Lectures on the eight-vertex model and bosonization

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Abstract

These are introductory lectures on application of the free field representation (bosonization) techniques to the solid-on-solid (SOS) and eight-vertex models. We start from the very beginnings, including the physical background of lattice models and some basic information on quantum integrability. After definitions of the eight-vertex and SOS models, we describe their relation known as the vertex–face correspondence. Then, skipping the Bethe ansatz solution, we turn to the problem of calculation of correlation functions by means of the free field representation. We explain, how the vertex–face correspondence works on the level of vertex operators and bosonization, making it possible to express the correlation functions of the eight-vertex model in terms of the free field representation aimed to describe the SOS model.

1 Eight-vertex model and commuting transfer matrices

Since we shall speak about some models of physical importance, let us formulate first the general physical framework. Consider a classical system of interacting particles. Let C be a state of this system, which can run some (generally infinite) set of admissible states. Let $E(C)$ be the energy of the state C, defined by usual Hamiltonian mechanics. Suppose that the system weakly interacts with a thermal bath of the temperature T. The most fundamental postulate of statistical mechanics, the *Gibbs law*, says that the probability of the state C is given by

$$w(C) = \frac{1}{Z}\, e^{-E(C)/T}. \tag{1.1}$$

It is easy to understand, what is the proportionality coefficient $1/Z$. As the total probability for the system to be in any state is unity, we have for the *partition functions* of the system

$$Z = \sum_C e^{-E(C)/T}. \tag{1.2}$$

Since the space of configurations can be continuous, the sum may turn out to be an integration, but we shall not consider this general case in these lectures.

The partition function encodes the most fundamental observable thermodynamic function of the model, the *free energy*:

$$F = E - TS = -T\log Z. \tag{1.3}$$

Here E is the *total energy* of the system,

$$E = \sum E(C)w(C) = \frac{T^2}{Z(T)}\frac{d}{dT}Z(T) = -T^2\frac{d}{dT}\frac{F}{T} = F - T\frac{dF}{dT},$$

while

$$S = -\sum w(C)\log w(C) = -\frac{dF}{dT}$$

is the *entropy*. The first equality in (1.3) is the thermodynamic definition of the free energy, which, in principle, was established before Gibbs, while the second equality provides its statistical interpretation.

Let $f_i(C)$ be some functions of the state of the system. The *correlation functions* are generic expectation values

$$\langle f_1 \dots f_n\rangle = \sum_C w(C) f_1(C)\dots f_n(C).$$

Note, that the correlation functions can be expressed in terms of derivatives of a generalized partition function of the system with external fields F_i coupled to the variables f_i:

$$Z(F_1, \ldots, F_N) = \sum_C e^{-E(C; F_1, \ldots, F_N)/T}, \qquad E(C; F_1, \ldots, F_N) = E(C) - \sum_i F_i f_i .$$

Usually, from the physical point of view the most interesting objects are local correlation functions, i.e. the correlation functions of the amounts $f_i(C)$ that can be measured at some space points. We shall discuss examples of such functions in detail later.

Naturally, evaluation of the partition function and correlation functions is a difficult task, except some trivial examples solved by classics of the science. Most these solved examples are systems of independent particles, each of which possesses a finite set or, at least, a finite-dimensional space of states and a simple function for $E(C)$ (like a quadratic function in the Boltzmann gas). But how to do in the case of interacting particles? The usual approach in physics is developing some approximate methods, based either on the perturbation theory or on some experimentally supported assumptions. It is a very effective way, but sometimes physicists need some additional support to their assumptions.

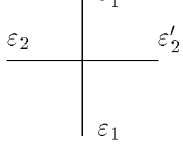
How mathematicians can help physicists? First, they can propose some rigorous estimates, which can prove the effects predicted by physicists. Many of such important estimates were proposed (see, e. g. [1, 2]). Second, they may propose some sophisticated methods to solve exactly some particular nontrivial examples. Though these methods can be not completely rigorous, they better convince physicists, because they are more in the way of physical thinking, and can be used by physicists themselves.

Let us now slightly specify the problem. Forget about motion of particles. We can do it for some problems, for example, if positions of particles are fixed by the crystalline lattice. The set of states of the system (*configurations*) is reduced to the direct product of sets of internal states of particles ('spins'). We shall assume these 'spins' to be discrete variables. Physically the discrete 'spins' may originate in quantum states of atoms (e.g. physical spin states) with diagonal interaction or in other sources, e.g. in types of atoms in the substitutional solutions or in positions of atoms in different quantum wells as in the ice-type systems.

In these lectures we shall discuss two-dimensional models of statistical mechanics. Why two-dimensional? Because in one dimension the lattice statistical models are trivially reduced to quantum mechanics of small systems without phase transitions or other interesting features. On the other hand, in three dimensions the problem is too complicated and the classes of solvable models are too narrow. We restrict our attention onto two classes of two-dimensional solvable lattice models, which are in a sense basic: eight-vertex model and solid-on-solid (SOS) solvable model.

Let us start with the ice model on the square lattice called also the six-vertex model [3]. Consider a square lattice made of oxygen ions (O^{2-}) in the vertices and hydrogen ions (H^+) on the vertical and horizontal bonds of the lattice, which we shall call edges. (Do not think of all this too seriously, because experimentalists are unable to produce any two-dimensional ice-like systems, but this picture of ice can help you both to remember the formulation of the model and to better understand the physical origin of such kind of problems.) We know that the hydrogen ions can form with the oxygen ions two types of bonds: strong and narrow polar bond and weak and long 'hydrogen' bond. It means that a hydrogen ion lying on an edge of the lattice must be positioned near one of the oxygen ion and far from another oxygen ion. There are two such positions on each edge. It forms discrete 'spin' state on each edge. We shall think that the 'spin' is equal to '+' or '+1' if the hydrogen ion is positioned near the right end of a horizontal edge or near the upper end of the vertical edge, and is equal to '−' or '−1' if it is positioned near the left or lower end of the edge.

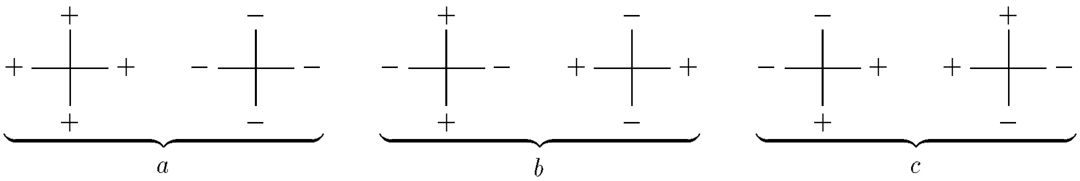
Further, from the neutrality condition we conclude, that in the vicinity of each oxygen ion there must be just two hydrogen ions. It imposes a restriction onto values of 'spins' at edges surrounding each vertex. Namely, denote these spins by ε_1, ε_2, ε'_1, ε'_2:



Then this restriction, called the *ice rule*, can be expressed as

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_1' + \varepsilon_2' \tag{1.4}$$

There are six configurations around each vertex that satisfy the ice rule:



Let us think that each of this vertex configuration may have its own energy $E_1, \ldots, E_6$. The energy of the system is equal to the sum of energies of vertices. It defines the model nearly completely. The only thing to specify is the boundary conditions. We shall consider three possibilities:

1. The model of M columns and N rows with cyclic (toroidal) boundary conditions.
2. The model with fixed spins at the boundary.
3. The model on an infinite lattice, considered as a limit of any of these models as $M, N \to \infty$.

The six-vertex model is known to be solvable if

$$E_1 = E_2, \qquad E_3 = E_4, \quad E_5 = E_6.$$

It means that the configurations braced together on the picture possess the same energies and the same Boltzmann weights:

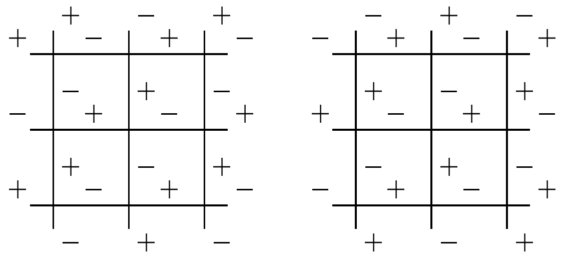
$$a = e^{-E_1/T} = e^{-E_2/T}, \qquad b = e^{-E_3/T} = e^{-E_4/T}, \qquad c = e^{-E_5/T} = e^{-E_5/T}. \tag{1.5}$$

This model is called the homogeneous six-vertex model without external field.

The disadvantage of the six-vertex model is that it possesses some pathologic physical properties, related to severeness of the ice condition. Namely, consider its phase diagram:

$$\begin{aligned} \mathrm{AF}: &\quad c > a + b, \\ \mathrm{F}_1: &\quad a > b + c, \\ \mathrm{F}_2: &\quad b > a + c, \\ \mathrm{D}: &\quad \tfrac{1}{2}(a + b + c) \geq a, b, c. \end{aligned}$$

The antiferroelectric (AF) region is the region of antiferroelectric order considered as excitations above the following ground states:



(1.6)

The excitations can be considered as some loops of changed spins and can have finite energy even on the infinite lattice. The correlation functions of spins decrease as $e^{-r/\xi}$ as the distance $r \to \infty$. The constant ξ is called *correlation length*. It is a normal behavior of correlation functions out of special point called critical points.

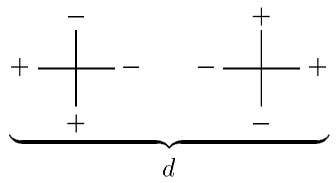
In the ferroelectric regions (F_1 and F_2) the situation is strange: all excitations consist of the lines of flipped spins in the SW–NE direction, which are infinite in the infinite volume limit! It means that these excitations possess large energy, and their contribution tends to zero as $M, N \to \infty$. We have the situation of a *frozen order*. The free energy of such system per site tends to zero.

Another pathological feature is related to the disordered (D) region. The whole region turns out to be *critical.* It means that the correlations between local variables like spins decrease with distance r like r^{-2d} with an appropriate *scaling dimension* d. From physics we know that critical points always lie on surfaces that separate phases in the system. But here we have a situation where the critical points form a region on the phase diagram. One can expect, that if we add to the model new configurations that break the ice conditions, this critical region will become a surface that separates two phases.

Unfortunately, physically reasonable solvable generalizations of the six-vertex model are unknown. If we relate some energy to disbalance of charge at a vertex, we shall lose solvability. But it is possible to generalize the model in a 'mathematical' way without lost of solvability as follows [4, 5]. Suppose the configurations around a vertex to be admissible if

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_1' + \varepsilon_2' \bmod 4. \tag{1.7}$$

It means that we admit two more vertex configurations:



Such model is called *eight-vertex.* If, in addition to (1.5), the corresponding energies E_7 and E_8 are equal,

$$d = e^{-E_7/T} = e^{-E_8/T},$$

the model turns out to be solvable [6]. The properties described below can be found in [7].

The phase diagram of the eight-vertex model looks like:

$$\begin{aligned}
\mathrm{AF}_1: &\quad c > a + b + d, \\
\mathrm{AF}_2: &\quad d > a + b + c, \\
\mathrm{F}_1: &\quad a > b + c + d, \\
\mathrm{F}_2: &\quad b > a + c + d, \\
\mathrm{D}: &\quad \tfrac{1}{2}(a + b + c + d) > a, b, c, d.
\end{aligned} \tag{1.8}$$

In this case the disordered region D is not critical. The correlation length is finite and correlation functions decrease exponentially. The critical points lie on the boundaries of the regions

$$a = b + c + d; \quad b = a + c + d; \quad c = a + b + d; \quad d = a + b + c$$

and at the special surfaces

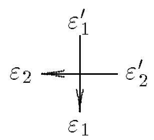
$$\begin{aligned}
a = 0, &\quad \tfrac{1}{2}(b + c + d) \geq b, c, d; \\
b = 0, &\quad \tfrac{1}{2}(a + c + d) \geq a, c, d; \\
c = 0, &\quad \tfrac{1}{2}(a + b + d) \geq a, b, d; \\
d = 0, &\quad \tfrac{1}{2}(a + b + c) \geq a, b, c;
\end{aligned}$$

There are special maps ('dualities') that can map each of the regions (1.8) on to another [5]. So it is enough to study only one region, e.g. AF_1. In this region the ground states again look like (1.6).

I repeated many times the words 'solvability', 'solvable'. What does they mean? Though there is no rigorous definition of this term, it must mean approximately the following: there are some quantities of physical importance (partition function, correlation functions) that could be found exactly in these models. What are the conditions for solvability of this or that model? Let us make several steps to see such (presumably sufficient) condition.

First introduce the weight matrix (called also R matrix) of the eight-vertex model:

$$R^{\varepsilon_1'\varepsilon_2'}_{\varepsilon_1\varepsilon_2} = \tag{1.9}$$

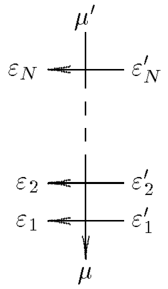


We added the arrows to the lines to define the orientation on the lattice, so that we could rotate or distort it. The R matrix can be written as

$$R = \begin{array}{c} \\ ++ \\ +- \\ -+ \\ -- \end{array} \!\! \begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{pmatrix} a & & & d \\ & b & c & \\ & c & b & \\ d & & & a \end{pmatrix} \end{array} . \tag{1.10}$$

Now introduce the L matrix

$$(L_\mu^{\mu'})_{\varepsilon_1\ldots\varepsilon_N}^{\varepsilon'_1\ldots\varepsilon'_N} =$$



$$= R_{\mu_{N-1}\,\varepsilon_N}^{\mu'\ \ \ \ \varepsilon'_N} \ldots R_{\mu_1\,\varepsilon_2}^{\mu_2\,\varepsilon'_2} R_{\mu\ \varepsilon_1}^{\mu_1\,\varepsilon'_1}. \tag{1.11}$$

We shall consider this object as a matrix in indices μ, μ' and an operator in the product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (N times) spanned on the vectors $v_{\varepsilon_1} \otimes v_{\varepsilon_2} \otimes \cdots \otimes v_{\varepsilon_N}$, where v_+ and v_- form the natural basis in $\mathbb{C}^2$. The product associated to $\varepsilon_1, \ldots, \varepsilon_N$ is called *quantum* space, while the space $\mathbb{C}^2$ associated to μ is called *auxiliary* space. We shall always omit the 'quantum' indices and sometimes omit the auxiliary indices, substituting them by a numeric subscript labeling the space, e.g. L_1.

Now we are ready to define the *transfer matrix*

$$T = \mathrm{tr}_1\, L_1 \equiv \sum_\mu L_\mu^\mu. \tag{1.12}$$

Consider the eight-vertex model of M columns and N rows with cyclic boundary conditions. It is easy to see that the partition function is equal to

$$Z = \mathrm{Tr}\, T^M,$$

where the trace Tr is taken over the quantum space. Let $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_{2^N}$ be the eigenvalues of the transfer matrix. Then

$$Z = \sum_i \Lambda_i^M.$$

In the large M limit the leading contribution is given by Λ_1^M and we have

$$\kappa \equiv Z^{1/MN} \to \Lambda_1^{1/N}, \qquad f \equiv \frac{F}{MN} \to -\frac{T}{N} \log \Lambda_1 \qquad \text{as } M \to \infty,$$

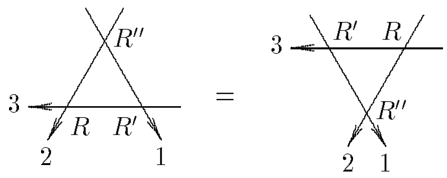
where the partition function per site κ and the partial free energy f are introduced. Surely, it is necessary to study the behavior of subleading contributions at large M and N to substantiate these formulas, and it can be done, but in these lectures we shall assume these formulas to be correct without a proof.

Anyhow the problem is reduced to that of the quantum mechanics: we have an operator T of evolution by one step in the 'time' direction and we have to diagonalize it. When this can be done exactly? From classical mechanics we know that the system is solvable (more precisely, integrable) if we have sufficiently many integrals of motion in involution (the Liouville theory). Though there is no 'quantum Liouville theorem' it can be expected that in quantum mechanics the situation is similar. We have to find some other commutative integrals of motion, i.e. operators that commute with the transfer matrix and with each other. Let us look for them in the same form as the transfer matrix. Namely, let R' be the matrix of the form (1.10) with some new 'weights' a', b', c', d'. Define the L operator L' and the transfer matrix T' in terms of R' according to (1.11) and (1.12). Besides, for any product $V_1 \otimes \cdots \otimes V_n = \mathbb{C}^2 \otimes \cdots \mathbb{C}^2$ we shall denote by R_{ij} the matrix R acting on the ith and jth components of the product. Then there is a

Theorem [8]. *If there exists an invertible matrix R'' of the form (1.10), such that the* ***Yang–Baxter equation*** *holds*

$$R''_{12}R'_{13}R_{23} = R_{23}R'_{13}R''_{12} \tag{1.13}$$

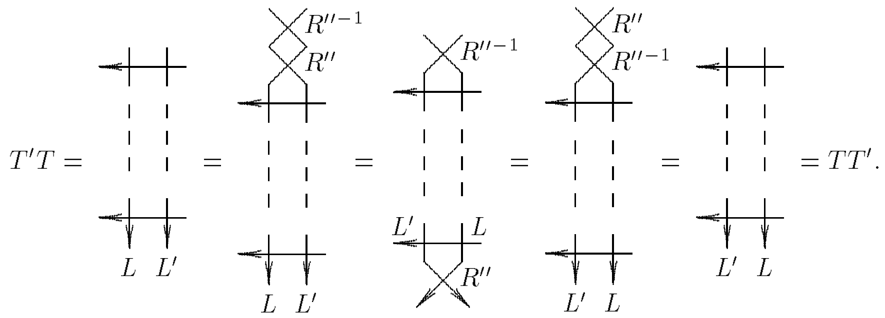
or, graphically,



then the transfer matrices T and T' commute.

$$TT' = T'T. \tag{1.14}$$

The proof is elementary in graphical form:



Note, that the third equality means that

$$R''_{12}L'_1L_2 = L_2L'_1R_{12}. \tag{1.15}$$

which generalizes the Yang–Baxter equation to the situation where the space 3 is substituted by the whole quantum space.

What are the solutions to the Yang–Baxter equation? It turns out that they can be written in the form

$$R = R(u_2 - u_3), \qquad R' = R(u_1 - u_3), \qquad R'' = R(u_1 - u_2)$$

with some analytic function $R(u)$. The *spectral parameters* u_i can be associated with the spaces V_i. The Yang–Baxter equation takes the form

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2). \tag{1.16}$$

The solution matrix elements $a(u), \ldots, d(u)$ of the matrix $R(u)$ are written in terms of the Jacobi theta functions $\theta_i(u;\tau)$ $(i = 1, \ldots, 4)$ with quasiperiods 1 and τ $(\operatorname{Im}\tau > 0)$. Namely, in the region AF_1 we have

$$\begin{aligned} a(u) &= \rho(u;\epsilon,r)s(1-u;\epsilon,r), \\ b(u) &= \rho(u;\epsilon,r)s(u;\epsilon,r), \\ c(u) &= \rho(u;\epsilon,r)s(1;\epsilon,r), \\ d(u) &= \rho(u;\epsilon,r)s(1-u;\epsilon,r)s(u;\epsilon,r)s(1;\epsilon,r), \end{aligned} \tag{1.17}$$

where

$$s(u;\epsilon,r) = \frac{\theta_1\left(\frac{u}{2r};\frac{i\pi}{2\epsilon r}\right)}{\theta_2\left(\frac{u}{2r};\frac{i\pi}{2\epsilon r}\right)}.$$

The parameters $\epsilon > 0$, $r > 1$ are fixed numbers. The scalar function $\rho(u;\epsilon,r)$ is arbitrary.

In the limit $r \to \infty$ we obtain the six-vertex model:

$$\begin{aligned} a(u) &\sim \operatorname{sh} \epsilon(1-u), \\ b(u) &\sim \operatorname{sh} \epsilon u, \\ c(u) &\sim \operatorname{sh} \epsilon, \\ d(u) &= 0. \end{aligned}$$

The presence of a continuous set of solutions $R(u)$ means that there is an infinite family of commuting transfer matrices:

$$T(u_1)T(u_2) = T(u_2)T(u_1). \tag{1.18}$$

The matrix $R(0)$ is proportional to the transposition matrix

$$P = \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix}.$$

It means that the transfer matrix $T(0)$ is proportional to the shift operator. Define now a set of Hamiltonians $H_1, H_2, \ldots$ as follows

$$T^{-1}(0)T(u) = 1 + H_1 u + H_2 u^2 + \ldots. \tag{1.19}$$

They all commute with each other and with the shift operator

$$[H_m, H_n] = 0, \qquad [H_m, T(0)] = 0.$$

Not all of them are independent. In a finite system only a finite number of them are independent. It turns out that the model is indeed solvable. It is not easy to prove this, and it demands some other ideas to find even the partition function. You can find the solution in Baxter's book [7].

What are the Hamiltonians H_n? The general H_n is a complicated operator, but the simplest one is given by

$$H_1 = \text{const} - \frac{1}{2}\sum_{k=1}^{N}(J_x \sigma_k^x \sigma_{k+1}^x + J_y \sigma_k^y \sigma_{k+1}^y + J_z \sigma_k^z \sigma_{k+1}^z)$$

with σ_k^a are the Pauli sigma-matrices,

$$\sigma^x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} & i \\ -i & \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$$

acting on the kth component of the tensor product. The coefficients J_a are functions of ϵ, r. The common factor is not so interesting, but the ratios of these coefficients are important u independent combinations of weights:

$$\Delta = \frac{2J_z}{J_x + J_y} = \frac{a^2 + b^2 - c^2 - d^2}{2ab}, \qquad \Gamma = \frac{J_x - J_y}{J_x + J_y} = \frac{cd}{ab}.$$

In the case of the six-vertex model $\Gamma = 0$ and we have the XXZ model ($J_x = J_y$).

It can be said that the XYZ chain is much more physical model than the eight-vertex model itself. The largest eigenvalue of T corresponds to the lowest eigenvalue of H_1. The next-to-large eigenvalues of T correspond to the first excitations above the ground state of the XYZ model. In the infinite-volume limit the lowest two states are degenerate, while the gap between these lowest states and the other excitations remains finite and only vanishes at the critical points. This gap is the inverse correlation length in the time dimension, while the spectrum above the gap defines the correlation length in the spatial dimension.

Generally, the two-dimensional lattice models of classical statistical mechanics are related to the one-dimensional models of quantum mechanics.

If the function ρ satisfies the conditions

$$\rho(u)\rho(-u) = (s(u;\epsilon,r)s(-u;\epsilon,r) + s^2(1;\epsilon,r))^{-1}, \qquad \rho(u) = \rho(1-u), \tag{1.20}$$

the R matrix satisfies two additional conditions:

$$R_{12}(u)R_{21}(-u) = \mathrm{id} \qquad \textbf{(Unitarity)}, \tag{1.21}$$

$$R(u)_{\varepsilon_1\varepsilon_2}^{\varepsilon'_1\varepsilon'_2} = R(1-u)_{\varepsilon'_2,-\varepsilon_1}^{\varepsilon_2,-\varepsilon'_1} \qquad \textbf{(Crossing symmetry)}. \tag{1.22}$$

It is possible to find the solution to the equations (1.20), such that $R(u)$ has the minimal number of poles on the strip $0 < \operatorname{Re} u < 1$. It reads

$$\begin{aligned} \rho(u;\epsilon,r) &= x^{1-r/2}\frac{(x^{2r+2};x^{4r})_\infty(x^{2r-2};x^{4r})_\infty}{(x^{2r};x^{4r})_\infty^2}\frac{(x^{2r}z;x^{4r})_\infty(x^{2r}z^{-1};x^{4r})_\infty}{(x^{4r-2}z;x^{4r})_\infty(x^2z^{-1};x^{4r})_\infty}\frac{g(z^{-1})}{g(z)}, \\ g(z) \equiv g(z;\epsilon,r) &= \frac{(x^2z;x^4,x^{2r})_\infty(x^{2r+2}z;x^4,x^{2r})_\infty}{(x^4z;x^4,x^{2r})_\infty(x^{2r}z;x^4,x^{2r})_\infty}, \end{aligned} \tag{1.23}$$

with the brace function

$$(z;p_1,\ldots,p_N) = \prod_{n_1,\ldots,n_N=0}^{\infty}(1-zp_1^{n_1}\ldots p_N^{n_N})$$

and the 'multiplicative' parameters x, p, z defined as

$$x = e^{-\epsilon}, \qquad p = x^{2r}, \qquad z = x^{2u}.$$

It turns out that this solution gives just the R matrix for which the partition function per site is equal to 1 according to Baxter's solution. It means that solutions of such simple 'reflection equations' (1.21) and (1.22) make it possible to easily reproduce the result of tedious and involved calculations based on the Bethe equations!

2 SOS model and vertex-face correspondence

To understand better the origin of the SOS model let us sketch the Bethe ansatz for the six-vertex model, where $d = 0$. In this case we can introduce the operator S^z of 'total spin', which counts the signs along a column:

$$S^z(v_{\varepsilon_1}\otimes\cdots\otimes v_{\varepsilon_N}) = \frac{1}{2}\sum_{i=1}^{N}\varepsilon_i(v_{\varepsilon_1}\otimes\cdots\otimes v_{\varepsilon_N}).$$

Due to the ice condition this operator commutes with the transfer matrix

$$[T(u), S^z] = 0.$$

This is a trivial fact: the number of 'minuses' is conserved.

So we can easily establish at least two eigenvectors (*pseudovacuums*)

$$|\Omega_\pm\rangle = v_\pm\otimes\cdots\otimes v_\pm$$

with the eigenvalue $a^N + b^N$. But this is generally (everywhere except in the ferroelectric regions F_i) NOT the largest one. How to find the other eigenvectors? Let us start from $|\Omega_+\rangle$ and flip spins one by one. Any state with the eigenvalue of S^z being $N/2 - n$ will be called a state of n pseudoparticles. Let $\sigma_k^- = (\sigma^x - i\sigma^y)/2$ be the operator that turns the kth spin down.

Consider the state of one pseudoparticle. From the translational invariance we conclude, that it has the form

$$|p\rangle = \sum_{k=1}^{N} e^{\mathrm{i}pk}\sigma_k^-|\Omega_+\rangle.$$

From cyclic boundary condition we conclude that

$$e^{ipN} = 1,$$

so that we have N states with $p_j = \frac{2\pi}{N}j$, $j = 0, \ldots, N-1$. You can easily find the corresponding eigenvalues. There are larger ones than $a^N + b^N$, but they are also do not contain the largest one.

Consider the state of 2 pseudoparticles. Substitute the following ansatz:

$$|p_1, p_2\rangle = \sum_{k_1 < k_2} (A_{12} e^{ip_1 k_1 + ip_2 k_2} + A_{21} e^{\mathrm{i}p_2 k_1 + \mathrm{i}p_1 k_2}) \sigma^-_{k_1} \sigma^-_{k_2} |\Omega_+\rangle.$$

Apply the operator T or, simpler, H_1. A miracle! This is an eigenvector if

$$\frac{A_{12}}{A_{21}} = z(p_1, p_2)$$

with some given function $z(p_1, p_2)$. The cyclic boundary condition imposes the restrictions

$$e^{\mathrm{i}p_1 N} = z(p_1, p_2), \qquad e^{\mathrm{i}p_2 N} = z(p_2, p_1).$$

In the general case of n pseudoparticles the same miracle takes place. The wave function can be made of plane waves. The cyclic boundary conditions impose the *Bethe equations*

$$e^{\mathrm{i}p_j N} = \prod_{j'(\neq j)}^{n} z(p_j, p_{j'}), \qquad j = 1, \ldots, n.$$

It is generally impossible to solve these equations analytically. But in the limit $N \to \infty$, $n/N = \text{const}$ they are reduced to an integral equation. The case $n = N/2$, corresponding to the largest eigenvalue, admits an analytic solution. This is how the six-vertex model is solved.

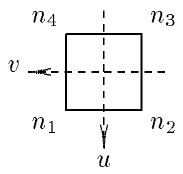
What is wrong with the general eight-vertex model? The obstacle is that

$$[S^z, T] \neq 0 \quad \text{for } d \neq 0.$$

It destroys the whole picture of pseudoparticles. There is a nice construction of the Q operator proposed by Baxter, that makes it possible to obtain the Bethe equations without any reference to the Bethe ansatz. Nevertheless, there is a question: is it possible to relate this model to another one that admits the whole construction of Bethe ansatz? Is it possible to construct something similar to the six-vertex model, but involving elliptic functions? The answer is YES.

Consider again the square lattice on the plane, but associate the variables to the vertices of the lattice and the Boltzmann weights to the plaquet or faces. Namely, associate to each vertex a variable $n \in \mathbb{Z} + \delta$, where the real shift δ is introduced for convenience. The partition function will be independent of this shift. Associate to each face of the lattice a weight:

$$e^{-E(n_1, n_2, n_3, n_4)/T} = W\left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix}\middle| u - v\right] =$$

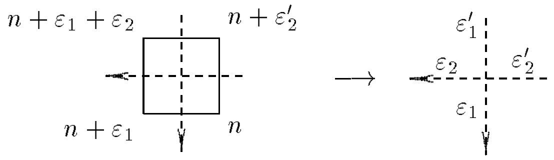


The dashed lines, first, denote the orientation and, second, carry the spectral parameters. The configuration sum is taken over all ns at all vertices such that

$$|n_i - n_j| = 1 \qquad \textbf{(admissibility condition)} \tag{2.1}$$

on the neighboring vertices.

What does the admissibility condition mean? Consider the dual (dashed) lattice. Define on each edge of this lattice a variable $\varepsilon = +1$ if the variable $n_i = n_j + 1$, if i denotes the vertex on the left or upper end of the edge, while j denotes the vertex on the right or lower end of the edge:



$(n + \varepsilon_1 + \varepsilon_2 = n + \varepsilon'_1 + \varepsilon'_2)$

In these notations the variables ε satisfy the ice condition by definition. But the weight W at each vertex of the dual lattice depend not only on the variables $\varepsilon_1, \varepsilon_2, \varepsilon'_1, \varepsilon'_2$, but also on the value of n at e.g. the right lower corner of the face, which is (up to δ) the sum of all εs on any path along the initial (solid) lattice from some fixed point at the lattice to this right lower corner of the face.

The Boltzmann weights, analogous to a, b and c of the six-vertex model, are given by

$$
\begin{aligned}
a_n^{\pm}(u) &= W\left[\begin{matrix} n\pm 2 & n\pm 1 \\ n\pm 1 & n \end{matrix}\middle| u\right] = R_0(u), \\
b_n^{\pm}(u) &= W\left[\begin{matrix} n & n\mp 1 \\ n\pm 1 & n \end{matrix}\middle| u\right] = R_0(u)\frac{[n\mp 1][u]}{[n][1-u]} \\
c_n^{\pm}(u) &= W\left[\begin{matrix} n & n\pm 1 \\ n\pm 1 & n \end{matrix}\middle| u\right] = R_0(u)\frac{[n\pm u][1]}{[n][1-u]},
\end{aligned}
\tag{2.2}
$$

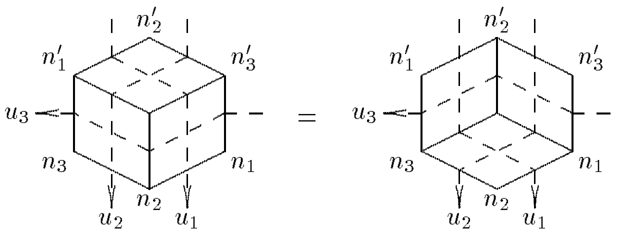
with an arbitrary function $R_0(u)$ and

$$
\begin{aligned}
[u]_i &= \sqrt{\frac{\pi}{\epsilon r}}\, e^{\frac{1}{4}\epsilon r}\, \theta_i\left(\frac{u}{r}; \frac{\mathrm{i}\pi}{\epsilon r}\right), \\
[u] &= [u]_1 = x^{u^2/r-u}(z;p)_\infty (pz^{-1};p)_\infty (p;p)_\infty .
\end{aligned}
$$

The weights W satisfy the Yang–Baxter equation of the form

$$
\begin{aligned}
&\sum_n W\left[\begin{matrix} n'_1 & n_2 \\ n & n'_3 \end{matrix}\middle| u_1-u_2\right] W\left[\begin{matrix} n & n'_3 \\ n_2 & n_1 \end{matrix}\middle| u_1-u_3\right] W\left[\begin{matrix} n'_1 & n \\ n_3 & n_2 \end{matrix}\middle| u_2-u_3\right] \\
&\qquad = \sum_n W\left[\begin{matrix} n'_2 & n'_3 \\ n & n_1 \end{matrix}\middle| u_2-u_3\right] W\left[\begin{matrix} n'_1 & n'_2 \\ n_3 & n \end{matrix}\middle| u_1-u_3\right] W\left[\begin{matrix} n_3 & n \\ n_2 & n_1 \end{matrix}\middle| u_1-u_2\right].
\end{aligned}
\tag{2.3}
$$

Graphically it looks like:



The dashed lines here play the role of solid lines in the Yang–Baxter equation for the eight-vertex model, while the solid lines here simply form the lattice dual to the dashed one.

If the function $R_0(u)$ satisfy the relations

$$
R_0(u)R_0(-u) = 1, \qquad R_0(u)[u] = R_0(1-u)[1-u],
$$

the weights satisfy the relations

$$
\sum_n W\left[\begin{matrix} n_4 & n \\ n_1 & n_2 \end{matrix}\middle| u\right] W\left[\begin{matrix} n_4 & n_3 \\ n & n_2 \end{matrix}\middle| -u\right] = \delta_{n_1,n_3}, \qquad \textbf{(Unitarity)}, \tag{2.4}
$$

$$
[n_3]^{-1} W\left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix}\middle| u\right] = [n_4]^{-1} W\left[\begin{matrix} n_1 & n_4 \\ n_2 & n_3 \end{matrix}\middle| 1-u\right] \qquad \textbf{(Crossing symmetry)}. \tag{2.5}
$$

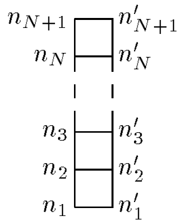
The solution

$$
R_0(u) \equiv R_0(u;\epsilon,r) = z^{(r-1)/2r}\frac{g(z^{-1})}{g(z)} \tag{2.6}
$$

makes the partition function per site equal to 1.

It is easy to define the L operator

$$L(u)^{n'_1\ldots n'_{N+1}}_{n_1\ldots n_{N+1}} =$$



$$= W\begin{bmatrix} n_{N+1} & n'_{N+1} \\ n_N & n'_N \end{bmatrix}\!\!\left|u\right] \ldots W\begin{bmatrix} n_3 & n'_3 \\ n_2 & n'_2 \end{bmatrix}\!\!\left|u\right] W\begin{bmatrix} n_2 & n'_2 \\ n_1 & n'_1 \end{bmatrix}\!\!\left|u\right] \tag{2.7}$$

and the transfer matrix

$$T(u)^{n'_1\ldots n'_N}_{n_1\ldots n_N} = L(u)^{n'_1\ldots n'_N n'_1}_{n_1\ldots n_N n_1}. \tag{2.8}$$

The transfer matrices form a commuting family,

$$T(u_1)T(u_2) = T(u_2)T(u_1),$$

and $T(0)$ is again the shift operator.

Now we formulate Baxter's fundamental statement about the relation between two models [9]. There exist functions $t_\varepsilon(u)^{n'}_n$ such that

$$\sum_{\varepsilon'_1\varepsilon'_2} R(u-v)^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2} t_{\varepsilon'_2}(v-u_0)^n_{s'} t_{\varepsilon'_1}(u-u_0)^{s'}_{n'} = \sum_{s\in\mathbb{Z}} t_{\varepsilon_1}(u-u_0)^n_s t_{\varepsilon_2}(v-u_0)^s_{n'}\, W\begin{bmatrix} n' & s' \\ s & n \end{bmatrix}\!\!\left|u-v\right] \tag{2.9}$$

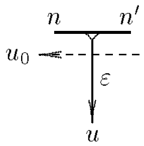
for arbitrary u_0. This relation is referred to as the *vertex–face correspondence*. Explicitly, these *intertwining* functions have the form

$$\begin{aligned} t_+(u)^{n'}_n &= (-1)^{(n-\delta)(n'-n-1)/2} e^{\mathrm{i}\pi/4} f(u)\theta_3\left(\frac{(n'-n)u+n'}{2r}; \mathrm{i}\frac{\pi}{2\epsilon r}\right), \\ t_-(u)^{n'}_n &= -(-1)^{(n-\delta)(n'-n+1)/2} e^{-\mathrm{i}\pi/4} f(u)\theta_4\left(\frac{(n'-n)u+n'}{2r}; \mathrm{i}\frac{\pi}{2\epsilon r}\right). \end{aligned} \tag{2.10}$$

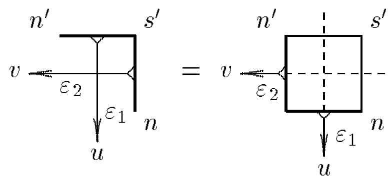
Here $f(u)$ is an arbitrary function and δ is the shift discussed above.

To understand better the fundamental identity (2.9), let us represent it graphically. Introduce the graphical representative of the intertwining functions:

$$t_\varepsilon(u-u_0)^{n'}_n \; =$$



With this notation the vertex–face correspondence looks like (u_0 line is not depicted)



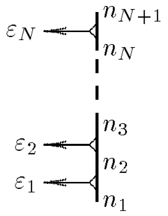
Note that this relation looks like the Yang–Baxter equation of mixed vertex–face type!

This means that if we take a square finite SOS lattice with open boundaries and attach intertwining functions to their left and lower boundaries summing over necessary boundary variables n, we can push the intertwining functions up and right using the vertex–face correspondence and obtain a square lattice of the eight-vertex model with the intertwining functions attached to the right and upper boundaries. In physics we usually expect that the contribution of boundaries to the partition function is neglectable in a large system.

It means that the large volume limit of the partition functions per site of the eight-vertex model and of the SOS model coincide.

Moreover, it can be rigorously derived that the spectra of eigenvalues of the transfer matrices of the eight-vertex and SOS models with the cyclic boundary condition coincide. First, introduce the L-type operator

$$\lambda(u_0)^{n_1\ldots n_N n_{N+1}}_{\varepsilon_1\ldots\varepsilon_N} =$$



$$= t_{\varepsilon_N}(-u_0)^{n_{N-1}}_{n_N}\ldots t_{\varepsilon_2}(-u_0)^{n_2}_{n_3} t_{\varepsilon_1}(-u_0)^{n_1}_{n_2} \qquad (2.11)$$

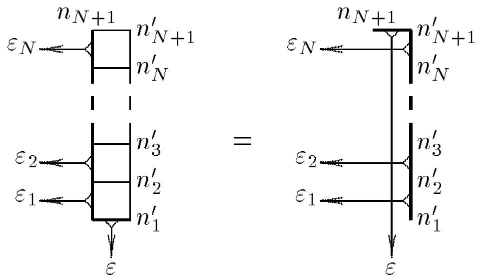
and the transfer matrix type operator

$$\tau(u_0)^{n_1\ldots n_N}_{\varepsilon_1\ldots\varepsilon_N} = \lambda(u_0)^{n_1\ldots n_N n_1}_{\varepsilon_1\ldots\varepsilon_N}.$$

Then

$$\sum_{n_2\ldots n_{N-1}} L(u)^{n'_1\ldots n'_{N+1}}_{n_1\ldots n_{N+1}}\lambda(u_0)^{n_1\ldots n_{N+1}} t_1(u-u_0)^{n_1}_{n'_1} = \sum_n \lambda(u_0)^{n'_1\ldots n'_{N+1}} t_1(u-u_0)^{n'_{N+1}}_n L_1(u)$$

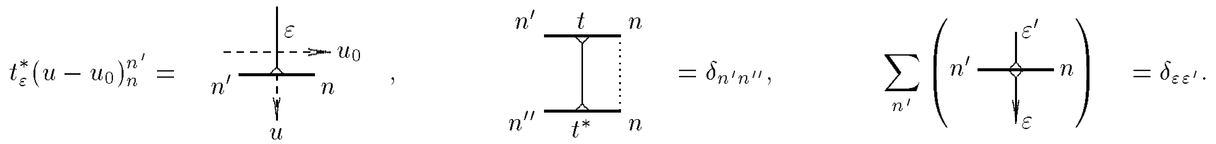
or, graphically,



Introduce now the object 'inverse' to the intertwining functions:

$$\sum_\varepsilon t_\varepsilon(u)^n_{n'} t^*_\varepsilon(u)^{n''}_n = \delta_{n'n''} \quad \text{or} \quad \sum_{n'} t^*_{\varepsilon'}(u)^{n'}_n t_\varepsilon(u)^n_{n'} = \delta_{\varepsilon\varepsilon'}. \qquad (2.12)$$

or, graphically,



Attaching these t^* functions to the upper boundary and imposing the cyclic boundary condition we obtain

$$\tau(u_0)T_{8\mathrm{v}}(u) = T_{\mathrm{SOS}}(u)\tau(u_0), \qquad (2.13)$$

where $T_{8\mathrm{v}}$ and $T_{\mathrm{SOS}}(u)$ are transfer matrices of the eight-vertex and SOS models respectively. Note that this equation has been obtained in the full analogy to the derivation of commutativity of transfer matrices.

Similarly, one can introduce the matrix

$$\tau^*(u)^{\varepsilon_1\ldots\varepsilon_N}_{n_1\ldots n_N} = t^*_{\varepsilon_N}(-u_0)^{n_1}_{n_N}\ldots t^*_{\varepsilon_1}(-u_0)^{n_2}_{n_1}$$

with the relation

$$T_{8\mathrm{v}}(u)\tau^*(u_0) = \tau^*(u_0)T_{\mathrm{SOS}}(u), \qquad (2.14)$$

Let $|\Lambda\rangle_{\mathrm{SOS}}$ be an eigenvector of $T_{\mathrm{SOS}}(u)$ with the eigenvalue function $\Lambda(u)$. Then

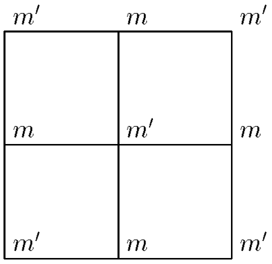
$$T_{8\mathrm{v}}(u)\tau^*(u_0)|\Lambda\rangle_{\mathrm{SOS}} = \tau^*(u_0)T_{\mathrm{SOS}}(u)|\Lambda\rangle_{\mathrm{SOS}} = \Lambda(u)\tau^*(u_0)|\Lambda\rangle_{\mathrm{SOS}}.$$

It means that $|\Lambda\rangle_{8\mathrm{v}} = \tau^*(u_0)|\Lambda\rangle_{\mathrm{SOS}}$ is an eigenvector of the operator $T_{8\mathrm{v}}(u)$ with the same eigenvalue function $\Lambda(u)$. It proves that the spectra of both models coincide.

To conclude, let us say something about the ground states in this theory. We shall consider the SOS model in the so called regime III region:

$$\epsilon > 0, \qquad r \geq 1, \qquad 0 < u < 1.$$

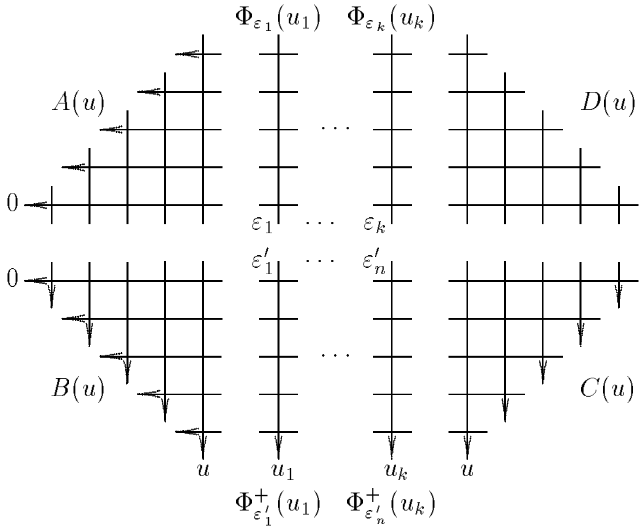
In this region the ground states (the states of maximal weight) are numerated by $m \in \mathbb{Z} + \delta$ and $m' = m \pm 1$, such that $(k-1)r < m, m' < kr$ for some integer k. The ground state (m, m') looks like



The conclusion is the following. There is a highly nonlocal transformation that relates the eight-vertex model to another model, the solid-on-solid one, which can be treated by means of the Bethe ansatz approach. Though this relation is not a direct one-to-one correspondence between configurations, it is nevertheless a 'detailed' correspondence that makes it possible to express any expectation value of the eight-vertex model to an expectation value defined in terms of the SOS model. We discuss this point in the Lecture 4.

3 Corner transfer matrices and vertex operators

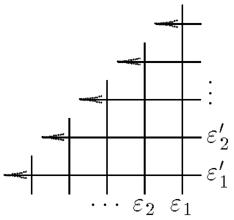
Consider the eight-vertex model on a large but finite lattice with fixed boundary condition. Set the spectral parameter on the horizontal lines to be equal to 0, while on the vertical lines to be equal to u always except k neighboring lines, where it will be equal to $u_1, \ldots, u_k$. Let us partition the lattice into several pieces as follows:



As it is shown in the picture at each of the k exceptional lines we indeed cut the bond in the very middle and fix the variables ε_i, ε'_i at the ends.

The pieces $A(u)$, $B(u)$, $C(u)$, $D(u)$ can be considered as matrices acting clockwise, e.g.

$$A(u)^{\varepsilon'_1\varepsilon'_2\ldots}_{\varepsilon_1\varepsilon_2\ldots} =$$



The matrices $A(u)$, $B(u)$, $C(u)$, $D(u)$ are called *corner transfer matrices.*

The pieces $\Phi_{\varepsilon_i}(u_i)$ are $\Phi^+_{\varepsilon_i}(u_i)$ act as matrices in the left-to-right and right-to-left directions respectively. They are called *half transfer matrices* and (because of some properties in the infinite volume limit) *vertex operators.*

We have to specify the boundary conditions at the outer boundary. We shall fix the spins at the boundary so as if they belong to one of the ground states described in the first lecture. We shall denote the boundary condition by the superscript (i) ($i \in \mathbb{Z}_2$), if $\varepsilon_1 = (-1)^i$ in the corresponding ground state. To avoid multiple usage of this superscript at any corner transfer matrix and vertex operator like $A^{(i)}$, $B^{(i)}$, $C^{(i+n)}$, $D^{(i+n)}$, $\Phi^{(i+j,i+j-1)}_{\varepsilon_j}(u_j)$, we shall put it at the trace signs below.

Let $Z^{(i)\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_k}$ be the partition function of the lattice with the given boundary conditions and fixed variables at the upper and lower banks of the cut. Let $Z^{(i)} = \sum_{\varepsilon_1\ldots\varepsilon_k} Z^{(i)\varepsilon_1\ldots\varepsilon_k}_{\varepsilon_1\ldots\varepsilon_k}$ be the partition functions of the lattice without the cut. Now consider the ratio $P^{(i)\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_k} = Z^{(i)\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_n}/Z^{(i)}$. In particular, $P^{(i)\varepsilon_1\ldots\varepsilon_k}_{\varepsilon_1\ldots\varepsilon_k}$ is the probability that the configuration of spins on the bonds in the middle of the exceptional lines is $\varepsilon_1, \ldots, \varepsilon_k$.

These quantities are basic for calculation of local correlation functions. For example, let us calculate the average $\langle\varepsilon_1\varepsilon_2\rangle \equiv \langle\sigma^z_1\sigma^z_2\rangle$ of the product of two neighboring spins on the lattice without any cut. It is given by

$$\langle\sigma^z_1\sigma^z_2\rangle^{(i)} = P^{(i)++}_{++} + P^{(i)--}_{--} - P^{(i)+-}_{+-} - P^{(i)-+}_{-+}.$$

Other local correlation functions are expressed similarly.

From the partition of the lattice described above it is easy to obtain

$$P^{(i)\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_n} = \frac{1}{Z^{(i)}}\,\mathrm{Tr}^{(i)}(\Phi^+_{\varepsilon'_1}(u_1)\ldots\Phi^+_{\varepsilon'_n}(u_k)C(u)D(u)\Phi_{\varepsilon_k}(u_k)\ldots\Phi_{\varepsilon_1}(u_1)A(u)B(u)). \tag{3.1}$$

Surely, we have not yet approached the exact solution to the problem. Nevertheless, in the large volume limit the objects defined above get remarkable properties.

First of all, not all of these objects are independent. From the crossing symmetry it is easy to find that

$$C(u) = QA(u)Q, \qquad B(u) = QD(u)Q = A(1-u)Q, \qquad \Phi^+_\varepsilon(u) = Q\Phi_\varepsilon(u)Q, \tag{3.2}$$

where $Q = \sigma^x \otimes \sigma^x \otimes \ldots$ is the operator that flips all spins.

Baxter observed [10] that in the large volume limit

$$A(u) = F(u)e^{-\epsilon u H} \tag{3.3}$$

with some scalar (not operator) function $F(u)$ and some constant operator H with the discrete spectrum $\{0, 1, 2, \ldots\}$.

Let us sketch Baxter's argumentation. Consider the product $A(u)B(u-v)$. On the infinite lattice this product, considered as a vector, must be an eigenvector corresponding to the largest eigenvalue of the transfer matrix of an inhomogeneous model. Hence, the product $A(u)A(1+v-u)$ as a function of u is a constant operator times a scalar function. As $A(0) = 1$, we have the equation

$$A(u)A(u') = g(u,u')A(u+u'). \tag{3.4}$$

Therefore

$$A(u)A'(0) = g_{u'}(u,0)A(u) + g(u,0)A'(u)$$

or

$$\frac{A'(u)}{A(u)} = f_1(u)A'(0) + f_2(u).$$

Solving this differential equation we obtain

$$A(u) = F_1(u)e^{F_2(u)A'(0)},$$

with some functions $F_1(u)$, $F_2(u)$, such that

$$F_1(0) = 1, \qquad F_2(0) = 0, \qquad F_2'(0) = 1.$$

Substituting this solution back into the difference equation (3.4), we obtain that $F_2(u) = u$. We obtain

$$A(u) = F_1(u)e^{uA'(0)}. \tag{3.5}$$

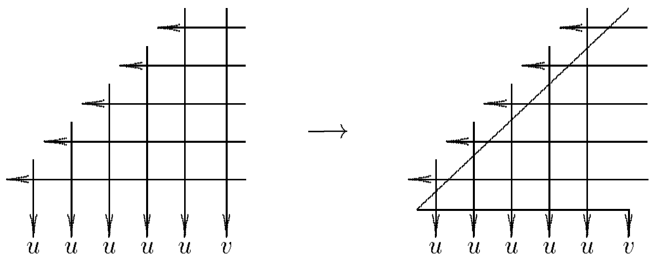
From the definition of the model we know that the Boltzmann weight without the function $\rho(u)$ are doubly periodic. They are unchanged after the substitutions $u \to u+2r$ and $u \to u+2i\pi/\epsilon$. The function $\rho(u)$ written out in the first lecture do not respect the first periodicity and respects the second. This is very important. The function $\rho(u)$ can be obtained directly from the Bethe ansatz, i.e. from first principles. Moreover, it can be concluded from the Bethe ansatz solution that all physical quantities respect this second periodicity. Therefore, impose this periodicity on the solution (3.5). We immediately obtain that the spectrum of $A'(u)$ is equidistant with the separation ϵ. This 'proves' (on the physical level of rigorousness) the equation (3.3). From now on we shall omit the factor $F(u)$ and write

$$A(u) = e^{-\epsilon u H} = z^{H/2}. \tag{3.6}$$

Now, on the physical level of rigorousness, it is easy to obtain the following two commutation relations [11]:

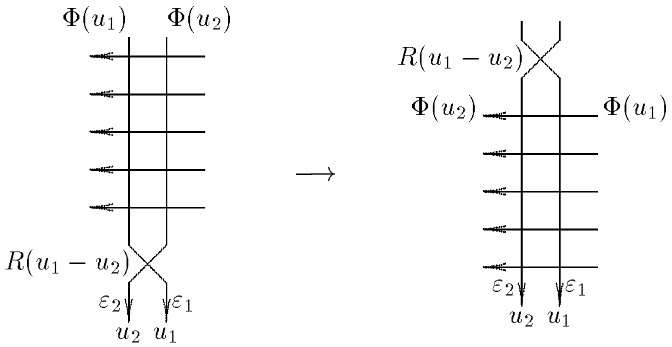
$$\begin{aligned} \Phi_\varepsilon(u)Q &= Q\Phi_{-\varepsilon}(u), \\ \Phi_\varepsilon(v)A(u) &= A(u)\Phi_\varepsilon(v-u), \\ \Phi_{\varepsilon_1}(u_1)\Phi_{\varepsilon_2}(u_2) &= \sum_{\varepsilon_1'\varepsilon_2'} R(u_1-u_2)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}\Phi_{\varepsilon_2'}(u_2)\Phi_{\varepsilon_1'}(u_1). \end{aligned} \tag{3.7}$$

The first equation is trivial. The second equation can be proven as follows. Take the product $\Phi_\varepsilon(v)A(u)$ and, using the Yang–Baxter equation, push the line corresponding to Φ_ε to the left:



Any physicist knows that the boundary does not affect essentially the bulk. So let us forget completely about the skew line on the right picture. Its only effect is the boundary condition. The lower horizontal line is just the operator $\Phi_\varepsilon(u-v)$.

The derivation of the last line is similar. Consider the product in the r.h.s. and push the R matrix upside:



Now the R matrix 'at the infinity' can be erased and we obtain the third equation.

From these equations we derive that

$$\Phi_\varepsilon^+(u_j)C(u)D(u) = Q\Phi_{-\varepsilon}(u_j)e^{-2\epsilon H} = Qe^{-2\epsilon H}\Phi_{-\varepsilon}(u_j-1) = C(u)D(u)\Phi_{-\varepsilon}(u_j-1).$$

Let us introduce the notation

$$\Phi_\varepsilon^*(u) = \Phi_{-\varepsilon}(u-1). \tag{3.8}$$

Then

$$\Phi_\varepsilon^+(u_j)C(u)D(u) = C(u)D(u)\Phi_\varepsilon^*(u_j). \tag{3.9}$$

It means that we can move the product $C(u)D(u)$ to the left simultaneously replacing $\Phi_\varepsilon^+(u_j)$ by $\Phi_\varepsilon^*(u_j)$. From the fact that $\sum_\varepsilon \Phi_\varepsilon(u)\otimes(\Phi_\varepsilon^+(u))^t$ is just the transfer matrix on the infinite lattice with the largest eigenvalue 1 we conclude that

$$\sum_\varepsilon \Phi_\varepsilon^*(u)\Phi_\varepsilon(u) = 1, \qquad \Phi_\varepsilon(u)\Phi_{\varepsilon'}^*(u) = \delta_{\varepsilon\varepsilon'}. \tag{3.10}$$

Now consider the product of the product of the corner transfer matrices. It is easy to find from (3.6) that

$$A(u)B(u)C(u)D(u) = e^{-4\epsilon H} = x^{2H}. \tag{3.11}$$

We can also specify what we mean under $\mathrm{Tr}^{(i)}$ in the infinite volume limit. Consider the consequences of spin variables $\varepsilon(1), \varepsilon(2), \ldots$ that stabilize to $\varepsilon(n) = (-)^{i+n}$. The space of such *paths* will be denoted by $\mathcal{H}^{(i)}$. Then $\mathrm{Tr}^{(i)} = \mathrm{Tr}_{\mathcal{H}^{(i)}}$.

Substituting (3.9) and (3.11) to (3.1) we obtain

$$P^{(i)\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_n} = \frac{1}{\chi^{(i)}}\mathrm{Tr}_{\mathcal{H}^{(i)}}(\Phi^*_{\varepsilon'_1}(u_1)\ldots\Phi^*_{\varepsilon'_n}(u_k)\Phi_{\varepsilon_k}(u_k)\ldots\Phi_{\varepsilon_1}(u_1)x^{2H}). \tag{3.12}$$

with

$$\chi^{(i)} = \mathrm{Tr}_{\mathcal{H}^{(i)}}\, x^{2H}. \tag{3.13}$$

It was shown [12], that

$$\mathrm{Tr}_{\mathcal{H}^{(i)}}\, q^H = \frac{1}{(q;q^2)_\infty}.$$

This result was proven in the limit $x\to 0$, $r\to\infty$, but, since degeneracy cannot change continuously, it holds in the whole AF_1 phase.

Generally, we can consider the *trace functions*

$$F^{(i)}_{\varepsilon_1\ldots\varepsilon_k}(u_1,\ldots,u_k) = \frac{1}{\chi^{(i)}}\mathrm{Tr}_{\mathcal{H}^{(i)}}(\Phi_{\varepsilon_k}(u_k)\ldots\Phi_{\varepsilon_1}(u_1)x^{2H}). \tag{3.14}$$

Evidently,

$$P^{(i)\varepsilon_1\ldots\varepsilon_k}_{\varepsilon'_1\ldots\varepsilon'_n} = F^{(i)}_{\varepsilon_1,\ldots,\varepsilon_k,-\varepsilon'_n,\ldots,-\varepsilon'_1}(u_1,\ldots,u_k,u_k-1,\ldots,u_1-1).$$

The functions $F^{(i)}(u_1,\ldots,u_k)$ satisfy a number of difference equations, that follow from the properties of the corner transfer matrices and vertex operators:

$$F^{(i)}_{\varepsilon_1\ldots\varepsilon_k}(u_1+v,\ldots,u_k+v) = F^{(i)}_{\varepsilon_1\ldots\varepsilon_k}(u_1,\ldots,u_k), \tag{3.15}$$

$$F^{(i)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}(u_1+2\mathrm{i}\pi/\epsilon,u_2,\ldots,u_k) = F^{(i)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}(u_1,u_2,\ldots,u_k), \tag{3.16}$$

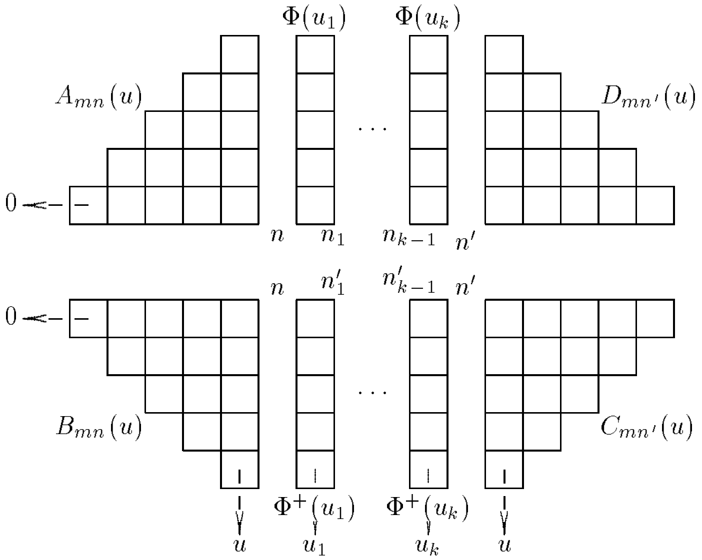
$$F^{(i)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}(u_1,u_2,\ldots,u_k) = F^{(i+1)}_{\varepsilon_2\ldots\varepsilon_k\varepsilon_1}(u_2,\ldots,u_k,u_1-2), \tag{3.17}$$

$$\sum_\varepsilon F^{(i)}_{\varepsilon_1,\ldots,\varepsilon_k,\varepsilon,-\varepsilon}(u_1,\ldots,u_k,u,u-1) = F^{(i)}_{\varepsilon_1\ldots\varepsilon_k}(u_1,\ldots,u_k), \tag{3.18}$$

$$F^{(i)}_{\ldots\varepsilon_j\varepsilon_{j+1}\ldots}(\ldots,u_j,u_{j+1},\ldots) = \sum_{\varepsilon'_j\varepsilon'_{j+1}} R(u_{j+1}-u_j)^{\varepsilon'_j\varepsilon'_{j+1}}_{\varepsilon_j\varepsilon_{j+1}} F^{(i)}_{\ldots\varepsilon'_{j+1}\varepsilon'_j\ldots}(\ldots,u_{j+1},u_j,\ldots). \tag{3.19}$$

In principle, it is possible to find the functions F and probabilities P by solving these equations under some analyticity conditions. In practice, the case $n=2$ in F only admits a direct solution. In more general case we need some additional ideas to solve the equations.

Consider now the SOS model. The necessary partition of the lattice looks like:



Here we introduce the corner transfer matrices $A_{mn}(u),\ldots,D_{mn}(u)$, which depend on the 'central site' variable n and the boundary condition $(m,m+1)$ or $(m+1,m)$ depending on the parity of $n-m$, and the vertex operators $\Phi(u_j)^{n_j}_{n_{j-1}}$ and $\Phi^+(u_j)^{n'_{j-1}}_{n'_j}$, which depend on the variables at their feet.

In the same way as for the eight-vertex model one can write the quantities

$$P_m\left(n\,{}^{n_1\ldots n_{k-1}}_{n'_1\ldots n'_{k-1}}\,n'\right)=\frac{1}{Z_m}\,\mathrm{Tr}(\Phi^+(u_1)^{n_1}_{n'_1}\ldots\Phi^+(u_k)^{n'_{k-1}}_{n'}C_{mn}(u)D_{mn}(u)$$
$$\times\,\Phi(u_k)^{n'}_{n_{k-1}}\ldots\Phi(u_1)^{n_1}_{n}A_{mn}(u)B_{mn}(u)).\qquad(3.20)$$

In particular, for $n'_i=n_i$ $(i=1,\ldots,k-1)$ they are the multipoint *local height probabilities*, which describe the probabilities of configurations along a finite line on the lattice.

In the infinite volume limit we have up to a scalar factor

$$A_{mn}(u)=e^{-2\epsilon uH_{mn}}.\qquad(3.21)$$

The product of the corner transfer matrices is given by

$$A_{mn}(u)B_{mn}(u)C_{mn}(u)D_{mn}(u)=[n]x^{4H_{mn}}.\qquad(3.22)$$

The additional factor 2 before the corner Hamiltonian in comparison with the eight-vertex model is related to the (quasi)periodicity of all quantities with the period $\mathrm{i}\pi/\epsilon$ instead of $2\pi\mathrm{i}/\epsilon$, while the factor $[n]$ in the product is related to the similar factors in the crossing property.

Another important property is

$$\Phi^+(u)^{n'}_nC_{mn}(v)D_{mn}(v)=C_{mn'}(v)D_{mn'}(v)\Phi^*(u)^{n'}_n,\qquad(3.23)$$

where

$$\Phi^*(u)^{n'}_n=[n]\Phi(u-1)^{n'}_n,\qquad\sum_{n'}\Phi^*(u)^n_{n'}\Phi(u)^{n'}_n=1,\qquad\Phi(u)^{n'}_{n''}\Phi^*(u)^{n''}_n=\delta^{n'}_n.\qquad(3.24)$$

The basic commutation relations look like

$$\Phi(u)^{n'}_nx^{2vH_{mn}}=x^{2vH_{mn'}}\Phi(u-v)^{n'}_n,\qquad(3.25)$$

$$\Phi(u_1)_s^{n'}\Phi(u_2)_n^s = \sum_{s'} W\begin{bmatrix} n & s' \\ s & n' \end{bmatrix} u_1-u_2 \Bigg] \Phi(u_2)_{s'}^{n'}\Phi(u_1)_n^{s'}. \tag{3.26}$$

In the infinite volume limit define the trace functions

$$F^m_{nn_1\dots n_k}(u_1,\dots,u_k) = \frac{[n]}{\chi_m}\operatorname{Tr}_{\mathcal{H}_{mn}}(\Phi(u_k)^n_{n_k}\dots\Phi(u_2)^{n_2}_{n_1}\Phi(u_1)^{n_1}_n x^{4H_{mn}}) \tag{3.27}$$

with

$$\chi_m = \sum_n [n]\chi_{mn}, \qquad \chi_{mn} = \operatorname{Tr}_{\mathcal{H}_{mn}} x^{4H_{mn}}. \tag{3.28}$$

Here $\mathcal{H}_{mn}$ is the space of paths $n(0)=n, n(1), n(2), \dots$ that stabilize to the sequence $\dots, m, m+1, m, m+1, \dots$. There is a remarkable result by Andrews, Baxter and Forrester [13] based on the $x\to 0$ limit (the *low temperature* limit) that

$$\operatorname{Tr}_{\mathcal{H}_{mn}} q^{H_{mn}} = \frac{q^{(rm-(r-1)n)^2/4r(r-1)}}{(q;q)_\infty}.$$

Notice an important property of χ_{mn}:

$$\sum_{n\in 2\mathbb{Z}+m+i} [n]\chi_{mn} = [m]'\chi^{(i)}, \qquad [u]' = [u]|_{r\to r-1}. \tag{3.29}$$

From the properties (3.22) and (3.23) we obtain

$$P_m\begin{pmatrix} n & n_1\dots n_{k-1} & n' \\ & n'_1\dots n'_{k-1} & \end{pmatrix} = F^m_{nn_1\dots n_{k-1}n'n'_{k-1}\dots n'_1}(u_1,\dots,u_k,u_k-1,\dots,u_1-1)[n']\prod_{j=1}^{k-1}[n'_k].$$

Using the commutation relations (3.25), (3.26) we obtain

$$F^m_{nn_1\dots n_{k-1}}(u_1+v,\dots,u_k+v) = F^m_{nn_1\dots n_{k-1}}(u_1,\dots,u_k), \tag{3.30}$$

$$F^m_{nn_1n_2\dots n_{k-1}}(u_1+\mathrm{i}\pi/\epsilon,u_2,\dots,u_k) = -e^{\mathrm{i}\pi(n_1^2-n^2)/2r}F^m_{nn_1n_2\dots n_k}(u_1,u_2,\dots,u_k), \tag{3.31}$$

$$F^m_{nn_1n_2\dots n_k}(u_1,u_2,\dots,u_k) = \frac{[n]}{[n_1]}F^m_{n_1n_2\dots n_kn}(u_2,\dots,u_k,u_1), \tag{3.32}$$

$$\sum_{n'}[n']F^m_{nn_1\dots n_{k-1}nn'}(u_1,\dots,u_k,u,u-1) = F^m_{nn_1\dots n_{k_1}}(u_1,\dots,u_k), \tag{3.33}$$

$$F^m_{\dots n_{k-1}n_kn_{k+1}\dots}(\dots,u_k,u_{k+1},\dots) = \sum_{n'_k} W\begin{bmatrix} n_{k-1} & n'_k \\ n_k & n_{k+1} \end{bmatrix} u_{k+1}-u_k \Bigg] F^m_{\dots n_{k-1}n'_kn_{k+1}\dots}(\dots,u_{k+1},u_k,\dots). \tag{3.34}$$

The problem of solving these equations in the SOS model will be discussed in the next lecture. The respective problem for the eight-vertex model is more difficult and will be the topic of the last lecture.

4 Free field representation: SOS model

The bosonization or free field representation appeared in conformal field theory in the works by Feigin and Fuchs [14] and by Dotsenko and Fateev [15] in 1983. It is no use to recall these papers for our purposes. The most important outcome of these papers for us is that some linear equations can be solved by representing the solution in terms of expectation values of some quantum operators. The trace form of the functions we want to obtain prompt us that it must be a thermal average. If the Hamiltonian H is quadratic in the bosonic field and the operators $\Phi(u)_n^{n'}$ is expressed as exponentials of this field, the problem will be solvable.

Let me first formally introduce the construction by Lukyanov and Pugai [16] and then to explain how it can be obtained.

Consider a Heisenberg algebra of operators a_k ($k \in \mathbb{Z} \setminus \{0\}$) and a pair of 'zero-mode' operators $\mathcal{P}$ and $\mathcal{Q}$ with the commutation relations

$$[\mathcal{P}, \mathcal{Q}] = -\mathrm{i}, \qquad [a_k, a_l] = k\frac{[\![k]\!]_x [\![(r-1)k]\!]_x}{[\![2k]\!]_x [\![rk]\!]_x}\delta_{k+l,0} \quad \text{with} \quad [\![u]\!]_x = \frac{x^u - x^{-u}}{x - x^{-1}}. \tag{4.1}$$

The 'q-number' $[\![u]\!]_x$ here should not be confused with the 'elliptic q-numbers' $[u]_i$ ($i = 1, \ldots, 4$). It is also useful to introduce the operators

$$\tilde{a}_k = \frac{[\![rk]\!]_x}{[\![(r-1)k]\!]_x} a_k. \tag{4.2}$$

The normal ordering operation $:\ldots:$ places $\mathcal{P}$ to the right of $\mathcal{Q}$ and a_k with positive k to the right of a_{-k}. It will be convenient to assign

$$\alpha_+ = \sqrt{a_+} = \sqrt{\frac{r}{r-1}}, \qquad \alpha_- = -\sqrt{a_-} = -\sqrt{\frac{r-1}{r}}, \qquad 2\alpha_0 = \alpha_+ + \alpha_- = \frac{1}{\sqrt{r(r-1)}}. \tag{4.3}$$

Now introduce the fields

$$\begin{aligned} \varphi(z) &= \frac{\alpha_-}{\sqrt{2}}(\mathcal{Q} - \mathrm{i}\mathcal{P}\log z) - \sum_{k \neq 0} \frac{a_k}{\mathrm{i}k} z^{-k}. \\ \tilde{\varphi}(z) &= \frac{\alpha_+}{\sqrt{2}}(\mathcal{Q} - \mathrm{i}\mathcal{P}\log z) + \sum_{k \neq 0} \frac{\tilde{a}_k}{\mathrm{i}k} z^{-k}. \end{aligned} \tag{4.4}$$

These fields enter the exponential operators

$$\begin{aligned} V(u) &= z^{(r-1)/4r} {:}e^{\mathrm{i}\varphi(z)}{:}, \qquad & \bar{V}(u) &= z^{(r-1)/r} {:}e^{-\mathrm{i}\varphi(x^{-1}z) - \mathrm{i}\varphi(xz)}{:}, \\ \tilde{V}(u) &= z^{r/4(r-1)} {:}e^{\mathrm{i}\tilde{\varphi}(z)}{:}, \qquad & \bar{\tilde{V}}(u) &= z^{r/(r-1)} {:}e^{-\mathrm{i}\tilde{\varphi}(x^{-1}z) - \mathrm{i}\tilde{\varphi}(xz)}{:}, \end{aligned} \tag{4.5}$$

and Lukyanov's screening operators

$$\begin{aligned} x(u, C) &= \frac{\epsilon}{\eta} \int_C \frac{dv}{\mathrm{i}\pi} \bar{V}(v) \frac{[v - u + \frac{1}{2} - \sqrt{2r(r-1)}\,\mathcal{P}]}{[v - u - \frac{1}{2}]}, \\ \tilde{x}(u, C) &= \frac{\epsilon}{\eta'} \int_C \frac{dv}{\mathrm{i}\pi} \bar{\tilde{V}}(v) \frac{[v - u - \frac{1}{2} + \sqrt{2r(r-1)}\,\mathcal{P}]'}{[v - u + \frac{1}{2}]'}. \end{aligned} \tag{4.6}$$

The constants η, η' will be fixed as

$$\begin{aligned} \eta^{-1} &= \mathrm{i}[1]\, x^{\frac{r-1}{2r}} \frac{(x^2; x^{2r})_\infty}{(x^{2r-2}; x^{2r})_\infty} \frac{(x^6; x^4, x^{2r})_\infty (x^{2r+2}; x^4, x^{2r})_\infty}{(x^4; x^4, x^{2r})_\infty (x^{2r+4}; x^4, x^{2r})_\infty}, \\ \eta'^{-1} &= -\frac{2\epsilon}{\pi}[1]' x^{-\frac{r}{2(r-1)}} \frac{(x^{2r-2}; x^{2r-2})^2_\infty}{(x^{2r}; x^{2r-2})^2_\infty} \frac{(x^4; x^4, x^{2r-2})_\infty (x^{2r+2}; x^4, x^{2r-2})_\infty}{(x^2; x^4, x^{2r-2})_\infty (x^{2r+4}; x^4, x^{2r-2})_\infty}. \end{aligned} \tag{4.7}$$

Now let us fix the contours. Let C_u^- and C_u^+ go from $u - \frac{\mathrm{i}\pi}{2\epsilon}$ to $u + \frac{\mathrm{i}\pi}{2\epsilon}$ to the left and to the right of u respectively.

(We assume that the contours $C_u^\pm$ go to the left of all poles in the 'main rectangle' related to the operators that are to the right of the screening operator and to the right of all poles related to the operators placed to the left of the screening operators. The 'main rectangle' is understood as a rectangle with sides r along the real axis and $\frac{\pi}{\epsilon}$ along the imaginary axis that contains all points u_i, v_i *etc.* It is well defined for large enough r and for points u_i, v_i,... close enough to each other. In the general case the operator products are considered as analytic continuation from this region.)

Then

$$\begin{aligned} X(u) &= x(u, C^-_{u+1/2}), \qquad & Y(u) &= x(u-1, C^+_{u-1/2}), \\ \tilde{X}(u) &= \tilde{x}(u, C^-_{u-1/2}), \qquad & \tilde{Y}(u) &= \tilde{x}(u+1, C^+_{u+1/2}). \end{aligned} \tag{4.8}$$

These operators satisfy the equations

$$Y(u)V(u)=V(u)X(u),\qquad \tilde Y(u)\tilde V(u)=\tilde V(u)\tilde X(u). \tag{4.9}$$

Define the Fock spaces $\mathcal{F}_{mn}$ generated by the operators a_{-k} $(k>0)$ from the highest weight vectors $|P_{mn}\rangle$ such that

$$a_k|P_{mn}\rangle=0\quad(k>0),\qquad \mathcal{P}|P_{mn}\rangle=P_{mn}|P_{mn}\rangle,\qquad P_{mn}=\frac{1}{\sqrt2}(\alpha_+m+\alpha_-n). \tag{4.10}$$

There are strong evidences that $\mathcal{F}_{mn}$ can be identified with $\mathcal{H}_{mn}$ for generic r.

The vertex operators are defined on $\mathcal{F}_{mn}$ as follows:

$$\begin{aligned}
\Phi(u)_n^{n+1}&=\frac{\mathrm{i}^{m-n}}{[n]}V(u),\\
\Phi(u)_n^{n-1}&=-\frac{\mathrm{i}^{m-n}}{[n]}V(u)X(u),\\
\Psi^*(u)_m^{m+1}&=\tilde V(u),\\
\Psi^*(u)_m^{m-1}&=(-1)^{m-n}\tilde Y(u)\tilde V(u).
\end{aligned} \tag{4.11}$$

The corner Hamiltonian H_{mn} is the restriction to $\mathcal{F}_{mn}$ of the operator

$$H=\frac{\mathcal{P}^2}{2}+\sum_{k=1}^{\infty}\frac{[\![2k]\!]_x[\![rk]\!]_x}{[\![k]\!]_x[\![(r-1)k]\!]_x}a_{-k}a_k. \tag{4.12}$$

Ooh! This is the end at last!

The operators H and $\Phi(u)_n^{n'}$ satisfy the necessary algebra of commutation relations. Besides, the operators $\Psi^*(u)_n^{n'}$ satisfy a similar algebra

$$\Psi^*(u)_m^{m'}x^{2vH_{mn}}=x^{2vH_{m'n}}\Psi^*(u-v)_m^{m'}, \tag{4.13}$$

$$\sum_{s'}\tilde W\left[\begin{matrix}m' & s\\ s' & m\end{matrix}\middle|u_1-u_2\right]\Psi^*(u_1)_{s'}^{m'}\Psi^*(u_2)_m^{s'}=\Psi^*(u_2)_s^{m'}\Psi^*(u_1)_m^s, \tag{4.14}$$

$$\Psi(u')_{m''}^{m'}\Psi^*(u)_m^{m''}=\frac{1}{\pi}\frac{\delta_m^{m'}}{u'-u}+O(1),\qquad \Psi(u)_m^{m'}=\frac{1}{[m]'}\Psi^*(u-1)_m^{m'}. \tag{4.15}$$

Here

$$\tilde W\left[\begin{matrix}m_4 & m_3\\ m_1 & m_2\end{matrix}\middle|u\right]=-\,W\left[\begin{matrix}m_4 & m_3\\ m_1 & m_2\end{matrix}\middle|u\right]\Bigg|_{r\to r-1}.$$

The operators $\Phi(u)$ are called the *type I* vertex operators, while $\Psi^*(u)$ are called the *type II* vertex operators. The difference between these operators is in their physical meaning. The type I operators are, as we already said, the half transfer matrices, while the type II vertex operators represent one-particle excitation states. Both types of operators satisfy the relation

$$\Phi(u_1)_n^{n'}\Psi^*(u_2)_m^{m'}=\tau(u_1-u_2)\Psi^*(u_2)_m^{m'}\Phi(u_1)_n^{n'},\qquad \tau(u)=\mathrm{i}\frac{\theta_1\left(\frac14-\frac u2;\frac{\mathrm{i}\pi}{2\epsilon}\right)}{\theta_1\left(\frac14+\frac u2;\frac{\mathrm{i}\pi}{2\epsilon}\right)}. \tag{4.16}$$

The function $\prod_{j=1}^{l}\tau(u-v_j)$ is the eigenvalue of the transfer matrix $T_{\mathrm{SOS}}(u)$ on the excited states of l particles. The functions $\tilde W\left[\begin{matrix}m_4 & m_3\\ m_1 & m_2\end{matrix}\middle|v_1-v_2\right]$ provide the scattering matrix of two excitations. A trace function with a product of $\Psi^*(v)$ and $\Psi(v)$ inserted represents a matrix element of a local operator described by Φs instead of its vacuum expectation value. Namely,

$$\left\langle \begin{matrix} mm'_1\ldots m'_{l'-1}m' \\ v'_1,\ldots,v'_{l'} \end{matrix} \middle| \mathcal{O}\begin{pmatrix} n_1\ldots n_{k-1} \\ n \qquad\quad n' \\ n'_1\ldots n'_{k-1} \end{pmatrix} \middle| \begin{matrix} mm_1\ldots m_{l-1}m' \\ v_1,\ldots,v_l \end{matrix} \right\rangle$$
$$= \frac{1}{\chi_m}\,\mathrm{Tr}_{\mathcal{F}_{mn}}\big(\Psi(v'_1)^m_{m'_1}\ldots\Psi(v'_{l'})^{m'_{l'-1}}_{m'}\Psi^*(v_l)^{m'}_{m_{l-1}}\ldots\Psi^*(v_1)^{m_1}_m$$
$$\times\,\Phi^*(u_1)^n_{n'_1}\ldots\Phi^*(u_k)^{n'_{k-1}}_{n'}\Phi(u_k)^{n'}_{n_{k-1}}\ldots\Phi(u_1)^{n_1}_n x^{4H}\big),$$

where $\mathcal{O}\begin{pmatrix} n_1\ldots n_{k-1} \\ n \qquad\quad n' \\ n'_1\ldots n'_{k-1} \end{pmatrix}$ is the operator corresponding to the picture in the last lecture. The corresponding vacuum expectation values are just the probabilities $P\begin{pmatrix} n_1\ldots n_{k-1} \\ n \qquad\quad n' \\ n'_1\ldots n'_{k-1} \end{pmatrix}$.

Now let us make some comments on the construction. Derivation of the construction above starts from the following observation. Consider first the commutation relation

$$\Phi(u_1)^{n+2}_{n+1}\Phi(u_2)^{n+1}_n = W\left[\begin{matrix} n+2 & n+1 \\ n+1 & n \end{matrix}\middle| u_1-u_2\right]\Phi(u_2)^{n+2}_{n+1}\Phi(u_1)^{n+1}_n.$$

Since

$$W\left[\begin{matrix} n+2 & n+1 \\ n+1 & n \end{matrix}\middle| u\right] = R_0(u) = z^{(r-1)/2r}\frac{g(z^{-1})}{g(z)}, \qquad z=x^{2u},$$

we can rewrite it as

$$(z_2/z_1)^{(r-1)/4r}g^{-1}(z_2/z_1)\Phi(u_1)^{n+2}_{n+1}\Phi(u_2)^{n+1}_n = (z_1/z_2)^{(r-1)/4r}g^{-1}(z_1/z_2)\Phi(u_2)^{n+2}_{n+1}\Phi(u_1)^{n+1}_n$$

This can be reproduced if $\Phi(u)^{n+1}_n \sim {:}e^{\mathrm{i}\varphi(u)}{:}$ with $\varphi(u)$ has the form (4.4). The pairs (a_n, a_{-n}) are supposed to form independent Heisenberg algebra, but the normalization from (4.1) is not supposed. It is known that

$${:}e^{\mathrm{i}\varphi_1}{:}{:}e^{\mathrm{i}\varphi_2}{:} = e^{-\langle 0|\varphi_1\varphi_2|0\rangle}{:}e^{\mathrm{i}\varphi_1+\mathrm{i}\varphi_2}{:},$$

where φ_1, φ_2 are any linear combinations of $\mathcal{P}$, $\mathcal{Q}$, a_k. It means that if

$$\langle 0|\varphi(u_1)\varphi(u_2)|0\rangle = -\log((z_2/z_1)^{-(r-1)/4r}g(z_2/z_1)),$$

we will be able to satisfy our equation.

Let us represent $\log g(z)$ in the form of a series in z. Namely, use the identity

$$\log(z;p_1,p_2)_\infty = \sum_{n_1,n_2=0}^\infty \log(1-zp_1^{n_1}p_2^{n_2}) = -\sum_{n_1,n_2=0}^\infty\sum_{m=1}^\infty \frac{z^m p_1^{mn_1}p_2^{mn_2}}{m} = -\sum_{m=1}^\infty \frac{z^m}{(1-p_1^m)(1-p_2^m)}.$$

Applying it to the definition of $g(z)$, we obtain

$$\log g(z) = -\sum_{m=1}^\infty \frac{(x^{2m}+x^{(2r+2)m}-x^{4m}-x^{2rm})z^m}{(1-x^{4m})(1-x^{2rm})}.$$

This reproduces the normalizations in (4.1).

Obtaining the other relations, e.g.

$$\Phi(u_1)^n_{n-1}\Phi(u_2)^{n-1}_n = (\text{something})\times\Phi(u_2)^n_{n-1}\Phi(u_1)^{n-1}_n + (\text{something})\times\Phi(u_2)^n_{n+1}\Phi(u_1)^{n+1}_n$$

Without the second term it could be reproduced by pure exponentials (but with a wrong coefficient!), but the second term spoils everything. From conformal field theory it is known that such commutation relations can by obtained by use of the *screening operators*, which are integral of exponentials. The particular form of the screening operator (4.6) was guessed after long attempts to use a simpler form without an elliptic function of the zero mode operator. The normalization constant η is extracted from the normalization property for the vertex operators.

The Ψ operators appeared as a natural generalizations of some operators in the conformal field theory. Their meaning as representatives of excited states was established by comparing with similar construction for the six-vertex model by Jimbo and Miwa and with Lukyanov's construction for form factors in quantum field theory.

Similar operators $\Psi^*_\varepsilon(u)$ must exist for the eight-vertex model. The algebra of these operators was established by Foda, Iohara, Jimbo, Miwa, and Yan in 1994 in the study of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}_2)$ [17]. Let us write down the result:

$$\sum_{\varepsilon'_1\varepsilon'_2} \tilde{R}(u_1-u_2)^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}\Psi^*_{\varepsilon'_1}(u_1)\Psi^*_{\varepsilon'_2}(u_2) = \Psi^*_{\varepsilon_2}(u_2)\Psi^*_{\varepsilon_1}(u_1), \tag{4.17}$$

$$\Psi_{\varepsilon_1}(u_1)\Phi_{\varepsilon_2}(u_2) = \tau(u_1-u_2)\Phi_{\varepsilon_2}(u_2)\Psi_{\varepsilon_1}(u_1), \tag{4.18}$$

$$\Psi_{\varepsilon_1}(u')\Psi^*_{\varepsilon_2}(u) = \frac{1}{\pi}\frac{\delta_{\varepsilon_1\varepsilon_2}}{u'-u} + O(1), \qquad \Psi_\varepsilon(u) = \Psi^*_{-\varepsilon}(u-1). \tag{4.19}$$

Here the $\tilde{R}(u)$ matrix is defined by the weights

$$\tilde{a}(u) = -a(u)|_{r\to r-1}, \quad \tilde{b}(u) = -b(u)|_{r\to r-1}, \quad \tilde{c}(u) = -c(u)|_{r\to r-1}, \quad \tilde{d}(u) = d(u)|_{r\to r-1} \tag{4.20}$$

and provides the S matrix of the eight-vertex model. This form of the S matrix was confirmed by Takebe by means of the Bethe ansatz [18].

The function $\tau(u)$ here is the same function as in the similar commutation relation (4.16) for the SOS model. It is not surprising, because we know that the spectra of transfer matrices of both models coincide.

How to calculate anything with these bosonic fields? Denote by Tr_* the trace over oscillator modes and by H^* the oscillator contribution to H_{mn}. Besides, let

$$\chi^* = \mathrm{Tr}_*(x^{4H^*}) = \frac{1}{(x^4;x^4)_\infty}.$$

Then, according to the Wick theorem,

$$\mathrm{Tr}_{\mathcal{F}_{mn}}(U_N(u_N)\dots U_1(u_1)) = \langle P_{mn}|U^0_N(u_N)\dots U^0_1(u_1)|P_{mn}\rangle\, \chi^* \prod_{i=1}^N c_i \prod_{i<j} g_{ij}(u_i-u_j)$$

with

$$\log c_i = \frac{1}{\chi^*}\,\mathrm{Tr}_*(\phi^+_i(0)\phi^-_i(0)x^{4H^*}),$$
$$\log g_{ij}(u) = \frac{1}{\chi^*}\,\mathrm{Tr}_*(\phi_i(0)\phi_j(u)x^{4H^*}), \qquad \phi_i(u) = \phi^+_i(u) + \phi^-_i(u).$$

The constants c_i and functions g_{ij} are expressed in terms of the infinite products $(z;p_1,\dots,p_k)_\infty$ defined above. The resulting form factors are expressed in terms of integrations of the products of these functions.

We know the vertex–face correspondence for the weights. Is it possible to relate somehow the vertex operator algebras? Surely, it is, but we need again a kind of 'physical reasoning'. You remember the relation

$$[m']\,\mathrm{Tr}_{\mathcal{H}^{(i)}}(x^{2H}) = \sum_{n\in 2\mathbb{Z}+m+i} \mathrm{Tr}_{\mathcal{H}_{mn}}\, x^{4H_{mn}}.$$

Analysis of the low temperature expansion really indicates the relation between ith ground state of the eight vertex model and the $n-m=i \pmod 2$ ground states of the SOS model. So, suppose that there exist the operators

$$T(u_0)_{mn} : \mathcal{H}_{mn}\to\mathcal{H}^{(i)}, \qquad T(u_0)^{mn} : \mathcal{H}^{(i)}\to\mathcal{H}_{mn}, \qquad i = n-m \pmod 2, \tag{4.21}$$

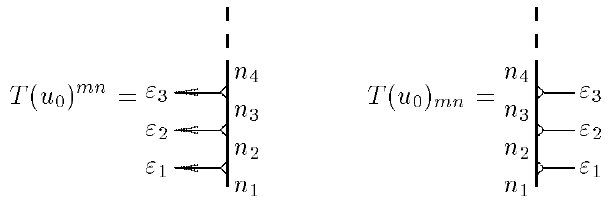
such that

$$[m]'x^{2H^{(i)}} = \sum_{n\in 2\mathbb{Z}+m+i} T(u_0)_{mn}x^{4H_{mn}}T(u_0)^{mn} \tag{4.22}$$

and

$$
\begin{aligned}
&\sum_n t_\varepsilon(u-u_0)_n^{n'}\Phi(u)_n^{n'}T(u_0)^{mn} = T(u_0)^{mn'}\Phi_\varepsilon(u) \\
&\sum_\varepsilon t^*_\varepsilon(u-u_0)^n_{n'}\Phi_\varepsilon(u)T(u_0)_{mn} = T(u_0)_{mn'}\Phi(u)^{n'}_n,
\end{aligned}
\tag{4.23}
$$

The operators $T(u_0)^{mn}$ and $T(u_0)_{mn}$ can be considered as a half of the $\tau(u_0)$ transfer matrix and its 'conjugate' $\tau^*(u_0)$ in the infinite volume limit:



How the operators $T(u_0)^{mn}$ and $T(u_0)_{mn}$ intertwine the type II operators? In the spirit of this algebraic approach, we can suggest that the commutation relation look like

$$
\begin{aligned}
\Psi^*_\varepsilon(u)T(u_0)_{mn} &= \sum_{m'} T(u_0)_{m'n}\Psi^*(u)^{m'}_m\tilde{t}^*_\varepsilon(u-u_0-\Delta u_0)^m_{m'}, \\
\Psi^*(u)^{m'}_m T(u_0)^{mn} &= \sum_\varepsilon T(u_0)^{m'n}\Psi^*_\varepsilon(u)\tilde{t}_\varepsilon(u-u_0-\Delta u_0)^{m'}_m,
\end{aligned}
\tag{4.24}
$$

where Δu_0 is some shift and $\tilde{t}_\varepsilon(u)^{m'}_m$ are intertwining functions after the substitution $r\to r-1$ and erasing $e^{\pm i\pi/4}$:

$$
\begin{aligned}
\tilde{t}_+(u)^{n'}_n &= \tilde{f}(u)\theta_3\left(\frac{(n'-n)u+n'}{2r'};\mathrm{i}\frac{\pi}{2\epsilon r'}\right), \\
\tilde{t}_-(u)^{n'}_n &= \tilde{f}(u)\theta_4\left(\frac{(n'-n)u+n'}{2r'};\mathrm{i}\frac{\pi}{2\epsilon r'}\right), \qquad r'=r-1.
\end{aligned}
\tag{4.25}
$$

The normalization conditions fix the overall factors $f(u)$ and $\tilde{f}(u)$ to be solutions of the equations

$$
\begin{aligned}
[u]f(u)f(u-1) = C &\equiv \frac{[0]_4^2}{2\theta_3(0;\mathrm{i}\pi/2\epsilon r)\theta_4(0;\mathrm{i}\pi/2\epsilon r)}, \\
[u]'\tilde{f}(u)\tilde{f}(u-1) = C' &\equiv \frac{[0]_4'^2}{2\theta_3(0;\mathrm{i}\pi/2\epsilon r')\theta_4(0;\mathrm{i}\pi/2\epsilon r')}.
\end{aligned}
$$

The function $f(u)$ is not essential for the answers since the type I vertex operators Φ are always accompanied by the 'conjugates' Φ^*. But the particular form of $\tilde{f}(u)$ is essential and will be fixed on the basis of the bosonization procedure as well as the value of the shift Δu_0.

Consider any form factor from the eight-vertex model

$$
\frac{1}{\chi^{(i)}}\mathrm{Tr}_{\mathcal{H}^{(i)}}(\Phi^*_{\varepsilon'_1}(u_1)\ldots\Phi^*_{\varepsilon'_k}(u_k)\Phi_{\varepsilon_k}(u_k)\ldots\Phi_{\varepsilon_1}(u_1)\Psi^*_{\alpha_l}(v_l)\ldots\Psi^*_{\alpha_1}(v_1)x^{2H^{(i)}})
$$

We can represent the operator $x^{2H^{(i)}}$ according to (4.22) and push $T(u_0)_{mn}$ to the left and $T(u_0)^{mn}$ to the right by use of the intertwining relations (4.23) and (4.24). We obtain an infinite liner combination of the traces

$$
\frac{[n]}{[m']'\chi^{(i)}}\mathrm{Tr}_{\mathcal{H}_{m'n}}(\Phi^*(u_1)^n_{n'_1}\ldots\Phi^*(u_k)^{n'_{k-1}}_{n''}\Phi(u_k)^{n''}_{n_{k-1}}\ldots\Phi(u_1)^{n_1}_{n'}\Lambda(u_0)^{m'n'}_{mn}\Psi(u_l)^m_{m_{l-1}}\ldots\Psi(u_1)^{m_1}_{m'}x^{4H_{m'n}})
$$

with

$$
\Lambda(u_0)^{m'n'}_{mn} = T(u_0)^{m'n'}T(u_0)_{mn} \;:\; \mathcal{H}_{mn}\to\mathcal{H}_{m'n'}. \tag{4.26}
$$

All ingredients in this trace are known from the free field representation except the operators $\Lambda(u_0)^{m'n'}_{mn}$. In the next lecture we fix the bosonic form of these operators using the commutation relations of this operator with the vertex operators.

5 Free field representation: eight-vertex model via SOS model

At the end of the last lecture we introduced the operator $\Lambda(u_0)_{mn}^{m'n'} = T(u_0)^{m'n'}T(u_0)_{mn}$. We said that, having a bosonization of the SOS model, this operator is the only thing to be bosonized. To obtain bosonization for the Λ operator let us consider its commutation relations with the vertex operators. From the commutation relations of the operators $T(u_0)^{mn}$ and $T(u_0)_{mn}$ we find

$$\Lambda(u_0)_{m\ s}^{m'n'}\Phi(u)_n^s = \sum_{s'} L\left[\begin{matrix} n' & s' \\ s & n \end{matrix}\middle| u-u_0\right]\Phi(u)_{s'}^{n'}\Lambda(u_0)_{m\ n}^{m's'}, \tag{5.1}$$

$$\Psi^*(u)_s^{m'}\Lambda(u_0)_{mn}^{s\ n'} = \sum_{s'} \Lambda(u_0)_{s'\ n}^{m'n'}\Psi^*(u)_m^{s'}\,\tilde{L}\left[\begin{matrix} m' & s \\ s' & m \end{matrix}\middle| u-u_0-\Delta u_0\right], \tag{5.2}$$

where

$$L\left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix}\middle| u\right] = \sum_\varepsilon t_\varepsilon^*(u)_{n_1}^{n_2} t_\varepsilon(u)_{n_3}^{n_4}, \qquad \tilde{L}\left[\begin{matrix} m_4 & m_3 \\ m_1 & m_2 \end{matrix}\middle| u\right] = \sum_\varepsilon \tilde{t}_\varepsilon^*(u)_{m_1}^{m_2}\tilde{t}_\varepsilon(u)_{m_3}^{m_4} = L\left[\begin{matrix} m_4 & m_3 \\ m_1 & m_2 \end{matrix}\middle| u\right]\Bigg|_{r\to r-1}. \tag{5.3}$$

Explicitly, we have

$$\begin{aligned} L\left[\begin{matrix} n' & n'\pm 1 \\ n & n\pm 1 \end{matrix}\middle| u\right] &= \frac{[u\pm\frac{1}{2}(n-n')][\frac{1}{2}(n+n')]}{[u][n]}, \\ L\left[\begin{matrix} n' & n'\mp 1 \\ n & n\pm 1 \end{matrix}\middle| u\right] &= \frac{[u\pm\frac{1}{2}(n+n')][\frac{1}{2}(n-n')]}{[u][n]} \end{aligned} \tag{5.4}$$

for $n'-n\in 2\mathbb{Z}$. Note, that the number of type I vertex operators and that of type II vertex operators in any meaningful trace is even. This means that considering the Λ operator for $n'-n\in 2\mathbb{Z}$ and $m'-m\in 2\mathbb{Z}$ is natural.

Evidently,

$$L\left[\begin{matrix} n & n'' \\ n & n' \end{matrix}\middle| u\right] = \delta_{n'n''}. \tag{5.5}$$

As the Λ operator is a 'half infinite product' of Ls, we easily conclude that

$$\Lambda(u)_{mn}^{m'n} = \delta_{m'm}. \tag{5.6}$$

Besides,

$$\Lambda(u_0)_{m\ n}^{m'n'} = 0 \quad \text{if} \quad m'<m,\ n'>n \text{ or } m'>m,\ n'<n. \tag{5.7}$$

Indeed, if, for example $m'\le m$, $n'>n$ for nonzero $\Lambda(u)_{mn}^{m'n'}$, there must be a point j at the paths where $n(j)=n'(j)$. But, due to (5.5), it means that $n(j+1)=n'(j+1)$ and, by induction $n(k)=n'(k)$ for any $k>j$. Therefore, $m'=m$.

Let us start derivation from the case $m'=m$. This case is sufficient for calculation of correlation functions. Consider the commutation relation (5.1) in the limit $u\to u_0$. In this limit $L(s,n;s',n|u-u_0)\to\infty$ so that

$$L(n\pm 1,n;n+1,n|u-u_0)/L(n\pm 1,n;n-1,n|u-u_0)\to 1.$$

We obtain the relation (we omit the indices m for simplicity)

$$\Phi(u)_{n'-1}^{n'}\Lambda(u)_n^{n'-1} = -\Phi(u)_{n'+1}^{n'}\Lambda(u)_n^{n'+1}$$

which amounts

$$V(u)\Lambda(u)_n^{n'-1} = \frac{[n'-1]}{[n'+1]}V(u)X(u)\Lambda(u)_n^{n'+1}.$$

We can conjecture that

$$\Lambda(u)_{m\ n}^{m\ n-2l} = \frac{[n-2l]}{[n]}X^l(u) \quad \text{for} \quad l\ge 0. \tag{5.8}$$

If we substitute this solution to the commutation relation (5.1) in its general form, we can make sure that this is indeed the solution to these commutation relations $n' \le n$.

But what to do in the case $n' > n$? Look at the weights of the SOS model. They are invariant with respect to the reflection

$$m \to -m, \qquad n \to -n.$$

It means that we can identify with $\mathcal{H}_{mn}$ not only $\mathcal{F}_{mn}$, but also $\mathcal{F}_{-m,-n}$. With this identification we have an alternate bosonization for the vertex operators of the SOS model:

$$\begin{aligned}
\Phi(u)_n^{n+1} &= \frac{\mathrm{i}^{n-m}}{[n]} V(u) X(u), \\
\Phi(u)_n^{n-1} &= -\frac{\mathrm{i}^{n-m}}{[n]} V(u), \\
\Psi^*(u)_m^{m+1} &= (-1)^{m-n} \tilde{Y}(u) \tilde{V}(u), \\
\Psi^*(u)_m^{m-1} &= \tilde{V}(u).
\end{aligned} \tag{5.9}$$

In this alternate bosonization we have

$$\Lambda(u)_{m\,n}^{m\,n+2l} = \frac{[n+2l]}{[n]} X^l(u) \quad \text{for} \quad l \ge 0. \tag{5.10}$$

We use different free field representation in different cases. This fact must not embarrass you, because the Λ operator enters the trace once. Hence, we simply need to use different bosonization for different traces.

This is sufficient for calculation of correlation functions. But the construction contains two free parameters u_0 and m. If there would be a rigorous proof of our construction, we would be sure that the answer is independent of these parameters. But from the mathematical point of view our construction is nothing but a conjecture. As a test we must check the u_0 and m independence. The u_0 independence can be proven from periodicity properties: it turns out that the correlation function constructed from this bosonization procedure must be periodic in u_0 with two periods, 2 and $2r$. As r is, generically, irrational, the answer is a constant. We have no proof of m independence, but the simplest examples of the one- and two-point functions demonstrate this independence. The answer for the one-point function is nothing but the famous Baxter's staggered spontaneous polarization:

$$\langle \sigma^z \rangle^{(i)} \equiv P^{(i)}{}^+_+ - P^{(i)}{}^-_- = (-)^i \frac{(x^2;x^2)^2_\infty (-x^{2r};x^{2r})^2_\infty}{(-x^2;x^2)^2_\infty (x^{2r};x^{2r})^2_\infty}.$$

Now turn our attention to the case $m' \ne m$. Due to the selection rule (5.7), we have to use the first bosonization in the case $n' \le n$, $m' \le m$ and the alternative bosonization in the case $n' \ge n$, $m' \ge m$. First of all, let us try to find $\Lambda(u)_{m\ \ n}^{m-2\ n-2}$. Take the commutation relation (5.2) for $s = m$, $m' = m-1$. In this case it becomes

$$\Psi^*(u)_m^{m-1} \Lambda(u_0)_{mn}^{mn'} = \Lambda(u_0)_{m-1\ n}^{m-1\ n'} \Psi^*(u)_m^{m-1} + \Lambda(u_0)_{m+1\ n}^{m-1\ n'} \Psi^*(u)_m^{m+1} \frac{[u-u_0-\Delta u_0 - m]'[1]'}{[u-u_0-\Delta u_0]'[m+1]'}.$$

For $n' = n-2$ we have

$$\frac{[n-2]}{[n]} \tilde{Y}(u) X(u_0) \tilde{V}(u) = \frac{[n-2]}{[n]} X(u_0) \tilde{Y}(u) \tilde{V}(u) + (-1)^{n-m} \Lambda(u_0)_{m-1\ n}^{m-1\ n-2} \tilde{V}(u) \frac{[u-u_0-\Delta u_0 - m]'[1]'}{[u-u_0-\Delta u_0]'[m+1]'}.$$

Erasing $\tilde{V}(u)$ we obtain

$$\Lambda(u)_{m\ \ n}^{m-2\ n-2} \tilde{V}(u_0) = (-)^{n-m} \frac{[u-u_0-\Delta u_0 - m+1]'[1]'}{[u-u_0-\Delta u_0]'[m]'} \frac{[n-2]}{[n]} [\tilde{Y}(u), X(u_0)] \tilde{V}(u). \tag{5.11}$$

What is the commutator $[\tilde{Y}(u), X(u_0)]$? If the contours of the commutator would not catch poles it would be zero. But it can be proven that this commutator is a simple combinations of exponentials:

$$[\tilde{Y}(u'), X(u)]|_{\mathcal{F}_{mn}} = \frac{\epsilon}{\eta\eta' \operatorname{sh} \epsilon} \left(\frac{[m-1]'}{\partial[0]'} \frac{[u'-u+\frac{1}{2}-n]}{[u'-u-\frac{1}{2}]} W_+(u') \right.$$

$$+\frac{[n-1]}{\partial[0]}\frac{[u'-u+\frac{3}{2}-m]'}{[u'-u+\frac{1}{2}]'}W_-(u)\Big). \tag{5.12}$$

Here $\partial[0]=d[u]/du|_{u=0}$, $\partial[0]'=d[u]'/du|_{u=0}$ and

$$W_+(u)=W(u+\tfrac{r}{2}),\qquad W_-(u)=W(u-\tfrac{r-1}{2}) \tag{5.13}$$

with

$$\begin{aligned}W(u)&=z^{1/r(r-1)}{:}e^{\mathrm{i}\varphi_0(z)}{:},\\ \varphi_0(z)&=-2\sqrt{2}\alpha_0(\mathcal{Q}-\mathrm{i}\mathcal{P}\log z)-\sum_{k\neq 0}\frac{[\![2k]\!]_x}{[\![(r-1)k]\!]_x}\frac{a_k}{\mathrm{i}k}z^{-k}.\end{aligned} \tag{5.14}$$

It can be checked that

$$W_+(u)\tilde{V}(u)=0.$$

Therefore the first term in the commutator does not affect the operator $\Lambda(u)$. But the second exponential W_- has the argument of the X screening operator. The difference $u-u_0$ drop out of (5.11) with these substitution, if we accept that

$$\Delta u_0=-1/2. \tag{5.15}$$

Finally, we obtain

$$\Lambda(u)_{m\ \ n}^{m-2\ n-2}=(-)^{n-m}\frac{\epsilon}{\eta\eta'\operatorname{sh}\epsilon}\frac{[m]'}{[1]'}\frac{[n-1][n-2]}{\partial[0]\,[n]}W_-(u). \tag{5.16}$$

We see that $\Lambda(u)_{m\ \ n}^{m-2\ n-2}$ is proportional to $W_-(u)$. If we return to the commutation relation $[\tilde{Y}(u'),X(u)]$ we shall see that at the point $u'=u-1/2$ the second term with $W_-(u)$ has a pole, while the first one remains finite. Therefore,

$$\lim_{u'\to u-1/2}[u'-u+1/2]'[\tilde{Y}(u'),X(u)]\sim W_-(u).$$

One can check that this pole is appears in the product $\tilde{Y}(u')X(u)$, while the product $X(u)\tilde{Y}(u')$ is regular at that point. It means that

$$\Lambda(u)_{m\ \ n}^{m-2\ n-2}\sim\lim_{u'\to u}[u'-u]'\tilde{Y}(u'-1/2)X(u).$$

We may conjecture that

$$\Lambda(u)_{m\ \ \ n}^{m-2k\ n-2l}\sim\lim_{u'\to u}[u'-u]'\tilde{Y}^k(u'-1/2)X^l(u)\quad\text{for } k,l>0. \tag{5.17}$$

It can be checked that the product in the right hand side has just a simple pole and that

$$\lim_{u'\to u}[u'-u]'\tilde{Y}^k(u'-1/2)X^l(u)\sim\tilde{Y}^{k-1}(u-1/2)W_-(u)X^{l-1}(u).$$

The coefficients in (5.17) can be obtained from the intertwining relation (5.2) in the limit $u\to u_0-1/2$ just in the same way as we obtained the coefficient in (5.8) using the equation

$$\Psi^*(u)_{m-2k}^{m-2k+1}\Lambda(u_0)_{m\ \ \ n}^{m-2k\ n-2l}=\Psi^*(u)_{m-2k+2}^{m-2k+1}\Lambda(u_0)_{m\ \ \ \ \ n}^{m-2k+2\ n-2l}+O(1)\quad\text{as}\quad u\to u_0-\tfrac{1}{2}$$

and one more equation that follows from (5.1) in the limit $u\to u_0$ (but written in the form $\Phi(u)\Lambda(u_0)=\ldots$).

Then we have to substitute the answer into the relations (5.1) and (5.2) in the general form and to check it. It is a very cumbersome calculation, but it was done. Now the answer is

$$\begin{aligned}\Lambda(u)_{m\ \ \ n}^{m-2k\ n-2l}\big|_{\mathcal{F}_{mn}}&=C_{m\ \ \ n}^{m-2k\ n-2l}\lim_{u'\to u}[u'-u]'\tilde{Y}^k(u'-\tfrac{1}{2})X^l(u)\\&=D_{m\ \ \ n}^{m-2k\ n-2l}\tilde{X}^{k-1}(u-\tfrac{1}{2})W_-(u)Y^{l-1}(u),\end{aligned} \tag{5.18}$$

$$\begin{aligned}\Lambda(u)_{m\ \ \ n}^{m+2k\ n+2l}\big|_{\mathcal{F}_{-m\,-n}}&=C_{m\ \ \ n}^{m+2k\ n+2l}\lim_{u'\to u}[u'-u]'\tilde{Y}^k(u'-\tfrac{1}{2})X^l(u)\\&=D_{m\ \ \ n}^{m+2k\ n+2l}\tilde{X}^{k-1}(u-\tfrac{1}{2})W_-(u)Y^{l-1}(u)\end{aligned} \tag{5.19}$$

for $k, l > 0$ with

$$\begin{aligned} D_{m\ \ n}^{m-2k\ n-2l} &= (-)^{(m-n+1)k+l+1} \frac{\epsilon}{\eta\eta' \,\mathrm{sh}\, \epsilon} \frac{[m]'}{[1]'} \frac{[l][n-l][n-2l]}{\partial[0]\,[1][n]}. \\ C_{m\ \ n}^{m-2k\ n-2l} &= (-)^{(m-n)k} \frac{[m]'}{[k]'[m-k]'} \frac{[n-2l]}{[n]}. \end{aligned} \tag{5.20}$$

The two-particle form factors of any of operators σ^a can be obtained without integrations. Any of these calculations (for example for σ^z) allows to fix the function $\tilde{f}(u)$ from the condition of u_0 independence of the answer. The explicit (and readable) answers for these quantities can be found in the paper by Lukyanov and Terras [19].

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