Lecture 1
Boson field on a cylinder and on a plane

Consider the cylinder with the coordinates ξ^0, ξ^1 with the circumference L:

$$\xi^1 \sim \xi^1 + L. \tag{1}$$

We assume the metrics $ds^2 = (d\xi^0)^2 - (d\xi^1)^2$. Also we will consider the Wick rotation given by $\xi^2 = \mathrm{i}\xi^0$. Define a scalar real boson field $\varphi(\xi)$ on the cylinder

$$S[\varphi] = \int d^2\xi \left(\frac{(\partial_\mu \varphi)^2}{16\pi} - U(\varphi) \right) = \int d\xi^0 \, \mathcal{L}[\varphi, \partial_0 \varphi]. \tag{2}$$

According to the usual rules the conjugate momentum field is

$$\rho(\xi) = \frac{\delta \mathcal{L}}{\delta \partial_0 \varphi(\xi)} = \frac{1}{8\pi} \partial_0 \varphi(\xi) \tag{3}$$

with the canonical bracket

$$\{\rho(\xi^0, \xi^1), \varphi(\xi^0, \xi'^1)\} = \delta(\xi^1 - \xi'^1), \qquad \{\rho(\xi^0, \xi^1), \rho(\xi^0, \xi'^1)\} = \{\varphi(\xi^0, \xi^1), \varphi(\xi^0, \xi'^1)\} = 0. \tag{4}$$

The Hamiltonian

$$H = \int d\xi^1 \, \rho \, \partial_0 \varphi - \mathcal{L} = \int d\xi^1 \left(4\pi\rho^2 + \frac{1}{16\pi} (\partial_1 \varphi)^2 + U(\varphi) \right). \tag{5}$$

On the cylinder it is convenient to use the Fourier expansion

$$\varphi(\xi^0, \xi^1) = \sum_{k \in \mathbb{Z}} q_k(\xi^0) \mathrm{e}^{2\pi \mathrm{i} k \xi^1 / L}, \qquad \rho(\xi^0, \xi^1) = \frac{1}{L} \sum_{k \in \mathbb{Z}} p_k(\xi^0) \mathrm{e}^{2\pi \mathrm{i} k \xi^1 / L}. \tag{6}$$

Since $\varphi(\xi), \rho(\xi) \in \mathbb{R}$, we have

$$q_k^* = q_{-k}, \qquad p_k^* = p_{-k}. \tag{7}$$

The Poisson bracket looks like

$$\{p_k, q_l\} = \delta_{kl}, \qquad \{p_k, p_l\} = \{q_k, q_l\} = 0. \tag{8}$$

The Hamiltonian reads

$$H = \frac{4\pi}{L} p_0^2 + \frac{1}{L} \sum_{k>0} \left(8\pi p_k p_{-k} + \frac{\pi}{2} k^2 q_k q_{-k} \right) + U[q]. \tag{9}$$

Define new variables

$$\begin{aligned} a_k &= \frac{\mathrm{i}k}{2} q_k - 2p_{-k}, \\ \bar{a}_k &= \frac{\mathrm{i}k}{2} q_{-k} - 2p_k. \end{aligned} \tag{10}$$

Their Poisson brackets read

$$\{a_k, a_l\} = \{\bar{a}_k, \bar{a}_l\} = 2\mathrm{i}k\delta_{k+l,0}, \qquad \{a_k, \bar{a}_l\} = 0. \tag{11}$$

In terms of the a-variables the fields read

$$\begin{aligned} \varphi(\xi) &= q_0(\xi^0) + \sum_{k \in \mathbb{Z}} \frac{a_k(\xi^0) - \bar{a}_{-k}(\xi^0)}{\mathrm{i}k} \mathrm{e}^{2\pi \mathrm{i} k \xi^1 / L}, \\ \rho(\xi) &= \frac{p_0(\xi^0)}{L} - \frac{1}{4L} \sum_{k \in \mathbb{Z}} \left(a_k(\xi^0) + \bar{a}_{-k}(\xi^0) \right) \mathrm{e}^{2\pi \mathrm{i} k \xi^1 / L}. \end{aligned} \tag{12}$$

The Hamiltonian is

$$H = \frac{4\pi}{L}p_0^2 + \frac{\pi}{L}\sum_{k>0}(a_{-k}a_k + \bar a_{-k}\bar a_k) + \int d\xi^1\, U(\varphi). \tag{13}$$

The equation of motion $\dot f \equiv \partial_0 f = \{H, f\}$ reduces to

$$\begin{aligned} \dot q_0 &= \frac{8\pi}{L}p_0, & \dot a_k &= -\mathrm{i}\frac{2\pi k}{L}a_k - 2F_k, \\ \dot p_0 &= F_0, & \dot{\bar a}_k &= -\mathrm{i}\frac{2\pi k}{L}\bar a_k + 2F_k, \end{aligned} \tag{14}$$

where F_k are components of the 'force':

$$F_k = -\int d\xi^1\, \frac{\partial U(\varphi)}{\partial\varphi}\mathrm{e}^{2\pi\mathrm{i}k\xi^1/L}.$$

Now I want to discuss a free massless boson: $U(\varphi) = 0$. In this case the equation of motion admit a simple solution:

$$p_0(\xi^0) = P, \qquad q_0(\xi^0) = Q + \frac{8\pi}{L}P\xi^0, \qquad a_k(\xi_0) = \alpha_k\mathrm{e}^{-2\pi\mathrm{i}k\xi^0/L}, \qquad \bar a_k(\xi_0) = \bar\alpha_k\mathrm{e}^{-2\pi\mathrm{i}k\xi^0/L}. \tag{15}$$

In terms of the basic Lagrangian field the solution looks like

$$\varphi(\xi) = Q + \frac{4\pi}{L}P(\bar\zeta - \zeta) + \sum_{k\neq 0}\left(\frac{\alpha_k}{\mathrm{i}k}\mathrm{e}^{2\pi\mathrm{i}k\zeta} + \frac{\bar\alpha_k}{\mathrm{i}k}\mathrm{e}^{-2\pi\mathrm{i}k\bar\zeta}\right), \tag{16}$$

where

$$\zeta = \xi^1 - \xi^0 = \xi^1 + \mathrm{i}\xi^2, \qquad \bar\zeta = \xi^1 + \xi^0 = \xi^1 - \mathrm{i}\xi^2$$

are light-cone (or holomorphic in the Euclidean space) variables.

As we expected the solution splits into right- and left-moving waves:

$$\varphi(\xi) = \varphi_R(\zeta) + \varphi_L(\zeta), \tag{17}$$

where

$$\begin{aligned} \varphi_R(\zeta) &= \frac{Q}{2} + \frac{4\pi}{L}P\zeta + \sum_{k\neq 0}\frac{\alpha_k}{\mathrm{i}k}\mathrm{e}^{2\pi\mathrm{i}k\zeta/L}, \\ \varphi_L(\zeta) &= \frac{Q}{2} - \frac{4\pi}{L}P\bar\zeta + \sum_{k\neq 0}\frac{\bar\alpha_k}{\mathrm{i}k}\mathrm{e}^{-2\pi\mathrm{i}k\bar\zeta/L}. \end{aligned} \tag{18}$$

There is an evident ambiguity in splitting Q since the constant contribution, which is neither right- nor left-moving wave. Besides the constantly increasing term in both chiral fields depends on the same constant p.

The action of the free field is invariant under the (pseudo)conformal transformation

$$\zeta = f(z), \qquad \bar\zeta = \bar f(\bar z). \tag{19}$$

The functions f and $\bar f$ are independent in the Minkowski space, while on the Euclidean plane they are related as

$$\bar f(z^*) = (f(z))^*.$$

Indeed, evidently

$$\int d^2\xi\, \frac{\partial\varphi}{\partial\xi^\mu}\frac{\partial\varphi}{\partial\xi_\mu} = -2\int d\zeta\, d\bar\zeta\, \frac{\partial\varphi}{\partial\zeta}\frac{\partial\varphi}{\partial\bar\zeta} = -2\int dz\, d\bar z\, f'(z)\bar f'(\bar z)\, \frac{\partial\varphi}{f'(z)\partial z}\frac{\partial\varphi}{\bar f'(\bar z)\partial\bar z} = \int d^2x\, \frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x_\mu}.$$

Consider the particular conformal transformation

$$\zeta = \mathrm{i}\frac{L}{2\pi}\log z, \qquad \bar\zeta = -\mathrm{i}\frac{L}{2\pi}\log\bar z \tag{20}$$

or

$$z = \mathrm{e}^{-2\pi\mathrm{i}\zeta/L}, \qquad \bar z = \mathrm{e}^{2\pi\mathrm{i}\bar\zeta/L}. \tag{21}$$

Evidently, this transformation only makes sense in the Euclidean field theory. It maps the Euclidean cylinder ξ^μ onto the Euclidean plane x^μ. In this plane the equations (16) provides the radial expansion:

$$\varphi(z) = Q - 2\mathrm{i}P\log(z\bar z) + \sum_{k\neq 0}\left(\frac{\alpha_k}{\mathrm{i}k}z^{-k} + \frac{\bar\alpha_k}{\mathrm{i}k}\bar z^{-k}\right). \tag{22}$$

The Hamiltonian H is proportional to the dilation operator on the plane:

$$H = \frac{2\pi}{L}D.$$

Now let us quantize the system. To do it let us use the standard rule $[f,g] = -\mathrm{i}\hbar\{f,g\}$, if f,g are linear in the basic oscillators. It means that

$$[\rho(\xi^0,\xi^1),\varphi(\xi^0,\xi'^1)] = -\mathrm{i}\hbar\delta(\xi^1-\xi'^1), \qquad [\rho(\xi^0,\xi^1),\rho(\xi^0,\xi'^1)] = [\varphi(\xi^0,\xi^1),\varphi(\xi^0,\xi'^1)] = 0 \tag{23}$$

in terms of fields, or

$$[p_k,q_l] = -\mathrm{i}\hbar\delta_{kl}, \qquad [p_k,p_l] = [q_k,q_l] = 0 \tag{24}$$

in terms of modes, or

$$[p_0,q_0] = -\mathrm{i}\hbar, \qquad [a_k,a_l] = [\bar a_k,\bar a_l] = 2k\hbar\delta_{k+l,0}, \qquad [a_k,\bar a_l] = 0 \tag{25}$$

in terms of a modes. By rescaling of the field we may set $\hbar = 1$. We will assume it throughout the lectures.

Now define the vacuum states $|p\rangle$ by the conditions

$$\alpha_k|p\rangle = \bar\alpha_k|p\rangle = 0 \quad (k>0), \qquad P|p\rangle = p|p\rangle. \tag{26}$$

We used the operators $P,Q,\alpha_k,\bar\alpha_k$ instead of $p_0,q_0,a_k,\bar a_k$ since the former are time-independent. The vacuums define the normal ordering product $:\cdots:$ according to the following rules: 1) it puts the operators α_k $(k>0)$ to the right of any operator α_{-k}; 2) it puts P to the right of Q.

We have to make a remark on the Hamiltonian. Let us define the Hamiltonian on the cylinder according to (5) with no normal ordering. It means that in (13) we must write symmetrized products $\frac{1}{2}(a_{-k}a_k + a_k a_{-k})$ instead of $a_{-k}a_k$. Then we have

$$H = {:}H{:} + \frac{\pi}{L}\sum_{k=1}^\infty\left(\frac12[a_k,a_{-k}] + \frac12[a_k,a_{-k}]\right) = {:}H{:} + 2\pi\sum_{k=0}^\infty\frac{k}{L} \overset{\mathrm{def}}{=} {:}H{:} + \left[-2\pi\frac{\partial}{\partial\varepsilon}\sum_{k=0}^\infty \mathrm{e}^{-\varepsilon k/L} + \mathrm{const}\cdot L\right]_{\varepsilon\to 0}$$
$$= {:}H{:} + \left[-2\pi\frac{\partial}{\partial\varepsilon}\frac{1}{1-\mathrm{e}^{-\varepsilon/L}} + \mathrm{const}\cdot L\right]_{\varepsilon\to0} = {:}H{:} + \left[\frac{2\pi L}{\varepsilon^2} - \frac{\pi}{6L} + \mathrm{const}\cdot L + O(\varepsilon)\right]_{\varepsilon\to0} = {:}H{:} - \frac{\pi}{6L}.$$

We conclude that

$$H = \frac{4\pi}{L}P^2 + \frac{\pi}{L}\sum_{k>0}(\alpha_{-k}\alpha_k + \bar\alpha_{-k}\bar\alpha_k) - \frac{\pi}{6L} = \frac{2\pi}{L}\left(D - \frac{1}{12}\right). \tag{27}$$

Here D is the dilation operator on the plane. Hence,

$$H|p\rangle = \frac{2\pi}{L}\left(2p^2 - \frac{1}{12}\right)|p\rangle. \tag{28}$$

The Fock space is spanned on the vectors

$$\alpha_{-k_1}\cdots\alpha_{-k_m}\bar\alpha_{-l_1}\cdots\bar\alpha_{-l_n}|p\rangle \tag{29}$$

The pair $(K,\bar K) = (\sum k_i, \sum l_j)$ is called the level of the state. The energy and momentum of such state are

$$E_{K\bar K}(p) = \frac{2\pi}{L}\left(2p^2 + K + \bar K - \frac{1}{12}\right), \qquad P_{K\bar K}(p) = \frac{2\pi}{L}(K - \bar K). \tag{30}$$

The true vacuum of the system corresponds to $p = 0$. Any vacuum $|p\rangle$ can be obtained from $|0\rangle$ by means of the q_0 operator:

$$P\mathrm{e}^{\mathrm{i}Qp}|0\rangle = \mathrm{e}^{\mathrm{i}Qp}(P + p)|0\rangle = p\mathrm{e}^{\mathrm{i}Qp}|0\rangle.$$

Hence, we may assume

$$|p\rangle = \mathrm{e}^{\mathrm{i}Qp}|0\rangle. \tag{31}$$

In what follows we will construct operators in terms of the field φ and, hence, we will need the pair correlation function $\langle\varphi(x')\varphi(x)\rangle$ (it is easier to calculate them on the plane). We have

$$\begin{aligned}
\langle\varphi(x')\varphi(x)\rangle &= \langle 0|\varphi(x')\varphi(x)|0\rangle \\
&= \langle 0|Q^2|0\rangle - 2\mathrm{i}\langle 0|(QP\log(z\bar{z}) + PQ\log(z'\bar{z}')|0\rangle + \sum_{k>0}\frac{1}{k^2}\left\langle 0\left|\alpha_k\alpha_{-k}\left(\frac{z}{z'}\right)^k + \bar{\alpha}_k\bar{\alpha}_{-k}\left(\frac{\bar{z}}{\bar{z}'}\right)^k\right|0\right\rangle \\
&= \langle Q^2\rangle - 2\log(z'\bar{z}') + 2\sum_{k>0}\frac{1}{k}\left(\left(\frac{z}{z'}\right)^k + \left(\frac{\bar{z}}{\bar{z}'}\right)^k\right) = \langle Q^2\rangle - 2\log(z'\bar{z}') - 2\log\left(1 - \frac{z}{z'}\right)\left(1 - \frac{\bar{z}}{\bar{z}'}\right) \\
&= \langle Q^2\rangle + 2\log\frac{1}{(z'-z)(\bar{z}'-\bar{z})}.
\end{aligned}$$

The term $\langle Q^2\rangle$ is strictly speaking infinite. Let us regularize it by substituting an arbitrary finite number, which we will write in the form

$$\langle Q^2\rangle = 2\log R^2.$$

Finally, we have

$$\langle\varphi(z')\varphi(z)\rangle = 2\log\frac{R^2}{(z'-z)(\bar{z}'-\bar{z})}. \tag{32}$$

If we consider the theory on a finite scrap of a plane, the number R will be a linear scale of the scrap. On the infinite plane the correlation function is translationally invariant, as expected. Due to the Wick theorem the pair correlation functions determines all other correlation functions

Problems

1. Generally the scalar boson with nonconstant potential $U(\varphi)$ is not conformally invariant. Nevertheless, in the case (**Liouville theory**)

$$U(\varphi) = \mu\mathrm{e}^{b\varphi}$$

the conformal invariance can be restored by a modification of the transformation of the field $\varphi(\xi)$. Find the modified transformation rule and, hence, prove the conformal invariance of the Liouville theory.

2. Suppose that the field $\varphi(\xi)$ is periodic itself, which means the equivalence

$$\varphi(\xi) \sim \varphi(\xi) + 2\pi R.$$

It can be taken into account by means of the 'winding' periodicity condition

$$\varphi(\xi^0, \xi^1 + L) = \varphi(\xi^0, \xi^1) + 2\pi R w, \qquad w \in \mathbb{Z}.$$

The integer w is called the winding number.

Rewrite the equations (12) and (16) with winding taken into account. Demonstrate that the periodicity of φ leads to quantization of the momentum zero mode p_0 and find its allowed eigenvalues.

3. ***T* duality.** Show that the Hamiltonian of the free massless boson is invariant under the transformation $R \to 4/R$ together with $w \leftrightarrow n$, where n is the momentum quantum number.