Lecture 2
SOS model and vertex-face correspondence

To understand better the origin of the SOS model let us sketch the Bethe ansatz for the six-vertex model, where $d = 0$. In this case we can introduce the operator S^z of 'total spin', which counts the signs along a column:

$$S^z(v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_N}) = \frac{1}{2}\sum_{i=1}^{N} \varepsilon_i (v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_N}).$$

Due to the ice condition this operator commutes with the transfer matrix

$$[T(u), S^z] = 0.$$

This is a trivial fact: the number of 'minuses' is conserved.

So we can easily establish at least two eigenvectors (*pseudovacuums*)

$$|\Omega_\pm\rangle = v_\pm \otimes \cdots \otimes v_\pm$$

with the eigenvalue $a^N + b^N$. But this is generally (everywhere except in the ferroelectric regions F_i) NOT the largest one. How to find the other eigenvectors? Let us start from $|\Omega_+\rangle$ and flip spins one by one. Any state with the eigenvalue of S^z being $N/2 - n$ will be called a state of n pseudoparticles. Let $\sigma_k^- = (\sigma^x - i\sigma^y)/2$ be the operator that turns the kth spin down.

Consider the state of one pseudoparticle. From the translational invariance we conclude, that it has the form

$$|p\rangle = \sum_{k=1}^{N} e^{\mathrm{i}pk} \sigma_k^- |\Omega_+\rangle.$$

From cyclic boundary condition we conclude that

$$e^{ipN} = 1,$$

so that we have N states with $p_j = \frac{2\pi}{N} j$, $j = 0, \ldots, N-1$. You can easily find the corresponding eigenvalues. There are larger ones than $a^N + b^N$, but they are also do not contain the largest one.

Consider the state of 2 pseudoparticles. Substitute the following ansatz:

$$|p_1, p_2\rangle = \sum_{k_1 < k_2} (A_{12} e^{ip_1k_1 + ip_2k_2} + A_{21} e^{\mathrm{i}p_2k_1 + \mathrm{i}p_1k_2}) \sigma_{k_1}^- \sigma_{k_2}^- |\Omega_+\rangle.$$

Apply the operator T or, simpler, H_1. A miracle! This is an eigenvector if

$$\frac{A_{12}}{A_{21}} = z(p_1, p_2)$$

with some given function $z(p_1, p_2)$. The cyclic boundary condition imposes the restrictions

$$e^{\mathrm{i}p_1 N} = z(p_1, p_2), \qquad e^{\mathrm{i}p_2 N} = z(p_2, p_1).$$

In the general case of n pseudoparticles the same miracle takes place. The wave function can be made of plane waves. The cyclic boundary conditions impose the *Bethe equations*

$$e^{\mathrm{i}p_j N} = \prod_{j'(\neq j)}^{n} z(p_j, p_{j'}), \qquad j = 1, \ldots, n.$$

It is generally impossible to solve these equations analytically. But in the limit $N \to \infty$, $n/N =$ const they are reduced to an integral equation. The case $n = N/2$, corresponding to the largest eigenvalue, admits an analytic solution. This is how the six-vertex model is solved.

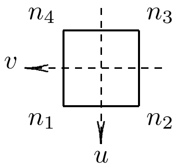
What is wrong with the general eight-vertex model? The obstacle is that

$$[S^z, T] \neq 0 \quad \text{for } d \neq 0.$$

It destroys the whole picture of pseudoparticles. There is a nice construction of the Q operator proposed by Baxter, that makes it possible to obtain the Bethe equations without any reference to the Bethe ansatz. Nevertheless, there is a question: is it possible to relate this model to another one that admits the whole construction of Bethe ansatz? Is it possible to construct something similar to the six-vertex model, but involving elliptic functions? The answer is YES.

Consider again the square lattice on the plane, but associate the variables to the vertices of the lattice and the Boltzmann weights to the plaquet or faces. Namely, associate to each vertex a variable $n \in \mathbb{Z} + \delta$, where the real shift δ is introduced for convenience. The partition function will be independent of this shift. Associate to each face of the lattice a weight:

$$e^{-E(n_1,n_2,n_3,n_4)/T} = W\left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix}\middle| u - v\right] =$$

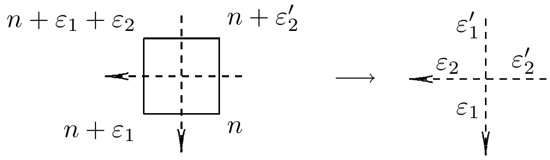


The dashed lines, first, denote the orientation and, second, carry the spectral parameters. The configuration sum is taken over all ns at all vertices such that

$$|n_i - n_j| = 1 \qquad \textbf{(admissibility condition)} \tag{1}$$

on the neighboring vertices.

What does the admissibility condition mean? Consider the dual (dashed) lattice. Define on each edge of this lattice a variable $\varepsilon = +1$ if the variable $n_i = n_j + 1$, if i denotes the vertex on the left or upper end of the edge, while j denotes the vertex on the right or lower end of the edge:



$(n + \varepsilon_1 + \varepsilon_2 = n + \varepsilon_1' + \varepsilon_2')$

In these notations the variables ε satisfy the ice condition by definition. But the weight W at each vertex of the dual lattice depend not only on the variables $\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2'$, but also on the value of n at e.g. the right lower corner of the face, which is (up to δ) the sum of all εs on any path along the initial (solid) lattice from some fixed point at the lattice to this right lower corner of the face.

The Boltzmann weights, analogous to a, b and c of the six-vertex model, are given by

$$\begin{aligned} a_n^{\pm}(u) &= W\left[\begin{matrix} n \pm 2 & n \pm 1 \\ n \pm 1 & n \end{matrix}\middle| u\right] = R_0(u), \\ b_n^{\pm}(u) &= W\left[\begin{matrix} n & n \mp 1 \\ n \pm 1 & n \end{matrix}\middle| u\right] = R_0(u)\frac{[n \mp 1][u]}{[n][1-u]} \\ c_n^{\pm}(u) &= W\left[\begin{matrix} n & n \pm 1 \\ n \pm 1 & n \end{matrix}\middle| u\right] = R_0(u)\frac{[n \pm u][1]}{[n][1-u]}, \end{aligned} \tag{2}$$

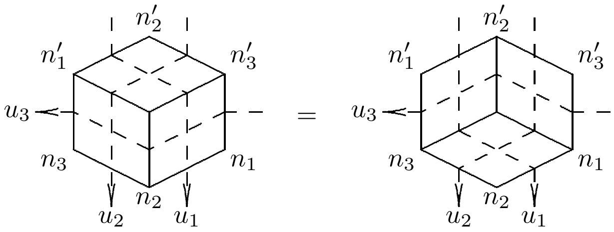
with an arbitrary function $R_0(u)$ and

$$\begin{aligned} [u]_i &= \sqrt{\frac{\pi}{\epsilon r}}\, e^{\frac{1}{4}\epsilon r}\, \theta_i\left(\frac{u}{r}; \frac{\mathrm{i}\pi}{\epsilon r}\right), \\ [u] &= [u]_1 = x^{u^2/r-u}(z;p)_\infty (pz^{-1};p)_\infty (p;p)_\infty. \end{aligned}$$

The weights W satisfy the Yang–Baxter equation of the form

$$\sum_n W\left[\begin{matrix} n_1' & n_2 \\ n & n_3' \end{matrix}\middle| u_1-u_2\right] W\left[\begin{matrix} n & n_3' \\ n_2 & n_1 \end{matrix}\middle| u_1-u_3\right] W\left[\begin{matrix} n_1' & n \\ n_3 & n_2 \end{matrix}\middle| u_2-u_3\right]$$
$$= \sum_n W\left[\begin{matrix} n_2' & n_3' \\ n & n_1 \end{matrix}\middle| u_2-u_3\right] W\left[\begin{matrix} n_1' & n_2' \\ n_3 & n \end{matrix}\middle| u_1-u_3\right] W\left[\begin{matrix} n_3 & n \\ n_2 & n_1 \end{matrix}\middle| u_1-u_2\right]. \quad (3)$$

Graphically it looks like:



The dashed lines here play the role of solid lines in the Yang–Baxter equation for the eight-vertex model, while the solid lines here simply form the lattice dual to the dashed one.

If the function $R_0(u)$ satisfy the relations

$$R_0(u)R_0(-u)=1, \qquad R_0(u)[u]=R_0(1-u)[1-u],$$

the weights satisfy the relations

$$\sum_n W\left[\begin{matrix} n_4 & n \\ n_1 & n_2 \end{matrix}\middle| u\right] W\left[\begin{matrix} n_4 & n_3 \\ n & n_2 \end{matrix}\middle| -u\right] = \delta_{n_1,n_3}, \qquad \textbf{(Unitarity)}, \quad (4)$$

$$[n_3]^{-1}\, W\left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix}\middle| u\right] = [n_4]^{-1}\, W\left[\begin{matrix} n_1 & n_4 \\ n_2 & n_3 \end{matrix}\middle| 1-u\right] \qquad \textbf{(Crossing symmetry)}. \quad (5)$$

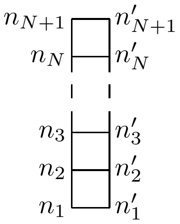
The solution

$$R_0(u)\equiv R_0(u;\epsilon,r)=z^{(r-1)/2r}\frac{g(z^{-1})}{g(z)} \quad (6)$$

makes the partition function per site equal to 1.

It is easy to define the L operator

$$L(u)^{n_1'\ldots n_{N+1}'}_{n_1\ldots n_{N+1}} =$$



$$= W\left[\begin{matrix} n_{N+1} & n_{N+1}' \\ n_N & n_N' \end{matrix}\middle| u\right]\ldots W\left[\begin{matrix} n_3 & n_3' \\ n_2 & n_2' \end{matrix}\middle| u\right] W\left[\begin{matrix} n_2 & n_2' \\ n_1 & n_1' \end{matrix}\middle| u\right] \quad (7)$$

and the transfer matrix

$$T(u)^{n_1'\ldots n_N'}_{n_1\ldots n_N} = L(u)^{n_1'\ldots n_N' n_1'}_{n_1\ldots n_N n_1}. \quad (8)$$

The transfer matrices form a commuting family,

$$T(u_1)T(u_2)=T(u_2)T(u_1),$$

and $T(0)$ is again the shift operator.

Now we formulate Baxter's fundamental statement about the relation between two models [1]. There exist functions $t_\varepsilon(u)^{n'}_n$ such that

$$\sum_{\varepsilon_1'\varepsilon_2'} R(u-v)^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2} t_{\varepsilon_2'}(v-u_0)^n_{s'} t_{\varepsilon_1'}(u-u_0)^{s'}_{n'} = \sum_{s\in\mathbb{Z}} t_{\varepsilon_1}(u-u_0)^n_s t_{\varepsilon_2}(v-u_0)^s_{n'}\, W\left[\begin{matrix} n' & s' \\ s & n \end{matrix}\middle| u-v\right] \quad (9)$$

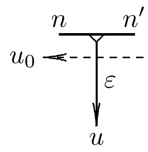
for arbitrary u_0. This relation is referred to as the *vertex–face correspondence*. Explicitly, these *intertwining* functions have the form

$$
\begin{aligned}
t_+(u)_n^{n'} &= (-1)^{(n-\delta)(n'-n-1)/2} e^{\mathrm{i}\pi/4} f(u)\theta_3\left(\frac{(n'-n)u+n'}{2r};\mathrm{i}\frac{\pi}{2\epsilon r}\right), \\
t_-(u)_n^{n'} &= -(-1)^{(n-\delta)(n'-n+1)/2} e^{-\mathrm{i}\pi/4} f(u)\theta_4\left(\frac{(n'-n)u+n'}{2r};\mathrm{i}\frac{\pi}{2\epsilon r}\right).
\end{aligned}
\tag{10}
$$

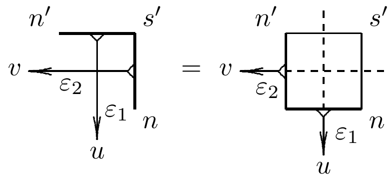
Here $f(u)$ is an arbitrary function and δ is the shift discussed above.

To understand better the fundamental identity (9), let us represent it graphically. Introduce the graphical representative of the intertwining functions:

$t_\varepsilon(u-u_0)_n^{n'} =$



With this notation the vertex–face correspondence looks like (u_0 line is not depicted)

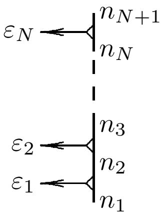


Note that this relation looks like the Yang–Baxter equation of mixed vertex–face type!

This means that if we take a square finite SOS lattice with open boundaries and attach intertwining functions to their left and lower boundaries summing over necessary boundary variables n, we can push the intertwining functions up and right using the vertex–face correspondence and obtain a square lattice of the eight-vertex model with the intertwining functions attached to the right and upper boundaries. In physics we usually expect that the contribution of boundaries to the partition function is neglectable in a large system. It means that the large volume limit of the partition functions per site of the eight-vertex model and of the SOS model coincide.

Moreover, it can be rigorously derived that the spectra of eigenvalues of the transfer matrices of the eight-vertex and SOS models with the cyclic boundary condition coincide. First, introduce the L-type operator

$\lambda(u_0)^{n_1\dots n_N n_{N+1}}_{\varepsilon_1\dots\varepsilon_N} =$



$$= t_{\varepsilon_N}(-u_0)_{n_N}^{n_{N-1}} \dots t_{\varepsilon_2}(-u_0)_{n_3}^{n_2} t_{\varepsilon_1}(-u_0)_{n_2}^{n_1} \tag{11}$$

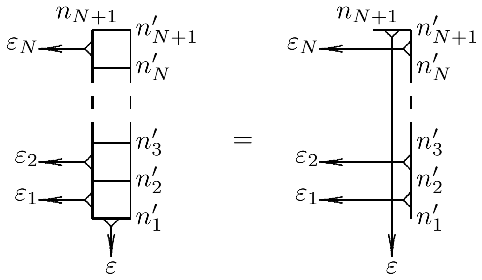
and the transfer matrix type operator

$$\tau(u_0)^{n_1\dots n_N}_{\varepsilon_1\dots\varepsilon_N} = \lambda(u_0)^{n_1\dots n_N n_1}_{\varepsilon_1\dots\varepsilon_N}.$$

Then

$$\sum_{n_2\dots n_{N-1}} L(u)^{n'_1\dots n'_{N+1}}_{n_1\dots n_{N+1}} \lambda(u_0)^{n_1\dots n_{N+1}} t_1(u-u_0)^{n_1}_{n'_1} = \sum_n \lambda(u_0)^{n'_1\dots n'_{N+1}} t_1(u-u_0)^{n'_{N+1}}_n L_1(u)$$

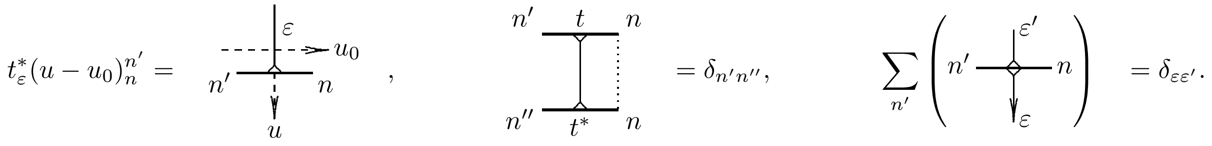
or, graphically,



Introduce now the object 'inverse' to the intertwining functions:

$$\sum_{\varepsilon} t_\varepsilon(u)^n_{n'} t^*_\varepsilon(u)^{n''}_n = \delta_{n'n''} \quad \text{or} \quad \sum_{n'} t^*_{\varepsilon'}(u)^{n'}_n t_\varepsilon(u)^n_{n'} = \delta_{\varepsilon\varepsilon'}. \tag{12}$$

or, graphically,



Attaching these t^* functions to the upper boundary and imposing the cyclic boundary condition we obtain

$$\tau(u_0)T_{8\mathrm{v}}(u) = T_{\mathrm{SOS}}(u)\tau(u_0), \tag{13}$$

where $T_{8\mathrm{v}}$ and $T_{\mathrm{SOS}}(u)$ are transfer matrices of the eight-vertex and SOS models respectively. Note that this equation has been obtained in the full analogy to the derivation of commutativity of transfer matrices.

Similarly, one can introduce the matrix

$$\tau^*(u)^{\varepsilon_1\ldots\varepsilon_N}_{n_1\ldots n_N} = t^*_{\varepsilon_N}(-u_0)^{n_1}_{n_N}\ldots t^*_{\varepsilon_1}(-u_0)^{n_2}_{n_1}$$

with the relation

$$T_{8\mathrm{v}}(u)\tau^*(u_0) = \tau^*(u_0)T_{\mathrm{SOS}}(u), \tag{14}$$

Let $|\Lambda\rangle_{\mathrm{SOS}}$ be an eigenvector of $T_{\mathrm{SOS}}(u)$ with the eigenvalue function $\Lambda(u)$. Then

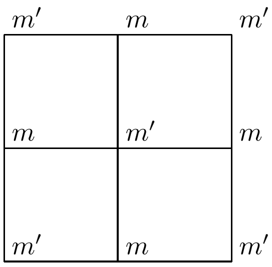
$$T_{8\mathrm{v}}(u)\tau^*(u_0)|\Lambda\rangle_{\mathrm{SOS}} = \tau^*(u_0)T_{\mathrm{SOS}}(u)|\Lambda\rangle_{\mathrm{SOS}} = \Lambda(u)\tau^*(u_0)|\Lambda\rangle_{\mathrm{SOS}}.$$

It means that $|\Lambda\rangle_{8\mathrm{v}} = \tau^*(u_0)|\Lambda\rangle_{\mathrm{SOS}}$ is an eigenvector of the operator $T_{8\mathrm{v}}(u)$ with the same eigenvalue function $\Lambda(u)$. It proves that the spectra of both models coincide.

To conclude, let us say something about the ground states in this theory. We shall consider the SOS model in the so called regime III region:

$$\epsilon > 0, \qquad r \geq 1, \qquad 0 < u < 1.$$

In this region the ground states (the states of maximal weight) are numerated by $m \in \mathbb{Z}+\delta$ and $m' = m \pm 1$, such that $(k-1)r < m, m' < kr$ for some integer k. The ground state (m, m') looks like



The conclusion is the following. There is a highly nonlocal transformation that relates the eight-vertex model to another model, the solid-on-solid one, which can be treated by means of the Bethe ansatz approach. Though this relation is not a direct one-to-one correspondence between configurations, it is nevertheless a 'detailed' correspondence that makes it possible to express any expectation value of the eight-vertex model to an expectation value defined in terms of the SOS model. We discuss this point in the Lecture 4.

References

[1] R. J. Baxter, Annals Phys. **76** (1973) 25–47