toroidal) boundary conditions.
2. The model with fixed spins at the boundary.
3. The model on an infinite lattice, considered as a limit of any of these models as $M, N \to \infty$.

The six-vertex model is known to be solvable if

$$E_1 = E_2, \qquad E_3 = E_4, \quad E_5 = E_6.$$

It means that the configurations braced together on the picture possess the same energies and the same Boltzmann weights:

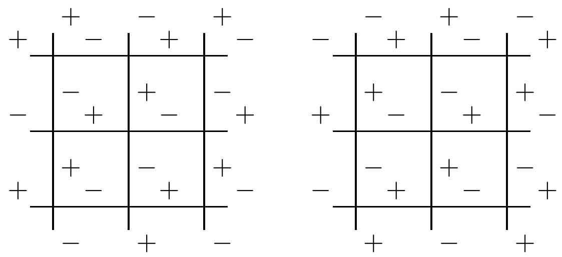
$$a = e^{-E_1/T} = e^{-E_2/T}, \qquad b = e^{-E_3/T} = e^{-E_4/T}, \qquad c = e^{-E_5/T} = e^{-E_5/T}. \tag{5}$$

This model is called the homogeneous six-vertex model without external field.

The disadvantage of the six-vertex model is that it possesses some pathologic physical properties, related to severeness of the ice condition. Namely, consider its phase diagram:

$$\begin{aligned} \mathrm{AF}: &\quad c > a + b, \\ \mathrm{F}_1: &\quad a > b + c, \\ \mathrm{F}_2: &\quad b > a + c, \\ \mathrm{D}: &\quad \tfrac{1}{2}(a + b + c) \geq a, b, c. \end{aligned}$$

The antiferroelectric (AF) region is the region of antiferroelectric order considered as excitations above the following ground states:



(6)

The excitations can be considered as some loops of changed spins and can have finite energy even on the infinite lattice. The correlation functions of spins decrease as $e^{-r/\xi}$ as the distance $r \to \infty$. The constant ξ is called *correlation length.* It is a normal behavior of correlation functions out of special point called critical points.

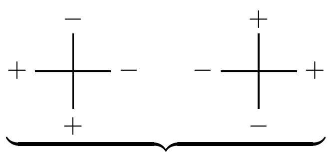
In the ferroelectric regions (F_1 and F_2) the situation is strange: all excitations consist of the lines of flipped spins in the SW–NE direction, which are infinite in the infinite volume limit! It means that these excitations possess large energy, and their contribution tends to zero as $M, N \to \infty$. We have the situation of a *frozen order.* The free energy of such system per site tends to zero.

Another pathological feature is related to the disordered (D) region. The whole region turns out to be *critical.* It means that the correlations between local variables like spins decrease with distance r like r^{-2d} with an appropriate *scaling dimension* d. From physics we know that critical points always lie on surfaces that separate phases in the system. But here we have a situation where the critical points form a region on the phase diagram. One can expect, that if we add to the model new configurations that break the ice conditions, this critical region will become a surface that separates two phases.

Unfortunately, physically reasonable solvable generalizations of the six-vertex model are unknown. If we relate some energy to disbalance of charge at a vertex, we shall lose solvability. But it is possible to generalize the model in a 'mathematical' way without lost of solvability as follows [4, 5]. Suppose the configurations around a vertex to be admissible if

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_1' + \varepsilon_2' \bmod 4. \tag{7}$$

It means that we admit two more vertex configurations:



d

Such model is called *eight-vertex*. If, in addition to (5), the corresponding energies E_7 and E_8 are equal,

$$d = e^{-E_7/T} = e^{-E_8/T},$$

the model turns out to be solvable [6]. The properties described below can be found in [7].

The phase diagram of the eight-vertex model looks like:

$$\begin{aligned} \mathrm{AF}_1: &\quad c > a+b+d, \\ \mathrm{AF}_2: &\quad d > a+b+c, \\ \mathrm{F}_1: &\quad a > b+c+d, \\ \mathrm{F}_2: &\quad b > a+c+d, \\ \mathrm{D}: &\quad \tfrac{1}{2}(a+b+c+d) > a,b,c,d. \end{aligned} \tag{8}$$

In this case the disordered region D is not critical. The correlation length is finite and correlation functions decrease exponentially. The critical points lie on the boundaries of the regions

$$a = b+c+d; \quad b = a+c+d; \quad c = a+b+d; \quad d = a+b+c$$

and at the special surfaces

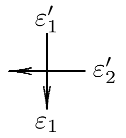
$$\begin{aligned} a = 0, &\quad \tfrac{1}{2}(b+c+d) \ge b,c,d; \\ b = 0, &\quad \tfrac{1}{2}(a+c+d) \ge a,c,d; \\ c = 0, &\quad \tfrac{1}{2}(a+b+d) \ge a,b,d; \\ d = 0, &\quad \tfrac{1}{2}(a+b+c) \ge a,b,c; \end{aligned}$$

There are special maps ('dualities') that can map each of the regions (8) on to another [5]. So it is enough to study only one region, e.g. AF_1. In this region the ground states again look like (6).

I repeated many times the words 'solvability', 'solvable'. What does they mean? Though there is no rigorous definition of this term, it must mean approximately the following: there are some quantities of physical importance (partition function, correlation functions) that could be found exactly in these models. What are the conditions for solvability of this or that model? Let us make several steps to see such (presumably sufficient) condition.

First introduce the weight matrix (called also R matrix) of the eight-vertex model:

$$R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2} = \tag{9}$$

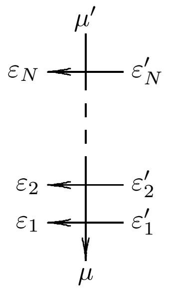


We added the arrows to the lines to define the orientation on the lattice, so that we could rotate or distort it. The R matrix can be written as

$$R = \begin{array}{c} \\ ++ \\ +- \\ -+ \\ -- \end{array} \!\!\begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{pmatrix} a & & & d \\ & b & c & \\ & c & b & \\ d & & & a \end{pmatrix} \end{array}. \tag{10}$$

Now introduce the L matrix

$$(L^{\mu'}_{\mu})^{\varepsilon'_1\ldots\varepsilon'_N}_{\varepsilon_1\ldots\varepsilon_N} = \quad = R^{\mu'}_{\mu_{N-1}}{}^{\varepsilon'_N}_{\varepsilon_N} \ldots R^{\mu_2}_{\mu_1}{}^{\varepsilon'_2}_{\varepsilon_2} R^{\mu_1}_{\mu}{}^{\varepsilon'_1}_{\varepsilon_1}. \tag{11}$$



We shall consider this object as a matrix in indices μ, μ' and an operator in the product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (N times) spanned on the vectors $v_{\varepsilon_1} \otimes v_{\varepsilon_2} \otimes \cdots \otimes v_{\varepsilon_N}$, where v_+ and v_- form the natural basis in $\mathbb{C}^2$. The product associated to $\varepsilon_1, \ldots, \varepsilon_N$ is called *quantum* space, while the space $\mathbb{C}^2$ assotiated to μ is called *auxiliary* space. We shall always omit the 'quantum' indeces and sometimes omit the auxiliar indeces, substituting them by a numeric subscript labelling the space, e.g. L_1.

Now we are ready to define the *transfer matrix*

$$T = \mathrm{tr}_1 L_1 \equiv \sum_\mu L^\mu_\mu. \tag{12}$$

Consider the eight-vertex model of M columns and N rows with cyclic boundary conditions. It is easy to see that the partition function is equal to

$$Z = \mathrm{Tr}\, T^M,$$

where the trace Tr is taken over the quantum space. Let $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_{2^N}$ be the eigenvalues of the transfer matrix. Then

$$Z = \sum_i \Lambda_i^M.$$

In the large M limit the leading contribution is given by Λ_1^M and we have

$$\kappa \equiv Z^{1/MN} \to \Lambda_1^{1/N}, \qquad f \equiv \frac{F}{MN} \to -\frac{T}{N} \log \Lambda_1 \qquad \text{as } M \to \infty,$$

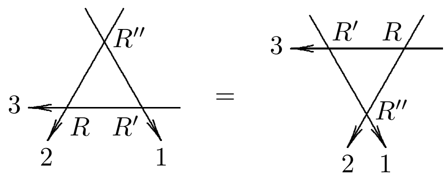
where the partition function per site κ and the partial free energy f are introduced. Surely, it is necessary to study the behavior of subleading contributions at large M and N to substantiate these formulas, and it can be done, but in these lectures we shall assume these formulas to be correct without a proof.

Anyhow the problem is reduced to that of the quantum mechanics: we have an operator T of evolution by one step in the 'time' direction and we have to diagonalize it. When this can be done exactly? From classical mechanics we know that the system is solvable (more precisely, integrable) if we have sufficiently many integrals of motion in involution (the Liouville theory). Though there is no 'quantum Liouville theorem' it can be expected that in quantum mechanics the situation is similar. We have to find some other commutative integrals of motion, i.e. operators that commute with the transfer matrix and with each other. Let us look for them in the same form as the transfer matrix. Namely, let R' be the matrix of the form (10) with some new 'weights' a', b', c', d'. Define the L operator L' and the transfer matrix T' in terms of R' according to (11) and (12). Besides, for any product $V_1 \otimes \cdots \otimes V_n = \mathbb{C}^2 \otimes \cdots \mathbb{C}^2$ we shall denote by R_{ij} the matrix R acting on the ith and jth components of the product. Then there is a

Theorem [8]. *If there exists an invertible matrix R'' of the form (10), such that the* ***Yang–Baxter eqaution*** *holds*

$$R''_{12} R'_{13} R_{23} = R_{23} R'_{13} R''_{12} \tag{13}$$

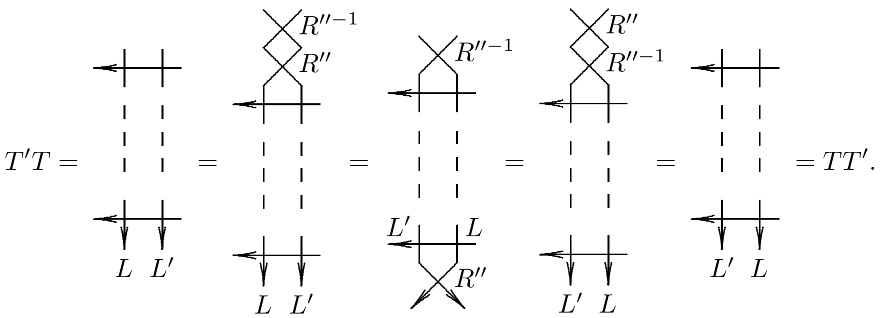
or, graphically,



then the transfer matrices T and T' commute.

$$TT' = T'T. \tag{14}$$

The proof is elementary in graphical form:



Note, that the third equality means that

$$R''_{12}L'_1L_2 = L_2L'_1R_{12}. \tag{15}$$

which generalizes the Yang–Baxter equation to the situation where the space 3 is substituted by the whole quantum space.

What are the solutions to the Yang–Baxter equation? It turns out that they can be written in the form

$$R = R(u_2 - u_3), \qquad R' = R(u_1 - u_3), \qquad R'' = R(u_1 - u_2)$$

with some analytic function $R(u)$. The *spectral parameters* u_i can be associated with the spaces V_i. The Yang–Baxter equation takes the form

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2). \tag{16}$$

The solution matrix elements $a(u), \dots, d(u)$ of the matrix $R(u)$ are written in terms of the Jacobi theta functions $\theta_i(u;\tau)$ $(i = 1, \dots, 4)$ with quasiperiods 1 and τ $(\operatorname{Im}\tau > 0)$. Namely, in the region AF_1 we have

$$\begin{aligned} a(u) &= \rho(u;\epsilon,r)s(1-u;\epsilon,r), \\ b(u) &= \rho(u;\epsilon,r)s(u;\epsilon,r), \\ c(u) &= \rho(u;\epsilon,r)s(1;\epsilon,r), \\ d(u) &= \rho(u;\epsilon,r)s(1-u;\epsilon,r)s(u;\epsilon,r)s(1;\epsilon,r), \end{aligned} \tag{17}$$

where

$$s(u;\epsilon,r) = \frac{\theta_1\left(\frac{u}{2r};\frac{i\pi}{2\epsilon r}\right)}{\theta_2\left(\frac{u}{2r};\frac{i\pi}{2\epsilon r}\right)}.$$

The parameters $\epsilon > 0$, $r > 1$ are fixed numbers. The scalar function $\rho(u;\epsilon,r)$ is arbitrary.

In the limit $r \to \infty$ we obtain the six-vertex model:

$$\begin{aligned} a(u) &\sim \operatorname{sh}\epsilon(1-u), \\ b(u) &\sim \operatorname{sh}\epsilon u, \\ c(u) &\sim \operatorname{sh}\epsilon, \\ d(u) &= 0. \end{aligned}$$

The presence of a continous set of soluitions $R(u)$ means that there is an infinite family of commuting transfer matrices:

$$T(u_1)T(u_2) = T(u_2)T(u_1). \tag{18}$$

The matrix $R(0)$ is proportional to the transposition matrix

$$P = \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix}.$$

It means that the transfer matrix $T(0)$ is proportional to the shift operator. Define now a set of Hamiltonians $H_1, H_2, \ldots$ as follows

$$T^{-1}(0)T(u) = 1 + H_1 u + H_2 u^2 + \ldots . \tag{19}$$

They all commute with each other and with the shift operator

$$[H_m, H_n] = 0, \qquad [H_m, T(0)] = 0.$$

Not all of them are independent. In a finite system only a finite number of them are independent. It turns out that the model is indeed solvable. It is not easy to prove this, and it demands some other ideas to find even the partition function. You can find the solution in Baxter's book [7].

What are the Hamiltonians H_n? The general H_n is a complicated operator, but the simplest one is given by

$$H_1 = \text{ const } - \frac{1}{2}\sum_{k=1}^{N}(J_x \sigma_k^x \sigma_{k+1}^x + J_y \sigma_k^y \sigma_{k+1}^y + J_z \sigma_k^z \sigma_{k+1}^z)$$

with σ_k^a are the Pauli sigma-matrices,

$$\sigma^x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} & i \\ -i & \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$$

acting on the kth component of the tensor product. The coefficients J_a are functions of ϵ, r. The common factor is not so interesting, but the ratios of these coefficients are important u independent combinations of weights:

$$\Delta = \frac{2J_z}{J_x + J_y} = \frac{a^2 + b^2 - c^2 - d^2}{2ab}, \qquad \Gamma = \frac{J_x - J_y}{J_x + J_y} = \frac{cd}{ab}.$$

In the case of the six-vertex model $\Gamma = 0$ and we have the XXZ model ($J_x = J_y$).

It can be said that the XYZ chain is much more physical model than the eight-vertex model itself. The largest eigenvalue of T corresponds to the lowest eigenvalue of H_1. The next-to-large eigenvalues of T correspond to the first excitations abouve the ground state of the XYZ model. In the infinite-volume limit the lowest two states are degenerate, while the gap between these lowest states and the other excitations remains finite and only vanishes at the critical points. This gap is the inverse correlation length in the time dimension, while the spectrum above the gap defines the corelation length in the spacial dimension.

Generally, the two-dimensional lattice models of classical statistical mechanics are related to the one-dimensional models of quantum mechanics.

If the function ρ satisfies the conditions

$$\rho(u)\rho(-u) = (s(u;\epsilon,r)s(-u;\epsilon,r) + s^2(1;\epsilon,r))^{-1}, \qquad \rho(u) = \rho(1-u), \tag{20}$$

the R matrix satisfies two additional conditions:

$$R_{12}(u)R_{21}(-u) = \text{id} \qquad \textbf{(Unitarity)}, \tag{21}$$

$$R(u)_{\varepsilon_1 \varepsilon_2}^{\varepsilon_1' \varepsilon_2'} = R(1-u)_{\varepsilon_2', -\varepsilon_1}^{\varepsilon_2, -\varepsilon_1'} \qquad \textbf{(Crossing symmetry)}. \tag{22}$$

It is possible to find the solution to the equations (20), such that $R(u)$ has the minimal number of poles on the strip $0 < \operatorname{Re} u < 1$. It reads

$$\begin{aligned} \rho(u;\epsilon,r) &= x^{1-r/2} \frac{(x^{2r+2};x^{4r})_\infty (x^{2r-2};x^{4r})_\infty}{(x^{2r};x^{4r})_\infty^2} \frac{(x^{2r}z;x^{4r})_\infty (x^{2r}z^{-1};x^{4r})_\infty}{(x^{4r-2}z;x^{4r})_\infty (x^2 z^{-1};x^{4r})_\infty} \frac{g(z^{-1})}{g(z)}, \\ g(z) \equiv g(z;\epsilon,r) &= \frac{(x^2 z;x^4,x^{2r})_\infty (x^{2r+2}z;x^4,x^{2r})_\infty}{(x^4 z;x^4,x^{2r})_\infty (x^{2r}z;x^4,x^{2r})_\infty}, \end{aligned} \tag{23}$$

with the brace function

$$(z;p_1,\ldots,p_N) = \prod_{n_1,\ldots,n_N=0}^{\infty} (1 - z p_1^{n_1} \ldots p_N^{n_N})$$

and the ‘multiplicative’ parameters x, p, z defined as

$$x = e^{-\epsilon}, \qquad p = x^{2r}, \qquad z = x^{2u}.$$

It turns out that this solution gives just the R matrix for which the partition function per site is equal to 1 according to Baxter’s solution. It means that solutions of such simple ‘reflection equations’ (21) and (22) make it possible to easily reproduce the result of tedious and involved calculations based on the Bethe equations!

References

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