Lecture 12
Factorized S matrix

What is integrability? In classical mechanics there is an exhaustive criterion: a conservative nondegenerate Hamiltonian system with n degrees of freedom (q_i, p_i) is integrable, if it satisfies the hypotheses of the Liouville theorem, i.e. it possesses exactly n independent first integrals of motion I_i, which are in involution, and the Jacobian $\partial(I)/\partial(p) \neq 0$. In classical field theory there is no universal criterion, since the Liouville theorem cannot be generalized to infinite number of degrees of freedom. There are, nevertheless, partial criteria, like existence of an LA pair, bi-Hamiltonian structure etc. In quantum mechanics we expect that the system is integrable, if the underlying classical system is integrable, and the integrals of motion preserve in the quantum case. Nevertheless, there are many quantum systems that do not have classical counterpart. Probably, the most general integrable systems are those that can be described by Skyanin's separation of variables procedure. Nevertheless, there are also limitations for this construction. In the quantum field theory there is no conventional criterion. A partial (necessary, but not sufficient) criterion is the existence of an infinite number of integrals of motion.

Now we will see that an infinite set of integrals of motion impose serious restriction on a relativistic field theory, which seem to be sufficient for integrability.

Any relativistic system in two dimensions, either integrable or non-integrable, possesses at least two integrals of motion: energy and momentum. In two dimensions it is more convenient to consider the currents $I_{\pm 1} = E \pm P$, which possess Lorentz spin ± 1. Consider an in- or out-state $|\alpha_1\theta_1, \ldots, \alpha_N\theta_N\rangle_{\text{in/out}}$, which consists of N stable particles with internal states α_i and rapidities θ_i. Then it is evident that

$$I_{\pm 1}|\alpha_1\theta_1, \ldots, \alpha_N\theta_N\rangle_{\text{in/out}} = \sum_{i=1}^{N} m_{\alpha_i} \mathrm{e}^{\pm\theta_i} |\alpha_1\theta_1, \ldots, \alpha_N\theta_N\rangle_{\text{in/out}}. \tag{1}$$

For simplicity consider a model, which contains particles of the same mass and, which forms a multiplet of the symmetry group of the theory. For example, sine-Gordon model with $\beta^2 \geq 1/2$ satisfies this condition. Consider a scattering process. The eigenvalues of integrals of motion on the allowed out- and in-state must coincide. Hence,

$$\sum_{i=1}^{N'} \mathrm{e}^{\pm\theta'_i} = \sum_{i=1}^{N} \mathrm{e}^{\pm\theta_i}, \tag{2}$$

where variables with primes correspond to the out-state. Let $N = N' = 2$ (the $2 \to 2$ scattering). Consider the equality (2) as an equation on θ_1, θ_2 with given θ'_1, θ'_2. This equation has just two solutions: $\theta'_1 = \theta_1, \theta'_2 = \theta_2$ and $\theta'_1 = \theta_2, \theta'_2 = \theta_1$. We may assume the first solution, since the second one can be reduce to the first one by exchanging $\alpha'_1 \leftrightarrow \alpha'_2$.

Suppose that there are more integrals of motion I_s, where s is the spin. In the case of sine-Gordon/Thirring model there are no integrals of motion of even values of spin, and there is just one integral of motion for each odd value of spin. Integrals of motion are supposed to be *local*, i.e. have the form

$$I_s = \int dx^1 j_s^0 = \int dx^1 (j_s^{\bar{z}} - j_s^z). \tag{3}$$

Note that the spin of $j_s^{\bar{z}}$ is $s+1$, while of j_s^z is $s-1$. In the in-state particles are placed far from each other and the eigenvalue of I_s splits into a sum of one-particle eigenvalues. But the one-particle eigenvalue must be proportional to $\mathrm{e}^{s\theta}$. We have

$$I_s|\alpha_1\theta_1, \ldots, \alpha_N\theta_N\rangle_{\text{in/out}} = I_s^{(0)} \sum_{i=1}^{N} \mathrm{e}^{s\theta_i} |\alpha_1\theta_1, \ldots, \alpha_N\theta_N\rangle_{\text{in/out}}. \tag{4}$$

Then we have for the $N \to N'$ process

$$\sum_{i=1}^{N'} \mathrm{e}^{s\theta'_i} = \sum_{i=1}^{N} \mathrm{e}^{s\theta_i}. \tag{5}$$

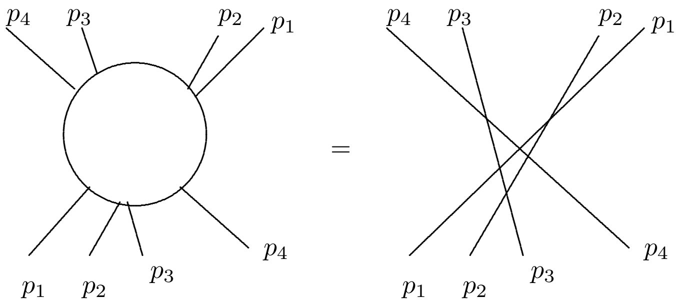
Again, consider this as a set of equations on $\{\theta_i\}$ for given $\{\theta'_i\}$. If $N' \geq N$ an infinite set of such equations has solutions for an N-dimensional region of values of θ'_i, if $N = N'$. The solutions are evident:

$$\theta_i = \theta'_{\sigma_i}, \qquad \sigma \in S^N. \tag{6}$$

Surely, for more general models we need more accurate consideration, but in general we may assume that a model is integrable, if it has an infinite number of commuting integrals of motion such that they do not allow arbitrary multiple particle production and uniquely define the momenta of ingoing particles by outgoing and vice versa. For relativistic systems it usually means that the numbers of particles of each mass conserve after scattering, and the momenta of particles remain unchanged.

Assumption of factorized scattering. *The amplitude of scattering of N particles into N particles factorizes in a product of pairwise scatterings of particles in any order with summing over intermediate states.*

Graphically we may represent it as follows:



(7)

In principle, the factorized scattering assumption can be checked order-by-order in perturbation theory. But we substantiate it in other way. Suppose that the interaction of particles is short-range in the following meaning. The stationary wave function of N particles (we will consider bosons) becomes nearly equal to a combination of products of plane wave if all particles are distant enough from each other: $|x_i - x_j| \gg R$ $(\forall i, j)$ with some interaction radius R. This wave function reads:

$$\psi_{\beta_1 p_1,\dots,\beta_n p_n}(\alpha_1 x_1,\dots,\alpha_n x_n) = \sum_{\tau\in S_n} A^{\alpha_{\sigma_1}\dots\alpha_{\sigma_n}}_{\beta_1\ \ \dots\beta_n}[\tau] e^{i\sum_{i=1}^n p_{\tau_i} x_{\sigma_i}},$$
$$\text{if}\quad x_{\sigma_1} < x_{\sigma_2} < \dots < x_{\sigma_n}, \quad |x_i - x_j| \gg R. \qquad (8)$$

It is easy to check that the function ψ is antisymmetric with respect to permutations of pairs (α_i, x_i). The parameters β_i numerate somehow internal states of particles.

Transposition of two particles corresponds to their scattering:

$$A^{\alpha_1\dots\alpha_{i+1}\alpha_i\dots\alpha_n}_{\beta_1\dots\beta_i\,\beta_{i+1}\dots\beta_n}[\tau s^i] = \sum_{\alpha'_i\alpha'_{i+1}} S(p_{\tau_i}, p_{\tau_{i+1}})^{\alpha_i\alpha_{i+1}}_{\alpha'_i\alpha'_{i+1}} A^{\alpha_1\dots\alpha'_i\alpha'_{i+1}\dots\alpha_n}_{\beta_1\dots\beta_i\,\beta_{i+1}\dots\beta_n}[\tau]. \qquad (9)$$

Here s^i is the permutation of two subsequent indexes and $S(p', p)^{\alpha_1\alpha_2}_{\alpha'_1\alpha'_2}$ is the two-particle scattering matrix. There are two equations the S matrix, which follow from this definition.

First, apply the equation (9) twice:

$$\begin{aligned} A^{\alpha_1\dots\alpha_{i+1}\alpha_i\dots\alpha_n}_{\beta_1\dots\beta_i\,\beta_{i+1}\dots\beta_n}[\tau] &= \sum_{\alpha'_i\alpha'_{i+1}} S(p_{\tau_i}, p_{\tau_{i+1}})^{\alpha_i\alpha_{i+1}}_{\alpha'_i\alpha'_{i+1}} A^{\alpha_1\dots\alpha'_i\alpha'_{i+1}\dots\alpha_n}_{\beta_1\dots\beta_i\,\beta_{i+1}\dots\beta_n}[\tau s^i] \\ &= \sum_{\alpha'_i\alpha'_{i+1}\alpha''_i\alpha''_{i+1}} S(p_{\tau_i}, p_{\tau_{i+1}})^{\alpha_i\alpha_{i+1}}_{\alpha'_i\alpha'_{i+1}} S(p_{\tau_{i+1}}, p_{\tau_i})^{\alpha'_{i+1}\alpha'_i}_{\alpha''_{i+1}\alpha''_i} A^{\alpha_1\dots\alpha''_{i+1}\alpha'_i\dots\alpha_n}_{\beta_1\dots\beta_i\,\beta_{i+1}\dots\beta_n}[\tau]. \end{aligned}$$

Hence, the S matrix must satisfy the **unitarity property**:

$$\sum_{\gamma_1\gamma_2} S(p_1, p_2)^{\beta_1\beta_2}_{\gamma_1\gamma_2} S(p_2, p_1)^{\gamma_2\gamma_1}_{\beta_2\beta_1} = \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2}. \qquad (10)$$

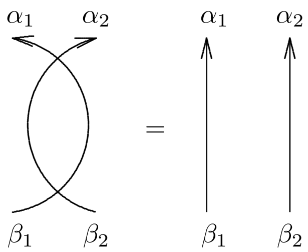
Note that the name 'unitarity' is lame: it does not mean the unitarity in the physical sense. It is simply a consistency condition, which must be satisfied even for a nonunitary theory.

Let us rewrite equation (10) in an indexless form. To do it let us assume that the matrix $S_{ij}(p, p')$ acts on the tensor product $V_1 \otimes V_2 \otimes \dots \otimes V_N$ of internal spaces of N particles as follows. It acts as the unit operator on V_k, $k \neq i, j$ and as $S(p, p')$ on the tensor product $V_i \otimes V_j$. Note that $S_{ij}(p, p')$ is generally not the same as $S_{ji}(p, p')$. If we introduce the transposition operator P, we may say $S_{ij}(p, p') = P_{ij} S_{ji}(p, p') P_{ij}$. Besides, below we will need the charge conjugation operator C, which will be denoted as C_i, if acting on V_i.

With this notation the unitarity condition reads

$$S_{12}(p_1, p_2) S_{21}(p_2, p_1) = 1. \tag{11}$$

It also can be expressed graphically:



(12)

Another equation is obtained from the following argument. Suppose that we want to change the order of the particles, e.g. $123 \to 321$. We can do it in two ways:

$$\begin{array}{ccccc} & & 132 \;\to\; 312 & & \\ & \nearrow & & \searrow & \\ 123 & & & & 321. \\ & \searrow & & \nearrow & \\ & & 213 \;\to\; 231 & & \end{array}$$

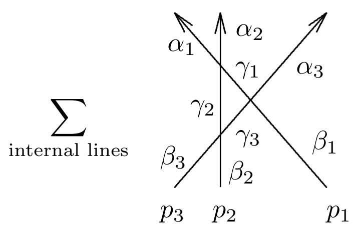
The first way leads to the equation

$$A^{\alpha_3\alpha_2\alpha_1\cdots}_{\cdots}[321\ldots] = \sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}(p_1, p_2) S^{\gamma_1\alpha_3}_{\beta_1\gamma_3}(p_1, p_3) S^{\gamma_2\gamma_3}_{\beta_2\beta_3}(p_2, p_3) A^{\beta_1\beta_2\beta_3\cdots}_{\cdots}[123\ldots], \tag{13}$$

or, in the indexless form,

$$A_{321\ldots} = S_{12}(p_1, p_2) S_{13}(p_1, p_3) S_{23}(p_2, p_3) A_{123\ldots}. \tag{14}$$

It can be presented in a graphic form:



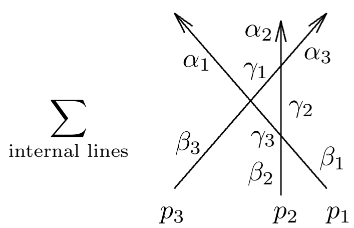
The second way leads to

$$A^{\alpha_3\alpha_2\alpha_1\cdots}_{\cdots}[321\ldots] = \sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_2\alpha_3}_{\gamma_2\gamma_3}(p_2, p_3) S^{\alpha_1\gamma_3}_{\gamma_1\beta_3}(p_1, p_3) S^{\gamma_1\gamma_2}_{\beta_1\beta_2}(p_1, p_2) A^{\beta_1\beta_2\beta_3\cdots}_{\cdots}[123\ldots], \tag{15}$$

or

$$A_{321\ldots} = S_{23}(p_2, p_3) S_{13}(p_1, p_3) S_{12}(p_1, p_2) A_{123\ldots}, \tag{16}$$

or graphically



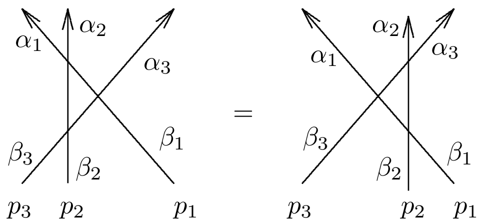
The condition that both way lead to the same result is called **Yang–Baxter equation**, which reads

$$\sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}(p_1,p_2)S^{\gamma_1\alpha_3}_{\beta_1\gamma_3}(p_1,p_3)S^{\gamma_2\gamma_3}_{\beta_2\beta_3}(p_2,p_3) = \sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_2\alpha_3}_{\gamma_2\gamma_3}(p_2,p_3)S^{\alpha_1\gamma_3}_{\gamma_1\beta_3}(p_1,p_3)S^{\gamma_1\gamma_2}_{\beta_1\beta_2}(p_1,p_2) \qquad (17)$$

or

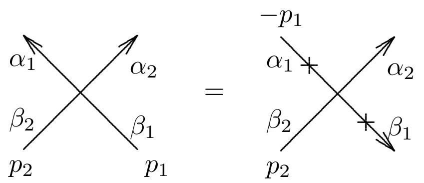
$$S_{12}(p_1,p_2)S_{13}(p_1,p_3)S_{23}(p_2,p_3) = S_{23}(p_2,p_3)S_{13}(p_1,p_3)S_{12}(p_1,p_2) \qquad (18)$$

or



(19)

There is one more equation, which follows from relativistic invariance and is well-known in the perturbation theory, the **crossing symmetry**. It is better to write it from the very beginning in the graphical form:



(20)

where crosses mean the charge conjugation C. Explicitly, we may write

$$S^{\alpha_1\alpha_2}_{\beta_1\beta_2}(p_1,p_2) = \sum_{\gamma_1\delta_1} C_{\beta_1\delta_1} S^{\alpha_2\delta_1}_{\beta_2\gamma_1}(p_2,-p_1)C^{\gamma_1\alpha_1} \qquad (21)$$

or

$$S_{12}(p_1,p_2) = C_2^{-1}S_{21}(p_2,-p_1)C_2. \qquad (22)$$

In the relativistic case it is convenient to use the rapidities θ_i, so that $S(p_1,p_2) = S(\theta_1-\theta_2)$. Then these properties read:

1. **Yang–Baxter equation**

$$S_{12}(\theta_1-\theta_2)S_{13}(\theta_1-\theta_3)S_{23}(\theta_2-\theta_3) = S_{23}(\theta_2-\theta_3)S_{13}(\theta_1-\theta_3)S_{12}(\theta_1-\theta_2). \qquad (23)$$

2. **Unitarity**

$$S_{12}(\theta)S_{21}(-\theta) = 1. \qquad (24)$$

3. **Crossing symmetry**

$$S_{12}(\theta) = C_2^{-1}S_{21}(i\pi-\theta)C_2. \qquad (25)$$

The *bootstrap equations* (23)–(25) are very restrictive. In many cases they make it possible to find the S matrix, maybe up to some parameters. The variable $\theta = \theta_1 - \theta_2$ is related to the Mandelstam variable s according to

$$s = m_1^2 + m_2^2 + 2m_1m_2 \operatorname{ch}\theta. \qquad (26)$$

It is easy to check that the physical sheet is

$$0 \le \operatorname{Im}\theta < \pi. \qquad (27)$$

The point $\theta = \mathrm{i}\pi$ corresponds to the branching point $s = (m_1-m_2)^2$, while the point $\theta = 0$ corresponds to the branching point $s = (m_1+m_2)^2$. The real axis and the line $\operatorname{Im}\theta = \pi$ correspond to the right and left cuts in the s plane correspondingly. On the imaginary axis $\theta = \mathrm{i}u$ the S matrix is real, and a pole on the segment $u \in (0,\pi)$ corresponds to a bound state, if its residue in u is positive.

In the case of sine-Gordon model the basic particles are soliton and antisoliton, which are charge and $U(1)$ symmetric. The only admissible form is

$$S(\theta)=\begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{pmatrix} a(\theta) & & & \\ & b(\theta) & c(\theta) & \\ & c(\theta) & b(\theta) & \\ & & & a(\theta) \end{pmatrix} \end{array} \begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array} \tag{28}$$

Here $a(\theta), b(\theta), c(\theta)$ are three analytic functions. By substituting this form into the Yang–Baxter equation (23) we obtain

$$b(\theta)=\frac{\operatorname{sh}\frac{\theta}{p}}{\operatorname{sh}\frac{\lambda-\theta}{\pi}}a(\theta), \qquad c(\theta)=\frac{\operatorname{sh}\frac{\lambda}{p}}{\operatorname{sh}\frac{\lambda-\theta}{\pi}}a(\theta). \tag{29}$$

Consistency with the crossing symmetry

$$a(\theta)=b(\mathrm{i}\pi-\theta), \qquad c(\theta)=c(\mathrm{i}\pi-\theta)$$

requires

$$\lambda=\mathrm{i}\pi \tag{30}$$

and

$$\frac{\operatorname{sh}\frac{\theta}{p}}{\operatorname{sh}\frac{\lambda-\theta}{\pi}}a(\theta)=a(\mathrm{i}\pi-\theta). \tag{31}$$

The unitarity condition reduces to

$$a(\theta)a(-\theta)=1. \tag{32}$$

Solution to the equations (31), (32) is not unique. Nevertheless, we may obtain a unique solution by imposing the following conditions

1. The solution is a meromorphic function of θ on the plane and of p on the half-plane $\operatorname{Re} p>0$.
2. There are no poles of the S matrix on the physical sheet except the segment $(0,\mathrm{i}\pi)$ of the imaginary axis, and there is no poles at all for $p\geq 1$.
3. $a(0)=-1$, which reflects the fact known from the classical theory that two solitons cannot have the same momentum. In this case $S(0)=-P$ and it can be shown that the wave function (8) vanishes.

This only solution is

$$a(\theta)=-\exp\left(-\int\frac{dt}{t}\frac{\operatorname{sh}\frac{\pi t}{2}\operatorname{sh}\frac{\pi(p-1)t}{2}}{\operatorname{sh}\pi t\operatorname{sh}\frac{\pi p t}{2}}\mathrm{e}^{-\mathrm{i}\theta t}\right)=\exp\left(2\mathrm{i}\int\frac{dt}{t}\frac{\operatorname{sh}\frac{\pi t}{2}\operatorname{sh}\frac{\pi(p-1)t}{2}}{\operatorname{sh}\pi t\operatorname{sh}\frac{\pi p t}{2}}\sin\theta t\right), \tag{33}$$

which differs from the function $S(\theta)$ of the Lecture 11 by the sign. It mean that the S matrix defined by (28)–(30), (33) is the S matrix of *bosons* corresponding to the Thirring fermions. In one spacial dimension there is no physical difference between bosons and fermions. But it does not mean that the interaction between fermions and between bosons must be the same. For example, the free Majorana fermion can be described as a system of interacting bosons, generated by the order (or disorder) parameter. The S matrices of bosons and fermions differ by the overall sign.

Bibliography

[1] A. Zamolodchikov and Al. Zamolodchikov, Factorized S Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models, *Annals Phys.* **120** (1979) 253.