Lecture 3
Free massless fermion on the cylinder and on the plane

Consider now a free massless Majorana fermion $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ on the cylinder:

$$S[\psi] = \frac{\mathrm{i}}{2\pi}\int d^2\xi\,\bar\psi\gamma^\mu\partial_\mu\psi = \frac{\mathrm{i}}{\pi}\int d^2\xi(\psi_1\bar\partial\psi_1 - \psi_2\partial\psi_2). \tag{1}$$

Here, as usual, $\bar\psi = \psi^+\gamma^0$. The gamma-matrices should satisfy the conditions

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \qquad \gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^+. \tag{2}$$

We choose them purely imaginary:

$$\gamma^0 = \begin{pmatrix} & -\mathrm{i}\\ \mathrm{i} & \end{pmatrix} = \sigma^2, \qquad \gamma^1 = \begin{pmatrix} & \mathrm{i}\\ \mathrm{i} & \end{pmatrix} = \mathrm{i}\sigma^1, \qquad \gamma^3 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & \\ & -1\end{pmatrix} = \sigma^3. \tag{3}$$

The Majorana condition in this case is

$$\psi^* = \psi. \tag{4}$$

The variables $\psi_\alpha(\xi)$ should be considered Grassmann (anticommuting): $\psi_\alpha(\xi)\psi_\beta(\xi') = -\psi_\beta(\xi')\psi_\alpha(\xi)$. The Hamiltonian reads[1]

$$H = -\frac{\mathrm{i}}{2\pi}\int_0^L d\xi^1\,\psi\gamma^3\partial_1\psi = -\frac{\mathrm{i}}{2\pi}\int_0^L d\xi^1(\psi_1\partial_1\psi_1 - \psi_2\partial_1\psi_2) \tag{5}$$

with the Poisson bracket

$$\{\psi_\alpha(\xi^0,\xi^1),\psi_\beta(\xi^0,\xi'^1)\} = \mathrm{i}\pi\delta_{\alpha\beta}\delta(\xi^1-\xi'^1). \tag{6}$$

Note that due to anticommutativity of the field the Poisson bracket is symmetric. The equation of motion looks like

$$\partial_0\psi = -\sigma^3\partial_1\psi \tag{7}$$

or

$$\bar\partial\psi_1 = \partial\psi_2 = 0, \tag{8}$$

where, as we remember, $\partial = \frac12(\partial_1 - \partial_0) = \partial_\zeta$, $\bar\partial = \frac12(\partial_1+\partial_0) = \partial_{\bar\zeta}$. We see that $\psi_1 = \psi_1(\zeta)$ is a right-moving Majorana–Weyl fermion wave, while $\psi_2 = \psi_2(\bar\zeta)$ is a left-moving Majorana–Weyl fermion wave.

Let us expand the ψ field into modes. Before doing it we have to fix the periodicity condition. There are two possibilities:

- Ramond (R) condition: $\psi(\xi^0,\xi^1+L) = \psi(\xi^0,\xi^1)$;
- Neveu–Schwarz (NS) condition: $\psi(\xi^0,\xi^1+L) = -\psi(\xi^0,\xi^1)$.

We have

$$\begin{aligned}\psi_1(\xi) &= \sqrt{\frac{\pi}{L}}\sum_{k\in\mathbb{Z}+\frac{\delta}{2}} b_k(\xi^0)\mathrm{e}^{2\pi\mathrm{i}k\xi^1/L},\\ \psi_2(\xi) &= \sqrt{\frac{\pi}{L}}\sum_{k\in\mathbb{Z}+\frac{\delta}{2}} \bar b_k(\xi^0)\mathrm{e}^{-2\pi\mathrm{i}k\xi^1/L},\end{aligned} \tag{9}$$

Here

$$\begin{aligned}\delta &= 0 \quad \text{in the R case;}\\ \delta &= 1 \quad \text{in the NS case.}\end{aligned} \tag{10}$$

Evidently, due to the Majorana condition we have

$$b_{-k} = (b_k)^*, \qquad \bar b_{-k} = \left(\bar b_k\right)^*. \tag{11}$$

[1] Accurate derivation of (5) and (6) demands careful taking into account constraints, but we omit these subtleties.

The Hamiltonian and momentum are

$$H = \frac{2\pi}{L} \sum_{k\in\mathbb{Z}_{\geq 0}+\frac{\delta}{2}} k(b_{-k}b_k + \bar{b}_{-k}\bar{b}_k), \qquad P = \frac{2\pi}{L} \sum_{k\in\mathbb{Z}_{\geq 0}+\frac{\delta}{2}} k(b_{-k}b_k - \bar{b}_{-k}\bar{b}_k). \tag{12}$$

The Poisson bracket is

$$\{b_k, b_l\} = \{\bar{b}_k, \bar{b}_l\} = \mathrm{i}\delta_{k+l,0}, \qquad \{b_k, \bar{b}_l\} = 0. \tag{13}$$

It is easy to check that the equations of motion for the modes $b_k, \bar{b}_k$ in both cases read

$$\partial_0 b_k = -\mathrm{i}\frac{2\pi}{L} k b_k, \qquad \partial_0 \bar{b}_k = -\mathrm{i}\frac{2\pi}{L} k b_k. \tag{14}$$

and have the following solution

$$b_k^+ = \beta_k \mathrm{e}^{-2\pi \mathrm{i} k\xi^0/L}, \qquad b_k^- = \bar{\beta}_k \mathrm{e}^{2\pi \mathrm{i} k\xi^0/L}. \tag{15}$$

We finally have

$$\begin{aligned} \psi_1(\zeta) &= \sqrt{\frac{\pi}{L}} \sum_{k\in\mathbb{Z}+\frac{\delta}{2}} \beta_k \mathrm{e}^{2\pi \mathrm{i} k\zeta/L}, \\ \psi_2(\bar{\zeta}) &= \sqrt{\frac{\pi}{L}} \sum_{k\in\mathbb{Z}+\frac{\delta}{2}} \bar{\beta}_k \mathrm{e}^{-2\pi \mathrm{i} k\bar{\zeta}/L}. \end{aligned} \tag{16}$$

Now let us quantize the system. The Poisson bracket is substituted by the anticommutator:

$$[\psi_\alpha(\xi^0, \xi^1), \psi_\beta(\xi^0, \xi'^1)]_+ = \pi\delta_{\alpha\beta}\delta(x - x') \tag{17}$$

or

$$[\beta_k, \beta_l]_+ = [\bar{\beta}_k, \bar{\beta}_l]_+ = \delta_{k+l,0}, \qquad [\beta_k, \bar{\beta}_l]_+ = 0. \tag{18}$$

We changed to constant operators $\beta_k, \bar{\beta}_k$. Define the vacuums by the conditions $\beta_k|0\rangle_{\mathrm{NS}} = \bar{\beta}_k|0\rangle_{\mathrm{R}} = \beta_k|0\rangle_{\mathrm{R}} = \bar{\beta}_k|0\rangle_{\mathrm{NS}} = 0$ for $k >$. But there will be some difference between the NS and R vacuums, which we specify later.

Let us write the Hamiltonian in terms of β-modes. As in the boson case we consider the products of operators as symmetrized ones:

$$\beta_{-k}\beta_k \mapsto \frac{1}{2}(\beta_{-k}\beta_k - \beta_k\beta_{-k}) = \beta_{-k}\beta_k - \frac{1}{2}.$$

In the R sector we obtain for the vacuum energy

$$E_0^{\mathrm{R}} = -2\pi \sum_{k=0}^{\infty} \frac{k}{L} = \frac{\pi}{6L}.$$

In the R sector we have

$$\begin{aligned} E_0^{\mathrm{NS}} = -2\pi \sum_{k=0}^{\infty} \frac{k+1/2}{L} &= \left[2\pi \frac{\partial}{\partial\varepsilon} \sum_{k=0}^{\infty} \mathrm{e}^{-\varepsilon(k+1/2)/L} + \text{ const } \cdot L \right]_{\varepsilon\to 0} \\ &= \left[\frac{\partial}{\partial\varepsilon} \frac{\pi}{\operatorname{sh}\frac{\varepsilon}{2L}} + \text{ const } \cdot L \right]_{\varepsilon\to 0} = \left[-\frac{2\pi L}{\varepsilon^2} - \frac{\pi}{12L} + \text{ const } \cdot L \right]_{\varepsilon\to 0} = -\frac{\pi}{12L}. \end{aligned}$$

The Hamiltonians read

$$\begin{aligned} H^{\mathrm{R}} &= \frac{2\pi}{L} \sum_{k\in\mathbb{Z}_{>0}} k(\beta_{-k}\beta_k + \bar{\beta}_{-k}\bar{\beta}_k) + \frac{\pi}{6L}, \\ H^{\mathrm{NS}} &= \frac{2\pi}{L} \sum_{k\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} k(\beta_{-k}\beta_k + \bar{\beta}_{-k}\bar{\beta}_k) - \frac{\pi}{12L}. \end{aligned} \tag{19}$$

The expressions for the momentum remain unchanged.

Note that the energy of the NS vacuum is larger than that of the R vacuum:

$$E_0^{\rm R} - E_0^{\rm NS} = \frac{\pi}{4L}. \tag{20}$$

It means that the R vacuum is an excited state. Moreover, since $[b_0, H^{\rm R}] = [\bar b_0, H^{\rm R}] = 0$ and $b_0^2 = \bar b_0^2 = \frac{1}{2}$, if $|v\rangle_{\rm R}$ is any eigenvector, we immediately obtain a quadruplet

$$|v\rangle_{\rm R}, \quad \beta_0|v\rangle_{\rm R}, \quad \bar\beta_0|v\rangle_{\rm R}, \quad \beta_0\bar\beta_0|v\rangle_{\rm R}. \tag{21}$$

Nevertheless, this representation is reducible. It splits into two equivalent irreducible representation, one of which we will associate with the fermion. Introduce two fermion operators

$$c_0 = \frac{{\rm i}^{-1/2}\beta_0 + {\rm i}^{1/2}\bar\beta_0}{\sqrt{2}}, \qquad c_0^+ = \frac{{\rm i}^{1/2}\beta_0 + {\rm i}^{-1/2}\bar\beta_0}{\sqrt{2}}. \tag{22}$$

They are mutually conjugate and satisfy the standard fermion relations

$$[c_0, c_0^+]_+ = 1, \qquad c_0^2 = (c_0^+)^2 = 0. \tag{23}$$

It is natural to construct any representation from the c_0-vacuum $|u\rangle$: $c_0|u\rangle = 0$. The representation is evidently two-dimensional: ${\rm span}\{|u\rangle, c_0^+|u\rangle\}$. It is easy to construct two such vectors in the representation (21):

$$|u_1\rangle = c_0\beta_0|v\rangle_{\rm R}, \qquad |u_2\rangle = c_0|v\rangle_{\rm R}.$$

We may remove doubling by choosing any of these representations.

Finally, define the vacuum vector $|0\rangle_{\rm R}$ by the conditions

$$\beta_k|0\rangle_{\rm R} = \bar\beta_k|0\rangle_{\rm R} = 0 \quad (k > 0), \qquad c_0|0\rangle_{\rm R} = 0. \tag{24}$$

The second orthogonal vector is

$$|1\rangle_{\rm R} = c_0^+|0\rangle_{\rm R} \tag{25}$$

satisfy the 'dual' condition: $c_0^+|1\rangle_{\rm R} = 0$.

The NS vacuum is nondegenerate and corresponds to the lowest energy in both sectors. The condition

$$\beta_k|0\rangle_{\rm NS} = \bar\beta_k|0\rangle_{\rm NS} = 0 \quad (k > 0) \tag{26}$$

defines it uniquely.

To understand this picture better, let us make the transformation to the plane $z = {\rm e}^{-2\pi{\rm i}\zeta/L}$, $\bar z = {\rm e}^{2\pi{\rm i}\bar\zeta/L}$. The action (1) is consistent with a conformal transformation, if the spinor field transforms as

$$\psi_1(\zeta,\bar\zeta) \to (f'(\zeta))^{1/2}\psi_1(f(\zeta),\bar f(\bar\zeta)), \qquad \psi_2(\zeta,\bar\zeta) \to (\bar f'(\bar\zeta))^{1/2}\psi_2(f(\zeta),\bar f(\bar\zeta)). \tag{27}$$

Hence on the plane we have

$$\begin{aligned} \psi_1(z) &= \frac{{\rm i}^{1/2}}{\sqrt{2}}\Psi(z) = \frac{{\rm i}^{1/2}}{\sqrt{2}}\sum_{k\in\mathbb{Z}+\frac{\delta}{2}}\beta_k z^{-1/2-k}, \\ \psi_2(\bar z) &= \frac{{\rm i}^{-1/2}}{\sqrt{2}}\bar\Psi(\bar z) = \frac{{\rm i}^{-1/2}}{\sqrt{2}}\sum_{k\in\mathbb{Z}+\frac{\delta}{2}}\bar\beta_k \bar z^{-1/2-k}. \end{aligned} \tag{28}$$

Note that the periodicity conditions on the plane are opposite to those on the cylinder:

$$\begin{aligned} &\text{R sector:} \quad \psi({\rm e}^{2\pi{\rm i}}z, {\rm e}^{-2\pi{\rm i}}\bar z) = -\psi(z,\bar z), \\ &\text{NS sector:} \quad \psi({\rm e}^{2\pi{\rm i}}z, {\rm e}^{-2\pi{\rm i}}\bar z) = \psi(z,\bar z). \end{aligned} \tag{29}$$

In the NS sector the fermion is well-defined in the vicinity of the origin, while in the R sector there is a singularity. The singularity is described by a special operator depending on the R vacuum:

$$|0\rangle_{\rm R} = \sigma(0)|0\rangle_{\rm NS}, \qquad |1\rangle_{\rm R} = \mu(0)|0\rangle_{\rm NS}. \tag{30}$$

The operator $\sigma(x)$ is called *spin operator* or *order parameter*, while the operator $\mu(x)$ is called *dual spin operator* or *disorder parameter*.[2]

From (19) we may conclude that the Hamiltonian on the cylinder H is related to the dilation operator D on the plane:

$$H = \frac{2\pi}{L} D - \frac{\pi}{12L}. \tag{31}$$

The vacuum energy term is twice smaller than in the boson case. From (20) we conclude that

$$D\sigma(z) = \frac{1}{8}\sigma(z), \qquad D\mu(z) = \frac{1}{8}\mu(z), \tag{32}$$

which means that their scaling dimensions $d_\sigma = d_\mu = \frac{1}{8}$.

Problems

1. Compute the correlation functions:

1. $\langle \Psi(z')\Psi(z) \rangle \stackrel{\text{def}}{=} {}_{\text{NS}}\langle 0|\Psi(z')\Psi(z)|0\rangle_{\text{NS}}$;
2. $\langle \mu(\infty)\Psi(z)\sigma(0) \rangle \stackrel{\text{def}}{=} {}_{\text{R}}\langle 1|\Psi(z)|0\rangle_{\text{R}}$;
3. $\langle \sigma(\infty)\Psi(z')\Psi(z)\sigma(0) \rangle \stackrel{\text{def}}{=} {}_{\text{R}}\langle 0|\Psi(z')\Psi(z)|0\rangle_{\text{R}}$.

2. The matrix element ${}_{\text{R}}\langle 0|0\rangle_{\text{R}}$ defines a pair correlation function on the plane:

$${}_{\text{R}}\langle 0|0\rangle_{\text{R}} = \langle \sigma(\infty)\sigma(0) \rangle = \lim_{z,\bar{z}\to\infty} z^{1/16}\bar{z}^{1/16}\langle \sigma(z,\bar{z})\sigma(0) \rangle.$$

By means of a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

calculate the correlation function

$$\langle \sigma(x')\sigma(x) \rangle = {}_{\text{NS}}\langle 0|\sigma(x')\sigma(x)|0\rangle_{\text{NS}}.$$

3. By applying a Möbius transformation to the second matrix element of Problem 1 find the correlation function

$$\langle \mu(z_1)\Psi(z_2)\sigma(z_3) \rangle.$$

[2] Their names are related to their role in the Ising model as the operators corresponding to the spin variable in two dual representations of the Ising model. The parameter σ becomes nonzero at low temperature $T < T_c$, while μ is only nonzero at high temperatures $T > T_c$. The free massless fermion describes the Ising model at the critical point, where there is no principal difference between these two objects.