Lecture 5
Free field representation: eight-vertex model via SOS model

At the end of the last lecture we introduced the operator $\Lambda(u_0)^{m'n'}_{mn} = T(u_0)^{m'n'}T(u_0)_{mn}$. We said that, having a bosonization of the SOS model, this operator is the only thing to be bosonized. To obtain bosonization for the Λ operator let us consider its commutation relations with the vertex operators. From the commutation relations of the operators $T(u_0)^{mn}$ and $T(u_0)_{mn}$ we find

$$\Lambda(u_0)^{m'\,n'}_{m\ s}\Phi(u)^s_n = \sum_{s'} L\begin{bmatrix} n' & s' \\ s & n \end{bmatrix}\!\!\left.\vphantom{\begin{bmatrix} n' \\ s\end{bmatrix}}\right| u-u_0\Big]\, \Phi(u)^{n'}_{s'}\Lambda(u_0)^{m'\,s'}_{m\ n}, \tag{1}$$

$$\Psi^*(u)^{m'}_s\Lambda(u_0)^{s\ n'}_{mn} = \sum_{s'} \Lambda(u_0)^{m'\,n'}_{s'\ n}\Psi^*(u)^{s'}_m\,\tilde L\left[\begin{matrix} m' & s \\ s' & m \end{matrix}\,\middle|\, u-u_0-\Delta u_0\right], \tag{2}$$

where

$$L\left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix}\,\middle|\, u\right] = \sum_\varepsilon t^*_\varepsilon(u)^{n_2}_{n_1}t_\varepsilon(u)^{n_4}_{n_3}, \qquad \tilde L\left[\begin{matrix} m_4 & m_3 \\ m_1 & m_2 \end{matrix}\,\middle|\, u\right] = \sum_\varepsilon \tilde t^*_\varepsilon(u)^{m_2}_{m_1}\tilde t_\varepsilon(u)^{m_4}_{m_3} = L\left[\begin{matrix} m_4 & m_3 \\ m_1 & m_2 \end{matrix}\,\middle|\, u\right]\Bigg|_{r\to r-1}. \tag{3}$$

Explicitly, we have

$$\begin{aligned} L\left[\begin{matrix} n' & n'\pm 1 \\ n & n\pm 1 \end{matrix}\,\middle|\, u\right] &= \frac{[u\pm\frac12(n-n')][\frac12(n+n')]}{[u][n]}, \\ L\left[\begin{matrix} n' & n'\mp 1 \\ n & n\pm 1 \end{matrix}\,\middle|\, u\right] &= \frac{[u\pm\frac12(n+n')][\frac12(n-n')]}{[u][n]} \end{aligned} \tag{4}$$

for $n'-n\in 2\mathbb{Z}$. Note, that the number of type I vertex operators and that of type II vertex operators in any meaningful trace is even. This means that considering the Λ operator for $n'-n\in 2\mathbb{Z}$ and $m'-m\in 2\mathbb{Z}$ is natural.

Evidently,

$$L\left[\begin{matrix} n & n'' \\ n & n' \end{matrix}\,\middle|\, u\right] = \delta_{n'n''}. \tag{5}$$

As the Λ operator is a 'half infinite product' of Ls, we easily conclude that

$$\Lambda(u)^{m'n}_{mn} = \delta_{m'm}. \tag{6}$$

Besides,

$$\Lambda(u_0)^{m'\,n'}_{m\ n} = 0 \quad \text{if} \quad m'<m,\ n'>n \text{ or } m'>m,\ n'<n. \tag{7}$$

Indeed, if, for example $m'\le m$, $n'>n$ for nonzero $\Lambda(u)^{m'n'}_{mn}$, there must be a point j at the paths where $n(j)=n'(j)$. But, due to (5), it means that $n(j+1)=n'(j+1)$ and, by induction $n(k)=n'(k)$ for any $k>j$. Therefore, $m'=m$.

Let us start derivation from the case $m'=m$. This case is sufficient for calculation of correlation functions. Consider the commutation relation (1) in the limit $u\to u_0$. In this limit $L(s,n;s',n|u-u_0)\to\infty$ so that

$$L(n\pm1,n;n+1,n|u-u_0)/L(n\pm1,n;n-1,n|u-u_0)\to 1.$$

We obtain the relation (we omit the indices m for simplicity)

$$\Phi(u)^{n'}_{n'-1}\Lambda(u)^{n'-1}_n = -\Phi(u)^{n'}_{n'+1}\Lambda(u)^{n'+1}_n$$

which amounts

$$V(u)\Lambda(u)^{n'-1}_n = \frac{[n'-1]}{[n'+1]}V(u)X(u)\Lambda(u)^{n'+1}_n.$$

We can conjecture that

$$\Lambda(u)_{m\ n}^{m\ n-2l} = \frac{[n-2l]}{[n]} X^l(u) \quad \text{for} \quad l \geq 0. \tag{8}$$

If we substitute this solution to the commutation relation (1) in its general form, we can make sure that this is indeed the solution to these commutation relations $n' \leq n$.

But what to do in the case $n' > n$? Look at the weights of the SOS model. They are invariant with respect to the reflection

$$m \to -m, \qquad n \to -n.$$

It means that we can identify with $\mathcal{H}_{mn}$ not only $\mathcal{F}_{mn}$, but also $\mathcal{F}_{-m,-n}$. With this identification we have an alternate bosonization for the vertex operators of the SOS model:

$$\begin{aligned} \Phi(u)_n^{n+1} &= \frac{\mathrm{i}^{n-m}}{[n]} V(u) X(u), \\ \Phi(u)_n^{n-1} &= -\frac{\mathrm{i}^{n-m}}{[n]} V(u), \\ \Psi^*(u)_m^{m+1} &= (-1)^{m-n} \tilde{Y}(u) \tilde{V}(u), \\ \Psi^*(u)_m^{m-1} &= \tilde{V}(u). \end{aligned} \tag{9}$$

In this alternate bosonization we have

$$\Lambda(u)_{m\ n}^{m\ n+2l} = \frac{[n+2l]}{[n]} X^l(u) \quad \text{for} \quad l \geq 0. \tag{10}$$

We use different free field representation in different cases. This fact must not embarrass you, because the Λ operator enters the trace once. Hence, we simply need to use different bosonization for different traces.

This is sufficient for calculation of correlation functions. But the construction contains two free parameters u_0 and m. If there would be a rigorous proof of our construction, we would be sure that the answer is independent of these parameters. But from the mathematical point of view our construction is nothing but a conjecture. As a test we must check the u_0 and m independence. The u_0 independence can be proven from periodicity properties: it turns out that the correlation function constructed from this bosonization procedure must be periodic in u_0 with two periods, 2 and $2r$. As r is, generically, irrational, the answer is a constant. We have no proof of m independence, but the simplest examples of the one- and two-point functions demonstrate this independence. The answer for the one-point function is nothing but the famous Baxter's staggered spontaneous polarization:

$$\langle \sigma^z \rangle^{(i)} \equiv P^{(i)}{}_{+}^{+} - P^{(i)}{}_{-}^{-} = (-)^i \frac{(x^2;x^2)_\infty^2 (-x^{2r};x^{2r})_\infty^2}{(-x^2;x^2)_\infty^2 (x^{2r};x^{2r})_\infty^2}.$$

Now turn our attention to the case $m' \neq m$. Due to the selection rule (7), we have to use the first bosonization in the case $n' \leq n$, $m' \leq m$ and the alternative bosonization in the case $n' \geq n$, $m' \geq m$. First of all, let us try to find $\Lambda(u)_{m\ \ n}^{m-2\ n-2}$. Take the commutation relation (2) for $s = m$, $m' = m-1$. In this case it becomes

$$\Psi^*(u)_m^{m-1} \Lambda(u_0)_{mn}^{mn'} = \Lambda(u_0)_{m-1\ n}^{m-1\ n'} \Psi^*(u)_m^{m-1} + \Lambda(u_0)_{m+1\ n}^{m-1\ n'} \Psi^*(u)_m^{m+1} \frac{[u-u_0-\Delta u_0-m]'[1]'}{[u-u_0-\Delta u_0]'[m+1]'}.$$

For $n' = n-2$ we have

$$\frac{[n-2]}{[n]} \tilde{Y}(u) X(u_0) \tilde{V}(u) = \frac{[n-2]}{[n]} X(u_0) \tilde{Y}(u) \tilde{V}(u) + (-1)^{n-m} \Lambda(u_0)_{m-1\ n}^{m-1\ n-2} \tilde{V}(u) \frac{[u-u_0-\Delta u_0-m]'[1]'}{[u-u_0-\Delta u_0]'[m+1]'}.$$

Erasing $\tilde{V}(u)$ we obtain

$$\Lambda(u)_{m\ \ n}^{m-2\ n-2} \tilde{V}(u_0) = (-)^{n-m} \frac{[u-u_0-\Delta u_0-m+1]'[1]'}{[u-u_0-\Delta u_0]'[m]'} \frac{[n-2]}{[n]} [\tilde{Y}(u), X(u_0)] \tilde{V}(u). \tag{11}$$

What is the commutator $[\tilde{Y}(u), X(u_0)]$? If the contours of the commutator would not catch poles it would be zero. But it can be proven that this commutator is a simple combinations of exponentials:

$$[\tilde{Y}(u'), X(u)]|_{\mathcal{F}_{mn}} = \frac{\epsilon}{\eta\eta' \operatorname{sh}\epsilon}\biggl(\frac{[m-1]'}{\partial[0]'}\frac{[u'-u+\frac{1}{2}-n]}{[u'-u-\frac{1}{2}]}W_+(u') + \frac{[n-1]}{\partial[0]}\frac{[u'-u+\frac{3}{2}-m]'}{[u'-u+\frac{1}{2}]'}W_-(u)\biggr). \tag{12}$$

Here $\partial[0] = d[u]/du|_{u=0}$, $\partial[0]' = d[u]'/du|_{u=0}$ and

$$W_+(u) = W(u+\tfrac{r}{2}), \qquad W_-(u) = W(u - \tfrac{r-1}{2}) \tag{13}$$

with

$$\begin{aligned} W(u) &= z^{1/r(r-1)}{:}e^{\mathrm{i}\varphi_0(z)}{:}, \\ \varphi_0(z) &= -2\sqrt{2}\alpha_0(\mathcal{Q} - \mathrm{i}\mathcal{P}\log z) - \sum_{k\neq 0}\frac{[\![2k]\!]_x}{[\![(r-1)k]\!]_x}\frac{a_k}{\mathrm{i}k}z^{-k}. \end{aligned} \tag{14}$$

It can be checked that

$$W_+(u)\tilde{V}(u) = 0.$$

Therefore the first term in the commutator does not affect the operator $\Lambda(u)$. But the second exponential W_- has the argument of the X screening operator. The difference $u - u_0$ drop out of (11) with these substitution, if we accept that

$$\Delta u_0 = -1/2. \tag{15}$$

Finally, we obtain

$$\Lambda(u)_{m\ \ n}^{m-2\ n-2} = (-)^{n-m}\frac{\epsilon}{\eta\eta' \operatorname{sh}\epsilon}\frac{[m]'}{[1]'}\frac{[n-1][n-2]}{\partial[0]\,[n]}W_-(u). \tag{16}$$

We see that $\Lambda(u)_{m\ \ n}^{m-2\ n-2}$ is proportional to $W_-(u)$. If we return to the commutation relation $[\tilde{Y}(u'), X(u)]$ we shall see that at the point $u' = u - 1/2$ the second term with $W_-(u)$ has a pole, while the first one remains finite. Therefore,

$$\lim_{u'\to u-1/2}[u'-u+1/2]'[\tilde{Y}(u'), X(u)] \sim W_-(u).$$

One can check that this pole is appears in the product $\tilde{Y}(u')X(u)$, while the product $X(u)\tilde{Y}(u')$ is regular at that point. It means that

$$\Lambda(u)_{m\ \ n}^{m-2\ n-2} \sim \lim_{u'\to u}[u'-u]'\tilde{Y}(u'-1/2)X(u).$$

We may conjecture that

$$\Lambda(u)_{m\ \ \ n}^{m-2k\ n-2l} \sim \lim_{u'\to u}[u'-u]'\tilde{Y}^k(u'-1/2)X^l(u) \quad \text{for } k,l>0. \tag{17}$$

It can be checked that the product in the right hand side has just a simple pole and that

$$\lim_{u'\to u}[u'-u]'\tilde{Y}^k(u'-1/2)X^l(u) \sim \tilde{Y}^{k-1}(u-1/2)W_-(u)X^{l-1}(u).$$

The coefficients in (17) can be obtained from the intertwining relation (2) in the limit $u \to u_0 - 1/2$ just in the same way as we obtained the coefficient in (8) using the equation

$$\Psi^*(u)_{m-2k}^{m-2k+1}\Lambda(u_0)_{m\ \ \ n}^{m-2k\ n-2l} = \Psi^*(u)_{m-2k+2}^{m-2k+1}\Lambda(u_0)_{m\ \ \ \ \ \ n}^{m-2k+2\ n-2l} + O(1) \quad \text{as} \quad u \to u_0 - \tfrac{1}{2}$$

and one more equation that follows from (1) in the limit $u \to u_0$ (but written in the form $\Phi(u)\Lambda(u_0) = \ldots$).

Then we have to substitute the answer into the relations (1) and (2) in the general form and to check it. It is a very cumbersome calculation, but it was done. Now the answer is

$$\begin{aligned}
\Lambda(u)_{m\ \ \ \ n}^{m-2k\ n-2l}\big|_{\mathcal{F}_{mn}} &= C_{m\ \ \ \ n}^{m-2k\ n-2l} \lim_{u'\to u}[u'-u]'\tilde{Y}^k(u'-\tfrac{1}{2})X^l(u) \\
&= D_{m\ \ \ \ n}^{m-2k\ n-2l}\tilde{X}^{k-1}(u-\tfrac{1}{2})W_-(u)Y^{l-1}(u), \qquad (18)\\
\Lambda(u)_{m\ \ \ \ n}^{m+2k\ n+2l}\big|_{\mathcal{F}_{-m\ -n}} &= C_{m\ \ \ \ n}^{m+2k\ n+2l} \lim_{u'\to u}[u'-u]'\tilde{Y}^k(u'-\tfrac{1}{2})X^l(u) \\
&= D_{m\ \ \ \ n}^{m+2k\ n+2l}\tilde{X}^{k-1}(u-\tfrac{1}{2})W_-(u)Y^{l-1}(u) \qquad (19)
\end{aligned}$$

for $k,l>0$ with

$$\begin{aligned}
D_{m\ \ \ \ n}^{m-2k\ n-2l} &= (-)^{(m-n+1)k+l+1}\frac{\epsilon}{\eta\eta'\,\mathrm{sh}\,\epsilon}\frac{[m]'}{[1]'}\frac{[l][n-l][n-2l]}{\partial[0]\,[1][n]}. \\
C_{m\ \ \ \ n}^{m-2k\ n-2l} &= (-)^{(m-n)k}\frac{[m]'}{[k]'[m-k]'}\frac{[n-2l]}{[n]}.
\end{aligned} \qquad (20)$$

The two-particle form factors of any of operators σ^a can be obtained without integrations. Any of these calculations (for example for σ^z) allows to fix the function $\tilde{f}(u)$ from the condition of u_0 independence of the answer. The explicit (and readable) answers for these quantities can be found in the paper by Lukyanov and Terras [1].

References

[1] S. Lukyanov, V. Terras, *Nucl. Phys.* **B654** (2003) 323 [hep-th/0206093]