MACROSCOPIC FLUCTUATION EFFECTS IN CONDENSED MATTER

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The role of long-range thermal fluctuations in the condensed matter physics is considered. As is known the most impressive effects related to fluctuations are observed near second order phase transitions. We give a theory of these transitions starting from the Landau expansion in the order parameter ψ. As an introduction we consider the mean field theory, then we take into account fluctuations the role of which can be examined in the framework of the perturbation theory formulated on the diagrammatic language. The ψ^4-model is examined in $d = 4$ using parquet summation methods, then the ϵ-expansion is developed enabling one to estimate the values of the critical indices for $d = 3$. The sheme can be generalized for the critical dynamics. The peculiarities of a weak crystallization transition where fluctuations qualitatively change the nature of the phase transition in comparison with the mean field picture are treated on the same diagrammatic language. The theoretical approach based on the Landau expansion is utilized to examine thermal fluctuation effects far from phase transition points. We consider the long-scale properties of smectics where fluctuations destroy the long-range order. Smectics are treated in the framework of the renorm-group approach. The same renorm-group technique is developed also for $2d$ ferromagnets where the effective coupling constant increases with increasing scale what drastically change long-scale properties of the system. Long-range fluctuations are also relevant for membranes which are two-dimensional objects immersed into a three-dimensional fluid. Elastic modules of a membrane are logarithmically renormalized, the renormalization law can be found by using renorm-group methods. Of special interest is Berezinskii-Kosterlitz-Thouless phase transition in superfluid, crystal or hexatic films which is related to appearing free point defects (vortices, dislocations or disclinations). The problem can be mapped into sine-Gordon model and then examined by renorm-group methods. We treat also dynamics of systems with strong fluctuations. We present some facts concerning critical dynamics and the so-called KPZ (Kardar-Parisi-Zhang) problem. Then we consider peculiarities of the 2d hydrodynamics.

Contents

1. MEAN FIELD THEORY

We will treat macroscopic processes which are described by quantities defined on scales $r \gg \Lambda^{-1}$ where Λ is a cutoff (Λ^{-1} is of the order of a molecular size). We will be interested mainly in effects associated with fluctuations which are variations with time of variables of a system. Near second order phase transitions and critical points fluctuations of a quantity φ, which is called the order parameter, are relevant. The order parameter can have different physical nature. Note, as examples, magnetization (near a ferromagnetic transition), Bose-condensate of atoms (near a superfluid transition) or of electronic pairs (near a superconductor transition), displacement of sublattices (near a ferroelectric transition), density of a fluid (near a critical point) etc. In the examples the number n of components of the order parameter varies from one to three. Namely, $n = 1$ for an uniaxial ferroelectric transition, $n = 2$ for a superfluid or for a superconductor phase transition, $n = 3$ for a ferromagnetic phase transition.

The order parameter is a macroscopic quantity, that is it is defined on scales, larger than the molecular size. Formally, it can be written in terms of the Fourier series

$$\varphi(\boldsymbol{r}) = \sum_q \varphi_q \exp(i\boldsymbol{q}\boldsymbol{r})\,, \tag{1.1}$$

where $\mid \boldsymbol{q} \mid < \Lambda$. For a given volume of a system the number of φ_q in the series (1.1) is large but finite. Both, $\varphi(\boldsymbol{r})$ and φ_q in Eq. (1.1) are functions of time t. All physical effects, related to the order parameter, can be expressed in terms of correlation functions of $\varphi(t, \boldsymbol{r})$. It is the main object of our consideration.

It is well known that simultaneous correlation functions of any equilibrium system can be described in terms of the Gibbs distribution. The microscopic Gibbs' partition function is

$$\exp\left(\frac{F - \hat{\mathcal{H}}}{T}\right), \tag{1.2}$$

where $\hat{\mathcal{H}}$ is the Hamiltonian of a system, F is its free energy and T is the temperature. Summing the partition function (1.2) over microscopic degrees of freedom at a given value of the order parameter $\varphi(\boldsymbol{r})$ we come to the macroscopic partition function

$$\mathcal{P} = \exp\left(\frac{F - \mathcal{F}_L}{T}\right), \tag{1.3}$$

where $\mathcal{F}_L$ is a functional of φ which is called the Landau functional. The partition function (1.3) determines simultaneous correlation functions of φ. For example, the average value $\langle\varphi\rangle$ of φ is

$$\langle\varphi\rangle = \int \mathcal{D}\varphi \, \exp\left(\frac{F - \mathcal{F}_L}{T}\right)\varphi\,. \tag{1.4}$$

Here $\int \mathcal{D}\varphi$ designates the functional (path) integration which can be treated as a multiple integral $\prod_q \int d\varphi_q$ over coefficients φ_q of the expansion (1.1). The normalization condition for the partition function (1.3) reads

$$\exp(-F/T) = \int \mathcal{D}\varphi \, \exp(-\mathcal{F}_L/T)\,. \tag{1.5}$$

The relation gives the principal method for calculating F.

Generally, $\mathcal{F}_L$ is the sum $\mathcal{F}_{reg} + \mathcal{F}_{add}$ where $\mathcal{F}_{reg}$ is a regular function of T independent of φ and $\mathcal{F}_{add}$ is an analytical function of φ. We are interested in the situation where the value of φ is small and we may expand $\mathcal{F}_{add}$ in φ. Let us suppose an absence of odd terms in the expansion what is characteristic of the second order phase transitions. This property is usually associated with symmetry reasons. Then the first term of the expansion is

$$\mathcal{F}_a = \int d\boldsymbol{r} \, \frac{a}{2} \, \varphi^2\,, \tag{1.6}$$

and the next term is

$$\mathcal{F}_\lambda = \int d\boldsymbol{r} \, \frac{\lambda}{24} \, \varphi^4\,. \tag{1.7}$$

This term (1.7) is relevant only if a is small. Just this case is realized near a second order phase transition, since the coefficient a is equal to zero at the transition temperature T_c. Near T_c the factor a can be expanded in $T - T_c$, the main term of the expansion is

$$a = \alpha(T - T_c)\,. \tag{1.8}$$

The expression (1.8) is correct if $\mid \tau \mid \ll 1$, where

$$\tau = (T - T_c)/T_c\,, \tag{1.9}$$

the quantity τ is usually called the reduced temperature. We see that a is positive above the transition temperature T_c and is negative below T_c. If the number n of the components of the order parameter differs from unity then φ^2 in (1.6,1.7) is $\varphi^2 = \varphi_1^2 + \varphi_2^2 + \dots$.

Consider the so-called mean field approximation which implies a smallness of fluctuations of φ. In this case the integral (1.5) is determined by a narrow vicinity of $\varphi = \langle\varphi\rangle$ what means $F \simeq \mathcal{F}_L(\langle\varphi\rangle)$. The expressions (1.6,1.7) give us [Landau, 1937]

$$F = \mathcal{F}_L(\langle\varphi\rangle) = \mathcal{F}_{reg} + V\left(\frac{a}{2}\langle\varphi\rangle^2 + \frac{\lambda}{24}\langle\varphi\rangle^4\right), \tag{1.10}$$

where V is the volume of the system. We know that in equilibrium the free energy of a system achieves a minimum value. Therefore to determine $\langle\varphi\rangle$ we should find the absolute minimum of F in terms of $\langle\varphi\rangle$. The minimum of the expression (1.10) corresponds to $\langle\varphi\rangle = 0$ if $a > 0$ and to $\langle\varphi\rangle = \sqrt{6 \mid a \mid /\lambda}$ if $a < 0$. We see that $\langle\varphi\rangle \propto \sqrt{\mid a \mid}$ and consequently the value of $\langle\varphi\rangle$ is small near the transition point. Therefore the condition $\tau \ll 1$ justifies the expansion of $\mathcal{F}_{add}$ in φ. The substitution of the above values of $\langle\varphi\rangle$ into (1.10) shows us that in the mean field approximation $F = \mathcal{F}_{reg}$, if $a > 0$ and

$$F = \mathcal{F}_{reg} - \frac{3a^2}{2\lambda} = \mathcal{F}_{reg} - \frac{3\alpha^2}{2\lambda}(T - T_c)^2\,, \tag{1.11}$$

if $a < 0$. Since the entropy is $S = -\partial F/\partial T$ and the heat capacity is $C_V = T\partial S/\partial T$ we conclude that the heat capacity experiences the jump

$$\Delta C_V = 3V\alpha^2 T_c/\lambda\,, \tag{1.12}$$

near the second order transition point.

The peculiarity of a critical point is the presence of terms odd in φ in the expansion of $\mathcal{F}_{add}$. First such terms can be written as

$$\int d\boldsymbol{r}\left(-h\varphi - \frac{\mu}{6}\varphi^3\right), \tag{1.13}$$

where h and μ are new phenomenological factors. The terms (1.13) have to be added to (1.6,1.7) what gives $\mathcal{F}_{add}$. It should be emphasized that the third order term determined by (1.13) can be eliminated from $\mathcal{F}_{add}$ by the shift $\varphi \to \varphi + \mu/\lambda$. Then we come to the Landau functional

$$\mathcal{F}_{add} = \int d\boldsymbol{r}\left(-h\varphi + \frac{a}{2}\varphi^2 + \frac{\lambda}{24}\varphi^4\right), \tag{1.14}$$

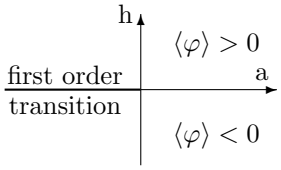
with the redefined values of a and h. The expansion (1.14) is correct if the characteristic value of φ is small, it is the reason why both a and h in (1.14) should be small. It is true in the vicinity of a single point on $P - T$ plane which is determined by the conditions $a = 0$ and $h = 0$ and is no other than the critical point. Note for comparison that a line of second order phase transitions on $P - T$ plane presents (determined by the condition $a = 0$).

Consider the average value $\langle\varphi\rangle$ near the critical point which is determined by the absolute minimum of $\mathcal{F}_{add}(\langle\varphi\rangle)$ where $\mathcal{F}_{add}$ is determined by (1.14). The extremum condition which can be derived at varying $\mathcal{F}_{add}(\langle\varphi\rangle)$ over $\langle\varphi\rangle$ is

$$a\langle\varphi\rangle + \lambda\langle\varphi\rangle^3/6 = h\,. \tag{1.15}$$

It is not very difficult to find the solution of (1.15) in different limit cases. If $a^3 \gg \lambda h^2$ then $\langle\varphi\rangle \simeq h/a$, if $\lambda h^2 \gg \mid a \mid^3$ then $\langle\varphi\rangle \simeq (6h/\lambda)^{1/3}$. If a is negative and $\mid a \mid^3 \gg \lambda h^2$ then $\langle\varphi\rangle \simeq \pm\sqrt{6 \mid a \mid /\lambda}$, where the sign of $\langle\varphi\rangle$ is determined by

the sign of h. We conclude that at $h = 0$ the average $\langle\varphi\rangle$ experiences the jump $\lfloor\varphi\rfloor = 2\sqrt{6 \mid a \mid /\lambda}$, what corresponds to a first order phase transition. Thus we come to the phase diagram



which is plotted on the $a - h$ plane. The phase diagram on the $P - T$ plane will have the same topology. We see that the critical point terminates the line of the first order transitions. Note that the analogous phase diagram can be constructed for a ferromagnetic in the external magnetic field, the role of the magnetic field is played by the quantity h.

Above we have treated the homogeneous average $\langle\varphi\rangle$. In a number of cases an inhomogeneous $\langle\varphi\rangle$ arises. It can be forced by the external influence, $\langle\varphi\rangle$ does depend on coordinates near boundaries or near defects. In these cases we should take into account the gradient term

$$\mathcal{F}_{grad} = \int d\boldsymbol{r} \frac{b}{2} (\nabla\varphi)^2 \,, \tag{1.16}$$

which have to be included into $\mathcal{F}_{add}$. The term $\mathcal{F}_{grad}$ is the first term of the expansion in q/Λ (where q is a characteristic wave vector), q/Λ is a small parameter since we consider macroscopic scales. The formal reason for introducing (1.16) besides (1.6) is the smallness of a. Comparing (1.16) and (1.6) we see that there exist the length

$$r_c = \sqrt{b/ \mid a \mid} \,, \tag{1.17}$$

which determines the characteristic scale in the mean field picture. The quantity r_c is called correlation length or correlation radius.

Consider now the peculiarities of the low-temperature phase if the number of the components of the order parameter is not equal to unity. If $n = 2$ then it is convenient to rewrite the order parameter in terms of the complex field

$$\psi = \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2) = \mid \psi \mid \exp(i\theta) \,, \tag{1.18}$$

where θ is the phase of ψ. The Landau functional $\mathcal{F}_{add}$ is determined by (1.6,1.7,1.16) (the odd in ψ terms are absent since $\mathcal{F}_{add}$ should be invariant under the transformation $\theta \to \theta + const$). After substitution of (1.18) into $\mathcal{F}_{add}$ we find

$$\mathcal{F}_{add} = \int d\boldsymbol{r} \Big(a \mid \psi \mid^2 + b(\nabla \mid \psi \mid)^2 + \lambda \mid \psi \mid^4 /6 + b \mid \psi \mid^2 (\nabla\theta)^2 \Big) \,. \tag{1.19}$$

Such expression arises e.g. in the theory of superfluidity.

If we consider the behavior of the order parameter ψ on scales $r \gg r_c$ then the term $b(\nabla \mid \psi \mid)^2$ in (1.19) does not play an essential role since it is negligible in comparison with the terms not containing $\nabla \mid \psi \mid$. It means that the absolute value of $\mid \psi \mid$ is practically equal to its equilibrium value in all points. We may say that $\mid \psi \mid$ is "frozen" on scales $r \gg r_c$. Besides the quantity θ is "soft" on scales $r \gg r_c$ because (1.19) contains only terms with $\nabla\theta$ (since $\mathcal{F}_{add}$ should be invariant under the transformation $\theta \to \theta + const$). Therefore on scales $r \gg r_c$ the energy (1.19) is reduced to

$$\mathcal{F}_{long} = \int d\boldsymbol{r} \frac{B}{2} (\nabla\theta)^2 \,, \tag{1.20}$$

where the module B is determined by the equilibrium value of the order parameter: $B = 2b \mid \psi \mid^2 = 6 \mid a \mid b/\lambda$. For superfluid ^4He instead of B the quantity $\rho_s = Bm^2/\hbar^2$ is introduced which is called the superfluid density (m being the mass of a ^4He atom). The analogous scheme can be applied to the multi-component order parameter. It can be represented in the following form

$$\varphi_\mu = \mid \varphi \mid n_\mu \,, \tag{1.21}$$

where n_μ is the unit vector. For a ferromagnetic n_μ is the unit vector in the direction of magnetization. On scales $r \gg r_c$ the Landau functional is reduced to

$$\mathcal{F}_{\rm long} = \int d\boldsymbol{r}\, \frac{B}{2} (\nabla n_\mu)^2 \,, \tag{1.22}$$

where $B = 6 \mid a \mid b/\lambda$.

In conclusion of this Lecture we will say some words about the applicability conditions of the mean field approximation. We see that the order parameter become "softer" with decreasing $\mid a \mid$ since the energy associated with a fluctuation of φ decreases. That means that in a vicinity of T_c the mean field approximation breaks since fluctuations of φ cannot be neglected. The quantitative criterium for the mean field theory to be applicable is

$$\mid \tau \mid \gg {\rm Gi} \,, \tag{1.23}$$

where Gi is the so-called Ginzburg number

$${\rm Gi} = \frac{T_c \lambda^2}{\alpha b^3} \,. \tag{1.24}$$

This criterium will be deduced at the next Lecture, at the same Lecture we will discuss the behavior of the system in the region $|\tau| \ll {\rm Gi}$. Here we would like to note that the region of applicability of the mean field theory (1.23) exists only if ${\rm Gi} \ll 1$. In the opposite case the mean field theory has no region of applicability, this case is realized e.g. for the superfluid phase transition in ^4He where ${\rm Gi} \sim 1$.

Problems

Problem 1.1

In the low-symmetry phase (for the one-component order parameter) there exist domain walls, which in the mean field approximation can be described by a solution $\varphi(z)$ with the asymptotics $\varphi \to \pm\varphi_0$ at $z \to \pm\infty$, where $\varphi_0 = \sqrt{6 \mid a \mid /\lambda}$. Find the function $\varphi(z)$.

Solution of the Problem 1.1

The function $\varphi(z)$ should correspond to an extremum of the Landau functional $\mathcal{F}_L$ which is the sum of the expressions (1.6,1.7,1.16). Therefore we get the equation

$$a\varphi + \lambda\varphi^3/6 - b\partial_z^2 \varphi = 0 \,. \tag{1.25}$$

Basing on an analogy with the Newton equation, one can find the first integral of the equation (1.25) which on the Newtonian language is energy

$$\frac{b}{2}(\partial_z \varphi)^2 - \frac{a}{2}\varphi^2 - \frac{\lambda}{24}\varphi^4 \,.$$

A value of the first integral can be found from the boundary values $\varphi = \pm\varphi_0$. Then we get

$$\partial_z \varphi = \frac{1}{\sqrt{2}\,\varphi_0 r_c} \left(\varphi_0^2 - \varphi^2\right) \,,$$

where r_c is defined by Eq. (1.17). A solution of this equation is

$$\varphi = \varphi_0 \tanh\left(\frac{z - z_0}{\sqrt{2}\, r_c}\right) \,,$$

where z_0 is an arbitrary constant determining a position of the domain wall.

2. FLUCTUATIONS, PERTURBATION THEORY

Here we are going to discuss quantitatively the role of fluctuations of the order parameter. Let us first discuss objects of the investigation. All observable parameters (e. g. heat capacity) can be expressed via correlation functions of the

order parameter. Therefore the basic problem of the theory is to determine correlation functions of φ. We will use the special designation for the pair correlation function

$$G(\boldsymbol{r}) = \langle \varphi(\boldsymbol{r})\varphi(0)\rangle \,, \tag{2.1}$$

the same designation G will be utilized for the Fourier transform

$$G(\boldsymbol{q}) = \int d\boldsymbol{r}\, \exp(-i\boldsymbol{q}r) G(\boldsymbol{r}) \,. \tag{2.2}$$

As previously

$$\langle \varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\rangle = \int \mathcal{D}\varphi \exp\left(\frac{F-\mathcal{F}_L}{T}\right)\varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2) \,. \tag{2.3}$$

where $\int \mathcal{D}\varphi$ designates the functional (path) integral, it can be treated as the multiple integral over φ_q which are coefficients of the expansion of $\varphi(\boldsymbol{r})$ in the Fourier series.

The quantity $\mathcal{F}_L$ can be taken as the sum $\mathcal{F}_{reg} + \mathcal{F}_{(2)} + \mathcal{F}_{int}$ (see Lecture 1), where

$$\mathcal{F}_{(2)} = \int d\boldsymbol{r}\left(\frac{a}{2}\varphi^2 + \frac{b}{2}(\nabla\varphi)^2\right), \tag{2.4}$$

$$\mathcal{F}_{int} = \int d\boldsymbol{r}\frac{\lambda}{24}\varphi^4 \,. \tag{2.5}$$

If to expand $\exp(-\mathcal{F}_{(2)}/T - \mathcal{F}_{int}/T)$ over $\mathcal{F}_{int}$ the integrals of the (2.3) type can be represented as a series over the interaction constant λ. The series is called the perturbation series. First of all consider expressions for correlation functions which can be derived neglecting $\mathcal{F}_{int}$, they being called bare correlation functions. The expressions for bare functions are calculated with the weight $\exp\big((F_0 - \mathcal{F}_{(2)})/T\big)$, where

$$\exp\left(-\frac{F_0}{T}\right) = \int \mathcal{D}\varphi \exp\left(-\frac{\mathcal{F}_{(2)}}{T}\right) \equiv \prod_q \int d\varphi_q \exp\left(-\frac{\mathcal{F}_{(2)}}{T}\right). \tag{2.6}$$

So the bare pair correlation function is

$$G_0(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \langle \varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\rangle_0 \equiv \int \mathcal{D}\varphi$$
$$\exp\left(\frac{F_0 - \mathcal{F}_{(2)}}{T}\right)\varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2) \,. \tag{2.7}$$

The explicit expression for $G_0(q)$ is (if $a > 0$)

$$G_0(\boldsymbol{q}) = \frac{T}{a + bq^2} \,. \tag{2.8}$$

It can be found easily since $\mathcal{F}_{(2)}$ is the sum of second order in φ_q terms, what means that the integral in (2.7) is reduced to a product of single Gaussian integrals and can be immediately found explicitly what leads to (2.8). In the $\boldsymbol{r}$-representation

$$G_0(\boldsymbol{r}) = \frac{T}{4\pi b r}\exp(-\sqrt{a/b}\, r) \,. \tag{2.9}$$

If $a < 0$ then the factor a in (2.8,2.9) should be substituted by $2 \mid a \mid$.

The rules for calculating $F_{sing} = F - F_{reg}$ can be formulated starting from the formally exact relation

$$\exp\left(-\frac{F_{sing}}{T}\right) = \int \mathcal{D}\varphi \exp\left(-\frac{\mathcal{F}_{(2)} + \mathcal{F}_{int}}{T}\right), \tag{2.10}$$

what leads to

$$\exp\left(\frac{F_0 - F_{sing}}{T}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle(\mathcal{F}_{int}/T)^n\rangle_0 \,,$$

where $\langle \ldots \rangle_0$ designates the average like in (2.7). Since $\mathcal{F}_{(2)}$ is of the second order in φ all averages in the above sum are determined by Gaussian integrals and can be found explicitly. Each such average is expressed through the pair correlation function, namely the average $\langle \varphi\varphi \ldots \varphi \rangle_0$ is equal to the sum of all possible terms which are products of the functions G_0 corresponding to the averages $\langle \varphi\varphi \rangle_0$. This assertion is known as Wick theorem. An example:

$$\langle \varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\varphi(\boldsymbol{r}_3)\varphi(\boldsymbol{r}_4)\rangle_0 = G_0(\boldsymbol{r}_1-\boldsymbol{r}_2)G_0(\boldsymbol{r}_3-\boldsymbol{r}_4)+$$

$$G_0(\boldsymbol{r}_1-\boldsymbol{r}_3)G_0(\boldsymbol{r}_2-\boldsymbol{r}_4) + G_0(\boldsymbol{r}_1-\boldsymbol{r}_4)G_0(\boldsymbol{r}_2-\boldsymbol{r}_3)\,.$$

The pair correlation function (2.3) can be rewritten as

$$G(\boldsymbol{r}_1-\boldsymbol{r}_2) = \exp\left(\frac{F_{sing}-\mathcal{F}_0}{T}\right)\times$$
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle(\mathcal{F}_{int}/T)^n\varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\rangle_0\,. \tag{2.11}$$

A correlation function of φ can be presented in the form analogous to (2.11).

The terms of the form $\langle \varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\ldots\rangle_0$ arising in the perturbation series for correlation functions can be depicted via Feynmann diagrams. Each point on a diagram corresponds to an argument $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, ... of the order parameter φ, each line on a diagram represents the bare correlation function G_0 taken as the function of the differences of coordinates of the two points which the line joints. The term of the n-th order in $\mathcal{F}_{int}$ for a correlation function of m fields φ is represented by diagrams with m "external" points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, ... and n "internal" points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, ... corresponding to the terms originating from (2.5). We see that there are four fields φ with an argument $\boldsymbol{x}_i$ and therefore four edges of the G_0-lines meets in $\boldsymbol{x}_i$. Since the factor λ corresponds to a point $\boldsymbol{x}_i$, λ is usually called the fourth order vertex. Note that the integration is performed over all "internal" points of a diagram $\boldsymbol{x}_i$.

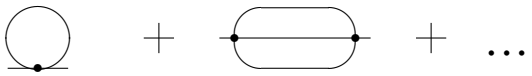
The diagrammatic series for the G-function is determined by (2.11). All the diagrams for G have two "external" points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ and a number of "internal" points. Consider the contribution to G of the first order in λ. It is determined by two terms one of which is

$$\langle \varphi(\boldsymbol{r}_1)(-\mathcal{F}_{int}/T)\varphi(\boldsymbol{r}_2)\rangle_0\,. \tag{2.12}$$

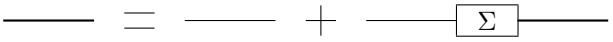
It can be depicted as the sum

The first (irreducible) diagram gives the actual contribution to G, the second (reducible) is the product of G_0 and $\langle(-\mathcal{F}_{int}/T)\rangle_0$. It is cancelled by the term originating from the expansion of $\exp((F_{sing}-F_0)/T)$ in (2.11) in λ. This observation can be generalized: only irreducible diagrams contribute to G.

We see that $G-G_0$ is determined by the sum of irreducible diagrams each containing two external G_0-lines and a block between these two lines. Let us take the sum of such blocks which cannot be cut along a single G_0-line:



We will designate this sum as Σ and depict it as a rectangular. Then the diagrammatic equation for G can be formulated



where a thick line designates G. In the analytical form it is

$$G(\boldsymbol{r}) = G_0(\boldsymbol{r}) + \int d\boldsymbol{r}_1\, d\boldsymbol{r}_2 G_0(\boldsymbol{r}-\boldsymbol{r}_1)\Sigma(\boldsymbol{r}_1-\boldsymbol{r}_2)G(\boldsymbol{r}_2)\,. \tag{2.13}$$

Taking into account (2.8) we conclude that in Fourier representation

$$G(\boldsymbol{q}) = \frac{T}{a+bq^2-T\Sigma(q)}\,. \tag{2.14}$$

The explicit expressions corresponding to the first diagrams depicted above are

$$\Sigma^{(1)}(\boldsymbol{r}) = -\frac{\lambda}{2T} G_0(\boldsymbol{r}=0)\delta(\boldsymbol{r})\,, \tag{2.15}$$

$$\Sigma^{(2)}(\boldsymbol{r}) = \frac{\lambda^2}{6T^2} G_0^3(\boldsymbol{r})\,. \tag{2.16}$$

In Fourier representation

$$\Sigma^{(1)}(\boldsymbol{k}) = -\frac{\lambda}{2}\int d\boldsymbol{q}\frac{1}{a+bq^2}\,. \tag{2.17}$$

where $d\boldsymbol{q} \equiv d^3q/(2\pi)^3$, actually $\Sigma^{(1)}$ not depending on the wave vector $\boldsymbol{k}$.

The integral in (2.17) formally diverges at large q, actually this divergence is cut at $q \sim \Lambda$ (Λ is the cutoff). The constant in $\Sigma^{(1)}(a=0)$ determined by $q \sim \Lambda$ cannot be found in the framework of the presented long wavelength theory, but this constant can be included into the redefinition of the transition temperature T_c. Namely, from the expression (2.14) it follows that this redefinition is

$$T_c \to T_c + (\partial a/\partial T)^{-1} T_c \Sigma^{(1)}(a=0)\,. \tag{2.18}$$

The difference $\Sigma^{(1)}(a) - \Sigma^{(1)}(a=0)$ is determined by large scales and can be consequently calculated in the framework of the presented theory:

$$\Sigma^{(1)}(a) - \Sigma^{(1)}(a=0) = -\frac{\lambda}{2}\int d\boldsymbol{q}\left(\frac{1}{a+bq^2} - \frac{1}{bq^2}\right) = \frac{\lambda a^{1/2}}{4\pi b^{3/2}}\,. \tag{2.19}$$

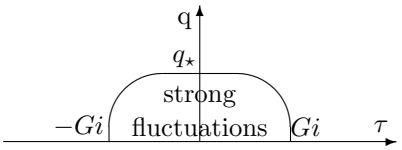
This correction can be neglected in comparison with the bare term a in (2.14) if $\mid \tau \mid \gg \lambda^2 T_c/\alpha b^3$ what is no other then Ginzburg criterium $\mid \tau \mid \gg Gi$ (see Lecture 1). The correction (2.19) tends to zero at $T \to T_c$, it is not so for the correction (2.16). Basing on the expression (2.9) we conclude that at $T = T_c$ (that is if $a = 0$) the Fourier transform of (2.16) can be estimated as

$$\Sigma^{(2)}(\boldsymbol{k}) \sim \frac{\lambda^2 T}{b^3}\ln(\Lambda/k)\,.$$

This value is less than the bare value bq^2 in (2.14) if $q \gg q_\star$ where

$$q_\star = T\lambda/b^2\,. \tag{2.20}$$

We conclude that the fluctuation contributions to Σ cannot be neglected if $\mid \tau \mid \ll Gi$ and $q \ll q_\star$. These two conditions determine the region near the origin on the $\tau - q$ plane where fluctuations essentially change the behavior of correlation functions of the order parameter. This region is depicted as



To examine the behavior of $G(q)$ in the region of strong fluctuations one should take into account all higher-order in λ contributions to G. It is a difficult task, we know only the main peculiarities of the behavior of correlation functions of φ in this region. The origin on the $\tau - q$ plane proves to be a singular point therefore we can expect a standard power-like behavior of all correlation functions near this point what is referred as scaling behavior. It means that a correlation function (in Fourier representation) is a product of a dimension factor, a power of τ and of a function of dimensionless combinations like qr_c. Here r_c is the correlation radius which behaves as

$$r_c \propto \mid \tau \mid^{-\nu}\,, \tag{2.21}$$

where ν is an exponent which depends on the number of components of the order parameter. So, the pair correlation function (2.2) is

$$G(q) \propto \mid \tau \mid^{-\gamma} g(qr_c)\,, \tag{2.22}$$

where γ is the new exponent. One extra exponent characterizes the correlation function

$$\int d\boldsymbol{r} \exp(-i\boldsymbol{q}r)\langle\langle\varphi^2(\boldsymbol{r})\varphi^2(0)\rangle\rangle \propto \mid \tau \mid^{-\alpha} g_1(qr_c)\,, \tag{2.23}$$

where double angular brackets designate an irreducible correlation function. There are some relations between the exponents, e.g. $\alpha = 2 - d\nu$ (where $d = 3$ is the dimensionality of the space), only two exponents are independent.

Let us demonstrate how is it possible to establish the scaling behavior of some observable quantities. Fist from (2.4,2.10) it follows

$$C_{sing} \simeq \frac{V}{4}\left(\frac{\partial a}{\partial T}\right)^2 \int d\boldsymbol{r}\langle\langle\varphi^2(\boldsymbol{r})\varphi^2(0)\rangle\rangle\,, \tag{2.24}$$

where V is the volume of the system. Comparing (2.23) and (2.24) we find that for $\tau \ll Gi$ the singular part of the heat capacity behaves as

$$C_{sing} \propto \mid \tau \mid^{-\alpha}\,, \tag{2.25}$$

what means that $F_{sing} \propto \mid \tau \mid^{2-\alpha}$. Suppose that an "external field" h is imposed on the system. Then the extra term

$$\mathcal{F}_h = -\int d\boldsymbol{r}\, h\varphi\,, \tag{2.26}$$

should be added to $\mathcal{F}_L$. Calculating now the average $\langle\varphi\rangle$ like (2.3) we can found the value of $\langle\varphi\rangle_h$ induced by the "external field" h. Expanding the exponent in $\mathcal{F}_h$ we get in the linear approximation

$$\langle\varphi(\boldsymbol{r}_1)\rangle_h = T^{-1}\int d\boldsymbol{r}_2\, G(\boldsymbol{r}_1 - \boldsymbol{r}_2)h(\boldsymbol{r}_2)\,. \tag{2.27}$$

Comparing this expression with (2.22) we conclude that for a homogeneous field

$$\langle\varphi(\boldsymbol{r}_1)\rangle_h = \chi h\,, \quad \chi \propto \mid \tau \mid^{-\gamma}\ . \tag{2.28}$$

The laws (2.25,2.28) can be observed experimentally.

Problems

Problem 2.1

Landau functional for a Heisenberg ferromagnetic is

$$\mathcal{F}_L = \int d\boldsymbol{r}\,\frac{B}{2}(\nabla n_i)^2\,,$$

where $\boldsymbol{n}$ is the unit vector along the direction of the magnetization. In the external magnetic field $\boldsymbol{H}$ there is the additional contribution to the energy of a magnetic

$$\mathcal{F}_H = -\int d\boldsymbol{r}\, M\boldsymbol{H}\boldsymbol{n}\,,$$

where M is the absolute value of the magnetization. Find the fluctuational contribution into the longitudinal magnetic susceptibility of a three-dimensional ferromagnetic.

Solution of the Problem 2.1

Suppose that $\boldsymbol{H}$ is directed along Z-axis. Then the magnetization $\langle\boldsymbol{M}\rangle$ is also directed along Z-axis. Therefore we should be interested in $M_z = M\langle n_z\rangle$. Fluctuations of $\boldsymbol{n}$ in $3d$ case are weak and we can expand $n_z \approx 1 - n_x^2/2 - n_y^2/2$. Therefore the longitudinal magnetic susceptibility $\chi_\parallel$ can be written as

$$\chi_\parallel \equiv \frac{\partial\langle M_z\rangle}{\partial H} \approx -M\frac{\partial}{\partial H}\langle n_x^2\rangle\,.$$

Next, using the same expansion $n_z \approx 1 - n_x^2/2 - n_y^2/2$ we find from $\mathcal{F}_L + \mathcal{F}_H$

$$\langle n_x n_x\rangle = \frac{T}{Bq^2 + MH}\,.$$

Then we find

$$\chi_\parallel = M^2\int\frac{d^3q}{(2\pi)^3}\frac{T}{(Bq^2 + MH)^2} = \frac{TM^{3/2}}{8\pi B^{3/2}H^{1/2}}\,.$$

3. DIMENSION 4, PARQUET DIAGRAMS

We have seen that the behavior of correlation functions of φ near T_c is complicated. It proves that the problem can be solved for the dimension $d = 4$ of the space. The main idea is that the dimension $d = 3$ is not far from $d = 4$. Then it is worthwhile to consider the problem in the arbitrary dimension $4 - \epsilon$ (where ϵ is a small parameter), and then to believe $\epsilon = 1$ [Wilson, 1972]. This procedure gives us physical quantities in the form of (asymptotic) series in ϵ and is called ϵ-expansion.

First we should consider the problem in $d = 4$. It is determined by the same Landau expansion $\mathcal{F}_{reg} + \mathcal{F}_{(2)} + \mathcal{F}_{int}$ (see Lectures 1,2), where

$$\mathcal{F}_{(2)} = \int d\boldsymbol{r} \left(\frac{a}{2}\varphi^2 + \frac{b}{2}(\nabla\varphi)^2 \right), \tag{3.1}$$

$$\mathcal{F}_{int} = \int d\boldsymbol{r}\, \frac{\lambda}{24}\varphi^4 , \tag{3.2}$$

but integration is now performed over the four-dimensional space. As previously (see Lecture 2) we can formulate the perturbation series. The bare pair correlation function G_0 in Fourier representation has the form (at $a > 0$)

$$G_0(\boldsymbol{q}) = \frac{T}{a + bq^2} , \tag{3.3}$$

coinciding formally with the expression for $d = 3$. In the $\boldsymbol{r}$-representation it is (for $d = 4$)

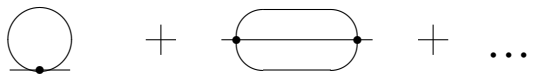
$$G_0(\boldsymbol{r}) \simeq \frac{T}{4\pi^2 b r^2} , \tag{3.4}$$

this expression is correct if $r \ll \sqrt{(b/a)}$, at $r \gg \sqrt{(b/a)}$ the function $G_0(\boldsymbol{r})$ decreases exponentially.

Consider the self-energy function Σ related to the dressed function G as

$$G(\boldsymbol{q}) = \frac{T}{a + bq^2 - T\Sigma(\boldsymbol{q})} . \tag{3.5}$$

The first contributions to the self-energy function are depicted in Figure



The one-loop contribution for $d = 4$ is formally the same as for $d = 3$

$$\Sigma^{(1)}(\boldsymbol{r}) = -\frac{\lambda}{2T} G_0(\boldsymbol{r} = 0)\delta(\boldsymbol{r}) . \tag{3.6}$$

Passing into Fourier representation and subtracting the constant which should be included into redefinition of the transition temperature we find

$$\Sigma^{(1)}(\boldsymbol{k}) - \Sigma^{(1)}(\boldsymbol{k} = 0, a = 0) = \\ -\frac{\lambda}{2}\int d\boldsymbol{q} \left(\frac{1}{a + bq^2} - \frac{1}{bq^2} \right) \simeq \frac{\lambda a}{16\pi^2 b^2} \ln\left(\frac{\Lambda}{\sqrt{a/b}} \right), \tag{3.7}$$

where $d\boldsymbol{q}$ designates $d^4q/(2\pi)^4$ and Λ is the cutoff wave vector. Comparing this expression with (3.5) we conclude that it gives the logarithmic renormalization of the coefficient α in the relation $a = \alpha(T - T_c)$.

The second contribution in $\boldsymbol{r}$-representation is

$$\Sigma^{(2)}(\boldsymbol{r}) = \frac{\lambda^2}{6T^2} G_0^3(\boldsymbol{r}) \simeq \frac{\lambda^2 T}{3 \cdot 2^7 \pi^6 b^3 r^6} , \tag{3.8}$$

the last expression being correct if $r \ll \sqrt{b/a}$. Passing into Fourier representation and subtracting the constant which should be included into redefinition of the transition temperature we find

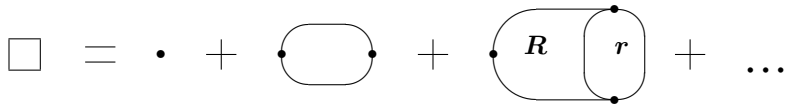
$$\Sigma^{(2)}(\boldsymbol{q}) - \Sigma^{(2)}(\boldsymbol{q} = 0, a = 0) \simeq \\ \frac{\lambda^2 T}{3 \cdot 2^7 \pi^6 b^3} \int \frac{d^4 r}{r^6} \Big(\exp(i\boldsymbol{q}r) - 1 \Big) \simeq -\frac{\lambda^2 T}{3 \cdot 2^9 \pi^4 b^3} q^2 \ln\left(\frac{\Lambda}{q} \right), \tag{3.9}$$

which is correct for $q \gg \sqrt{a/b}$. Comparing Expression (3.9) with (3.5) we conclude that it gives the logarithmic renormalization of the coefficient b.

Generalization – all higher order contributions to Σ give logarithmic corrections to α, b. These corrections should be taken into account if

$$\xi > b^2/(T\lambda)\,,\,, \tag{3.10}$$

where $\xi = \ln(\Lambda/q)$. To sum the main series over ξ we should introduce the new object which is the renormalized vertex λ_r defined as the sum of four-legs irreducible diagrams which cannot be cut along one line. The series for λ_r (which we will designate as a square) can be depicted as



We will denote this renormalized vertex as λ_r, which is $\lambda+\lambda^{(1)}+\lambda^{(2)}+\dots$. The term $\lambda^{(1)}$ determined by the one-loop diagram is

$$\lambda^{(1)}(r) = -\frac{3\lambda^2}{2T} G_0^2(r)\,. \tag{3.11}$$

We should compare with λ the Fourier-transform $\lambda^{(1)}(q)$ of (3.11)

$$\lambda^{(1)}(q) = -\frac{3\lambda^2 T}{2^4\pi^2 b^2}\xi\,, \tag{3.12}$$

where we have used the expression (3.4) and the condition $q \gg \sqrt{a/b}$ is implied. We again see that the logarithmic corrections are relevant if (3.10) is satisfied.

Consider now the part of $\lambda^{(2)}(\boldsymbol{q})$ determined by the second diagram in Figure. This term is of the order of $(\lambda^2 T\xi/b^2)^2$ and is therefore essential if (3.10) is satisfied. It is not very difficult to understand that the second power of ξ originates from the region of integration $R \gg r$ where $\boldsymbol{R}$ and $\boldsymbol{r}$ are the separation distances depicted in Figure. Generalization – the leading terms over ξ are determined by diagrams where separation distances can be ordered as $r_1 \gg r_2 \gg r_3 \dots$ (which is known as the parquet sequence). The value of λ_r is actually a function of the largest separation distance or (in Fourier representation) a function of the corresponding wave vector q.

Now we can find an equation for λ_r. Let us take the diagrammatic series for λ_r and mark at each diagram the loop with the largest separation distance R. Then the blocks at the right and at the left sides after summation will give λ_r again which is the function of R. The diagrammatic representation of this relation is

In the analytical form it can be written as

$$\lambda_r(k) = \lambda - \frac{3T}{2^4\pi^2 b^2}\int\limits_0^{\xi} d\xi' \lambda_r^2(q)\,, \tag{3.13}$$

where $\xi' = \ln(\Lambda/q)$ and the integral in (3.13) is taken in the limits 0 and $\xi = \ln(\Lambda/k)$. The integral equation (3.13) is equivalent to the differential one

$$d\lambda_r/d\xi = -\frac{3T}{2^4\pi^2 b^2}\lambda_r^2\,, \tag{3.14}$$

which has the following solution

$$\lambda_r(k) = \lambda\left(1 + \frac{3T\lambda}{2^4\pi^2 b^2}\ln(\Lambda/k)\right)^{-1} \to \frac{2^4\pi^2 b^2}{3T\xi}\,, \tag{3.15}$$

where the final expression is correct if $\xi \gg b^2/(T\lambda)$.

Now we can determine the behavior of the renormalized values $a_r = \alpha_r(T - T_c)$ and b_r which are the functions of the wave vector k. To find the equation for α_r we have to perform the same procedure – to take the loop with the

largest separation in the diagrammatic series for a_r an to sum left and right blocks what gives renormalized values. The result can be extracted from (3.7) where the bare quantities λ and a should be substituted by their renormalized values, it is

$$\alpha_r(k) = \alpha - \frac{T}{2^4\pi^2 b^2}\int\limits_0^\xi d\xi' \lambda_r \alpha_r \,, \tag{3.16}$$

where $\xi = \ln(\Lambda/k)$. The solution of (3.16) at $\xi \gg b^2/(T\lambda)$ is

$$\alpha_r(k) \propto \xi^{-1/3} \,, \tag{3.17}$$

what is the consequence of (3.15). To find the equation for b_r we should substitute in (3.9) the bare quantities λ and b by their renormalized values. As a result we find the equation

$$b_r - b = \int\limits_0^\xi d\xi' \frac{T^2}{3\cdot 2^9\pi^4}\frac{\lambda_r^2}{b_r^3} \,, \tag{3.18}$$

which has the solution $b_r \to$ const where $\xi \to \infty$. The point is that due to (3.15) the integral over ξ' in (3.18) converges at $b_r = const$. It justifies actually all above calculations since they implied the condition $b = const$. We conclude that in the explicit expression of the (3.15) type is correct only in the asymptotic region $\xi \gg b^2/(T\lambda)$ where we should substitute b by its asymptotic value $b_r(\infty)$. Let us stress that the relation (3.17) is not sensitive to the value of b.

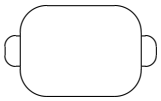
All said above is correct if we have possibility to neglect a_r in comparison with $b_r q^2$ in the expression

$$G(\boldsymbol{q}) = T/(a_r + b_r q^2) \,, \tag{3.19}$$

for the dressed pair correlation function. It is correct if $q > \sqrt{a_r/b_r}$ or, by other words, on scales $r < \sqrt{b_r/a_r}$. On larger scales the renormalization is finished. That implies e. g. that on these scales G will have the form (3.19) with the values a_r and b_r which were reached on scales where $a_r \sim b_r q^2$. Let us find the singular part of the heat capacity which can be written as

$$C_{sing} = \frac{V}{4}\int d\boldsymbol{r}\langle\langle \alpha^2\varphi^2(\boldsymbol{r})\varphi^2(0)\rangle\rangle \,. \tag{3.20}$$

The expression (3.20) can be deduced by direct differentiation of the free energy. The main contribution to the average (3.20) will be produced as previously by parquet diagrams. If to take the loop with the largest separation distance then the blocks at the right and at the left will give after summation the renormalized value of the quantity α. This result is presented by the diagram



where the semicircles correspond to α_r. In the analytical form

$$C_{sing} = \frac{V}{2}\int d\boldsymbol{r}\alpha_r^2 G^2(\boldsymbol{r}) \,. \tag{3.21}$$

After substitution of (3.17,3.19) we find

$$C_{sing} \propto \xi^{1/3} \,, \tag{3.22}$$

where $\xi \simeq \ln(\Lambda\sqrt{b/a})$.

Some words about dimensions $4-\epsilon$. If we consider small ϵ then the proposed scheme will be practically the same. We can derive the equations of (3.13,3.16,3.18) type but now an additional factor $N_\epsilon q^{-\epsilon}$ appears in (3.13,3.16) where N_ϵ is a geometrical factor, the additional factor in (3.18) being $(N_\epsilon q^{-\epsilon})^2$. Then we find instead of (3.15)

$$\lambda_r(k) \to \frac{2^4\pi^2 b^2\epsilon}{3N_\epsilon T q^\epsilon} \,. \tag{3.23}$$

The solutions of (3.16,3.18) will be now powerlike:

$$\alpha_r(k) \propto k^{\epsilon/3}\,, \quad b_r(k) \propto k^{-\epsilon^2/54}\,. \tag{3.24}$$

Comparing (3.24) with the definition of the exponents ν and γ (see Lecture 2) we conclude that the first terms of their expansion in ϵ are $\nu = 1/2 + \epsilon/12$ and $\gamma = 1 + \epsilon/6$. The index α of the heat capacity can be found if to use (3.21,3.24), the result is $\alpha = \epsilon/6$. If the number of the components n of the order parameter is taken into account the expressions for the exponents in the first ϵ approximation are

$$\nu = \frac{1}{2} + \frac{n+2}{4(n+8)}\epsilon\,, \quad \gamma = 1 + \frac{n+2}{2(n+8)}\epsilon\,, \quad \alpha = \frac{4-n}{2(n+8)}\epsilon\,. \tag{3.25}$$

Of course these values coincide with the given above for $n = 1$.

Despite the proposed procedure is rough (it gives asymptotical series in the parameter $\epsilon = 1$) the derived expressions for the critical exponents are in a reasonable agreement with experiment.

4. RENORM-GROUP, ϵ-EXPANSION

Here, we start from rederiving the results obtained in the previous section using another language which is more convenient and permits wide generalization. Remind, that the problem under consideration is described by Landau functional

$$\mathcal{F} = \int \mathrm{d}\boldsymbol{r} \left\{ \frac{a}{2}\varphi^2 + \frac{b}{2}(\nabla\varphi)^2 + \frac{\lambda}{24}\left(\varphi^2\right)^2 \right\}, \tag{4.1}$$

where $\boldsymbol{\varphi}$ is a n-component order parameter and

$$\varphi^2 \equiv \sum_1^n \varphi_b^2\,.$$

As previously $a \propto T - T_c$. Remind also that the problem implies the presence of the ultraviolet cutoff Λ.

As we saw at examining the diagrammatic expansion for the dimensionality $d = 4$ variables characterizing order parameter correlation functions at a given scale r are strongly renormalized due to the interaction of fluctuations of $\boldsymbol{\varphi}$ at the scale r with fluctuations with the wave vectors $r^{-1} < q < \Lambda$. Generally the renormalization is strong. Nevertheless the coupling constant is small. Therefore one expects that the interaction with fluctuations in the restricted phase volume will produce only a small renormalization of the variables characterizing the order parameter correlation functions at the scale r. That is the motivation for the following multistep procedure.

Let us devide the order parameter field into the 'slow' part φ' and the 'fast' part $\tilde{\varphi}$:

$$\varphi = \varphi' + \tilde{\varphi}\,, \tag{4.2}$$

where $\tilde{\varphi}$ contains the fastest Fourier harmonics of φ. Next, we want to exclude from the consideration φ integrating the probability distribution function $\exp(-\mathcal{F}/T)$ over $\tilde{\varphi}$:

$$\exp\left[-\frac{\mathcal{F}'(\varphi')}{T}\right] = \int \mathcal{D}\tilde{\varphi}\, \exp\left[-\frac{\mathcal{F}(\varphi'+\tilde{\varphi})}{T}\right], \tag{4.3}$$

where we introduced the designation $\mathcal{F}'$ for Landau functional of the slow field φ'. The functional contains the complete information about the correlation functions of the slow part of φ. Say, knowing $\mathcal{F}'$ we can principally calculate

$$G(r) = \int \mathcal{D}\tilde{\varphi}'\, \exp\left[-\frac{\mathcal{F}'(\varphi')}{T}\right] \varphi'(\boldsymbol{r})\varphi'(\boldsymbol{0})\,, \tag{4.4}$$

if $\Lambda r \gg 1$ since the contribution to $G(r)$ from fast degrees of freedom is negligible. Producing the procedure (4.3) again and again we can exclude from the consideration Fourier harmonics of φ with wave vectors $r^{-1} < q < \Lambda$. Then it will be possible to calculate the correlation function (4.4) in terms of the conventional perturbation expansion.

Of course $\mathcal{F}'$ differs from $\mathcal{F}$. But if the phase volume of $\tilde{\varphi}$ is small enough then it is possible to calculate the difference in terms of the conventional perturbation expansion. Then the parameters describing $\mathcal{F}$ will be changed

gradually at the introduced multistep procedure. The variation can be described in terms of the corresponding differential equations which are called renorm-group (RG) equations. Let us find RG-equations for Landau functional (4.1) regarding the space of dimensionality $d = 4$.

First, we should define more presicely the decomposition (4.2) producing at a single step of the introduced multistep procedure. We will believe that, say, the fast component $\tilde{\varphi}$ is the sum of Fourier harmonics with wave vectors $\Lambda' < q < \Lambda$, then the slow component φ' will be the sum of Fourier harmonics with wave vectors $q < \Lambda'$. We will suggest two conditions:

$$\Lambda \gg \Lambda' \,, \tag{4.5}$$

$$g \ln(\Lambda/\Lambda') \ll 1 \,, \tag{4.6}$$

where we introduced the dimensionless coupling constant g the exact definition of which will be done below. Of course the conditions (4.5) and (4.6) are compatible only if $g \ll 1$ what is the applicability condition of RG technique. The condition (4.6) permits to produce the conventional perturbation expansion and the condition (4.5) permits to keep terms containing the large logarithms $\ln(\Lambda/\Lambda')$ only what is an effective selection rule.

Substituting the decomposition (4.2) into (4.1) we get

$$\mathcal{F} = \mathcal{F}(\varphi') + \mathcal{F}^{(2)}_{\rm int} + \mathcal{F}^{(2)}(\tilde{\varphi}) + \dots \,, \tag{4.7}$$

$$\mathcal{F}^{(2)}_{\rm int} = \int \mathrm{d}\boldsymbol{r} \left\{ \frac{\lambda}{12} \varphi'^2 \tilde{\varphi}^2 + \frac{\lambda}{6} \left(\boldsymbol{\varphi}' \tilde{\boldsymbol{\varphi}}\right)^2 + \frac{a}{2} \tilde{\varphi}^2 \right\} , \tag{4.8}$$

$$\mathcal{F}^{(2)}(\tilde{\varphi}) = \int \mathrm{d}\boldsymbol{r} \, \frac{b}{2} (\nabla \tilde{\varphi})^2 \,, \tag{4.9}$$

where $\dots$ mean omitted terms. We neglected the term

$$\int \mathrm{d}\boldsymbol{r} \, \frac{\lambda}{6} \left(\tilde{\boldsymbol{\varphi}} \boldsymbol{\varphi}'\right) \varphi'^2 \,,$$

since it cannot produce the corrections to $\mathcal{F}$ proportional to the large logarithm $\ln(\Lambda/\Lambda')$. The reason is that the term is non-zero only for harmonics in $\tilde{\varphi}$ with wave vectors q close to Λ'. We neglected in (4.7) also terms of the third and of the fourth order over $\tilde{\varphi}$ since they produce only a small contribution into the renormalization of $\mathcal{F}$ due to the condition (4.6).

Substituting (4.8) into (4.3) we get

$$\exp\left[-\frac{\mathcal{F}'(\varphi') - \mathcal{F}(\varphi')}{T}\right] = \int \mathcal{D}\tilde{\varphi} \, \exp\left[-\frac{\mathcal{F}(\tilde{\varphi}) + \mathcal{F}_{\rm int}}{T}\right] . \tag{4.10}$$

Due to the condition (4.6) the difference $\mathcal{F}'(\varphi') - \mathcal{F}(\varphi')$ is small and we can expand the exponent in LHS of (4.10) over the difference. The same reasoning enables to expand RHS of (4.10) over $\mathcal{F}^{(2)}_{\rm int}$. As a result we obtain

$$\mathcal{F}'(\varphi') - \mathcal{F}(\varphi') = \left\langle \mathcal{F}^{(2)}_{\rm int} \right\rangle - \frac{1}{2T} \left\langle\!\!\left\langle \left(\mathcal{F}^{(2)}_{\rm int}\right)^2 \right\rangle\!\!\right\rangle . \tag{4.11}$$

Here, the angular brackets designate averaging over the statistics of the fast variable. We have kept in (4.11) two first terms of the expansion of the exponent over $\mathcal{F}^{(2)}_{\rm int}$. The general rule is that we should keep all the terms containing the large logarithm $\ln(\Lambda/\Lambda')$ and producing a renormalization of different terms in $\mathcal{F}$. Note that we should take the irreducible average of the second term in (4.11) (designating by double angular brackets) since the corresponding reducible part contributes really to the second-order term of the expansion of LHS of the relation (4.10).

In the main approximation statistics of the fast degrees of freedom is determed by the probability distribution function

$$\mathcal{P}(\tilde{\varphi}) \propto \exp\left[-\frac{\mathcal{F}^{(2)}(\tilde{\varphi})}{T}\right] . \tag{4.12}$$

Since the probability distribution function (4.12) is Gaussian all the correlation functions of $\tilde{\varphi}$ are reduced to $\tilde{G}$ where by definition

$$\langle \tilde{\varphi}_a(\boldsymbol{r}_1) \tilde{\varphi}_b(\boldsymbol{r}_2) \rangle = \tilde{G}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \delta_{ab} \,.$$

The function can be calculated starting from (4.12):

$$\tilde{G}(\boldsymbol{r}) = \int\limits_{\Lambda'}^{\Lambda} \frac{d^4q}{(2\pi)^4} \frac{T}{bq^2} \exp(i\boldsymbol{qr}) \approx \frac{T}{4\pi^2 br^2}\,, \tag{4.13}$$

The last expression is correct for $\Lambda'^{-1} > r > \Lambda^{-1}$.

Since the probability distribution function (4.12) is Gaussian we can easily find the averages in (4.11) explicitly. Say,

$$\left\langle \mathcal{F}_{\text{int}}^{(2)} \right\rangle = \int \mathrm{d}\boldsymbol{r}\, \frac{\lambda}{12}(n+2)\varphi'^2 \left\langle \tilde{\varphi}_1^2 \right\rangle + \dots\,, \tag{4.14}$$

where we have taken into account that $\langle \tilde{\varphi}_a \tilde{\varphi}_b \rangle \propto \delta_{ab}$. The average $\left\langle \tilde{\varphi}_1^2 \right\rangle$ figuring in (4.14) can be found using the expression (4.13)

$$\left\langle \tilde{\varphi}_1^2 \right\rangle = \int\limits_{\Lambda'}^{\Lambda} \frac{d^4q}{(2\pi)^4} \frac{T}{bq^2}\,.$$

The integral is determined by the upper limit, therefore the term (4.14) should be incorporated into the redefinition of the transition temperature T_c. Analogously one can recognize that the term proportional to a and designated by dots in (4.14) is also irrelevant. Now, let us turn to the second term in (4.11). Substituting (4.8) we find the following expression

$$\mathcal{F}'(\varphi') - \mathcal{F}(\varphi') \leftarrow -\frac{1}{T} \int \mathrm{d}\boldsymbol{r}_1\, d\boldsymbol{r}_2\, \tilde{G}^2(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$
$$\left\{ \frac{(n+8)\lambda^2}{4\cdot 27} \varphi'^2(\boldsymbol{r}_1)\varphi'^2(\boldsymbol{r}_2) + \frac{(n+2)\lambda a}{12} \varphi'^2(\boldsymbol{r}_1) + \frac{na^2}{4} \right\}. \tag{4.15}$$

The characteristic scale where the function $\tilde{G}(r)$ decays is Λ'^{-1} whereas the field φ' varies on larger scales. Therefore in the main approximation we can substitute $\varphi'^2(\boldsymbol{r}_2) \to \varphi'^2(\boldsymbol{r}_1)$ in the first term in (4.15) and we get the contribution

$$\mathcal{F}' - \mathcal{F} \leftarrow -\frac{\lambda^2}{36T} \int \mathrm{d}\boldsymbol{r}\, \varphi'^4(\boldsymbol{r}) \int \mathrm{d}\boldsymbol{R}\, \tilde{G}^2(\boldsymbol{R})\,. \tag{4.16}$$

The integral here can be found from (4.13):

$$\int \mathrm{d}\boldsymbol{R}\, \tilde{G}^2(\boldsymbol{R}) \approx \frac{T^2 S_4}{(2\pi)^4 b^2} \Delta\xi\,, \tag{4.17}$$

where $S_4 = 2\pi^2$ is the surface of the four-dimensional sphere. We see that the contribution (4.16) contains the large logarithm and produces the renormalization of the fourth-order term in $\mathcal{F}$. The renormalization can be written in terms of the correction to the factor λ:

$$\Delta\lambda = -\frac{n+8}{6} \frac{S_4}{(2\pi)^4} \frac{T}{b^2} \lambda^2 \Delta\xi\,, \tag{4.18}$$

where $\Delta\xi = \ln(\Lambda/\Lambda')$. Next, using (4.17) we can find the second and the third contributions to $\mathcal{F}' - \mathcal{F}$ in (4.15). The second contribution gives the renormalization of a which can be written as

$$\Delta a = -\frac{n+2}{6} \frac{S_4}{(2\pi)^4} \frac{T}{b^2} \lambda a \Delta\xi\,. \tag{4.19}$$

And the last term gives the contribution to the free energy

$$\Delta F = -\frac{na^2}{4} \frac{S_4}{(2\pi)^4} \frac{T}{b^2} \Delta\xi\,. \tag{4.20}$$

As follows from (4.19,4.18) the relative corrections to a and λ at the step are small if the condition (4.6) is satisfied with

$$g = \frac{n+8}{6}\frac{S_4}{(2\pi)^4}\frac{T}{b^2}\lambda\,. \tag{4.21}$$

The quantity (4.21) plays the role of the dimensionless coupling constant, it is called usually the invariant charge.

We have considered a single step of the multistep procedure. If to produce the noted multistep procedure then Landau functional (4.1) will keep its form (as follows from the analysis done) but the coefficients a and λ varies gradually at subsequent excluding fast variables. Since the variations (4.19,4.18) are small at each step of the procedure we can describe the variation in terms of the differential equations

$$\frac{d\lambda}{d\xi} = -\frac{n+8}{6}\frac{S_4}{(2\pi)^4}\frac{T}{b^2}\lambda^2\,, \tag{4.22}$$

$$\frac{da}{d\xi} = -\frac{n+2}{6}\frac{S_4}{(2\pi)^4}\frac{T}{b^2}\lambda a\,, \tag{4.23}$$

$$\frac{dF}{d\xi} = -\frac{na^2}{4}\frac{S_4}{(2\pi)^4}\frac{T}{b^2}\,, \tag{4.24}$$

following from Eqs. (4.19,4.18,4.20). Here, $\xi = \ln(\Lambda/\Lambda')$ where Λ' is the current ultraviolet cutoff (that is the maximal wave vector of φ'). Let us stress that the equations (4.22,4.23) does not imply the condition (4.6) and works at arbitrary ξ if only $g \ll 1$. It is instructive to rewrite the equations (4.22,4.23) in terms of the invariant charge:

$$\frac{dg}{d\xi} = -g^2\,, \tag{4.25}$$

$$\frac{d\lambda}{d\xi} = -g\lambda\,, \tag{4.26}$$

$$\frac{da}{d\xi} = -\frac{n+2}{n+8}ga\,. \tag{4.27}$$

Just the equations of (4.25-4.27) type containing the invariant charge are usually called RG-equations.

In the approximation used above the coefficient b does not vary. To find its renormalization one should take into account high-order terms of $\mathcal{F}_{\rm int}$. Namely we should take the third-order term

$$\mathcal{F}^{(3)}_{\rm int} = \frac{\lambda}{6}\int \mathrm{d}\boldsymbol{r}\,(\varphi'\tilde{\varphi})\tilde{\varphi}^2\,. \tag{4.28}$$

It is clear that $\langle\mathcal{F}^{(3)}_{\rm int}\rangle = 0$ and therefore only the contribution into $\mathcal{F}' - \mathcal{F}$ determined by the second term in the right-hand side of (4.11) should be taken into account. This term gives

$$\mathcal{F}' - \mathcal{F} \leftarrow -\frac{(n+2)\lambda^2}{36T}\int \mathrm{d}\boldsymbol{r}_1\,d\boldsymbol{r}_2\,\varphi'(\boldsymbol{r}_1)\varphi'(\boldsymbol{r}_2)\tilde{G}^3(\boldsymbol{r})\,,$$

where $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$. Again, the characteristic value of $\boldsymbol{r}$ is Λ'^{-1} whereas we are interested in slow φ'. Therefore in the main approximation we should substitute $\varphi'(\boldsymbol{r}_2)$ by $\varphi'(\boldsymbol{r}_1)$. Then we get a contribution determined by the integral sitting on the upper limit (in q-representation), the term should be incorporated into the redefinition of the transition temperarute T_c. To obtain a logarithmic integral giving a contribution to the renormalization of b we should expand $\varphi'(\boldsymbol{r}_2)$ over $\boldsymbol{r}$:

$$\varphi'(\boldsymbol{r}_2) = \varphi'(\boldsymbol{r}_1) + r_\alpha\nabla_\alpha\varphi'(\boldsymbol{r}_1) + \frac{1}{2}r_\alpha r_\beta\nabla_\alpha\nabla_\beta\varphi'(\boldsymbol{r}_1) + \ldots$$

The first term of the expansion does not produce a contribution into $\mathcal{F}' - \mathcal{F}$ (the corresponding integral is equal to zero because of integration of an odd function over angles). Thus the second term of the expansion is relevant. It produces the contribution

$$\mathcal{F}' - \mathcal{F} \leftarrow \frac{(n+2)\lambda^2}{9\cdot 32T}\int \mathrm{d}\boldsymbol{r}_1\,d\boldsymbol{r}\,[\nabla\varphi'(\boldsymbol{r}_1)]^2\tilde{G}^3(\boldsymbol{r})r^2\,,$$

where we have performed averaging over angles and taken the integral once in part. Substituting here (4.13) we find

$$\Delta b = \frac{(n+2)\lambda^2 T^2}{9\cdot 2^9\pi^4 b^3}\Delta\xi\,. \tag{4.29}$$

The corresponding renorm-group equation is

$$\frac{db}{d\xi} = \frac{n+2}{2(n+8)^2}g^2 b\,. \tag{4.30}$$

We see that the right-hand side of (4.30) is proportional to the second power of g what is the manifestation of the fact that the renormalization of b appears only in the second (two-loop) order of the renorm-group procedure.

Let us now consider the dimensionality $d = 4-\epsilon$ where $\epsilon \ll 1$. Then all the scheme will be the same if to substitute (4.21) by

$$g = \frac{n+8}{6}\frac{S_4}{(2\pi)^4}\frac{T}{b^2}\lambda\Lambda'^{-\epsilon}\,. \tag{4.31}$$

Then the renorm-group equations (4.26,4.27,4.30) keep their shape whereas (4.25) is rewritten as

$$\frac{dg}{d\xi} = \epsilon g - g^2\,, \tag{4.32}$$

The equation has the stable fix point $g = \epsilon$ and therefore $g \to \epsilon$ at $\xi \to \infty$. Substituting the value into (4.26,4.27,4.30) we get a power behavior of a, λ, b. Particularly,

$$a \propto (T - T_c)(\Lambda')^{(n+2)/(n+8)\epsilon}\,. \tag{4.33}$$

Let us find the exponents α and ν in the first ϵ approximation. Their definition is

$$C_V \propto |T - T_c|^{-\alpha}\,, \qquad R_c \propto |T - T_c|^{-\nu}\,, \tag{4.34}$$

where C_V is the heat capacity and R_c is the critical length. The definition of R_c can be extracted if to compare bR_c^{-2} and a which can be taken at $\Lambda' \sim R_c^{-1}$. In the first ϵ approximation b can be regarded to be a constant and we get (in the same approximation) from (4.33)

$$\nu = \frac{1}{2} + \frac{\epsilon}{4}\frac{n+2}{n+8}\,. \tag{4.35}$$

To find the exponent α we should use the modification of (4.24):

$$\frac{dF}{d\xi} \propto a^2(\Lambda')^{-\epsilon}\,. \tag{4.36}$$

Substituting here (4.33), integrating up to $\Lambda' \sim R_c^{-1}$ we get the singular contribution to the free energy

$$F_{\rm sing} \propto (T - T_c)^2 R_c^{(4-n)/(n+8)\epsilon}\,.$$

Taking now the second derivative over T and using (4.35) we get

$$\alpha = \frac{4-n}{2(n+8)}\epsilon\,. \tag{4.37}$$

Problems

Problem 4.1
Find the large-scale behavior of the vertices λ and λ_1 ($d = 4$) figuring in Landau functional

$$\mathcal{F} = \int \mathrm{d}\boldsymbol{r}\,\left\{\frac{b}{2}(\nabla\varphi_1)^2 + \frac{b}{2}(\nabla\varphi_2)^2 + \frac{\lambda}{24}(\varphi_1^2+\varphi_2^2)^2 + \frac{\lambda_1}{24}(\varphi_1^2-\varphi_2^2)^2\right\}\,. \tag{4.38}$$

Problem 4.2
Find the renorm-group equations in the space of dimensionality $d = 4$ for the coefficients of Landau expansion

$$\mathcal{F}_{\rm high} = \int {\rm d}\boldsymbol{r} \sum_{n>2} \frac{1}{(2n)!} \lambda_n \varphi^{2n} \,. \tag{4.39}$$

Solution of the Problem 4.2

$$\frac{d\lambda_n}{d\xi} = -\frac{1}{3} n(2n-1) g \lambda_n \,.$$

5. WEAK CRYSTALLIZATION THEORY

The theory of weak crystallization is constructed in terms of the Landau phase transition theory. Therefore, in the first place, one should introduce the order parameter, associated with the transition. For this purpose, define the following quantity:

$$\varphi = \rho_{short}/\rho. \tag{5.1}$$

Here ρ is a long-wavelength component of the density and ρ_{short} is a short-wavelength component of the density. In virtue of the definition, the field φ contains Fourier components with the wave vectors of the order of the inverse molecular size. In the liquid phase the average $\langle\varphi\rangle = 0$, in the crystalline phase there emerges a non-zero average $\langle\varphi\rangle$. Thus, the field φ can be regarded as the crystallization order parameter.

As follows from the definition (5.1), the average $\langle\varphi\rangle$ characterizes the amplitude of the short-wavelength density modulation of the crystalline phase. At the weak crystallization there should appear a non-zero average, satisfying the condition:

$$\langle\varphi\rangle \ll 1 \,. \tag{5.2}$$

Practically in all known crystals $\langle\varphi\rangle \sim 1$, but in some liquid crystilline phases $\langle\varphi\rangle$ satisfying the inequality (5.2) is observed. Generally speaking, the average $\langle\varphi\rangle$ is a sum of an infinite number of spatial Fourier harmonics. Yet, under the condition (5.1), out of these harmonics one can single out principal harmonics, whose number is finite. The remaining harmonics will have amplitudes much smaller than the amplitudes of the principal harmonics.

In the theoretical investigation of weak crystallization we should start from the Landau functional $\mathcal{F}_L(\varphi)$. We will, as previously, confine ourselves to a few first terms of the expansion of the Landau functional in the order parameter, this expansion is justified by the inequality (5.2). The first terms of the expansion of $\mathcal{F}_L$ in φ can be written as

$$\mathcal{F}_L/V = \sum_{\boldsymbol{q}} \frac{\tau(\boldsymbol{q})}{2} \varphi(\boldsymbol{q})\varphi(-\boldsymbol{q}) - \sum_{\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3=0} \frac{\mu(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)}{6} \varphi(\boldsymbol{q}_1)\varphi(\boldsymbol{q}_2)\varphi(\boldsymbol{q}_3)$$
$$+ \sum_{\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3+\boldsymbol{q}_4=0} \frac{\lambda(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3,\boldsymbol{q}_4)}{24} \varphi(\boldsymbol{q}_1)\varphi(\boldsymbol{q}_2)\varphi(\boldsymbol{q}_3)\varphi(\boldsymbol{q}_4) \,, \tag{5.3}$$

where V is the volume of the system. There is no linear term in (5.3) since φ is a short-wavelength field and, consequently, does not involve a zero Fourier harmonic.

The phase transition, associated with the emergence of the average $\langle\varphi\rangle$, occurs when the parameter τ in (5.3) diminishes. Due to the presence of the cubic term in the expansion (5.3) this transition is a first-order transition [Landau, 1937]. Thus, for this theory to hold, it is necessary that the additional condition of the small value of the coefficient μ in the expansion (5.3) should be fulfilled. This condition could be expressed via the inequality:

$$\mu/\lambda \ll 1 \qquad . \tag{5.4}$$

Since the field φ is short-wavelength, the dependence of the coefficients of the expansion of the Landau functional $\mathcal{F}_L$ in φ on the wave vectors of the field φ is fairly important. So, the coefficient τ in (5.3) is a function of module of the wave vector $\boldsymbol{q}$. We will have in mind the situation when $\tau(\boldsymbol{q})$ reaches the minimum on a certain sphere of the radius q_0 in the reciprocal space. We will be interested in fluctuations of the Fourier harmonics of the field φ with the wave vectors in the vicinity of this sphere. Expanding τ near the $|\boldsymbol{q}| = q_0$, we find with the necessary accuracy

$$\tau(\boldsymbol{q}) = a + b(|\boldsymbol{q}| - q_0)^2. \tag{5.5}$$

Here the coefficients a and b do not any longer contain the dependence on $\boldsymbol{q}$. Note that $a = \tau(q_0)$. The vertex μ in (5.3) can be regarded as constant. Also assume that the condition $\lambda = \text{const}$ is fulfilled.

The parameter a in (5.5) changes its sign in the vicinity of the transition point. Therefore in the case when the phase transition takes place at a variation of the temperature, for the parameter a one can use the standard expression:

$$a = \alpha(T - T^*). \tag{5.6}$$

Here α is the constant, T^* is the temperature at which a becomes zero. Since the phase transition under consideration is a first-order transition, the temperature T^* does not coincide with the crystallization temperature, although it is close to it as long as the average $\langle\varphi\rangle$, emerging at the crystallization, is small. The applicability condition of Expr. (5.5) is an inequality:

$$|q - q_0| \ll q_0 \qquad . \tag{5.7}$$

In the same approximation the second order term of the expansion of the Landau functional in φ can be represented as

$$\mathcal{F}_L^{(2)} = \int \mathrm{d}\boldsymbol{r}\,.\left(a\varphi^2/2 + b\left[(\nabla^2 + q_0^2)\varphi\right]^2/8q_0^2\right). \tag{5.8}$$

This expression is handy since it is written in a local form.

Consider first the weak crystallization theory in the mean field approximation. Starting from (5.3) we can find the energies of different crystalline phases taking as $\langle\varphi\rangle$ sums of principal harmonics of the corresponding symmetry. The comparison of energy values of different phases shows that only the smectic phase SA, the columnar phase D_h and the body-centered cubic phase BCC can be absolutely stable at the condition $\lambda = \text{const}$. At increasing temperature the following cascade of phases is realized in the model:

$$SA - D_h - BCC - I\,, \tag{5.9}$$

where I designates the isotropic liquid.

The Landau functional $\mathcal{F}_L$ determines the energy, related to fluctuations of the order parameter $\varphi(\boldsymbol{r})$. Therefore in conformity with the Gibbs distribution, the probability of emergence of such fluctuations is

$$\exp(F - \mathcal{F}_L/T)\,. \tag{5.10}$$

As previously, we will designate correlation functions calculated with the weight (5.10) by angular brackets. Introduce the special notation for the pair correlation function

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle\varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\rangle - \langle\varphi(\boldsymbol{r}_1)\rangle\langle\varphi(\boldsymbol{r}_2)\rangle. \tag{5.11}$$

For the correlation function (5.11) there is a standard relation

$$\hat{\tau}G(\boldsymbol{r}, \boldsymbol{r}_1) - \int d\boldsymbol{r}_2\Sigma(\boldsymbol{r}, \boldsymbol{r}_2)G(\boldsymbol{r}_2, \boldsymbol{r}_1) = T\delta(\boldsymbol{r} - \boldsymbol{r}_1). \tag{5.12}$$

Here the operator

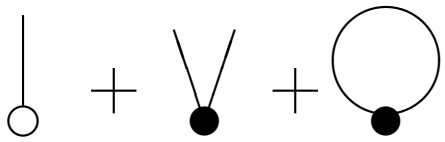
$$\hat{\tau} = a + b(\nabla^2 + q_0^2)^2/4q_0^2 \tag{5.13}$$

acts upon the argument $\boldsymbol{r}$. The bare value G_0 of the correlation function (5.11) is deduced if to put in (5.12) $\Sigma = 0$. It is clear that G_0 depends only on the difference $\boldsymbol{r} - \boldsymbol{r}_1$. In the Fourier representation the expression for the bare value G_0 will read

$$G_0(\boldsymbol{q}) = \int \mathrm{d}\boldsymbol{r}\,\exp(-i\boldsymbol{q}\boldsymbol{r})G_0(\boldsymbol{r}, 0) = \frac{T}{a + b(q - q_0)^2}\,. \tag{5.14}$$

Here we have used the inequality (5.7).

To calculate correlation functions of the field φ with fluctuations taken into account, one can make use of the diagram technique where the bare Green's function is determined by Expr. (5.14) and the bare vertices are determined by the interaction terms in (5.3). The perturbation series for such quantities as G, Σ can be constructed starting from the representation (5.10). It proves that in the weak crystallization theory the one-loop approximation is the main one [Brazovskii, 1975]. In this approximation Σ is determined by the diagram given in figure



In this figure the circle denotes the triple vertex μ, and the disk denotes the quartic vertex λ. The solid line in the loop is the Green's function (5.11) and the line with a free edge is the average $\langle\varphi\rangle$. This diagram representation can be easily written out analytically if to assume that $\lambda = \text{const}$. Bearing in mind that also $\mu = \text{const}$, we find

$$\Sigma(\boldsymbol{r}, \boldsymbol{r}_1) = \Big(\mu\langle\varphi(\boldsymbol{r})\rangle - \lambda\langle\varphi(\boldsymbol{r})\rangle^2/2 - \lambda G(\boldsymbol{r}, \boldsymbol{r})/2\Big)\delta(\boldsymbol{r} - \boldsymbol{r}_1) \tag{5.15}$$

Let us introduce the designation:

$$\Delta = a + \lambda\overline{\langle\varphi(\boldsymbol{r})\rangle^2}/2 + \lambda\overline{G(\boldsymbol{r}, \boldsymbol{r})}/2. \tag{5.16}$$

where the line above the function denotes the spatial averaging. In other words, the line above the function implies that one should retain only the zero Fourier harmonics in it. Now the equation (5.12) can be written as

$$\Big(\Delta + b(\nabla^2 + q_0^2)^2/4q_0^2 - \Theta(\boldsymbol{r})\Big)G(\boldsymbol{r}, \boldsymbol{r}_1) = T\delta(\boldsymbol{r} - \boldsymbol{r}_1). \tag{5.17}$$

where $\overline{\Theta(\boldsymbol{r})} = 0$. The term with Θ in (5.17) produces small corrections and consequently can be omitted. Then the function G will be dependent only on the difference of the coordinates $\boldsymbol{r} - \boldsymbol{r}_1$. In the Fourier representation, introduced similarly to (5.14), we get

$$G(\boldsymbol{q}) = T/(\Delta + b(q - q_0)^2) \qquad . \tag{5.18}$$

This expression differs from the bare expression (5.14) by the replacement $a \to \Delta$. We will refer to the quantity Δ as the gap, which is justified by the form of the function (5.18).

Now calculate the single-point correlation function $G(\boldsymbol{r}, \boldsymbol{r})$ figuring in the equation (5.15):

$$G(\boldsymbol{r}, \boldsymbol{r}) = \int \mathrm{d}\boldsymbol{q} G(\boldsymbol{q})/(2\pi)^3 = Tq_0^2/2\pi(b\Delta)^{1/2}. \tag{5.19}$$

Here we have used the inequality (5.7) and confined ourselves to the vicinity of the sphere $|\boldsymbol{q}| = q_0$ in the reciprocal space. For characteristic vectors, determining the integral (5.19), in virtue of (5.18) we have an estimate:

$$|q - q_0| \sim (\Delta/b)^{1/2} \qquad . \tag{5.20}$$

Therefore for the constraint (5.7) to hold, it is necessary that the inequality

$$\Delta \ll bq_0^2 \tag{5.21}$$

should be fulfilled. Employing the expression (5.19), from (5.16) we get an equation for the gap Δ

$$\Delta = a + \lambda\overline{\langle\varphi(\boldsymbol{r})\rangle^2}/2 + \beta\Delta^{-1/2}, \tag{5.22}$$

where

$$\beta = \lambda T q_0^2/4\pi b^{1/2} \qquad . \tag{5.23}$$

The first two terms in the r.h.s. of (5.22) are the mean field terms and the last term emerges due to fluctuations. Note that for the liquid phase (i.e., at $\langle\varphi\rangle = 0$) the equation (5.22) has a solution for Δ at an arbitrary value of a. In other words, fluctuation effects in the model under study prove to be so strong that they stabilise the liquid phase (i.e., render this phase metastable) even at $a < 0$.

An amazing property of the equation (5.22) is that at $a \to 0$ the gap Δ does not tend to zero but remains a constant of the order of

$$\Delta \sim (\lambda^2 T^2 q_0^4/b)^{1/3}, \tag{5.24}$$

which becomes particularly evident for the liquid phase, where $\langle\varphi\rangle = 0$. This property testifies to a large strength of fluctuations of φ in the weak crystallization theory, which is accounted for by a large phase volume of fluctuations, distributed near the sphere in the reciprocal space. For comparison note that at an conventional second-order transition fluctuations are concentrated in the vicinity of one or several points in the reciprocal space.

Let us explain how the transition points between different phases can be found. We will regard $\lambda = \text{const}$, then Landau functional can be written as

$$\mathcal{F}_L = \int \mathrm{d}\boldsymbol{r} \left\{ \frac{a}{2}\varphi^2 + \frac{b}{8q_0^2}\left[(\nabla^2 + q_0^2)\varphi\right]^2 - \frac{\mu}{6}\varphi^3 + \frac{\lambda}{24}\varphi^4 \right\} . \tag{5.25}$$

The field φ can be devided into the average part (condensate) $\langle\varphi\rangle$ and $\tilde{\varphi}$ which describes fluctuations near the condensate. The structure of the condensate $\langle\varphi\rangle$ is determined by the symmetry of the phase under consideration. In the main approximation it can be written as

$$\langle\varphi\rangle = \sqrt{A}\sum_a \exp\left(i\phi_a + i\boldsymbol{k}_a\boldsymbol{r}\right) , \qquad |\boldsymbol{k}_a| = q_0 . \tag{5.26}$$

Here, A is the amplitude of the condensate and the set of the wave vectors $\boldsymbol{k}_a$ is related to the symmetry of the phase. Say, for the simple cubic phase $\boldsymbol{k}_a$ are directed along the edges of a cube. Of course $\boldsymbol{k}_a$ is a set of pairs with opposite values to ensure the real meaning of (5.26), so their number is $2N$. For the simple qubic case $N = 3$ and therefore there are six $\boldsymbol{k}_a$. For the body-centered cubic crystal or for the face-centered cubic crystal $\boldsymbol{k}_a$ are directed along diagonals of a cube and $N = 6$; the difference between the phases is in different relations between values of ϕ_a in (5.26). It is clear that various sets $\boldsymbol{k}_a$ cover different crystalline phases and also quasicrystalline, columnar and smectic ones. Say, a well known quasicrystalline phase corresponds to the set $\boldsymbol{k}_a$ directed along edges of an icosahedron. For a smectic phase there are only two opposite wave vectors in the set $\boldsymbol{k}_a$ (that is $N = 1$) and we can write

$$\langle\varphi\rangle = 2\sqrt{A}\,\cos\left(\phi + q_0 z\right) , \tag{5.27}$$

where $\boldsymbol{k}_a$ are supposed to be directed along Z-axis. Note that generally

$$\overline{\langle\varphi\rangle^2} = 2NA , \tag{5.28}$$

as follows from (5.26).

To compare energies of different phases we should start from the definition

$$\exp\left(-\frac{F}{T}\right) = \int \mathcal{D}\tilde{\varphi}\exp\left(-\frac{\mathcal{F}_L}{T}\right) , \tag{5.29}$$

determining the free energy F for a phase characterized by the condensate (5.26). Substituting the decomposition $\varphi = \langle\varphi\rangle + \tilde{\varphi}$ into (5.25) and keeping only relevant terms we get

$$\mathcal{F}_L = \mathcal{F}_L(\langle\varphi\rangle) + \int \mathrm{d}\boldsymbol{r} \left\{ \frac{a}{2}\tilde{\varphi}^2 + \frac{b}{8q_0^2}\left[(\nabla^2 + q_0^2)\tilde{\varphi}\right]^2 + \frac{\lambda}{2}NA\tilde{\varphi}^2 + \frac{\lambda}{24}\tilde{\varphi}^4 \right\} , \tag{5.30}$$

where we used the relation (5.28). The quantity $\mathcal{F}_L(\langle\varphi\rangle)$ in (5.30) is a function of A only. Say, for the smectic

$$\mathcal{F}_L(\langle\varphi\rangle) = aA + \frac{\lambda}{4}A^2 . \tag{5.31}$$

Then, examining in the one-loop approximation (the validity of which was established above) correlation functions of $\tilde{\varphi}$ at a given $\langle\varphi\rangle$ we get from (5.30) the expression (5.18) for Fourier transform of the pair correlation function where the gap Δ obeys the equation (5.22). The equation can be rewritten using the relation (5.28) as

$$\Delta = a + \lambda NA + \beta\Delta^{-1/2} . \tag{5.32}$$

Next, it is possible to find an explicit expression for $\mathrm{d}F/\mathrm{d}A$ taking a derivative of (5.29):

$$\frac{dF}{dA} = \frac{d}{dA}\mathcal{F}_L(\langle\varphi\rangle) + \frac{N\beta}{\sqrt{\Delta}} , \tag{5.33}$$

where we substituted the expression for $\langle(\tilde{\varphi})^2\rangle$ following from (5.18).

Of course for a stable (or, more precisely, metastable) phase dF/dA should be equal to zero and we find from (5.33) the condition

$$\frac{d}{dA}\mathcal{F}_L(\langle\varphi\rangle) + \frac{\beta N}{\sqrt{\Delta}} = 0\,, \tag{5.34}$$

relating A to the value of gap Δ. The equation (5.34) together with (5.32) determines the values of A and Δ for a given phase. Say, for the smectic phase they read

$$\Delta = a + \lambda A + \frac{\beta}{\sqrt{\Delta}}\,,$$
$$a + \frac{\lambda A}{2} + \frac{\beta}{\sqrt{\Delta}} = 0\,.$$

Besides, the expression (5.33) formally defines dF/dA for any A and therefore we can find the difference between the free energies of a given phase and of the liquid phase (corresponding to $A = 0$) as the following integral

$$F - F_{\rm liq} = \mathcal{F}_L(\langle\varphi\rangle) + \int\limits_0^A dA\,\frac{\beta N}{\sqrt{\Delta}}\,. \tag{5.35}$$

Expressing dA via $d\Delta$ from (5.32) and substituting into (5.35) we obtain

$$F - F_{\rm liq} = \mathcal{F}_L(\langle\varphi\rangle) + \frac{2\beta}{\lambda}(\sqrt{\Delta} - \sqrt{\Delta_0}) - \frac{\beta^2}{2\lambda}\left(\Delta^{-1} - \Delta_0^{-1}\right)\,, \tag{5.36}$$

where the subscript 0 designates the value of the gap in the liquid phase. Since the liquid phase is metastable at any a (5.36) defines the energy difference for any possible phase. Then using (5.36) we can find the energy difference between two arbitrary phases at a given a. The stable phase corresponds to the minimum free energy and phase transitions (of the first order) occur when free energies of two phases coincide.

The last condition can be used to establish the phase diagram of the system. Fluctuations change the phase cascade found in the mean field approximation. At small μ ($\mu \ll (\lambda^5 T^2 q_0^4/b)^{1/6}$) only the direct $I \to SA$ transition occurs. Note that although in the mean field theory (if to neglect μ) this transition should be continuous, fluctuations render this transition a first-order transition.

Problems

Problem 5.1
Regarding $\mu = 0$ find the terminating point $a_{\max}$ for existing the (metastable) smectic phase.
Problem 5.2
Regarding $\mu = 0$ find the value a at which the phase transition liquid–smectic occurs. Fluctuations of φ should be taken into account.

Solution of the Problem 5.2
Landau functional and the condensate for this case are

$$\mathcal{F}_L = \int \mathrm{d}\boldsymbol{r}\,\left\{\frac{a}{2}\varphi^2 + \frac{b}{8q_0^2}\left[(\nabla^2 + q_0^2)\varphi\right]^2 + \frac{\lambda}{24}\varphi^4\right\}\,. \tag{5.37}$$

$$\langle\varphi\rangle = 2\sqrt{A}\,\cos(q_0 z)\,. \tag{5.38}$$

The equations for the gap in the liquid phase Δ_0 and in the smectic phase Δ are

$$\Delta_0 = a + \frac{\beta}{\sqrt{\Delta_0}}\,, \tag{5.39}$$

$$\Delta = a + \lambda A + \frac{\beta}{\sqrt{\Delta}}\,. \tag{5.40}$$

The condition $dF/dA = 0$ gives

$$a + \frac{\lambda A}{2} + \frac{\beta}{\sqrt{\Delta}} = 0\,, \tag{5.41}$$

$$\Delta = -a - \frac{\beta}{\sqrt{\Delta}}\,, \tag{5.42}$$

the last relation being a consequence of (5.40). The condition following from

$$F_{\rm sm} - F_{\rm liq} = \mathcal{F}_L(\langle\varphi\rangle) + \frac{2\beta}{\lambda}(\sqrt{\Delta} - \sqrt{\Delta_0}) - \frac{\beta^2}{2\lambda}\left(\Delta^{-1} - \Delta_0^{-1}\right) = 0\,, \tag{5.43}$$

is written as

$$\sqrt{\Delta} + \sqrt{\Delta_0} = -3\frac{\beta}{a}\,. \tag{5.44}$$

Note that since $\mathcal{F}_L(\langle\varphi\rangle) < 0$ then it follows from (5.43) that $\Delta > \Delta_0$. Let us introduce designations

$$x = \frac{\sqrt{\Delta_0}}{\beta^{1/3}}\,, \qquad y = \frac{\sqrt{\Delta}}{\beta^{1/3}}\,. \tag{5.45}$$

Then it follows from (5.44)

$$a = -\frac{3\beta^{2/3}}{x+y}\,. \tag{5.46}$$

Substituting (5.46) into (5.39,5.42) we get

$$x^2 = -\frac{3}{x+y} + \frac{1}{x}\,, \tag{5.47}$$

$$y^2 = \frac{3}{x+y} - \frac{1}{y}\,. \tag{5.48}$$

One can obtain from the system (5.47,5.48) the relation

$$x^4 - 2x^3y - 2xy^3 + y^4 = 0\,.$$

It is reduced to the square equation for $\xi + \xi^{-1}$ where $\xi = y/x$. Taking the positive root of the equation we find

$$\xi^2 - (\sqrt{3}+1)\xi + 1 = 0\,.$$

Recalling that $\xi > 1$ (since $\Delta > \Delta_0$) we find

$$\xi = \xi = \frac{1+\sqrt{3}}{2} + \sqrt{\frac{\sqrt{3}}{2}}\,. \tag{5.49}$$

Next, we obtain from (5.47)

$$(x+y)^3 = (1+\xi)^2(\xi - 2) = 2^{1/2}3^{3/4}\,.$$

And finally we get from (5.46)

$$a = -3^{3/4}2^{-1/6}\beta^{2/3}\,. \tag{5.50}$$

6. SMECTICS, RG-EQUATIONS

As we have noted a smectic is a matter with an one-dimensional short-wavelength modulation of the density $\langle\varphi\rangle$ (see Lecture 4). To investigate the role of fluctuations in a smectic we should consider the density modulation φ as a fluctuating variable, which can be taken in the form

$$\varphi(\boldsymbol{r}) = \psi\cos(\phi)\,, \tag{6.1}$$

where ψ and ϕ are slow functions of coordinates. If we are interested in large-scale fluctuations, then ψ can be considered as a "frozen" variable and consequently only fluctuations of ϕ are relevant. Strictly speaking the representation (6.1) is correct only in the case of weak crystallization. In the general case we would take the function

$$\varphi(\boldsymbol{r}) = \sum_{n=0}^{\infty} \psi_n \cos(\phi_n)\,.$$

But at considering large-scale fluctuations ψ_n can be assumed to be "frozen" and we can believe $\phi_n = n\phi$. Therefore we return to the conclusion that in a smectic there exists only one soft (on large scales) variable ϕ related to the short-scale density modulation, ϕ has the meaning of the phase of the density modulation.

Let us now determine the principal terms of the expansion of the Landau functional in ϕ. First of all we should recall that the average $\langle\varphi\rangle$ in a smectic is proportional to $\cos(q_0 z)$ and therefore the equilibrium value of $\nabla\phi$ is not equal to zero, it should be

$$\mid \nabla\phi \mid = q_0\,. \tag{6.2}$$

Next, the Landau functional should be invariant under the transformation

$$\phi \to \phi + \mathrm{const}\,, \tag{6.3}$$

since it corresponds to the translation of the system of the smectic layers as a whole which cannot change the energy of the system. Therefore the main terms of the expansion of the Landau functional $\mathcal{F}_L$ in ϕ are

$$\mathcal{F}_\phi = \int \mathrm{d}^3 r\, \frac{B}{8}\left(q_0^{-2}(\nabla\phi)^2 - 1\right)^2, \tag{6.4}$$

where B is an elasticity module. The minimum of (6.4) is achieved on $\phi = q_0 z$, this function corresponds to the unperturbed system of smectic layers perpendicular to the Z-axis.

To describe fluctuations of the smectic layers we introduce the function $u(\boldsymbol{r})$ determining deviations of ϕ from its equilibrium value

$$\phi = q_0(z - u(\boldsymbol{r}))\,. \tag{6.5}$$

The quantity u can be considered as the local displacement of the smectic layers along the Z-axis. Now the term (6.4) can be rewritten as

$$\mathcal{F}_\phi = \int \mathrm{d}^3 r\, \frac{B}{2}\left(\nabla_z u - \frac{1}{2}(\nabla u)^2\right)^2. \tag{6.6}$$

This expression enables us to examine the correlation function

$$G(\boldsymbol{r}) = \langle u(\boldsymbol{r})u(0)\rangle\,. \tag{6.7}$$

The second order term of the expansion of (6.6) in u is

$$\mathcal{F}_{(2)} = \int \mathrm{d}^3 r\, \frac{B}{2}(\nabla_z u)^2\,. \tag{6.8}$$

It gives the following bare value of (6.7)

$$\int \frac{d^3 q}{(2\pi)^3} \exp(i\boldsymbol{q}r) T/(Bq_z^2)\,,$$

what is an indefinite expression. It causes us to take into account the additional term

$$\mathcal{F}_K = \int \mathrm{d}^3 r\, \frac{K}{2} q_0^2 (\nabla^2\phi)^2 = \int \mathrm{d}^3 r\, \frac{K}{2}(\nabla^2 u)^2\,. \tag{6.9}$$

Then the bare value of the correlation function (6.7) determined by (6.8,6.9) is

$$G_0(\boldsymbol{q}) = \frac{T}{Bq_z^2 + Kq^4}\,. \tag{6.10}$$

It is not very difficult to check that the integral

$$G_0(\boldsymbol{r}=0)=\int\frac{d^3q}{(2\pi)^3}\frac{T}{Bq_z^2+Kq^4} \tag{6.11}$$

diverges at small q and is consequently determined by the size of the smectic. Therefore it is more reasonable to consider the quantity

$$G_0(\boldsymbol{r}=0)-G_0(\boldsymbol{r})=\int\frac{d^3q}{(2\pi)^3}\frac{T}{Bq_z^2+Kq^4}\times$$
$$(1-\exp(i\boldsymbol{q}r))\simeq\frac{1}{4\pi}\frac{T}{\sqrt{BK}}\xi\,, \tag{6.12}$$

where

$$\xi=\ln\max\left(\Lambda r_\perp,\Lambda\left(\frac{K}{B}\right)^{1/4}\sqrt{\mid z\mid}\right),\quad r_\perp=\sqrt{x^2+y^2}\,, \tag{6.13}$$

and Λ is the cutoff wave vector.

Let us consider the influence of fluctuations of u on the density modulation φ. If to neglect selfinteraction of u (i.e. to take into account only the terms (6.8,6.9) in the Landau expansion) then averaging over fluctuations of u is reduced to Gaussian integrals which can be calculated explicitly. So the density modulation (6.1) averaged over fluctuations is determined by

$$\langle\cos\phi\rangle_0=\cos(q_0z)\exp\left(-\frac{1}{2}q_0^2\langle u^2\rangle_0\right). \tag{6.14}$$

Since the value $\langle u^2\rangle_0=G_0(\boldsymbol{r}=0)$ diverges if the size of the system tends to infinity the average φ in accordance with (6.1) is zero for an infinite system. We may say that fluctuations of u destroy the density modulation in the smectic. Consider now the pair correlation function of φ determined by

$$\langle\cos\phi(\boldsymbol{r})\cos\phi(0)\rangle_0=1/2\cdot\cos(q_0z)\exp\left(-q_0^2(G_0(0)-G_0(\boldsymbol{r})\right)\propto r_\perp^{-\zeta}\,, \tag{6.15}$$

where

$$\zeta=\frac{Tq_0^2}{4\pi\sqrt{BK}}\,. \tag{6.16}$$

The last relation in (6.15) is a consequence of (6.12), it is correct if $r_\perp^2\gg\mid z\mid\sqrt{K/B}$.

Consider now fluctuational corrections to (6.10) caused by the interaction terms which are the third and the fourth order terms of the expression (6.6)

$$\mathcal{F}_{(3)}=-\int\mathrm{d}^3r\,\frac{B}{2}\nabla_z u(\nabla u)^2\,, \tag{6.17}$$

$$\mathcal{F}_{(4)}=\int\mathrm{d}^3r\,\frac{B}{8}(\nabla u)^4\,. \tag{6.18}$$

Starting from these expressions it is possible to develop the perturbation series. Corrections to the bare expressions of correlation functions will be determined by diagrams where both triple and quartic vertices defined by (6.17,6.18) figure. An analysis shows that fluctuational corrections to these vertices are logarithmic and therefore we encounter the problem of summation of the main series of logarithmic diagrams similar to ones examined for the phase transition theory in four dimensions (see Lecture 3). For smectics it is a more bulky procedure because of the presence of two types of vertices. Therefore we consider an alternative method known as renorm-group (RG) method which proves to be useful in different contexts. This method works effectively if the main fluctuational corrections to the objects defined on a scale R are determined by fluctuations of the scales $r\ll R$. This is just the property characteristic of the logarithmic situation.

Let us devide the field u on the slow u' and the fast $\tilde{u}$ parts: $u=u'+\tilde{u}$. The quantity $\tilde{u}$ is

$$\tilde{u}(\boldsymbol{r})=\sum_q u_q exp(i\boldsymbol{q}r)\,,$$

where $\Lambda' < q_\perp < \Lambda$, the constant Λ' being the new cutoff wave vector. Then it is possible to introduce Landau functional $\mathcal{F}'_L(u')$ corresponding to this cutoff

$$\exp(-F'_L/T) = \int \mathcal{D}\tilde{u} \exp(-\mathcal{F}_L/T)\,. \tag{6.19}$$

Then all correlation functions of u determined on scales larger than $(\Lambda')^{-1}$ can be calculated starting from F', the expressions are the same as for F (see Lecture 2). To find F' the perturbation theory can be used since the number of degrees of freedom of $\tilde{u}$ is restricted (nevertheless we should believe $\Lambda' \ll \Lambda$ to have the possibility to calculate with the logarithmic accuracy). To develop this perturbation series we should expand (6.6) in $\tilde{u}$. The first term of this expansion is equal to zero since $\tilde{u}$ is the fast field. Therefore the main term of this expansion will be of the second order in $\tilde{u}$:

$$\mathcal{F}_L(u) \simeq \mathcal{F}_L(u') + \tilde{\mathcal{F}}_{(2)} + \mathcal{F}_{int}\,, \tag{6.20}$$

where

$$\tilde{\mathcal{F}}_{(2)} = \int \mathrm{d}^3 r \left(\frac{B}{2} (\nabla_z \tilde{u})^2 + \frac{K}{2} (\nabla^2 \tilde{u})^2 \right), \tag{6.21}$$

$$\mathcal{F}_{int} = -\int \mathrm{d}^3 r \left(\frac{B}{2} (\nabla_z u - (\nabla u)^2/2)(\nabla \tilde{u})^2 + B \nabla u \nabla \tilde{u} \nabla_z \tilde{u} \right). \tag{6.22}$$

Expanding now (6.19) in $\mathcal{F}_{int}$ we find

$$\mathcal{F}'_L(u') - \mathcal{F}_L(u') \simeq \langle \mathcal{F}_{int} \rangle_0 - \frac{1}{2T} \langle\langle \mathcal{F}_{int} \mathcal{F}_{int} \rangle\rangle_0\,, \tag{6.23}$$

where the subscript zero denotes averaging determined by $\tilde{\mathcal{F}}_{(2)}$.

The first term in (6.23) is

$$\langle \mathcal{F}_{int} \rangle_0 = \int \mathrm{d}^3 r\, B (\nabla u)^2 \langle (\nabla \tilde{u})^2 \rangle_0\,. \tag{6.24}$$

The same term originates from (6.4) at shifting q_0. Therefore (6.24) should be included into redefinition of q_0. The average $\langle\langle \mathcal{F}_{int} \mathcal{F}_{int} \rangle\rangle_0$ produces two terms. The first term is

$$-\frac{B^2}{8T} \int \mathrm{d}^3 r_1\, d^3 r_2\, (\nabla_z u_1 - (\nabla u_1)^2/2)(\nabla_z u_2 - (\nabla u_2)^2/2) \times \\ \langle (\nabla \tilde{u}_1)^2 (\nabla \tilde{u}_2)^2 \rangle_0\,. \tag{6.25}$$

The average here has a characteristic scale $1/\Lambda'$ and therefore we can substitute $u_2 \to u_1$ since u is the slow field. Then we conclude that (6.25) gives the renormalization of the module B. It is not very difficult to calculate the integral explicitly, the result is

$$B' - B = -\frac{T B^{3/2}}{32\pi K^{3/2}} \ln(\Lambda/\Lambda')\,. \tag{6.26}$$

The second term produced by $\langle\langle \mathcal{F}_{int} \mathcal{F}_{int} \rangle\rangle_0$ is

$$-\frac{B^2}{8T} \int \mathrm{d}^3 r_1\, d^3 r_2\, \nabla_i u_1 \nabla_k u_2 \langle \nabla_z \tilde{u}_1 \nabla_i \tilde{u}_1 \nabla_z \tilde{u}_2 \nabla_k \tilde{u}_2 \rangle_0\,. \tag{6.27}$$

If to substitute $u_2 \to u_1$ we arrive at the term of the (6.24) type which should be included into redefinition of q_0. It means that we must expand $u_2 - u_1$ in $\boldsymbol{r}_2 - \boldsymbol{r}_1$. The first term gives a contribution vanishing after averaging over angles and the second term gives the integral over $\boldsymbol{r}_1$ with the integrand proportional to $(\nabla^2 u)^2$. Therefore this term gives the renormalization of the module K, $K' - K$ being proportional to

$$\int \mathrm{d}^3 r_2\, (\boldsymbol{r}_2 - \boldsymbol{r}_1)^2 \langle \nabla_z \tilde{u}_1 \nabla_i \tilde{u}_1 \nabla_z \tilde{u}_2 \nabla_i \tilde{u}_2 \rangle_0\,.$$

After calculating this integral we find

$$K' - K = \frac{TB^{1/2}}{64\pi K^{1/2}} \ln(\Lambda/\Lambda') \,. \tag{6.28}$$

All said above is correct if $B - B' \ll B$ and $K' - K \ll K$. That means that

$$TB^{1/2} \ll K^{3/2} \,, \tag{6.29}$$

then the conditions $B - B' \ll B$ and $K' - K \ll K$ are compatible with $\Lambda \gg \Lambda'$.

Let us now produce a multi-step procedure of excluding fast variables shifting the cutoff Λ at each step on a value which guaranties small values of the corrections to B, K (at one step). Then instead of the relations (6.26,6.28) we can formulate the differential equations

$$dB/d\xi = -\frac{TB^{3/2}}{32\pi K^{3/2}} \,, \quad dK/d\xi = \frac{TB^{1/2}}{64\pi K^{1/2}} \,, \tag{6.30}$$

where $\xi = \ln(\Lambda/\Lambda')$. The equations of (6.30) type are called RG-equations. To solve the equations let us introduce

$$g = \frac{5TB^{1/2}}{128\pi K^{3/2}} \,, \tag{6.31}$$

which satisfies the equation

$$dg/d\xi = -g^2 \tag{6.32}$$

with the solution

$$g = \frac{g_0}{1 + g_0 \xi} \,. \tag{6.33}$$

Here g_0 is the bare value of the constant g that is the value of g on scales $1/\Lambda$. We see that if $\xi \gg g_0^{-1}$ then $g \simeq \xi^{-1} \to 0$. This property which is referred as "zero-charge" justifyes the above consideration for large ξ. After substitution of (6.33) into (6.30) we find that at large ξ [Nelson and Pelkovits, 1982]

$$B \propto \xi^{-4/5} \,, \quad K \propto \xi^{2/5} \,. \tag{6.34}$$

Now we can determine the form of correlation functions of u. If we are interested in the correlation function on a scale r we should exclude from the integration over u harmonics with the wave vectors larger then r^{-1} using (6.19). It is reduced to the renormalization $B \to B'$, $K \to K'$ where B', K' are solutions of (6.30) taken at $\xi \simeq \ln(r\Lambda)$. After this exclusion we can neglect selfinteraction of u on scales less or of the order of r. Then for the pair correlation function we return to the expression (6.11) but with renormalized values of the modules B, K. Since $\ln(r\Lambda)$ grows with increasing scale this procedure is justified on large scales. Above we have implied that $r_\perp^2 \gg\mid z \mid \sqrt{K/B}$. In the general case the renormalization of B, K is determined by the same relations (6.30,6.34) where ξ is defined by (6.13).

Problems

Problem 6.1
Find the behavior of the pair correlation function of the density fluctuations

$$S(\mathbf{r}_\perp) = \langle \delta\rho(\mathbf{r}_\perp, z = 0) \delta\rho(0,0) \rangle \,,$$

in a smectic in the region of scales where the moduli B and K are strongly renormalized by fluctuations of u.

Solution of the Problem 6.1

$$
\begin{aligned}
S(\mathbf{r}_\perp) &\propto \langle \cos(q_0(u(\mathbf{r}_\perp,0)-u(0,0))\rangle \\
S = Z(\xi)\langle \cos(q_0(u(\mathbf{r}_\perp,0)-u(0,0))\rangle', &\quad \xi = \ln(\Lambda/\Lambda') \\
u \to u+\tilde{u}\,, &\quad \xi \to \xi + \Delta\xi\,, \\
Z \to Z(1-q_0^2\langle(\tilde{u}(\mathbf{r}_\perp,0)-\tilde{u}(0,0))^2\rangle/2) & \\
dZ = -Zq_0^2(dG(0)-dG(\mathbf{r})) & \\
Z = Z_0\exp(-q_0^2(G(0)-G(\mathbf{r}))) & \\
S(\mathbf{r}_\perp) \propto \exp\left(-C(\ln(\Lambda r_\perp))^{6/5}\right) & \\
C = \frac{5}{6}\frac{Tq_0^2}{4\pi}\left(\frac{5T}{128\pi K_0^4 B_0^2}\right)^{1/5} &
\end{aligned}
$$

Problem 6.2
Besides the leading term

$$\mathcal{F} = \int \mathrm{d}^3 r \left\{ \frac{B}{2}\left[\nabla_z u - (\nabla u)^2/2\right]^2 + \frac{K}{2}(\nabla^2 u)^2 \right\}, \tag{6.35}$$

in Landau expansion for a smectic one can consider also the subleading term

$$\mathcal{F}_{\mathrm{sub}} = \int \mathrm{d}^3 r \left\{ \frac{K_1}{2} l_i l_j \nabla_i \nabla_m u \nabla_j \nabla_m u + \frac{K_2}{2} l_i l_j l_m l_n \nabla_i \nabla_m u \nabla_j \nabla_n u \right\}. \tag{6.36}$$

Here, $\boldsymbol{l}$ is the unit vector, perpendicular to smectic layers. In components

$$l_\alpha = -\frac{\nabla_\alpha u}{\sqrt{1-2\nabla_z u + (\nabla u)^2}}\,, \qquad l_z = \frac{1-\nabla_z u}{\sqrt{1-2\nabla_z u + (\nabla u)^2}}\,. \tag{6.37}$$

Find RG-equations for the coefficients K_1 and K_2 figuring in (6.36).

Solution of the Problem 6.2
Fluctuations of the unit vector $\boldsymbol{l}$ are irrelevant. Therefore $\mathcal{F}_{\mathrm{sub}}$ is reduced simply to

$$\mathcal{F}_{\mathrm{sub}} = \int \mathrm{d}^3 r \left\{ \frac{K_1}{2} \nabla_z \nabla_m u \nabla_z \nabla_m u + \frac{K_2}{2} \nabla_z \nabla_z u \nabla_z \nabla_z u \right\}.$$

Therefore the pair correlation function of u is now

$$G(\boldsymbol{q}) = \frac{T}{Bq_z^2 + Kq^4 + K_1 q^2 q_z^2 + K_2 q_z^4}\,.$$

Next, renormalization of the coefficients K, K_1, K_2 is determined by the same diagram with the same vertex B and therefore we can write

$$
\begin{aligned}
&\Delta K q^4 + \Delta K_1 q^2 q_z^2 + \Delta K_2 q_z^4 \Leftarrow -\frac{B^2}{2T} q_i q_j \int \frac{d^3k}{(2\pi)^3} G(\boldsymbol{k}+\boldsymbol{q}/2)G(\boldsymbol{k}-\boldsymbol{q}/2) \\
&\left\{(k_z+q_z/2)^2(k_i-q_i/2)(k_j-q_j/2) + (k_z^2-q_z^2/4)(k_i k_j - q_i q_j/4)\right\} \\
&\Rightarrow -\frac{B^2}{T}\int \frac{d^3k}{(2\pi)^3} k_z^2 (\boldsymbol{kq})^2 G(\boldsymbol{k}+\boldsymbol{q}/2)G(\boldsymbol{k}-\boldsymbol{q}/2) + \frac{B^2}{2T} q^2 q_z^2 \int \frac{d^3k}{(2\pi)^3} k_z^2 G^2(\boldsymbol{k})\,.
\end{aligned}
$$

Here, the first term produces a logarithm only for $q_\perp$ and is consequently equal to $\Delta K q_\perp^4$ and the second term is equal to

$$\frac{TB^{1/2}}{16\pi K^{1/2}} q^2 q_z^2 \Delta\xi\,.$$

Recalling the expression

$$\Delta K = \frac{TB^{1/2}}{64\pi K^{1/2}}\Delta\xi\,,$$

we obtain

$$\Delta K_1 = \frac{TB^{1/2}}{32\pi K^{1/2}}\Delta\xi\,, \qquad \Delta K_2 = \frac{TB^{1/2}}{64\pi K^{1/2}}\Delta\xi\,.$$

And finally

$$\frac{dK_1}{d\xi} = \frac{4}{5}gK\,, \qquad \frac{dK_2}{d\xi} = \frac{2}{5}gK\,.$$

Note, that asymptotically at $\xi \to \infty$: $K_1 = 2K$, $K_2 = K$.

7. NONLINEAR SIGMA-MODEL

The problem originates from the physics of $2d$ ferromagnetics, where magnetization is a $3d$ vector whereas it depends on $2d$ radius-vector $\boldsymbol{r}$. If we are outside the critical region the absolute value of the magnetization is "frozen" but fluctuations of its direction which we will characterize by the unit vector $\boldsymbol{n}$ should be taken into account. The energy associated with fluctuations of $\boldsymbol{n}$ in the exchange approximation is not equal to zero only for nonhomogeneous fluctuations and can be written in the following form

$$\mathcal{F}_{\boldsymbol{n}} = \int \mathrm{d}^2 r\, b/2\, \nabla n_\mu \nabla n_\mu\,. \tag{7.1}$$

If we want to calculate something associated with fluctuations of $\boldsymbol{n}$, then we should accept a parameterization of $\boldsymbol{n}$. The simplest parameterization is

$$\begin{aligned} n_\mu &= (\varphi_1, \varphi_2, \sqrt{1-\varphi_1^2-\varphi_2^2}) \\ &\simeq (\varphi_1, \varphi_2, 1-(\varphi_1^2+\varphi_2^2)/2)\,, \end{aligned} \tag{7.2}$$

where we have expanded $\sqrt{1-\varphi_1^2-\varphi_2^2}$ in φ. It is useful if to believe that the problem is solved in terms of small fluctuations of $\boldsymbol{n}$ near its equilibrium value (along Z-axis). Let us examine this possibility.

First we should find the expansion of $\mathcal{F}_{\boldsymbol{n}}$ in φ, the lowest order terms are

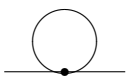
$$\mathcal{F}_{(2)} = \int \mathrm{d}^2 r\, b/2 \left((\nabla\varphi_1)^2 + (\nabla\varphi_2)^2 \right), \tag{7.3}$$

$$\mathcal{F}_{(4)} = \int \mathrm{d}^2 r\, b/2\, (\varphi_1\nabla\varphi_1 + \varphi_2\nabla\varphi_2)^2\,. \tag{7.4}$$

Now we can develop the perturbation series where the bare values of the correlation functions are determined by (7.3) and the vertices are determined by higher order terms of the expansion of $\mathcal{F}_{\boldsymbol{n}}$ in φ starting from (7.4). The bare value of the pair correlation function is

$$\begin{aligned} &G_0(\boldsymbol{r}) = \langle\varphi_1(\boldsymbol{r})\varphi_1(0)\rangle_0 = \langle\varphi_2(\boldsymbol{r})\varphi_2(0)\rangle_0 = \\ &\int \frac{d^2q}{(2\pi)^2}\exp(i\boldsymbol{q}r)\frac{T}{bq^2}\,. \end{aligned} \tag{7.5}$$

We see that this expression diverges logarithmically at small q, this divergence being cut by the size of the system. The first fluctuation correction to (7.5) is determined by the following diagram



where the lines correspond to (7.5) and the vertex is determined by (7.4). The loop in Figure represents the first contribution to the self-energy function which is determined by the averages

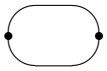
$$\langle\varphi_1^2\rangle_0\,,\quad \langle\varphi_1\nabla\varphi_1\rangle_0\,,\quad \langle(\nabla\varphi_1)^2\rangle_0\,.$$

The term $\langle\varphi_1\nabla\varphi_1\rangle_0$ is equal to zero due to the symmetry, the term $\langle(\nabla\varphi_1)^2\rangle_0$ is determined by the integral over the wave vectors $\boldsymbol{q}$ diverging at large q. Therefore the value of this integral cannot be actually calculated in the framework of the large-scale theory. Nevertheless the constant $\langle(\nabla\varphi_1)^2\rangle_0$ should be regarded as zero because of the symmetry reasons. The problem is that this constant should be included into redefinition of the bare pair correlation function, namely it will be $G_0(q) = T/(bq^2 + m)$, what contradicts to the structure of (7.1). Therefore we conclude that only the term $\langle\varphi_1^2\rangle_0$ will determine the first contribution to the self-energy function. This term implies the redefinition $G_0 \to G$ where G differs from G_0 in the value of b, the new value of b being $b + \Delta b$ where

$$\Delta b = b\int \frac{d^2q}{(2\pi)^2}\frac{T}{bq^2} = \frac{T}{2\pi}\ln(L\Lambda)\,. \tag{7.6}$$

Here L is the size of the specimen and Λ is the ultraviolet cutoff.

Since the bare pair correlation function is determined by the second order term (7.3) we may say that due to the presence of $\mathcal{F}_{(4)}$ a logarithmic renormalization of $\mathcal{F}_{(2)}$ occurs. Analogously due to the presence of $\mathcal{F}_{(6)}$ a logarithmic renormalization of $\mathcal{F}_{(4)}$ occurs and so further. Moreover there exist nonlinear in vortices terms producing logarithmic renormalization of the vertices. An example is presented by the diagram



where both vertices are determined by (7.4). This diagram describes the fluctuational contribution $\mathcal{F}_{(4)}\cdot\mathcal{F}_{(4)} \to \mathcal{F}_{(4)}$. So we encounter the problem of investigating the behavior of an infinite number of vertices which cannot be solved in the closed form. Nevertheless we understand that the presence of an infinite number of vertices with different laws of renormalization is an artifact. The problem is that there exist only one relevant term (7.1) in the Landau expansion (which is a consequence of the symmetry) and therefore the problem should be reduced to the renormalization of a single coefficient b. The appearance of an infinite number of independent parameters is the consequence of our choice of parameterization (7.2) which breaks the explicit rotational symmetry of (7.1). Therefore we should seek an alternative way.

First of all due to the logarithmic character of the renormalization it is worthwhile to use from the beginning the RG scheme (see Lecture 5). Next the fast and slow degrees of freedom should be divided in a way conserving the explicit rotational symmetry. It is possible to do if to suggest [Polyakov, 1975]

$$n_\mu = R_{\mu\nu}\tilde{n}_\nu\,, \tag{7.7}$$

where $\tilde{\boldsymbol{n}}$ is the unit vector and R is the orthogonal matrix

$$R_{\mu\nu}R_{\lambda\nu} = \delta_{\mu\lambda}\,, \tag{7.8}$$

what ensures the property $\boldsymbol{n}^2 = 1$. We believe that $\tilde{\boldsymbol{n}}$ is the "fast" unit vector and that R is the "slow" matrix. The relation (7.7) guaranties that all relations will be invariant under the transformation $R_{\mu\nu} \to R_{0\mu\rho}R_{\rho\nu}$, where R_0 is any orthogonal matrix not depending on coordinates x, y. It implies that all expressions derived at exclusion of fast degrees of freedom will be rotationally invariant. This property does not depend on the fashion of the parameterization of $\tilde{\boldsymbol{n}}$. Therefore we can accept an arbitrary parameterization, actually we will use for $\tilde{\boldsymbol{n}}$ the parameterization (7.2). We suggest that the field φ is the sum of Fourier harmonics with the wave vectors $\Lambda' < q < \Lambda$ where Λ is the cutoff and Λ' is the new cutoff.

After substitution of (7.7) into (7.1) we find

$$\mathcal{F}_{\boldsymbol{n}} = \frac{b}{2}\int \mathrm{d}^2r\Big(\nabla\tilde{n}_\mu\nabla\tilde{n}_\mu + 2\nabla R_{\mu\lambda}R_{\mu\nu}\tilde{n}_\lambda\nabla\tilde{n}_\nu + \nabla R_{\mu\lambda}\nabla R_{\mu\nu}\tilde{n}_\lambda\tilde{n}_\nu\Big)\,. \tag{7.9}$$

Substituting here instead of $\tilde{\boldsymbol{n}}$ the expression (7.2) and expanding (7.9) up to the second order in φ we find

$$\mathcal{F}_{\boldsymbol{n}} \to \mathcal{F}_{(2)} + \mathcal{F}_{(R)} + \mathcal{F}_{1int} + \mathcal{F}_{2int}\,,$$

where $\mathcal{F}_{(2)}$ is determined by (7.3) and

$$\mathcal{F}_R = \frac{b}{2}\int \mathrm{d}^2 r (\nabla R_{\mu 3})^2 \,, \tag{7.10}$$

$$\mathcal{F}_{1int} = 2b\int \mathrm{d}^2 r \nabla R_{\mu 1} R_{\mu 2}\varphi_1 \nabla\varphi_2 + \dots \,, \tag{7.11}$$

$$\mathcal{F}_{2int} = \frac{b}{2}\int \mathrm{d}^2 r \Big((\nabla R_{\mu 1})^2\varphi_1^2 + (\nabla R_{\mu 2})^2\varphi_2^2 - (\nabla R_{\mu 3})^2(\varphi_1^2+\varphi_2^2)\Big) \,, \tag{7.12}$$

where $\dots$ designates some irrelevant terms. Comparing (7.10) with (7.1) we conclude that the role of the "slow" vector $\boldsymbol{n}$ is played by the quantity

$$n'_\mu = R_{\mu 3} \,, \tag{7.13}$$

which is a unit vector as a consequence of (7.8).

As previously (see Lecture 5) we have to introduce the "slow" part of the Landau functional in accordance with the relation

$$\exp(-\mathcal{F}'_L/T) = \int \mathcal{D}\varphi \, \exp(-\mathcal{F}_L/T) \,.$$

In the approximation we have accepted it is

$$\mathcal{F}'_{\boldsymbol{n}'} = \mathcal{F}_{(R)} + \langle \mathcal{F}_{2int}\rangle_0 - \langle \mathcal{F}_{1int}\mathcal{F}_{1int}\rangle_0/2T \,, \tag{7.14}$$

where $\langle \dots \rangle_0$ denotes averaging determined by $\mathcal{F}_{(2)}$ defined by (7.3). The averages in (7.14) can be represented by the two diagrams drawn above. The corresponding analytical expressions are proportional to $G(\boldsymbol{r}=0)$ and to $\int \mathrm{d}^2 r \, (\nabla G(\boldsymbol{r}))^2$. Both these values up to factors are reduced to $\xi = \ln(\Lambda/\Lambda')$. Calculating all factors in (7.14) and transforming the result using (7.8) we find

$$\mathcal{F}'_{\boldsymbol{n}'} = \int \mathrm{d}^2 r \, b'/2 \, \nabla n'_\mu \nabla n'_\mu \,. \tag{7.15}$$

Here $b' = b - bg\xi$ where $g = T/(2\pi b)$, the expression for b' is correct at $g\xi \ll 1$. Otherwise we should produce the multi-step procedure of excluding fast degrees of freedom so that at each step $g\Delta\xi \ll 1$. Then the behavior of g at decreasing Λ' will be described by the differential equation

$$dg/d\xi = g^2 \,, \tag{7.16}$$

which is no other than RG-equation. It has the solution

$$g = \frac{g_0}{1 - g_0\xi} \,, \tag{7.17}$$

where g_0 is the bare value of the constant g that is the value which g has on scales of the order of Λ^{-1}.

It is not very difficult to understand that the quantity g plays the role of the dimensionless coupling constant, only the small value of g justifies the expansion in φ made above. Therefore we conclude that the condition $g_0 \ll 1$ should be satisfied for the presented theory to be correct. Let us compare the expressions (7.16,7.17) with ones for the coupling constant g given in Lecture 5. We see that the sign in the right-hand of (7.16) is opposite what leads to growing the coupling constant g with increasing scale. In the quantum field theory this situation is called "asymptotic freedom". Because of growing the coupling constant the applicability condition of the presented theory is violated on large scales. As it follows from (7.17) it happens at scales

$$R_c \sim \Lambda^{-1}\exp(1/g_0) \,, \tag{7.18}$$

exponentially large in $1/g_0$. As the exact solution of the problem shows [Wiegmann, 1985] the correlation function $G(\boldsymbol{q})$ at $q < R_c^{-1}$ remains a finite value of the order of R_c^2 what can be interpreted as a consequence of generating the spontaneous gap in the spectrum of excitations related to $\boldsymbol{n}$, this gap having a purely fluctuational nature.

To recognize the behavior of the correlation functions of $\boldsymbol{n}$ one can consider the case of large number N of the components of the vector $\boldsymbol{n}$. First, instead of the explicit account the condition $n^2 = 1$ one can integrate over all N-components vectors $\boldsymbol{n}$ enforcing the condition $n^2 = 1$ by introducing the corresponding δ-function. The δ-function can be written as the integral over an auxiliary field μ of the corresponding exponent. Then e.g. the generating functional of the correlation functions of $\boldsymbol{n}$ can be written as

$$\mathcal{Z}(\boldsymbol{y}) = \int \mathcal{D}\boldsymbol{n}\, \mathcal{D}\mu\, \exp\left(-\mathcal{H} + \int \mathrm{d}^2 r\, \boldsymbol{y}\boldsymbol{n}\right), \tag{7.19}$$

$$\mathcal{H} = \frac{1}{4\pi g_0} \int \mathrm{d}^2 r\, \left[(\nabla \boldsymbol{n})^2 + \mu \boldsymbol{n}^2 - \mu\right]. \tag{7.20}$$

Strictly speaking the integration in (7.19) is performed over imaginary fields μ.

Let us first integrate in (7.19) over $\boldsymbol{n}$ and then over μ. The first integration is Gaussian and therefore the answer is expressed via the pair correlation function of $\boldsymbol{n}$ the equation for which (at a given μ) is

$$[\mu(\boldsymbol{r}_1) - \nabla_1^2]\langle n_a(\boldsymbol{r}_1) n_b(\boldsymbol{r}_2)\rangle = 2\pi g_0 \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\delta_{ab}\,. \tag{7.21}$$

As we will see the field μ fluctuates weakly over its average value μ_0. Therefore in the main approximation we can substitute μ in (7.21) by μ_0. Then we find

$$\langle n_a(\boldsymbol{r}) n_b(0)\rangle = \int \frac{d^2 q}{2\pi} \frac{g_0}{q^2 + \mu_0} \delta_{ab} \exp(i\boldsymbol{q}\boldsymbol{r}) = g_0 \delta_{ab} K_0(\sqrt{\mu_0} r)\,. \tag{7.22}$$

Now we should enforce the condition $n^2 = 1$. Using (7.22) we get the relation

$$g_0 N \ln \frac{\Lambda}{\sqrt{\mu_0}} = 1\,, \tag{7.23}$$

determining μ_0. Since the answer (7.22) is obtained "in the mean field approximation" (with fluctuations of μ neglected) it really represents the expression for the true pair correlation function of $\boldsymbol{n}$. We see the exponential attenuation of the correlation function at $r > 1/\sqrt{\mu_0}$ and therefore $1/\sqrt{\mu_0}$ plays the role of the correlation length. Sometimes one says about spontaneous mass having in mind the exponential decay.

Now we should justify the weakness of the field μ fluctuations. For this one should introduce the functional $\mathcal{S}$:

$$\exp(-\mathcal{S}) = \int \mathcal{D}\boldsymbol{n}\, \exp(-\mathcal{H})\,, \tag{7.24}$$

determining the statistics of the field μ. The functional $\mathcal{S}$ is expressed via the pair correlation function of $\boldsymbol{n}$ taken at a given μ. Say, the second order term is

$$\mathcal{S}^{(2)} = -\frac{1}{16\pi^2 g_0^2} \int \mathrm{d}\boldsymbol{r}_1\, \mathrm{d}\boldsymbol{r}_2\, \mu(\boldsymbol{r}_1)\mu(\boldsymbol{r}_2)\langle n_a(\boldsymbol{r}_1) n_b(\boldsymbol{r}_2)\rangle^2\,. \tag{7.25}$$

As follows from Eq. (7.22) the characteristic $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ in (7.25) is μ_0^{-1}. Therefore the contribution can be estimated as $N \int \mathrm{d}^2\boldsymbol{r}\, \mu^2/\mu_0$. We see the factor N in front of the integral. The same is correct also for high-order terms in $\mathcal{S}$. Therefore at $N \gg 1$ fluctuations of μ are really suppressed.

Problems

Problem 7.1

In the external magnetic field $\boldsymbol{H}$ there is the contribution to the energy of a magnetic

$$\mathcal{F}_H = -\int \mathrm{d}\boldsymbol{r}\, M \boldsymbol{H}\boldsymbol{n}\,, \tag{7.26}$$

where M is the absolute value of the magnetization. Find RG-equations for the coefficient M figuring in (7.26) for a (two-dimensional) ferromagnetic layer ($\boldsymbol{n}$ is a three-dimensional vector). The answer should be expressed via the invariant charge.

8. MEMBRANES

In this Section we will study properties of membranes arising in lyotropic liquid crystals and apparently in many microemulsion phases. A membrane is a film of a molecular thickness which is, as a rule, made from lipid molecules and is a double layer of such molecules. Membranes are also characteristic of various biological systems.

To avoid confusion let us stress that we will study a membrane taken separately. This is justified by the fact that in real systems (e.g., lyotropic liquid crystals) membranes are positioned far one from another, and therefore the theory of macroscopic systems can be constructed in two stages: first, consider characteristics of a membrane and then take into account the interaction of membranes. The study of the interaction between membranes is a problem beyond the scope of this book.

Membranes of a small (molecular) thickness can be of substantial longitudinal sizes. This makes it possible, while studying long-scale properties of a membrane, to regard it (like a free-suspended film or a Langmuir film) as a two-dimensional object, i.e., neglecting the thickness of the membrane. This, in particular, means that characteristics of the film such as its energy can be set as an integral over a surface, determining the position of the membrane. All integrals presented below are believed to be taken over this surface.

A characteristic peculiarity of the membrane is the fact that its surface tension is zero, which, actually, is one of the conditions of the thermodynamic equilibrium of the membrane with the solution of molecules it is made from. By virtue of this condition the surface energy of the membrane is determined by its curvature. In the main approximation this energy can be written as the following surface integral (Canham, 1970; Helfrich, 1975)

$$\mathcal{H}_{\rm s} = \int {\rm d}S \left(\frac{\kappa}{2} (R_1^{-1} + R_2^{-1})^2 + \bar{\kappa} R_1^{-1} R_2^{-1} \right) \quad . \tag{8.1}$$

Here R_1 and R_2 are local radii of the membrane curvature and the coefficients κ and $\bar{\kappa}$ are called bending modules. The quantity $R_1^{-1}R_2^{-1}$ is the Gaussian curvature of the surface, and the combination $R_1^{-1} + R_2^{-1}$ is usually called its mean curvature.

In addition to the energy (8.1) related to the curvature of the surface, we should also introduce the energy, related to variations of the surface density of molecules $n_{\rm s}$, constituting the film. In the approximation we need, the energy can be written as

$$\mathcal{H}_{\rm n} = \frac{1}{2} \int {\rm d}S \, B (n_{\rm s}'/n_{\rm s})^2 \quad . \tag{8.2}$$

Here n_s' is the deviation of the surface density from its equilibrium value, and the coefficient B has the meaning of the inverse compressibility of the film.

In the conditions where the surface tension coefficient of the film is zero (or sufficiently small), thermal fluctuations of the shape of the membrane are relevant (De Gennes and Taupin, 1982). These fluctuations give rise to the destruction of the correlation between orientations of sufficiently distant pieces of the membrane. The scale, starting from when this is happening, is called persistent length. A membrane of sizes, exceeding the persistent length cannot in any approximation be treated as flat.

Fluctuations of the shape of the membrane lead to logarithmic renormalization of the modules κ and $\bar{\kappa}$. First an attempt to calculate the renormalization of the module κ was taken by Helfrich (1985) and later by Förster (1986). The correct renormalization group equation for the module κ in an one-loop approximation was derived in papers by Peliti and Leibler (1985), Kleinert (1986a) and Polyakov (1986), and the equation for $\bar{\kappa}$ and for the spontaneous curvature in the same approximation were found by Kleinert (1986b). Besides, the modulus B introduced in (8.2) is logarithmically renormalized.

To describe a shape of a membrane it is necessary to introduce a certain parametrization of the surface setting its position in space. As in the description of other films, we will assume that this surface is set by an equation

$$\Phi(\boldsymbol{r}) = 0 \quad ,$$

where Φ is a function of three coordinates. For a unit vector, normal to the surface, there is an expression

$$l_i = \frac{\nabla_i \Phi}{\mid \nabla \Phi \mid} \quad . \tag{8.3}$$

The variable $\boldsymbol{l}$, formally defined in the whole space, is meaningful of course only on the surface $\Phi = 0$. With the use of the introduced quantities, the energy (8.1) can be rewritten as

$$\Omega_{\rm s} = \int {\rm d}S \left(\frac{\kappa}{2} (\nabla_i l_i)^2 + \frac{\bar{\kappa}}{2} \left((\nabla_i l_i)^2 - \nabla_i l_k \nabla_k l_i \right) \right) \quad . \tag{8.4}$$

Let us give the one-loop renormalization group equations for the modules, we are interested in

$$\frac{d\kappa}{d\xi} = -\frac{3T}{4\pi} \quad , \tag{8.5}$$

$$\frac{d\bar{\kappa}}{d\xi} = \frac{5T}{6\pi} \quad , \tag{8.6}$$

$$\frac{dB}{d\xi} = \frac{TB}{4\pi\kappa} \quad . \tag{8.7}$$

Here T is the temperature, $\xi = \ln(\Lambda/q)$, where q is a characteristic wave vector and Λ is the cutoff parameter (wave vector of the order of the inverse molecular size). These equations describe the behavior of the modules κ, $\bar{\kappa}$, B at a variation of a characteristic scale q^{-1}.

The right-hand sides of renormalization group equations, found in the framework of perturbation theory, always contain the module κ but not $\bar{\kappa}$. The thing is that the term in the energy (8.1) or (8.4), proportional to the module $\bar{\kappa}$ (i.e., the surface integral of its Gaussian curvature), is a topological invariant, therefore it does not change at small perturbations of the shape of the membrane.

As follows from (8.5) the role of the "invariant charge" (dimensionless coupling constant) is played by the quantity

$$g = \frac{3T}{4\pi\kappa} \quad . \tag{8.8}$$

For the perturbation theory to be applicable and, consequently, for (8.5-8.7) to hold, the coupling constant g must be small. Henceforth we will treat g as a small quantity, which enables us to employ the perturbation theory.

A consequence of (8.5,8.8) is an expression

$$g = \frac{g_0}{(1 - g_0\xi)} \quad . \tag{8.9}$$

Here g_0 is the short-wavelength coupling constant. We see that with increasing scale (i.e., with increasing ξ) the coupling constant grows. Thus we come to a situation which in the quantum field theory is called "asymptotic freedom". The expression (8.9) determines a behavior of the coupling constant g up to scales where $g_0 L \sim 1$ (but of course $g \ll 1$). The scale on which g reaches a value of the order of unity, and the perturbation theory becomes unapplicable, is, in fact, the persistent length.

The behavior of the modules κ, $\bar{\kappa}$, and B at increasing scale is determined by (8.5-8.7) from which it follows that at increasing scale the module κ becomes smaller whereas the modules $\bar{\kappa}$, B increase together with the coupling constant. Using the relations (8.8,8.9), we find the following proportionality laws

$$\kappa \propto g^{-1} \quad , \qquad B \propto g^{1/3} \quad . \tag{8.10}$$

As above these laws are correct up to scales where $g_0\xi \sim 1$ but $g \ll 1$. We can say that due to thermal fluctuations the membrane becomes less compressible and softer with respect to fluctuations of the shape.

Renorm-Group Equation for the Helfrich Modules

Here we will derive the RG-equations (8.5,8.6) for κ and $\bar{\kappa}$. We know that the contribution to the Landau functional proportional to $\bar{\kappa}$ is the topological invariant. To establish the RG-equation it is impossible to consider an infinite membrane since the contribution proportional to $\bar{\kappa}$ is equal to zero in the case. Therefore we must take a closed membrane. Below we will use a membrane with the topology of a sphere. In this case the contribution the contribution proportional to $\bar{\kappa}$ to the Landau functional is equal to $4\pi\bar{\kappa}$.

Below we will believe that the membrane fluctuates near a sphere of the radius R. Then it is natural to parametrize the shape of the membrane by the dependence $r = R + u(\theta, \varphi)$, where r is the distance between the origin and a point of the membrane and θ and φ are angles characterizing the corresponding direction. By another words, u is the displacement of the membrane in the radial direction. We develop the perturbation series over u. The unit vector orthogonal to the membrane is expressed in terms of u as

$$\boldsymbol{l} = |\nabla\Phi|^{-1}\left(\frac{\boldsymbol{r}}{r} - \frac{\boldsymbol{\partial} u}{r}\right), \quad |\nabla\Phi| = \sqrt{1 + \frac{(\partial u)^2}{r^2}}\,, \tag{8.11}$$

where we introduced the designation $\boldsymbol{\partial}$ for the derivative 'along the angles':

$$(\boldsymbol{\partial}u)^2 \equiv \left(\frac{\partial u}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial u}{\partial\varphi}\right)^2 ,$$
$$\boldsymbol{\partial}^2 u \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} .$$

Then we can express the bending energy

$$\mathcal{F}_\kappa = \frac{\kappa}{2}\int \mathrm{d}o\, r^2|\nabla\Phi|\,(\nabla\boldsymbol{l})^2 , \tag{8.12}$$

where we should substitute $r = R + u$. Calculating the integrand up to the fourth order over u (at a given r) we get

$$\mathcal{F}_\kappa \approx 8\pi\kappa + \frac{\kappa}{2}\int \mathrm{d}o\left\{-\frac{2}{r^2}(\boldsymbol{\partial}u)^2 + \frac{1}{r^2}(\boldsymbol{\partial}^2u)^2 - \frac{2}{r^3}(\boldsymbol{\partial}u)^2\boldsymbol{\partial}^2u \right.$$
$$\left. +\frac{5}{2r^4}(\boldsymbol{\partial}u)^4 + \frac{1}{2r^4}(\boldsymbol{\partial}u)^2(\boldsymbol{\partial}^2u)^2 + \frac{1}{r^4}\boldsymbol{\partial}\boldsymbol{\partial}^2u\boldsymbol{\partial}u(\boldsymbol{\partial}u)^2\right\},$$

where we integrated in part once in some terms. Then we find the terms of the second, of the third and of the fourth order over u

$$\mathcal{F}_\kappa^{(2)} = \frac{\kappa}{2R^2}\int \mathrm{d}o\,\left[(\boldsymbol{\partial}^2u)^2 - 2(\boldsymbol{\partial}u)^2\right] , \tag{8.13}$$
$$\mathcal{F}_\kappa^{(3)} = \frac{\kappa}{R^3}\int \mathrm{d}o\,\left[-u(\boldsymbol{\partial}^2u)^2 - (\boldsymbol{\partial}u)^2\boldsymbol{\partial}^2u\right] , \tag{8.14}$$
$$\mathcal{F}_\kappa^{(4)} = \frac{\kappa}{2R^4}\int \mathrm{d}o\left\{3u^2(\boldsymbol{\partial}^2u)^2 + 6u(\boldsymbol{\partial}u)^2\boldsymbol{\partial}^2u \right.$$
$$\left. +\frac{5}{2}(\boldsymbol{\partial}u)^4 + \frac{1}{2}(\boldsymbol{\partial}u)^2(\boldsymbol{\partial}^2u)^2 + \boldsymbol{\partial}\boldsymbol{\partial}^2u\boldsymbol{\partial}u(\boldsymbol{\partial}u)^2\right\}, \tag{8.15}$$

where we kept only the main terms over the derivative $\boldsymbol{\partial}$.

Now we will start the renorm-group procedure deviding $u = u' + \tilde{u}$ and integrating over the fast field $\tilde{u}$. Expanding up to the second order over $\tilde{u}$ we get

$$\mathcal{F}_{\mathrm{int}}^{(3)} = \frac{\kappa}{R^3}\int \mathrm{d}o\,\left\{\boldsymbol{\partial}^2u'(\boldsymbol{\partial}\tilde{u})^2 - u'(\boldsymbol{\partial}^2\tilde{u})^2 - 2\boldsymbol{\partial}u'\boldsymbol{\partial}\tilde{u}\boldsymbol{\partial}^2\tilde{u}\right\} , \tag{8.16}$$

and an analogous expression for $\mathcal{F}_{\mathrm{int}}^{(4)}$. Next we calculate the correction to the slow part of the Landau functional

$$\Delta\mathcal{F}_\kappa^{(2)} = \left\langle\mathcal{F}_{\mathrm{int}}^{(4)}\right\rangle - \frac{1}{2T}\left\langle\left\langle\mathcal{F}_{\mathrm{int}}^{(3)}\mathcal{F}_{\mathrm{int}}^{(3)}\right\rangle\right\rangle ,$$

where the average is calculated in accordance with (8.13). Calculating the integrals we obtain

$$\Delta\mathcal{F}_\kappa^{(2)} = \frac{\kappa}{2R^4}\int \mathrm{d}o\left\{-\frac{3}{2}\langle(\boldsymbol{\partial}\tilde{u})^2\rangle(\boldsymbol{\partial}^2u')^2 - \frac{3}{2}\langle(\boldsymbol{\partial}^2\tilde{u})^2\rangle(\boldsymbol{\partial}u')^2 + (10-8)(\boldsymbol{\partial}u')^2\langle(\boldsymbol{\partial}\tilde{u})^2\rangle\right\} . \tag{8.17}$$

The first two terms and the term with the coefficient 10 in the right-hand side of (8.17) originate from $\left\langle\mathcal{F}_{\mathrm{int}}^{(4)}\right\rangle$ and the term with the coefficient -8 in the right-hand side of (8.17) originates from $-\left\langle\left\langle\mathcal{F}_{\mathrm{int}}^{(3)}\mathcal{F}_{\mathrm{int}}^{(3)}\right\rangle\right\rangle/(2T)$. The expression (8.17) must reproduce (8.13).

One can easily calculate

$$\langle(\boldsymbol{\partial}\tilde{u})^2\rangle = \frac{R^2T}{2\pi\kappa}\ln\frac{\Lambda}{\Lambda'} . \tag{8.18}$$

Substituting the expression (8.18) into (8.17) and comparing the result with (8.13) we find that the first term of the integrand of (8.17) gives $\Delta\kappa = -3T/(4\pi)\ln(\Lambda/\Lambda')$ in accordance with (8.5). To reproduce the same result for the second term of the integrand of (8.13) one should demand

$$\langle(\boldsymbol{\partial}^2\tilde{u})^2\rangle = -\frac{R^2T}{3\pi\kappa}\ln\frac{\Lambda}{\Lambda'} . \tag{8.19}$$

This unusual rule is related to the fact that the decomposition $u = u' + \tilde{u}$ is made really in angular harmonics l, m and the restriction condition $R\Lambda' < l < R\Lambda$ knows about R.

Now we find the correction to the free energy which in accordance with (8.13) is equal to

$$\Delta F = \frac{2\pi\kappa}{R^2}\left[\langle(\boldsymbol{\partial}^2\tilde{u})^2\rangle - 2\langle(\boldsymbol{\partial}\tilde{u})^2\rangle\right] .$$

Substituting here (8.18) and (8.19) we get $\Delta F = -(8/3)T\ln(\Lambda/\Lambda')$. From the other hand, the correction can be written as $\Delta F = 4\pi(2\Delta\kappa + \Delta\bar{\kappa})$. Substituting here $\Delta\kappa = -3T/(4\pi)\ln(\Lambda/\Lambda')$ we find $\Delta\bar{\kappa} = 5T/(6\pi)\ln(\Lambda/\Lambda')$ in accordance with (8.6). Thus the answer (8.6) is related to the rule (8.19).

Problems

Problem 8.1

In the external magnetic field $\boldsymbol{H}$ there is the contribution to the energy of a membrane

$$\mathcal{F}_H = -\int \mathrm{d}S\,\frac{\alpha}{2}\,(\boldsymbol{H}\boldsymbol{l})^2 , \tag{8.20}$$

where $\alpha > 0$ and $\boldsymbol{l}$ is the unit vector perpendicular to the membrane. If $u(x, y)$ is the displacement of the membrane along the Z-axis then

$$l_\alpha = -\frac{\nabla_\alpha u}{\sqrt{1 + (\nabla u)^2}} , \qquad l_z = \frac{1}{\sqrt{1 + (\nabla u)^2}} . \tag{8.21}$$

Find RG-equation for the coefficient α (the answer should be expressed via the invariant charge g).

Problem 8.2

Find RG-equation for the coefficient α (the answer should be expressed via the invariant charge g) for the opposite sign of the magnetic energy

$$\mathcal{F}_H = \int \mathrm{d}S\,\frac{\alpha}{2}\,(\boldsymbol{H}\boldsymbol{l})^2 , \tag{8.22}$$

where $\alpha > 0$.

Problem 8.3

If the membrane has a finite surface tension σ then one should take into account the energy

$$\mathcal{F}_\sigma = \int \mathrm{d}S\,\sigma . \tag{8.23}$$

Find RG-equation for the surface tension σ.

Problem 8.4

The elastic energy of the membrane can be written as

$$\mathcal{F}_{\mathrm{el}} = \int \mathrm{d}S\,\frac{B}{2}\,\frac{(n_s - n_0)^2}{n_0^2} \tag{8.24}$$

where B is the elastic module, n_s is the surface density of molecules of the membrane and n_0 is the equilibrium density. Find RG-eqution for the elastic module B.

Solution of the Problem 8.4

The quantity $n_0 = N/S$ where N is the number of molecules of the membrane and S is its area. The area is changed at the renormalization according to RG-equation

$$\frac{dS}{d\xi} = -\frac{1}{3}gS .$$

Therefore

$$\frac{dn_0}{d\xi} = \frac{1}{3}gn_0 .$$

This value determines the equilibriun meaning of the quantity

$$n_s = \frac{\Delta N}{\Delta S} = \frac{\Delta N}{\sqrt{1+(\nabla u)^2}\,\Delta x\,\Delta y}\,.$$

So, the factor $(n_s - n_0)^2/n_0^2$ is not renormalized and we get for B the same RG-equation as for σ:

$$\frac{dB}{d\xi} = \frac{1}{3}gB\,.$$

9. BEREZINSKII-KOSTERLITZ-THOULESS PHASE TRANSITION

Defects like quantum vortices, dislocations, disclinations in thin films (which are two-dimensional systems) are point objects and can therefore be excited at finite temperatures. The energy of a single defect is proportional to the logarithm of the size of the specimen. Therefore at low temperatures only bounded defect-antidefect pairs are excited since the energy of the pair is finite (does not depend on the size of the specimen). One can think that at low temperatures besides the phonon gas there is also the gas of the defect-antidefect pairs in the films. At increasing temperature the entropy associated with the random distribution of the defects grows. For a single defect the entropy is also proportional to the logarithm of the size of the specimen. Therefore at a temperature T_c the product TS (where S is the entropy of the defect) becomes larger than the energy of the defect. Starting from the temperature unbounded defects appear in the film, the defects kill long-scale correlations. Therefore at a temperature the phase transition like superfluid-normal liquid occurs. The transition was first examined by Berezinskii (1971) and then treated by Kosterlitz and Thouless (1973). Below we will consider the superfluid-normal phase transition. Note, that phase transitions of the same type are observed in two-dimensional crystals. Namely, at increasing temperature a crystal melts into the so-called hexatic phase, then the transition of the hexatic into the liquid phase occurs (?? and Nelson, 1980).

It is well known that a vortex (which is labelled by the subscript j) in a superfluid film is characterized by the superfluid velocity

$$n_i\hbar\frac{\epsilon_{\beta\alpha}(r_\beta - r_{j,\beta})}{m|\boldsymbol{r}-\boldsymbol{r}_j|^2}\,. \tag{9.1}$$

Here, $\boldsymbol{r}_i$ is the position of the vortex, m is the mass of the atom, and n_i is the quantum number of the vortex (usually only the vortices with $n = \pm 1$ are excited). The superfluid velocity in the film is the sum of the expressions (9.1) for all vortices and of the irrotational part:

$$v_{s,\alpha}(\boldsymbol{r}) = \sum_j \frac{n_j\hbar}{m}\frac{\epsilon_{\beta\alpha}(r_\beta - r_{j,\beta})}{|\boldsymbol{r}-\boldsymbol{r}_j|^2} + \frac{\hbar}{m}\nabla_\alpha\varphi\,. \tag{9.2}$$

The vorticity of the superfluid velocity ω is

$$\omega = \epsilon_{\gamma\alpha}\nabla_\gamma v_{s,\alpha} = \sum_j 2\pi n_j\frac{\hbar}{m}\delta(\boldsymbol{r}-\boldsymbol{r}_j)\,. \tag{9.3}$$

The energy associated with the superfluid velocity, can be written as

$$\mathcal{F}_\mathrm{s} = \int \mathrm{d}^2 r\,\frac{\rho_s}{2}v_s^2\,, \tag{9.4}$$

where ρ_s is the so-called superfluid density. The quantity is a function of temperature: at $T = 0$ the value of ρ_s is the complete two-dimensional density of the film and at increasing temperature ρ_s decreases due to excitations like phonons. Neglecting fluctuations of ρ_s we get from (9.4)

$$\mathcal{F}_\mathrm{s} = \int \mathrm{d}^2 r\,\frac{\rho_s\hbar^2}{2m^2}(\nabla\varphi)^2 + \mathcal{F}_\mathrm{vort}\,, \tag{9.5}$$

$$\mathcal{F}_\mathrm{vort} = -\pi\sum_{i\neq j}\frac{\rho_s\hbar^2}{m^2}n_i n_j\ln(\Lambda r_{ij}) + \sum_i \mu(n_i)\,, \tag{9.6}$$

that is the energy $\mathcal{F}_{\rm vort}$ associated with vortices is separated. In (9.6) $\mu(n_j)$ is the energy of the core of the j-th vortex, the size of the core is of the order of Λ^{-1}, and

$$\alpha = \Big(\frac{m}{2\pi\hbar}\Big)^2 \frac{T}{\rho_s}\,. \tag{9.7}$$

Below we treat only the solenoidal contribution to the superfluid velocity $\boldsymbol{v}_s$ determined by the first term in (9.2). The contribution can be written as $\epsilon_{\alpha\beta}\nabla_\beta\Phi$ and then the condition (9.3) gives

$$\nabla^2\Phi = -\sum_j 2\pi n_j \frac{\hbar}{m}\delta(\boldsymbol{r}-\boldsymbol{r}_j)\,. \tag{9.8}$$

Introducing the auxiliary field ϑ ensuring the condition (9.8) we can rewrite the vortex part of the free energy (9.6) as

$$\exp\left(-\mathcal{F}_{\rm vort}/T\right) = \int \mathcal{D}\vartheta\,\mathcal{D}\Phi$$
$$\exp\left\{-\frac{1}{T}\int \mathrm{d}^2r\,\frac{\rho_s}{2}(\nabla\Phi)^2 + i\int \mathrm{d}^2r\,\frac{m}{2\pi\hbar}\vartheta\nabla^2\Phi + i\sum_j n_j\vartheta(\boldsymbol{r}_j) - T^{-1}\sum_j \mu(n_j)\right\}.$$

Taking here the integral over Φ we get

$$\exp\left(-\mathcal{F}_{\rm vort}/T\right) = \int \mathcal{D}\vartheta\,\exp\left\{-\int \mathrm{d}^2r\,\frac{\alpha}{2}(\nabla\vartheta)^2 + i\sum_j n_j\vartheta(\boldsymbol{r}_j) - T^{-1}\sum_j \mu(n_j)\right\}. \tag{9.9}$$

Let us now write the generating functional for the vorticity (9.3):

$$\mathcal{Z}(\sigma) = \left\langle \exp\left\{i\frac{m}{2\pi\hbar}\int \mathrm{d}^2r\,\sigma\omega\right\}\right\rangle = \int \mathcal{D}\vartheta\,\exp(-\mathcal{H}_\sigma)\,, \tag{9.10}$$

where

$$\exp(-\mathcal{H}_\sigma) = \exp\left\{-\int \mathrm{d}^2r\,\frac{\alpha}{2}(\nabla\vartheta)^2\right\}$$
$$\times\sum\frac{1}{N!}\prod_j\int \mathrm{d}^2r_j\,\exp\left\{in_j\left[\vartheta(\boldsymbol{r}_j)+\sigma(\boldsymbol{r}_j)\right] - \frac{\mu(n_j)}{T}\right\}. \tag{9.11}$$

The sum here is performed over all sequences n_1, n_2 $\ldots n_N$ and over N from zero to ∞. In the long-scale limit $\mathcal{H}_\sigma$ is a local functional which can be written as

$$\mathcal{H}_\sigma = \int \mathrm{d}^2r\,\left\{\frac{\alpha}{2}(\nabla\vartheta)^2 + F(\vartheta+\sigma)\right\}, \tag{9.12}$$

where $F(\vartheta)$ is an even function. It follows from (9.11) that $\mathcal{H}$ is invariant under the transformation $\vartheta\to\vartheta+2\pi$ since n_j are integer numbers. Therefore $F(\vartheta)$ in (9.12) is a periodical function of ϑ with the period 2π and can be consequently expanded into Fourier series. As we will see only the first term of the expansion is relevant near the transition point and therefore we get

$$\mathcal{H} = \int \mathrm{d}^2r\,\left\{\frac{\alpha}{2}(\nabla\vartheta)^2 - \beta\cos(\vartheta)\right\}, \tag{9.13}$$

where $\mathcal{H}$ is equal to $\mathcal{H}_\sigma$ at $\sigma=0$.

The expression (9.13) can be deduced from (9.11) if to keep there only terms with $n_j=\pm1$. Then

$$\exp(-\mathcal{H}) = \exp\left\{-\int \mathrm{d}^2r\,\frac{\alpha}{2}(\nabla\vartheta)^2\right\}$$
$$\times\sum_{N=0}^{\infty}\frac{1}{N!}\left\{\int \mathrm{d}^2r\,2\exp(-\mu/T)\cos\left[\vartheta(\boldsymbol{r})\right]\right\}^N,$$

what leads to (9.13) with $\beta = 2\exp(-\mu/T)$. The procedure is justified if $\mu/T \gg 1$, then the vortices with $n_j = \pm 1$ are excited only. The above scheme is based on the "canonical" distribution of vortices. It is instructive to obtain the same result using the "microcanonical" distribution taking into account that in the closed system the number of positive vortices is equal to the number of negative ones. Then we get instead of (9.11)

$$\exp(-\mathcal{H}_{\rm mc}) = \exp\left\{-\int \mathrm{d}^2 r\, \frac{\alpha}{2}(\nabla\vartheta)^2\right\}$$
$$\times \sum_{N=0}^{\infty} \frac{1}{(N!)^2}\left\{\int \mathrm{d}^2 r\, \exp\left[i\vartheta(\boldsymbol{r}) - \mu/T\right]\right\}^N \left\{\int \mathrm{d}^2 r\, \exp\left[-i\vartheta(\boldsymbol{r}) - \mu/T\right]\right\}^N ,$$

where we keep again the vortices with $n_j = \pm 1$ only. Let us rewrite the expression as

$$\exp(-\mathcal{H}_{\rm mc}) = \int_0^{2\pi} \frac{ds}{2\pi} \exp\left\{-\int \mathrm{d}^2 r\, \frac{\alpha}{2}(\nabla\vartheta)^2\right\} \sum_{N_1,N_2=0}^{\infty} \frac{1}{(N_1!)(N_2!)} \exp(isN_1 - isN_2)$$
$$\times \left\{\int \mathrm{d}^2 r\, (\beta/2) \exp\left[i\vartheta(\boldsymbol{r})\right]\right\}^{N_1} \left\{\int \mathrm{d}^2 r\, (\beta/2) \exp\left[-i\vartheta(\boldsymbol{r})\right]\right\}^{N_2} ,$$

where we used

$$\int_0^{2\pi} \frac{ds}{2\pi} \exp(isN_1 - isN_2) = \delta_{N_1,N_2} .$$

Performing the summation over N_1 and N_2 we get

$$\exp(-\mathcal{H}_{\rm mc}) = \int_0^{2\pi} \frac{ds}{2\pi} \exp\left\{\int \mathrm{d}^2 r\, \left[-\frac{\alpha}{2}(\nabla\vartheta)^2 + \beta\cos(s+\vartheta)\right]\right\} .$$

All answers can be written as functional integrals over the field ϑ including homogeneous shifts. Therefore all the integrals do not depend on the parameter s and

$$\int \mathcal{D}\vartheta\, \exp(-\mathcal{H}_{\rm mc}) \cdots = \int \mathcal{D}\vartheta\, \exp(-\mathcal{H}) \ldots .$$

Fluctuations play an essential role at forming correlation functions of ϑ. To take the role into account we will use the renorm-group method. As usual we divide the field ϑ into the slow ϑ' and the fast $\tilde{\vartheta}$ parts:

$$\vartheta = \vartheta' + \tilde{\vartheta}\,, \qquad \vartheta' = \sum_{q<\Lambda'} \vartheta_q \exp(i\boldsymbol{q}\boldsymbol{r})\,, \qquad \tilde{\vartheta} = \sum_{\Lambda'<q<\Lambda} \vartheta_q \exp(i\boldsymbol{q}\boldsymbol{r})\,. \tag{9.14}$$

Then we introduce the 'slow' functional $\mathcal{H}'$:

$$\exp\left[-\mathcal{H}'(\vartheta')\right] = \int \mathcal{D}\tilde{\vartheta}\, \exp\left[-\mathcal{H}(\vartheta' + \tilde{\vartheta})\right] . \tag{9.15}$$

Substituting here (9.12) and expanding the exponent over β we get

$$\exp\left[-\mathcal{H}'(\vartheta') + \int \mathrm{d}^2 r\, \frac{\alpha}{2}\left(\nabla\vartheta'\right)^2\right] = \mathcal{Z} \tag{9.16}$$

$$\mathcal{Z} = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \left\langle \prod_{i=1}^{n} \int \mathrm{d}^2 r_i\, \cos\left[\vartheta'(\boldsymbol{r}_i) + \tilde{\vartheta}(\boldsymbol{r}_i)\right]\right\rangle_0 , \tag{9.17}$$

$$\langle Y(\tilde{\vartheta})\rangle_0 \equiv \int \mathcal{D}\tilde{\vartheta}\, \exp\left[-\int \mathrm{d}^2 r\, \frac{\alpha}{2}\left(\nabla\tilde{\vartheta}\right)^2\right] Y(\tilde{\vartheta})\,. \tag{9.18}$$

Thus the angular brackets denote averaging with Gaussian weight which is characterized by the pair correlation function

$$G(r) = \langle\tilde{\vartheta}(\boldsymbol{r})\tilde{\vartheta}(0)\rangle_0 = \int_{\Lambda'}^{\Lambda} \frac{d^2 q}{(2\pi)^2} \exp(i\boldsymbol{q}\boldsymbol{r}) \frac{1}{\alpha q^2} = \frac{1}{2\pi\alpha} \int_{\Lambda'}^{\Lambda} \frac{dq}{q} J_0(qr)\,. \tag{9.19}$$

Note that it follows from (9.19)

$$G(0) = \langle\tilde{\vartheta}^2\rangle_0 = \frac{1}{2\pi\alpha}\ln\left(\frac{\Lambda}{\Lambda'}\right) . \tag{9.20}$$

Principally, each average in the sum figuring in the right-hand side of the expression (9.17) can be explicitly expressed in terms of the correlation function (9.19), for example

$$\left\langle\cos\left[\vartheta'(\boldsymbol{r}_i)+\tilde{\vartheta}(\boldsymbol{r}_i)\right]\right\rangle_0 = \exp\left[-\frac{1}{2}\langle\tilde{\vartheta}^2\rangle_0\right]\cos\left[\vartheta'(\boldsymbol{r}_i)\right] = \left(\frac{\Lambda'}{\Lambda}\right)^{1/(4\pi\alpha)}\cos\left[\vartheta'(\boldsymbol{r}_i)\right] . \tag{9.21}$$

Really we are interested in scales larger than $(\Lambda')^{-1}$. Since the function $G(r)$ (9.19) decreases at $r\Lambda' > 1$ the main contribution into $\mathcal{Z}$ (9.17) can be found if to substitute the average of the product by the following product

$$\prod_{i=1}^{n}\int \mathrm{d}^2 r_i \left\langle\cos\left[\vartheta'(\boldsymbol{r}_i)+\tilde{\vartheta}(\boldsymbol{r}_i)\right]\right\rangle_0 ,$$

what gives the result

$$\mathcal{Z}_{(0)} = \exp\left(\int \mathrm{d}^2 r\,\beta'\cos\vartheta'\right) , \qquad \beta' = \left(\frac{\Lambda'}{\Lambda}\right)^{1/(4\pi\alpha)}\beta . \tag{9.22}$$

Besides the contribution $\mathcal{Z}_{(0)}$ the term $\mathcal{Z}_{(1)}$ where the correlation between pairs of $\cos\vartheta_i$ is taken into account will also be needed for us. The term $\mathcal{Z}_{(1)}$ is a small correction to $\mathcal{Z}_{(0)}$ and can be consequently found in the linear approximation over the correlation:

$$\mathcal{Z}_{(1)} = \sum_{n=2}^{\infty}\frac{1}{2(n-2)!}\left[\int \mathrm{d}^2 r\,\beta'\cos\vartheta'\right]^{n-2}\beta^2\int \mathrm{d}^2 r_1\, d^2 r_2\left[\left\langle\cos(\vartheta'_1+\tilde{\vartheta}_1)\cos(\vartheta'_2+\tilde{\vartheta}_2)\right\rangle_0\right.$$
$$\left.-\left\langle\cos(\vartheta'_1+\tilde{\vartheta}_1)\right\rangle_0\left\langle\cos(\vartheta'_2+\tilde{\vartheta}_2)\right\rangle_0\right] = -\mathcal{Z}_{(0)}\mathcal{H}_{(1)} , \tag{9.23}$$

$$\mathcal{H}_{(1)} = -\frac{\beta'^2}{2}\int \mathrm{d}^2 r_1\, d^2 r_2\left\{\frac{1}{2}\cos(\vartheta'_1+\vartheta'_2)\left[\exp\left(-G\right)-1\right]+\frac{1}{2}\cos(\vartheta'_1-\vartheta'_2)\left[\exp\left(G\right)-1\right]\right\} , \tag{9.24}$$

where $G = G(r_{12})$. Regarding $\ln(\Lambda/\Lambda')$ small enough we obtain $G \ll 1$ and can consequently expand $\exp(\pm G)-1$ in (9.24). Then

$$\mathcal{H}_{(1)} = -\frac{\beta'^2}{2}\int \mathrm{d}^2 r_1\, d^2 r_2\left[G(r_{12})\sin\vartheta'_1\sin\vartheta'_2+\frac{1}{2}G^2(r_{12})\cos\vartheta'_1\cos\vartheta'_2\right] . \tag{9.25}$$

The characteristic scale of $G(r)$ is Λ^{-1} whereas the field ϑ' is smooth. Therefore passing to the variables $\boldsymbol{R} = \boldsymbol{r}_1/2+\boldsymbol{r}_2/2$, $\boldsymbol{r} = \boldsymbol{r}_1-\boldsymbol{r}_2$ one can expand $\vartheta'(\boldsymbol{R}\pm\boldsymbol{r}/2)$ (and then $\cos\vartheta'_{1,2}$, $\sin\vartheta'_{1,2}$) into the series over $\boldsymbol{r}$. Using the expression (9.19) we get

$$\int \mathrm{d}^2 r\, r^{2n} G(r) = 0 ,$$

and therefore the first term in the expression (9.25) doesn't produce anything. In the main approximation (9.25) can be rewritten as

$$\mathcal{H}_{(1)} = -\frac{\beta'^2}{8}\int \mathrm{d}^2 r\, d^2 R\, G^2(r)\left\{\cos(2\vartheta')+\frac{1}{4}r^2(\nabla\vartheta')^2\cos(2\vartheta')-\frac{1}{4}r^2(\nabla\vartheta')^2\right\} , \tag{9.26}$$

where $\vartheta \equiv \vartheta(\boldsymbol{R})$. The coefficients in the expression (9.26) are determined by the integrals

$$\int \mathrm{d}^2 r\, G^2(r) = \frac{1}{2\pi\alpha^2}\int_{\Lambda'}^{\Lambda}\frac{dq}{q^3} \approx \frac{\Lambda-\Lambda'}{2\pi\alpha^2\Lambda^3} ,$$

$$\int \mathrm{d}^2 r\, r^2 G^2(r) = \frac{2}{\pi\alpha^2}\int\limits_{\Lambda'}^{\Lambda} \frac{dq}{q^5} \approx \frac{2(\Lambda-\Lambda')}{\pi\alpha^2\Lambda^5}\,,$$

where we believed that Λ' is close to Λ. Therefore

$$\mathcal{H}_{(1)} = -\frac{\beta'^2(\Lambda-\Lambda')}{16\pi\alpha^2\Lambda^5}\int \mathrm{d}^2 R\,\left\{\Lambda^2\cos(2\vartheta') + \cos(2\vartheta')(\nabla\vartheta')^2 - (\nabla\vartheta')^2\right\}\,. \tag{9.27}$$

Actually only the last term in the integrand in (9.27) is relevant.

Substituting into the right-hand side of (9.16)

$$\mathcal{Z} \approx \mathcal{Z}_{(0)} - \mathcal{H}_{(1)}\mathcal{Z}_{(0)} \approx \mathcal{Z}_{(0)}\exp[-\mathcal{H}_{(1)}]\,,$$

we conclude that $\mathcal{H}'$ has the form (9.13) with $\alpha \to \alpha'$ and $\beta \to \beta'$ where

$$\alpha' = \alpha + \frac{\beta'^2(\Lambda-\Lambda')}{8\pi\alpha^2\Lambda^5}\,, \tag{9.28}$$

and β' is determined by (9.22). We see that it is worth to introduce the quantity $\gamma' = \beta'/(\Lambda')^2$. Then (9.22,9.28) lead to the following renorm-group equations for the parameters

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\xi} = \left(2 - \frac{1}{4\pi\alpha}\right)\gamma\,, \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}\xi} = \frac{\gamma^2}{8\pi\alpha^2}\,, \tag{9.29}$$

where $\xi = \ln(\Lambda/\Lambda')$. We see that there exist the fix point of the system $\alpha = \alpha_c$, $\gamma = 0$ where the critical value $\alpha_c = 1/(8\pi)$.

Let us analyze the behavior of a solution of (9.29) near the fixed point. There the equations (9.29) can be rewritten as

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\xi} = 16\pi(\alpha-\alpha_c)\gamma\,, \qquad \frac{\mathrm{d}(\alpha-\alpha_c)}{\mathrm{d}\xi} = 8\pi\gamma^2\,. \tag{9.30}$$

There exists the first integral

$$C = \gamma^2/2 - (\alpha-\alpha_c)^2\,, \tag{9.31}$$

of the system (9.30). In terms of the quantity we can rewrite the equation for α as

$$\frac{d(\alpha-\alpha_c)}{d\xi} = 16\pi\left[C + (\alpha-\alpha_c)^2\right]\,. \tag{9.32}$$

We conclude from (9.32) that if $\alpha_{\mathrm{short}} < \alpha_c$ and $C < 0$ then at $\xi \to \infty$ the parameter α tends to a finite value $\alpha = \alpha_c - \sqrt{|C|}$, as to the parameter γ it tends to zero at $\xi \to \infty$. If $C = 0$ then at $\xi \to \infty$ α tends to its critical value α_c and γ tends to zero. If $C > 0$ or if the short-scale value $\alpha > \alpha_c$ then in accordance with (9.32) α tends to ∞ where $\xi \to \infty$, γ also tends to ∞ where $\xi \to \infty$. Strictly speaking, the last possibility is not proved since the region of large α, γ is without the scope of our consideration since it corresponds to the strong coupling region where one cannot use the expression (9.13) (corrections to the expression are relevant).

Nevertheless physically it seems reasonable that α tends to infinity at the conditions and at $\xi \to \infty$. The point is that the constant α is related to the superfluid density as (9.7) (and the renormalization of α at increasing scale can be interpreted as the renormalization of the large-scale superfluid density due to the presence of vortices on short scales). Then the tendency of α to a finite value means that the superfluid density is also finite at large scales. That corresponds to the superfluid phase. If α tends to infinity then ρ_s tends to zero, that corresponds to the normal phase. We see that the transition temperature between the superfluid and the normal states is determined by the condition $C = 0$ (and $\alpha_{\mathrm{short}} < \alpha_c$). Remind that at the transition point where $C = 0$ α tends at the critical value $\alpha_c = 1/(8\pi)$. Substituting the value in (9.7) we get

$$\rho_s = \frac{2m^2 T}{\pi\hbar^2}\,, \tag{9.33}$$

at the transition temperature T_c. Since $\rho_s = 0$ at $T > T_c$ we conclude that at $T = T_c$ the superfluid density ρ_s experiences the jump determined by (9.33) [Nelson and Kosterlitz, 1977].

Let us return to correlation functions of the vorticity which are determined by the generating functional (9.10). Substituting there (9.13) we get

$$\mathcal{Z}(\sigma) = \int \mathcal{D}\vartheta \exp\left\{ - \int \mathrm{d}^2 r \, \left[\frac{\alpha}{2} (\nabla\vartheta)^2 - \beta \cos(\vartheta + \sigma) \right] \right\} . \tag{9.34}$$

Expanding the right-hand side of (9.34) up to the second order over σ we get the pair correlation function

$$\left(\frac{m}{2\pi\hbar} \right)^2 \langle \omega_1 \omega_2 \rangle = \beta \langle \cos\vartheta \rangle \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) - \beta^2 \langle \sin\vartheta_1 \sin\vartheta_2 \rangle \, , \tag{9.35}$$

where averaging in the right-hand side of (9.35) is performed with the weight $\exp(-\mathcal{H})$. We are interested in the second term in the right-hand side of (9.35) which can be rewritten as

$$-\frac{1}{2}\beta^2 \langle \cos(\vartheta_1 - \vartheta_2) \rangle + \frac{1}{2}\beta^2 \langle \cos(\vartheta_1 + \vartheta_2) \rangle \, . \tag{9.36}$$

Let us now start the renorm-group procedure. Then averages in the right-hand side of (9.36) conserve their form except for the factors β which are renormalized in accordance with (9.22)

$$\frac{\mathrm{d}\beta^2}{\mathrm{d}\xi} = -\frac{1}{2\pi\alpha}\beta^2 \, . \tag{9.37}$$

There is an essential difference: the factor β^2 at $\cos(\vartheta_1 - \vartheta_2)$ in the right-hand side of (9.36) is renormalized up to the scale $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ whereas the factor β^2 at $\cos(\vartheta_1 + \vartheta_2)$ in the right-hand side of (9.36) is renormalized up to the size of the specimen. Thus the second contribution can be neglected. We conclude that

$$\langle \omega_1 \omega_2 \rangle \propto (\beta')^2 \, , \tag{9.38}$$

where $(\beta')^2$ is the solution of the equation (9.37) taken at $\xi = \ln(\Lambda|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$. We know that below the transition temperature α tends to a constant in the long-scale limit. In the region we conclude from (9.37,9.38)

$$\langle \omega_1 \omega_2 \rangle \propto |\boldsymbol{r}_1 - \boldsymbol{r}_2|^{-1/(2\pi\alpha)} \, . \tag{9.39}$$

Problems

Problem 9.1
Besides the leading term (9.13) in the effective Landau expansion there are also subleading terms

$$\mathcal{H}_{\mathrm{sub}} = -\int \mathrm{d}^2 r \sum_{n=2}^{\infty} \beta_n \cos(n\phi) \, . \tag{9.40}$$

Find RG-equations for the coefficients β_n.

Problem 9.2
Besides the leading term (9.13) in the effective Landau expansion there are also subleading terms

$$\mathcal{H}_{\mathrm{sub}} = -\int \mathrm{d}^2 r \sum_{n=1}^{\infty} \lambda_n \cos(n\phi) (\nabla\phi)^2 \, . \tag{9.41}$$

Find RG-equations for the coefficients λ_n.

10. CRITICAL DYNAMICS

In this Section we will study the dynamics of the order parameter near a second order phase transition. The critical dynamics is less universal than the static properties of the system: For different physical systems the dynamics of the order parameter with the same number of the components could be quite different. Below we will treat the simplest case: purely relaxational dynamics of the order parameter; the situation is widely spread.

The dynamical equation for the purely relaxational dynamics of the order parameter φ is

$$\partial_t \varphi = f\,, \qquad f = \Gamma^{-1}\left(-\frac{\delta \mathcal{F}}{\delta \varphi} + \xi\right)\,, \tag{10.1}$$

where Γ is the kinetic coefficient, $\mathcal{F}$ is the Landau functional and ξ are Langeven forces (white noise) which represent the influence of short-scale (microscopic) degrees of freedom on the dynamics of the order parameter φ (which is a macroscopic degree of freedom). Correlation functions of ξ have characteristic atomic scales and times. Since we are interested in the macroscopic scales and macroscopic times correlation function of ξ can be regarded δ-correlated both in time and in space. That means also that the statistics of ξ can be treated as Gaussian and is consequently determined only by the pair correlation function. For our case the correlation function is

$$\langle \xi(t_1, \boldsymbol{r}_1)\xi(t_2, \boldsymbol{r}_2)\rangle = 2T\Gamma\,\delta(t_1 - t_2)\,\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\,, \tag{10.2}$$

where T is the temperature. The correlation function (10.2) enforces the Gibbs distribution $\exp[(F - \mathcal{F})/T]$ for the order parameter fluctuations. The expression (10.2) can be considered also as a manifestation of the fluctuation-dissipation theorem.

Recall that the Landau functional $\mathcal{F}$ for the order parameter φ is

$$\mathcal{F} = \int \mathrm{d}\boldsymbol{r}\,\left\{\frac{a}{2}\varphi^2 + \frac{b}{2}(\nabla\varphi)^2 + \frac{\lambda}{24}\varphi^4 - h\varphi\right\}\,, \tag{10.3}$$

where h is an external "magnetic field" and we omitted the third-order term over φ assuming that it is identically equal to zero (or small). Then the equation (10.1) is written as

$$\Gamma\partial_t\varphi = -a\varphi + b\nabla^2\varphi - \frac{\lambda}{6}\varphi^3 + h + \xi\,. \tag{10.4}$$

Note that the second-order terms in the Landau functional (10.3) produce the linear terms in the equation (10.4) whereas the fourth-order term in the Landau functional (10.3) produces the non-linear term (of the third order) in the equation (10.4).

We will be interested in non-simultaneous correlation functions of the order parameter φ. A possible way to find the correlation functions is to solve the equation (10.1) that is to express φ via the noise ξ. Then, to calculate, say, the pair correlation function $\langle\varphi(t_1, \boldsymbol{r}_1)\varphi(t_2, \boldsymbol{r}_2)\rangle$ one should take the product $\varphi(t_1, \boldsymbol{r}_1)\varphi(t_2, \boldsymbol{r}_2)$ (expressed via ξ) and average the product over the statistics of ξ. Unfortunately, this straightforward way is inconvenient since it is impossible to express φ via ξ, say, from the equation (10.4) explicitly. One can do it only as a series over the non-linearity that is one can find φ as a series over λ. Then the correlation functions of φ can also be calculated as a series over λ. First such procedure (in the context of turbulence) was suggested by Wyld (1961).

Below we will use another way which permits to examine besides the perturbative series non-perturbative effects also (Martin, Siggia, Rose, 1973; Dominicis, 1976; Janssen, 1976). Namely, let us write a product of the solutions of the equation (10.1) as the following functional integral

$$N^{-1}\int \mathcal{D}\varphi\,\delta\left(\partial_t\varphi - f\right)\varphi(t_1, \boldsymbol{r}_1)\ldots\varphi(t_n, \boldsymbol{r}_n)\,. \tag{10.5}$$

The integration here is performed over all functions of t and $\boldsymbol{r}$ and the δ-function is functional. Its principal property is that

$$\int \mathcal{D}\varphi\,\delta(\varphi - \psi)\Phi(\varphi) = \Phi(\psi)\,,$$

where $\Phi(\varphi)$ is an arbitrary functional.

The factor N in (10.5) is the normalization constant

$$N = \int \mathcal{D}\varphi\,\delta\left(\partial_t\varphi - f\right)\,. \tag{10.6}$$

Let us calculate the constant assuming, say, a discretization over time and space. Then instead of the equation (10.1) one should write

$$\frac{\varphi_{n+1}(\boldsymbol{r}_j) - \varphi_n(\boldsymbol{r}_j)}{\epsilon} - f_n(\boldsymbol{r}_j) = 0\,, \tag{10.7}$$

where $\boldsymbol{r}_j$ are points of the space lattice, the subscript n labels time points and ϵ is the step over time. Note, that we accepted in Eq. (10.7) the retarded regularization: The "force" $f+\xi$ determining the difference $\varphi_{n+1}-\varphi_n$ is taken at the time moment t_n. This regularization will be implied below. The functional integral is now a multiple integral taken over values of the field φ at the $N_1 \times N_2$ points where N_1 is the number of points in the spce lattice and N_2 is the number of steps in time. And the functional δ-function is now a product of conventional δ-functions (taken in all $N_1 \times N_2$ points) with arguments determining by Eq. (10.7). Then

$$N = \prod_{n,j} \int \mathrm{d}\varphi_n(\boldsymbol{r}_j) \delta \left[\frac{\varphi_{n+1}(\boldsymbol{r}_j) - \varphi_n(\boldsymbol{r}_j)}{\epsilon} - f_n(\boldsymbol{r}_j) \right] = \epsilon^{N_1 N_2} , \tag{10.8}$$

since the integral in Eq. (10.8) can be calculated step-by-step.

Let us convert the functional δ-function in Eq. (10.5) into an exponent. For this we use the well-known relation for the conventional δ-function:

$$\delta(x) = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \exp(ipx) .$$

Representing the δ-functions as such integrals in all sites of our $N_1 \times N_2$ space-time lattice we get

$$N^{-1} \delta \left(\partial_t \varphi - f \right) = \prod_{n,j} \int \frac{\Delta^d}{2\pi} \mathrm{d}p_n(\boldsymbol{r}_j) \exp \left\{ i \Delta^d p_n(\boldsymbol{r}_j) \left[\varphi_{n+1}(\boldsymbol{r}_j) - \varphi_n(\boldsymbol{r}_j) - \epsilon f_n(\boldsymbol{r}_j) \right] \right\} ,$$

where Δ is the step of the space lattice and d is the dimensionality of space. In the continuous limit we deal with a new field $p(t, \boldsymbol{r})$, the product $\prod \int \mathrm{d}p_{nj}$ can be written as the functional integral $\int \mathcal{D}p$ and we get

$$N^{-1} \delta \left(\partial_t \varphi - f \right) = \mathcal{N}^{-1} \int \mathcal{D}p \, \exp \left\{ i \int \mathrm{d}t \, \mathrm{d}\boldsymbol{r} \, \left[p \partial_t \varphi - p f \right] \right\} , \tag{10.9}$$

where the normalization constant is $\mathcal{N} = (\Delta^d / 2\pi)^{N_1 N_2}$. We see that the normalization constant $\mathcal{N}$ is indepenent of all parameters determining the dynamics of φ (including the fields h and ξ).

Returning now to the expression (10.5) we conclude that the correlation function of φ can be written as

$$\langle \varphi(t_1, \boldsymbol{r}_1) \dots \varphi(t_n, \boldsymbol{r}_n) \rangle = \mathcal{N}^{-1} \int \mathcal{D}\varphi \, \mathcal{D}p \left\langle \exp \left\{ i \int \mathrm{d}t \, \mathrm{d}\boldsymbol{r} \, \left[p \partial_t \varphi - p f \right] \right\} \right\rangle \varphi(t_1, \boldsymbol{r}_1) \dots \varphi(t_n, \boldsymbol{r}_n) ,$$

where averaging in the right-hand side of the relation is performed over the statistics of ξ. This averaging with the function (10.1) can be performed explicitly and we get

$$\langle \varphi(t_1, \boldsymbol{r}_1) \dots \varphi(t_n, \boldsymbol{r}_n \rangle = \mathcal{N}^{-1} \int \mathcal{D}\varphi \, \mathcal{D}p \, \exp \left\{ i \int \mathrm{d}t \, \mathrm{d}\boldsymbol{r} \, \left[p \partial_t \varphi + \frac{1}{\Gamma} p \frac{\delta \mathcal{F}}{\delta \varphi} + i \frac{T}{\Gamma} p^2 \right] \right\} \varphi(t_1, \boldsymbol{r}_1) \dots \varphi(t_n, \boldsymbol{r}_n) \tag{10.10}$$

It will be convenient for us to pass to a new field $p \to \Gamma p$. Including the factor $\Gamma^{N_1 N_2}$ into a redinition of the normalization constant $\mathcal{N}$ we get finally

$$\langle \varphi(t_1, \boldsymbol{r}_1) \dots \varphi(t_n, \boldsymbol{r}_n \rangle = \mathcal{N}^{-1} \int \mathcal{D}\varphi \, \mathcal{D}p \, \exp \left(i \mathcal{I} \right) \varphi(t_1, \boldsymbol{r}_1) \dots \varphi(t_n, \boldsymbol{r}_n) , \tag{10.11}$$

$$\mathcal{I} = \int \mathrm{d}t \, \mathrm{d}\boldsymbol{r} \, \left[\Gamma p \partial_t \varphi + p \frac{\delta \mathcal{F}}{\delta \varphi} + i T \Gamma p^2 \right] . \tag{10.12}$$

By analogy with the quantum field theory we will call the quantity $\mathcal{I}$ effective action or simply action. For our particular problem

$$\mathcal{I} = \int \mathrm{d}t \, \mathrm{d}\boldsymbol{r} \, \left[\Gamma p \partial_t \varphi + a p \varphi + b \nabla p \nabla \varphi + \frac{\lambda}{6} p \varphi^3 - p h + i T \Gamma p^2 \right] . \tag{10.13}$$

It is worth to consider correlation functions including besides the order parameter φ the auxiliary field p. For example we will treat the pair correlation function

$$G(t_1, t_2, \boldsymbol{r}_1, \boldsymbol{r}_2) = \langle \varphi(t_1, \boldsymbol{r}_1) p(t_2, \boldsymbol{r}_2) \rangle = \mathcal{N}^{-1} \int \mathcal{D}\varphi \, \mathcal{D}p \, \exp \left(i \mathcal{I} \right) \varphi(t_1, \boldsymbol{r}_1) p(t_2, \boldsymbol{r}_2) , \tag{10.14}$$

which we will call Green function. The function determines the linear response of the system to the external "magnetic field" h. Namely, at a variation of the field δh the average $\langle\varphi\rangle$ varies also and the variations are related as

$$\delta\langle\varphi(t_1,\boldsymbol{r}_1)\rangle = -i\int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,G(t_1,t_2,\boldsymbol{r}_1,\boldsymbol{r}_2)\,\delta h(t_2,\boldsymbol{r}_2)\,, \tag{10.15}$$

as is follows from Eqs. (10.13,10.14). Now causality dictates $G = 0$ if $t_1 < t_2$. Of course for $h = 0$ (or for h homogeneous in space and time) the Green function G is a function of the differences $t_1 - t_2$ and $\boldsymbol{r}_1 - \boldsymbol{r}_2$ only. Note that the correlation functions of the auxiliary field p are zero. To prove the property let us return to the representation of the correlation functions before the explicit averaging over the thermal noise ξ:

$$\langle p(t_1,\boldsymbol{r}_1)\dots p(t_n,\boldsymbol{r}_n)\rangle = \mathcal{N}^{-1}\int \mathcal{D}\varphi\,\mathcal{D}p\,\left\langle \exp\left\{ i\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left[\Gamma p\partial_t\varphi - pf\right]\right\}\right\rangle p(t_1,\boldsymbol{r}_1)\dots p(t_n,\boldsymbol{r}_n)\,.$$

The integration of the exponent over φ gives the functional $\delta(p)$ and consequently the integral in the right-hand side is equal to zero. Particularly, the average $\langle p\rangle$ and the pair correlation function $\langle p(t_1,\boldsymbol{r}_1)p(t_2,\boldsymbol{r}_2)\rangle$ are zero.

Let us now prove the fluctuation-dissipation theorem (FDT) (assuming $h = 0$). For the purpose we introduce a new field

$$\tilde{p} = p - \frac{i}{2T}\partial_t\varphi\,. \tag{10.16}$$

Expressing the effective action (10.12) via the new field we get

$$\mathcal{I} = \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left[\frac{i\Gamma}{4T}(\partial_t\varphi)^2 + \tilde{p}\frac{\delta\mathcal{F}}{\delta\varphi} + iT\Gamma\tilde{p}^2\right]\,, \tag{10.17}$$

where we omitted the boundary term originating from $\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\delta\mathcal{F}/\delta\varphi\,\partial_t\varphi$. We see that in terms of the fields $\varphi, \tilde{p}$ the effective action (10.17) is invariant under time reversing $t \to -t$. That means particurlarly that the correlation function $\langle\tilde{p}(t,\boldsymbol{r})\tilde{p}(0,\boldsymbol{0})\rangle$ is invariant under $t \to -t$. Writing the condition via the initial fields φ and p we get

$$\partial_t F(t,\boldsymbol{r}) = iT\left[G(t,\boldsymbol{r}) - G(-t,\boldsymbol{r})\right]\,, \tag{10.18}$$

where F is the pair correlation function

$$F(t_1 - t_2, \boldsymbol{r}_1 - \boldsymbol{r}_2) = \langle\varphi(t_1,\boldsymbol{r}_1)\varphi(t_2,\boldsymbol{r}_2)\rangle\,. \tag{10.19}$$

Passing to Fourier components we obtain

$$F(\omega) = -\frac{T}{\omega}\left[G(\omega) - G(-\omega)\right]\,. \tag{10.20}$$

This is just the classical variant of the FDT since F is the pair correlation function and the difference in the right-hand side gives the imaginary part of the linear responce function in accordance with Eq. (10.15).

Since (due to causality) $G(t)$ is zero for negative t the Fourier transform $G(\omega)$ is a function analytical in the upper half-plane. The property enables one to obtain the explicit expression for the simultaneous pair correlation function in terms of the Green function. Indeed, the simultaneous correlation function can be written as

$$F(t=0) = \int\frac{\mathrm{d}\omega}{2\pi}F(\omega) = -T\int\frac{\mathrm{d}\omega}{2\pi\omega}\left[G(\omega) - G(-\omega)\right]\,.$$

Let us shift the integration contour into the lower semiplane. Since $G(-\omega)$ is analytic there, the integral of the second term in the right-hand side is equal to zero. So, $G(-\omega)$ can be omitted. Next, let us shift the contour into the upper semiplane. Since $G(\omega)$ is analytic in the upper semiplane the contribution to the integral will be produced by the residue in the pole $\omega = 0$ only. Therefore

$$F(t=0) = -iTG(\omega=0)\,. \tag{10.21}$$

We see from Eq. (10.13) that the effective action $\mathcal{I}$ is a sum of the second-order term over the fields φ and p and of the fourth-order term proportional to λ. Therefore one can formulate a perturbation series for calculating the correlation functions if to expand $\exp(i\mathcal{I}$ in the relations like Eq. (10.11) into the series over λ. Each term of the

FIG. 1: First correction to the normalization constant $\mathcal{N}$.

FIG. 2: First corrections to the Green function

expansion is reduced to a Gaussian integral and can be consequently calculated explicitly. The answers are expressed via the bare correlation functions (10.14,10.19). In the Fourier representation

$$G_0(\omega, q) = -\frac{1}{\Gamma\omega + ia + ibq^2}\,, \tag{10.22}$$

$$F_0(\omega, q) = \frac{2T\Gamma}{\Gamma^2\omega^2 + (a + bq^2)^2}\,. \tag{10.23}$$

Of course the integral

$$\int \frac{\mathrm{d}\omega}{2\pi} F_0(\omega, q) = \frac{T}{a + bq^2} = -iTG_0(\omega = 0, q)\,,$$

reproduces the bare simultaneous correlation function.

Let us first consider the perturbation series for the normalization constant $\mathcal{N}$. The series can be represented on the diagrammatic language. The first digram contributing to $\mathcal{N}$ is drawn in Fig. 1. There the solid line represents the pair correlation function (10.23), the mixed line represents the Green function (10.22) (the dashed side corresponding to the p-field) and the black point (the vertex) represents the interaction constant λ. The analytic expression corresponding to the diagram contains the factor $G(t = 0, \boldsymbol{r} = 0)$ which is badly defined since the Green function $G(t)$ has a jump at $t = 0$. To recognize what is the value of $G(t = 0, \boldsymbol{r} = 0)$ we should return to the discrete version of our theory. Then due to the retarded regularization assumed the simultaneous meaning of the Green function is equal to zero. Thus we must take $G(t = 0) = 0$. Therefore the considered contribution to $\mathcal{N}$ is equal to zero. Higher contributions to $\mathcal{N}$ are zero also since all the contributions contain the factor $G(t = 0) = 0$. Thus only the zero contribution to $\mathcal{N}$ is non-zero which is in accordance with the property established above that $\mathcal{N}$ is independent of the parameters of the effective action.

Next, let us examine the perturbation series for the Green function G. Diagrams corresponding to first corrections to G are drawn in Fig. 2. As usual disconnected diagrams do not contribute into G. Really the disconnected diagrams are zero because of the property $G(t = 0) = 0$. The same property leads to the conclusion that diagrams containing closed G-function loops are zero. That is the reason why we present in Fig. 2 the diagram containing the closed F-loop only. One can perform a partial summation corresponding to extracting self-energy blocks. Then the Green function is written as

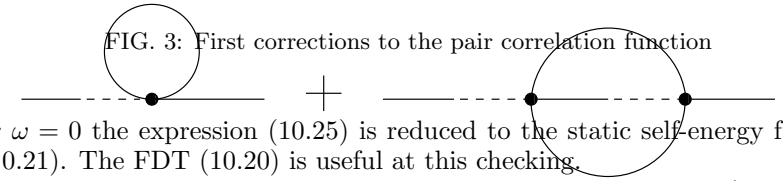
$$G(\omega, q) = -\frac{1}{\Gamma\omega + ia + ibq^2 + \Sigma}\,, \tag{10.24}$$

where Σ can be called the self-energy function. Two first contributions to Σ corresponding to diagrams drawn in Fig. 2 are

$$\Sigma(\omega, q) = \frac{i\lambda}{2} F_0(t = 0, r = 0) \tag{10.25}$$

$$-\frac{\lambda^2}{2} \int \frac{\mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\boldsymbol{q}_1\, \mathrm{d}\boldsymbol{q}_2}{(2\pi)^{2+2d}} F_0(\omega_1, \boldsymbol{q}_1) F_0(\omega_2, \boldsymbol{q}_2) G_0(\omega + \omega_1 + \omega_2, \boldsymbol{q} + \boldsymbol{q}_1 + \boldsymbol{q}_2)\,.$$

FIG. 3: First corrections to the pair correlation function



One can check that for $\omega = 0$ the expression (10.25) is reduced to the static self-energy function as it should be in accordance with Eq. (10.21). The FDT (10.20) is useful at this checking.

One can investigate also the perturbation series for the pair correlation function (10.19). Note that due to the FDT (10.20) the perturbation series for F is reduced to the one for G. Nevertheless, it is instructive to treat the perturbation series for F separately. First corrections to F is drawn in Fig. 3. Extracting the "polarization" blocks (like in the last diagram in Fig. 3) and producing the corresponding summation we get

$$F(\omega, q) = -G(\omega, q)[2T\Gamma + 2\Pi(\omega, q)]G(-\omega, q)\,. \tag{10.26}$$

The first contribution to the "polarization function" Π corresponding to diagrams drawn in Fig. 3 is

$$\Pi(\omega, q) = \frac{\lambda^2}{12}\int \frac{\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\boldsymbol{q}_1\,\mathrm{d}\boldsymbol{q}_2}{(2\pi)^{2+2d}} F_0(\omega_1, \boldsymbol{q}_1)F_0(\omega_2, \boldsymbol{q}_2)F_0(\omega+\omega_1+\omega_2, \boldsymbol{q}+\boldsymbol{q}_1+\boldsymbol{q}_2)\,. \tag{10.27}$$

Comparing Eq. (10.24) and Eq. (10.26) we conclude that the FDT (10.20) leads to the relation

$$\Sigma(\omega, q) - \Sigma(-\omega, q) = \frac{2\omega}{T}\Pi(\omega, q)\,. \tag{10.28}$$

The relation can be directly checked for the expressions (10.25) and (10.27).

One believes that near the second-order phase transitions the correlation functions like (10.24) or (10.26) exhibit a scaling behavior that is t-dependence does not destroy power-like behavior of the functions. Say, at $T = T_c$ one expects

$$F(t, q) = \frac{1}{q^{2-\eta}}\phi_1(tq^z)\,, \quad G(\omega, q) = \frac{1}{q^{2-\eta}}\phi_2(\omega/q^z)\,, \tag{10.29}$$

where η is the anomalous exponent of the static pair correlation function, z is the new dynamic exponent and ϕ_1, ϕ_2 are some functions of the homogeneous arguments.

As in statics, for the relaxation dynamics the dimension $d = 4$ is marginal. All corrections in this case (excluding ultraviolet terms) have logarithmic character. Therefore one can try to investigate the correlation functions like (10.24) or (10.26) by extracting the main sequences of diagrams. As in statics, the main sequence in this case is the parquett one which has the same topological structure. Besides, the effective action (10.13) is renormalizable. Therefore it will be more convenient for us to use the renorm-group methods to investigate the situation in $4d$.

As previously, to get renorm-group equations we should devide our field into slow and fast components and integrate the probability disribution function over the fast component. In our case we have two fields: φ and p and both fields should be devided: $\varphi = \varphi' + \tilde{\varphi}$ and $p = p' + \tilde{p}$ where $\tilde{\varphi}$ and $\tilde{p}$ are the fast components containing the wave vetors $\Lambda' < q < \Lambda$. The probability distribution function in our case is $\exp(i\mathcal{I})$. Therefore an elementary step of the renorm-group procedure is a transformation $\mathcal{I}(\varphi, p) \to \mathcal{I}'(\varphi', p')$ where

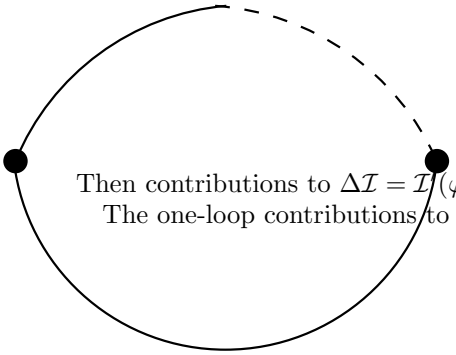
$$\exp\left[i\mathcal{I}'(\varphi', p')\right] = \int \mathcal{D}\tilde{\varphi}\,\mathcal{D}\tilde{p}\,\exp\left[i\mathcal{I}(\varphi'+\tilde{\varphi}, p'+\tilde{p})\right]\,. \tag{10.30}$$

FIG. 4: The $F-G$ loop.

The effective action here by analogy with the Landau functional can be written as

$$\mathcal{I}(\varphi'+\tilde{\varphi},p'+\tilde{p})=\mathcal{I}(\varphi',p')+\mathcal{I}(\tilde{\varphi},\tilde{p})+\mathcal{I}_{\mathrm{i}nt}\,,$$
$$\mathcal{I}^{(2)}_{\mathrm{i}nt}=\frac{\lambda}{2}\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left[p'\varphi'\tilde{\varphi}^2+\tilde{p}\tilde{\varphi}(\varphi')^2\right]\,, \tag{10.31}$$
$$\mathcal{I}^{(3)}_{\mathrm{i}nt}=\frac{\lambda}{6}\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left[p'\tilde{\varphi}^3+3\tilde{p}\tilde{\varphi}^2\varphi'\right]\,. \tag{10.32}$$

In spirit of the conventional procedure for the statics we start from the correlation functions which are determined by the second-order effective action



$$\tilde{\mathcal{I}}_0=\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left[\Gamma\tilde{p}\partial_t\tilde{\varphi}++b\nabla\tilde{p}\nabla\tilde{\varphi}+iT\Gamma\tilde{p}^2\right]\,,$$
$$\tilde{G}_0(\omega,q)=-\frac{1}{\Gamma\omega+bq^2}\,,\qquad \tilde{F}_0(\omega,q)=\frac{2T\Gamma}{\Gamma^2+b^2q^4}\,. \tag{10.33}$$

Then contributions to $\Delta\mathcal{I}=\mathcal{I}(\varphi',p')-\mathcal{I}(\varphi',p')$ can be calculated perturbatively.

The one-loop contributions to $\Delta\mathcal{I}$ can be written as

$$\Delta_1\mathcal{I}=\frac{i}{2}\left\langle\left[\mathcal{I}^{(2)}_{\mathrm{i}nt}\right]^2\right\rangle\,,\quad \Delta_2\mathcal{I}=i\left\langle\mathcal{I}^{(2)}_{\mathrm{i}nt}\tilde{\mathcal{I}}_a\right\rangle\,, \tag{10.34}$$
$$\tilde{\mathcal{I}}_a=a\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\tilde{p}\tilde{\varphi}\,,$$

where averaging is performed with the weight $\exp\left(i\tilde{\mathcal{I}}_0\right)$. For both contributions this averaging is reduced to calculating a $F-G$ loop depicted in Fig. (4). Analytically, the loop corresponds to the expression

$$\int\frac{\mathrm{d}\omega\,\mathrm{d}\boldsymbol{q}}{(2\pi)^{1+d}}\tilde{G}_0(\omega,q)\tilde{F}_0(\omega,q)\,.$$

Using the FDT (10.20) and taking the integral over ω we arrive at the same expression as in statics for a loop constituted fro two pair correlation functions. Next, calculating the coefficients we get corrections to λ (from $\Delta_1\mathcal{I}$) and to a (from $\Delta_2\mathcal{I}$) which coincide with ones calculated in statics. Therefore in dynamics we have the same renorm-group equations for λ and a as in statics.

There are no corrections to the parameters b and Γ in the one-loop approximation. Again, the property is identical to one occurring in statics. Thus to find principal contributions to b and Γ one should consider two-loop contributions. Namely, the main correction to the effective action responsible for the renormalization of b and Γ is

$$\Delta_3\mathcal{I}=\frac{i}{2}\left\langle\left[\mathcal{I}^{(3)}_{\mathrm{i}nt}\right]^2\right\rangle=\frac{i\lambda^2}{12}\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1\int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,\left\langle\tilde{\varphi}_1^3\tilde{p}_2\tilde{\varphi}_2^2\right\rangle p'_1\varphi'_2$$
$$+\frac{i\lambda^2}{72}\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1\int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,\left\langle\tilde{\varphi}_1^3\tilde{\varphi}_2^3\right\rangle p'_1p'_2\,. \tag{10.35}$$

The first contribution in Eq. (10.35) gives the renormalization both of the terms $\Gamma p\partial_t\varphi$ and $b\nabla p\nabla\varphi$ in the effective action whereas the second term gives the renormalization of the term $T\Gamma p^2$ in the effective action. Extracting the correction to b from the first term we find that it is the same as in statics. Therefore the renorm-group equation for b is identical to one occurring in statics. Next, one can extract from Eq. (10.35) corrections to the factors at $p\partial_t\varphi$ and p^2 and check that they give the same renormalization of Γ. The simplest way to check the property is in using the FDT (10.20) (without explicit finding the integrals). The property means that the effective action (10.13) is renormalizable. It can be treated also as a manifestation of the fluctuation-dissipation theorem.

Below we will be interested in the renormalization of Γ and therefore we focus on the second term in Eq. (10.35). We can substitute there p_2' by p_1' and a a result we find the following expression for the correction to Γ:

$$\Delta\Gamma = \frac{\lambda^2}{12T}\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\tilde{F}_0^3(t,r)\,, \tag{10.36}$$

where the function $\tilde{F}_0$ can be found as the Fourier transform of Eq. (10.33):

$$\tilde{F}_0(t,r) = \frac{T}{4\pi^2 br^2}\left[1-\exp\left(-\frac{\Gamma r^2}{4bt}\right)\right]. \tag{10.37}$$

Of course the expression (10.37) is correct only if $\Lambda' \ll r^{-1} \ll \Lambda$. Substituting Eq. (10.37) into Eq. (10.36) we get

$$\Delta\Gamma = \frac{\lambda^2 T^2\Gamma}{3\cdot 2^8\pi^4 b^4}\int\frac{\mathrm{d}r}{r}\int\limits_0^\infty\frac{\mathrm{d}\eta}{\eta^2}\left[1-\exp(-\eta)\right]^3 .$$

The integral over η here is equal to $3\ln(4/3)$. Next, the integral $\int \mathrm{d}r/r$ produces $\ln(\Lambda/\Lambda')$. Thus we get

$$\Delta\Gamma = \frac{1}{9}\ln(4/3)g^2\Gamma\ln(\Lambda/\Lambda')\,.$$

Therefore finally we get the following renorm-group equation for Γ

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi} = \frac{1}{9}\ln(4/3)g^2\Gamma\,,\quad \xi = \ln(\Lambda/\Lambda')\,, \tag{10.38}$$

since the invariant charge (for the one-component order parameter) is

$$g = \frac{3T\lambda}{16\pi^2 b^2}$$

The equation (10.38) can easily be generalized for the n-component order parameter:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi} = \frac{3(n+2)}{(n+8)^2}\ln(4/3)g^2\Gamma\,. \tag{10.39}$$

Now we can pass to dimensionality $d = 4-\epsilon$. In the leading order over ϵ we can substitute $g=\epsilon$ into Eq. (10.39). Then one gets $\Gamma \propto (\Lambda')^{-3(n+2)/(n+8)^2\ln(4/3)\epsilon^2}$. Remind that in the same approximation $b \propto (\Lambda')^{-(n+2)\epsilon^2/2(n+8)^2}$. Substituting here $\Lambda' = q$ and equating the terms $\Gamma\omega$ and bq^2 (figuring, say, in the Green function) we get $\omega \propto q^z$ where the dynamic critical exponent is

$$z = 2 + \frac{(n+2)\epsilon^2}{2(n+8)^2}\left[6\ln(4/3)-1\right].$$

The expression can be rewritten as (Halperin and Hohenberg, 1972)

$$z = 2 + \left[6\ln(4/3)-1\right]\eta\,, \tag{10.40}$$

where η is the anomalous exponent of the pair correlation function in statics.

Problems

Problem 10.1

Prove that the expressions (10.1,10.2) lead to the Gibbs simultaneous probability distribution function $\exp[(F-\mathcal{F})/T]$ for the order parameter fluctuations.

11. KPZ PROBLEM

The KPZ (Kardar-Parisi-Zhang) problem is associated with processes like flame propagation or roughning. The KPZ equation is formulated for the quantity h which describes fluctuations of the moving interface near its equilibrium (flat) shape. On the other hand, the problem is related to the properties of the mixed state of high-temperature superconductors. The same equation determines a distribution of the quantum vortices in a random potential (then the role of time is played by the coordinate along the magnetic field). Thus the equation is quite universal. The universality is accounted for by the fact that the KPZ equation describes the long-scale dynamics of any scalar degree of freedom h provided its dynamics is invariant under a shift $h \to h + const$ and is not invariant under $h \to -h$.

The KPZ equation is written as

$$\partial_t h = \lambda(\nabla h)^2 + D\nabla^2 h + \zeta\,. \tag{11.1}$$

The coefficient D in (11.1) could be interpreted as the diffusion coefficient, λ is the non-linearity coefficient, and ζ is the random force which is the white noise in time:

$$\langle \zeta(t_1, \boldsymbol{r}_1)\zeta(t_2, \boldsymbol{r}_2)\rangle = 2T\delta(t_1 - t_2)\delta_\Lambda(\boldsymbol{r}_1 - \boldsymbol{r}_2)\,. \tag{11.2}$$

Here, T characterizes the strength of the force (T can be called the effective temperature) and $\delta_\Lambda(\boldsymbol{r})$ is a function normalized as

$$\int \mathrm{d}\boldsymbol{r}\, \delta_\Lambda(\boldsymbol{r}) = 1\,, \tag{11.3}$$

and with the support of the size Λ^{-1}. The KPZ equation (11.1) describes processes occurring on scales much larger than Λ^{-1} that is the wave number Λ plays the role of the ultraviolet cutoff. Thus the driving force ζ in the KPZ equation (11.1) is short correlated both in space and time what resembles critical dynamics or inverse cascades in turbulence.

It is possible to treat the equation (11.1) for any dimension d of space. The most interesting physical dimension for the KPZ problem is $d = 2$, it corresponds both to roughning and to the vortices in the random potential. The dimension $d = 2$ is very interesting also from the theoretical point of view since in this case one encounters the "asymptotic freedom". Namely, a logarithmic renormalization takes place up to a "confinement length" R_c starting from which the strong coupling regime occurs.

We will examine correlation functions of h

$$\langle h(t_1, \boldsymbol{r}_1) \dots h(t_n, \boldsymbol{r}_n)\rangle\,, \tag{11.4}$$

on scales much larger than Λ^{-1}. The correlation functions are expected to be insensitive to the concrete shape of the function $\delta_\Lambda(\boldsymbol{r})$. One is interested in stationary distribution when the correlation function (11.4) are homogeneous in time and in space. Generally, the average $\langle \partial_t h\rangle$ is nonzero. Physically it describes a regular drift of the surface for flame propagation or roughning. The quantity is determined by the ultraviolet properties of the system and is consequently non-universal (has no relation to long-scale behavior of the system). The drift can be excluded by a transformation $h \to h + const \cdot t$. Thus we assume $\langle \partial_t h\rangle = 0$ (formally for the purpose we should accept a non-zero average $\langle \zeta\rangle$). We assume also $\langle h\rangle = 0$. Note that there is no Gibbs distribution for simultaneous correlation functions (except for $1d$ case). In that sense the problem is close to turbulence.

To calculate the correlation functions (11.4) one must solve the equation (11.1) at a given "driving force" ζ, take the product written in Eq. (11.4) and average over the statistics of ζ (the last procedure is designated by the angular brackets there). Unfortunately, it is impossible to realize the scheme explicitly. Therefore it is convenient to reformulate the problem. Instead of averaging over the solutions of (11.1) let us integrate over all functions $h(t, \boldsymbol{r})$ taking the equation (11.1) into account by introducing the corresponding (functional) δ-function. Converting then the δ-function into an exponent by introducing an auxiliary field p and averaging the result over the statistics of ζ we get a formulation close to one used in the quantum field theory. For example, the pair correlation function is written as the following functional integral

$$F(t_1 - t_2, \boldsymbol{r}_1 - \boldsymbol{r}_2) = \langle h(t_1, \boldsymbol{r}_1)h(t_2, \boldsymbol{r}_2)\rangle = \int \mathcal{D}h\, \mathcal{D}p\, \exp(i\mathcal{I})h(t_1, \boldsymbol{r}_1)h(t_2, \boldsymbol{r}_2)\,, \tag{11.5}$$

where the effective action $\mathcal{I}$ is

$$\mathcal{I} = \int \mathrm{d}t\, \mathrm{d}\boldsymbol{r}\, \left[p\partial_t h - Dp\nabla^2 h - \lambda p(\nabla h)^2 + iTp^2\right]\,. \tag{11.6}$$

More precisely, the last term in (11.6) should be written as

$$iT\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2\,\delta_\Lambda(\boldsymbol{r}_1-\boldsymbol{r}_2)p(t,\boldsymbol{r}_1)p(t,\boldsymbol{r}_2)\,,$$

what reveals the character of the ultraviolet behavior.

Analogously to (11.5) high-order correlation functions of h can be presented. One can incorporate into consideration also correlation functions containing the auxiliary field p. Say, the correlation function

$$G(t_1-t_2,\boldsymbol{r}_1-\boldsymbol{r}_2)=\langle h(t_1,\boldsymbol{r}_1)p(t_2,\boldsymbol{r}_2)\rangle=\int \mathcal{D}h\,\mathcal{D}p\,\exp(i\mathcal{I})h(t_1,\boldsymbol{r}_1)p(t_2,\boldsymbol{r}_2)\,, \tag{11.7}$$

has the meaning of the response function of the system. Namely, if an "external force" $f(t,\boldsymbol{r})$ is added to the right-hand side of the equation (11.1) then a non-zero value of $\langle h\rangle$ appears. In the linear approximation

$$\langle h(t_1,\boldsymbol{r}_1)\rangle=-i\int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,G(t_1-t_2,\boldsymbol{r}_1-\boldsymbol{r}_2)f(t_2,\boldsymbol{r}_2)\,. \tag{11.8}$$

Thus due to causality $G(t,\boldsymbol{r})=0$ if $t<0$. Note also that the correlation function $\langle p_1p_2\rangle$ is equal to zero.

Some words about the perturbation theory.

Bare values of the correlation functions can be found if to omit in Eq. (11.6) the third order term (proportional to λ). That corresponds to the linearization of the equation (11.1). Then functional integrals (11.5,11.7) and so further (determining the correlation functions) are Gaussian and can be calculated explicitly. All answers are expressed via the bare pair correlation functions. Explicit expressions for the functions are

$$F_0(t,\boldsymbol{r})=\int\frac{\mathrm{d}\omega\,\mathrm{d}\boldsymbol{q}}{(2\pi)^{d+1}}\exp(-i\omega t+i\boldsymbol{q}\boldsymbol{r})\frac{2T}{\omega^2+D^2q^4}=\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d}\exp(i\boldsymbol{q}\boldsymbol{r})\frac{T}{Dq^2}\exp(-Dq^2|t|)\,, \tag{11.9}$$

where ω is frequency and $\boldsymbol{q}$ is wave vector. The analogous expression for the responce function is

$$G_0(t,\boldsymbol{r})=-\int\frac{\mathrm{d}\omega\,\mathrm{d}\boldsymbol{q}}{(2\pi)^{d+1}}\exp(-i\omega t+i\boldsymbol{q}\boldsymbol{r})\frac{1}{\omega+iDq^2}=i\theta(t)\frac{1}{(4\pi Dt)^{d/2}}\exp\left(-\frac{r^2}{4Dt}\right)\,, \tag{11.10}$$

where $\theta(t)$ is the step function. The expressions (11.9,11.10) are correct provided $r\gg\Lambda^{-1}$ or $Dt\gg\Lambda^{-2}$. If $d>2$ then the simultaneous pair correlation function is

$$F_0(t=0,\boldsymbol{r})\sim\frac{T}{D}r^{2-d}\,. \tag{11.11}$$

If $d\le 2$ then at $t=0$ the integral (11.9) over q diverges at small q. That means that the integral is determined by scales of the order of the size of the system. In this case it is more reasonable to introduce the quantity

$$F_0(t=0,0)-F_0(t=0,\boldsymbol{r})\sim\frac{T}{D}r^{2-d}\,. \tag{11.12}$$

Then one should analyze corrections to the expressions (11.9,11.10) related to the third order term in (11.6). The corrections can be examined in terms of the perturbation series generated by (11.5). Actually the expansion is performed over h. It can be seen directly from Eq. (11.1) since the non-linear term there contains an additional power of h. Thus the expressions (11.11,11.12) multiplied by λ^2/D^2 can serve as a measure of the non-linearity level for typical fluctuations. If $d>2$ then the strength of the fluctuations diminishes with increasing scale. Therefore one should estimate the level of the non-linearity at $r\sim\Lambda^{-1}$. We conclude that we are in the weak coupling regime for $T\lambda^2\Lambda^{d-2}\ll D^3$ and in the strong coupling regime otherwise. If $d\le 2$ then we are always in the strong coupling regime since the expression (11.12) increases as the scale grows. More precisely, if $T\lambda^2\ll D^3\Lambda^{2-d}$ then there exists the region of scales

$$[D^3/(T\lambda^2)]^{1/(2-d)}\gg r\gg\Lambda^{-1}\,,$$

where corrections to the expression (11.9) are negligible. For larger scales fluctuations make the correlation function (11.5) strongly different from Eq. (11.9). We see that the case $d=2$ is marginal. Then the expression (11.12) is proportional to $\ln(\Lambda r)$. In the case fluctuation corrections to the pair correlation function (11.9) are small provided $\Lambda^{-1}\ll r\ll R_c$ where

$$R_c\sim\Lambda^{-1}\exp\left(\frac{2\pi D^3}{T\lambda^2}\right)\,, \tag{11.13}$$

which is just the "confinement length" noted above. This region of scales (where the perturbation theory works) exists if $T\lambda^2 \ll D^3$. If $r \gtrsim R_c$ then the expressions (11.9,11.10) for the correlation functions (11.5,11.7) are essentially modified. Calculation of the correlation functions in the region of scales $r \gtrsim R_c$ is one of the main challenges to the theory.

We see that for $d = 2$ the logarithmic situation is realized where the renorm-group approach could be useful. Let us formulate the conventional renorm-group procedure for the case. First of all, we should devide the fields h and p on the fast and the slow parts:

$$h = h' + \tilde{h}\,, \qquad p = p' + \tilde{p}\,, \tag{11.14}$$

where the fast parts $\tilde{h}$, $\tilde{p}$ contain Fourier harmonics with wave vectors from Λ' up to Λ. Then one should integrate over $\tilde{h}$, $\tilde{p}$ the partition function $\exp(i\mathcal{I})$ obtaining the new partition functions for the slow fields:

$$\exp(i\mathcal{I}') = \int \mathcal{D}\tilde{h}\,\mathcal{D}\tilde{p}\,\exp(i\mathcal{I})\,. \tag{11.15}$$

Substituting the representation (11.14) into (11.6) we get

$$\mathcal{I} = \mathcal{I}(h', p') + \mathcal{I}(\tilde{h}, \tilde{p}) + \mathcal{I}_{\mathrm{i}nt}\,,$$

where

$$\mathcal{I}_{\mathrm{i}nt} = -\lambda \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left\{p'(\nabla\tilde{h})^2 + 2\tilde{p}\nabla h'\nabla\tilde{h}\right\}\,. \tag{11.16}$$

Next, expanding the right-hand side of the relation (11.15) over $\mathcal{I}_{\mathrm{i}nt}$ we get

$$\mathcal{I}'(h', p') - \mathcal{I}(h', p') = \langle\mathcal{I}_{\mathrm{i}nt}\rangle + \frac{i}{2}\langle\langle\mathcal{I}^2_{\mathrm{i}nt}\rangle\rangle - \frac{1}{6}\langle\langle\mathcal{I}^3_{\mathrm{i}nt}\rangle\rangle + \dots\,, \tag{11.17}$$

where angular brackets mean averaging over the statistics of fast variables and double angular brackets mean irreducible correlation functions.

Really only terms of the second and of the third order in the right-hand side of the relation (11.17) could contain logarithms in one-loop terms and therefore only these terms should be kept in the approximation. Substituting the expression (11.16) into Eq. (11.17) we get in the approximation

$$\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\Delta\lambda p'(\nabla h')^2 = -2\lambda^3 \int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1 \int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2 \int \mathrm{d}t_3\,\mathrm{d}\boldsymbol{r}_3\,\langle\langle p'_1(\nabla\tilde{h}_1)^2\tilde{p}_2\nabla h'_2\nabla\tilde{h}_2\tilde{p}_3\nabla h'_3\nabla\tilde{h}_3\rangle\rangle\,,$$
$$\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\Delta T p'^2 = \frac{\lambda^2}{2}\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1 \int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,\langle\langle p'_1(\nabla\tilde{h}_1)^2 p'_2(\nabla\tilde{h}_2)^2\rangle\rangle\,,$$
$$\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\Delta D\nabla p'\nabla h' = 2i\lambda^2 \int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1 \int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,\langle\langle p'_1(\nabla\tilde{h}_1)^2\tilde{p}_2\nabla h'_2\nabla\tilde{h}_2\rangle\rangle\,.$$

Note that there are no corrections to the term $p\partial_t h$ in the effective action (11.6) since $\mathcal{I}_{\mathrm{i}nt}$ (11.16) contains only gradients of h. Calculating the above averages in accordance with the Wick theorem and the expressions (11.9,11.10) we find

$$\Delta\lambda = 0\,, \quad \Delta D = 0\,, \quad \Delta T = \frac{\lambda^2 T^2}{2\pi D^3}\ln(\Lambda/\Lambda')\,,$$

what corresponds to the following renorm-group equations

$$\frac{\mathrm{d}T}{\mathrm{d}\ln(\Lambda/\Lambda')} = \frac{\lambda^2 T^2}{2\pi D^3}\,, \quad \frac{\mathrm{d}\lambda}{\mathrm{d}\ln(\Lambda/\Lambda')} = 0\,, \quad \frac{\mathrm{d}D}{\mathrm{d}\ln(\Lambda/\Lambda')} = 0\,. \tag{11.18}$$

Of course zeros in the right-hand side of the RG equation for λ and D are not casual. They are consequences of some symmetries of the KPZ problem which are discussed below.

We conclude from Eqs. (11.18) that the dimensionless coupling constant g for the KPZ problem is

$$g = \frac{\lambda^2 T}{2\pi D^3}\,.$$

The equation (11.18) is rewritten as

$$\frac{\mathrm{d}g}{\mathrm{d}\ln(\Lambda/\Lambda')} = g^2\,,$$

with the well-known solution

$$g = \frac{g_0}{1 - g_0\ln(\Lambda/\Lambda')}\,,$$

where g_0 is the value of the coupling constant on scales $\sim \Lambda^{-1}$. If $g_0 \ll 1$ (what is equivalent to $T\lambda^2 \ll D^3$) then the renormalized coupling constant g is small up to the scale where $\ln(\Lambda/\Lambda') \approx g_0^{-1}$ what gives for Λ'^{-1} just the estimate (11.13). On scales larger than R_c we cannot apply the renorm-group method, it is the region of strong coupling. Thus the situation resembles asymptotic freedom known from the theory of strong interactions. It is very hard to establish analytically the behavior of the correlation functions of h on scales larger than R_c. Numerics exhibits power-like behavior of the average $\langle(h_1 - h_2)^2\rangle$. The theoretical problem is to confirm (or to reject) the scaling behavior and to find the corresponding exponent. There is also a question concerning anomalous scaling that is exponents of the averages $\langle(h_1 - h_2)^{2n}\rangle$.

Below we establish some properties which would help in examining the non-perturbative region.

By rescaling the field h and the time t we can change the coefficients λ and D. It will be convenient for us to choose $\lambda = 1/2$ and $D = 1$. Then the equation (11.1) is rewritten as

$$\partial_t h = \frac{1}{2}(\nabla h)^2 + \nabla^2 h + \zeta\,. \tag{11.19}$$

If $d \neq 2$ then the constant T can also be redefined if to include into rescaling coordinates. Then the only (dimensionless) parameter in the problem will be the ultraviolet cutoff Λ. Below we keep the parameter T.

Note that the equation (11.19) is invariant under the transformation

$$h(t,\boldsymbol{r}) \to h(t,\boldsymbol{r} - \boldsymbol{v}t) - \boldsymbol{v}\boldsymbol{r}\,, \quad \zeta(t,\boldsymbol{r}) \to \zeta(t,\boldsymbol{r} - \boldsymbol{v}t)\,, \tag{11.20}$$

where $\boldsymbol{v}$ is an arbitrary infinitesemal parameter. The relation (11.2) is also invariant under the transformation (11.20). Therefore our problem has symmetry which can be called Galilean invariance since (11.20) resembles the Galilean transformation. The symmetry under (11.20) shows that the terms with $\partial_t h$ and with $(\nabla h)^2$ in the KPZ equation must have the same transformation law at a renormalization. That explains zero in the right-hand side of the RG equation (11.18) for λ since there are no (logarithmic) corrections to the term with $\partial_t h$ in the effective action.

In the dimension one the KPZ problem is in some sence exactly solvable. The peculiarity of the $1d$ case is that the non-linear term in the equation (11.19) conserves the energy

$$\mathcal{H} = \frac{1}{2}\int \mathrm{d}x\,(\partial_x h)^2\,, \tag{11.21}$$

what can be checked directly. Therefore the stationary distribution in the case is the Gibbs distribution

$$\mathcal{P}(h) \propto \exp\left(-\mathcal{H}/T\right)\,, \tag{11.22}$$

where $\mathcal{P}$ is the probability distribution function for simultaneous fluctuations of the field h. It can be checked, say, if to solve the Focker-Plank equation

$$\partial_t \mathcal{P} = -\int \mathrm{d}x\,\frac{\partial}{\partial h(x)}\left\{\left[\frac{1}{2}(\partial_x h)^2 + \partial_x^2 h\right]\mathcal{P}\right\} + T\int \mathrm{d}x_1\,\mathrm{d}x_2\,\delta_\Lambda(x_1 - x_2)\frac{\partial^2\mathcal{P}}{\partial h(x_1)\partial h(x_2)}\,. \tag{11.23}$$

Strictly speaking, the Gibbs distribution (11.22) is a solution of the equation (11.23) if to substitute $\delta_\Lambda(x_1 - x_2) \to \delta(x_1 - x_2)$ into Eq. (11.23). That means that the equipartition (11.22) is correct only for scales $r \gg \Lambda^{-1}$.

The distribution (11.22) is Gaussian one and therefore the simultaneous statistics of h is determined solely by the pair correlation function

$$F(x_1 - x_2) = \langle h(t,x_1)h(t,x_2)\rangle\,. \tag{11.24}$$

The correlation function (11.24) is determined by the integral

$$F(x) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}q}{2\pi}\exp(iqx)\frac{T}{q^2}\,,$$

sitting on the size of the system. Therefore a more reasonable quantity is

$$\langle (h_1 - h_2)^2 \rangle = T|x_1 - x_2|\,, \tag{11.25}$$

where $h_1 = h(t, x_1)$, $h_2 = h(t, x_2)$. The quantity (11.25) is analogous to the so-called second-order structure function in hydrodynamic turbulence.

There is no information about non-simultaneous correlation functions of h in $1d$. It is possible to say something definite only for wave vectors $T \ll q \ll \Lambda$ (if the region exists) where the expression (11.9) is correct. For $q \lesssim T$ the non-simultaneous statistics of h is strongly non-Gaussian since we deal with strong interaction there. One cannot say anything definite about correlation functions of h in the region. Moreover, a character of scaling for the correlation functions is unknown.

Let us introduce the quantity

$$\Psi = \exp(h/2)\,. \tag{11.26}$$

Then the equation (11.19) is rewritten as

$$\partial_t \Psi = \nabla^2 \Psi + (\zeta/2)\Psi\,. \tag{11.27}$$

Thus instead of the nonlinear equation (11.19) with the additive noise we obtain the linear equation with the multiplicative noise. Unfortunately, it does not lead to solving the problem. Moreover, correlation function of Ψ does not possess a simple scaling behavior (as expected for the correlation functions of h). Next, to restore the correlation function of h we should know an infinite series of the correlation functions of Ψ.

We see that if omit the term with the pumping ζ in Eq. (11.27) then the equation will be invariant under the transformation

$$\Psi \to \Psi + \boldsymbol{u}\boldsymbol{r}\,,$$

where $\boldsymbol{u}$ is an arbitrary parameter. In terms of the field h that is a nonlinear transformation which is satisfied due to an interplay of the terms in (11.1) proportional to λ and D. Therefore renormalization laws of the terms are identical. That is the reason why besides zero in the right-hand side of the RG equation (11.18) for λ we see zero in the right-hand side of the RG equation (11.18) for D.

The correlation functions of the quantity Ψ can be written like in Eq. (11.5) as functional integrals:

$$\langle \Psi_1 \Psi_2 \dots \rangle = \int \mathcal{D}\Psi\, \mathcal{D}P\, \exp\left(i\mathcal{I}_{CH}\right) \Psi_1 \Psi_2 \dots\,, \tag{11.28}$$

where P is a new auxiliary field. The effective action corresponding to Eq. (11.27) is

$$\begin{aligned} \mathcal{I}_{CH} &= \int \mathrm{d}t\, \mathrm{d}\boldsymbol{r}\ (P\partial_t \Psi + \nabla P \nabla \Psi) \\ &+ \frac{iT}{4} \int \mathrm{d}t\, \mathrm{d}\boldsymbol{r}_1\, \mathrm{d}\boldsymbol{r}_2\, P(t, \boldsymbol{r}_1)\Psi(t, \boldsymbol{r}_1)\delta_\Lambda(\boldsymbol{r}_1 - \boldsymbol{r}_2)P(t, \boldsymbol{r}_2)\Psi(t, \boldsymbol{r}_2)\,. \end{aligned} \tag{11.29}$$

Note that the correlation functions (11.28) are time-dependent. That is accounted for by the fact that at the accepted initial condition ($h = 0$ at $t = 0$) in an unbound specimen there is a contribution to $\langle h^2 \rangle$ linearly growing with increasing t. Besides, the distribution of $h(t, \boldsymbol{r}_1) - h(t, \boldsymbol{r}_2)$ is stationary.

Let us introduce the following simultaneous correlation functions

$$S_N(t, \boldsymbol{r}_1, \dots, \boldsymbol{r}_N) = \langle \Psi(t, \boldsymbol{r}_1) \dots \Psi(t, \boldsymbol{r}_N) \rangle \exp\left[-\frac{T}{4} N \delta_\Lambda(0) \right]\,. \tag{11.30}$$

Then we obtain from (11.27) closed equations

$$\partial_t S_N = \sum_{k=1}^{N} \nabla_k^2 S_N + \frac{T}{2} \sum_{k>j} \delta_\Lambda\left(\boldsymbol{r}_k - \boldsymbol{r}_j\right) S_N\,. \tag{11.31}$$

Particularly for $N = 2$ we get

$$\partial_t S_2(t, \boldsymbol{r}) = 2\nabla^2 S_2 + \frac{T}{2} \delta_\Lambda\left(\boldsymbol{r}\right) S_2\,, \tag{11.32}$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$.

One can get from (11.31,11.32) an integral equation for S_N. Assuming that $h = 0$ at $t = 0$ we obtain for dimensionality $d = 2$

$$S_N(t, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = 1 + \frac{T}{2} \sum_{k>j} \int_0^t \frac{\mathrm{d}\tau}{(4\pi\tau)^N} \int \prod_i \mathrm{d}^2 r_i'$$

$$\exp\left[-\frac{1}{4\tau} \sum_n (\boldsymbol{r}_n - \boldsymbol{r}_n')^2 \right] \delta_\Lambda\left(\boldsymbol{r}_k' - \boldsymbol{r}_j'\right) S_N(t-\tau, \boldsymbol{r}_1', \ldots, \boldsymbol{r}_N') \,, \tag{11.33}$$

$$S_2(t, \boldsymbol{r}) = 1 + \frac{T}{4} \int_0^t \frac{\mathrm{d}\tau}{4\pi\tau} \int \mathrm{d}^2 r' \exp\left[-\frac{(\boldsymbol{r} - \boldsymbol{r}')^2}{8\tau} \right] \delta_\Lambda\left(\boldsymbol{r}'\right) S_2(t-\tau, \boldsymbol{r}') \,. \tag{11.34}$$

If $T \ll 1$ then one can construct a perturbation series over T for S_N. For example, the pair correlation function is written as

$$S_2(t, \boldsymbol{r}) = 1 + \frac{T}{16\pi} \ln \frac{t}{r^2} + \ldots \,, \tag{11.35}$$

the expression implies $t \gg r^2$. We see that for large enough t the perturbation theory fails.

The equations (11.31,11.32) can be written as

$$\partial_t S_N = -\hat{H}_N S_N \,,$$

where "Hamiltonians" $\hat{H}_N$ contain kinetic and potential parts, the latter correspond to potential wells. Then at least for $N = 2$ there exists a bound state (remember that $d = 2$). We believe that bound states exist also for any N. In this case the large t asymptotics of S_N will be determined by the ground state of the Hamiltonian $\hat{H}_N$ with the energy $E_N = -|E_N|$, that is

$$S_N(t, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \approx \exp(|E_N| t) \Phi_N(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \,, \tag{11.36}$$

where Φ_N are wave functions corresponding to the ground states. An equation for Φ_N can be found from (11.31) and has the form:

$$\left(E_N + \sum_{i=1}^N \nabla_i^2 \right) \Phi_N = -\frac{T}{2} \sum_{i<j} \delta_\Lambda(\boldsymbol{r}_i - \boldsymbol{r}_j) \Phi_N \,. \tag{11.37}$$

Actually, we should find a solution of Eq. (11.37) independent of the center mass coordinate.

One can easily find the solution of the equation (11.37) if $N = 2$:

$$\Phi_2(\boldsymbol{r}_1, \boldsymbol{r}_2) = K_0\left(\sqrt{\frac{|E_2|}{2}} |\boldsymbol{r}_1 - \boldsymbol{r}_2| \right) \,, \tag{11.38}$$

$$\frac{8\pi}{T} = \ln \frac{\Lambda}{\sqrt{|E_2|}} \,. \tag{11.39}$$

The expression (11.38) is correct if $\Lambda |\boldsymbol{r}_1 - \boldsymbol{r}_2| \gg 1$ and the relation (11.39) is correct if $T \ll 1$. We see that the function is logarithmic for $R_c > r > \Lambda^{-1}$ and exponentially decays for $r > R_c$. The same behavior is expected for the high-order functions Φ_N.

12. TWO-DIMENSIIONAL HYDRODYNAMICS

Here, we are going to consider the role of thermal fluctuations in two-dimensional hydrodynamics. We start from the incompressible case where fluctuations were examined by Forster, Nelson, and Stephen (1978).

The incompressible hydrodynamics is described by the Navier-Stokes equation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = \nu \nabla^2 \boldsymbol{v} - \nabla P/\rho + \boldsymbol{\xi} \,, \tag{12.1}$$

supplemented by the incompressibility condition

$$\nabla \boldsymbol{v} = 0 \,. \tag{12.2}$$

Here $\boldsymbol{v}$ is the velocity, ν is the kinematic viscosity coefficient, P is the pressure, ρ is the 2d mass density (which is treated as homogeneous one), and $\boldsymbol{\xi}$ is the thermal noise. It is characterized by the correlation function

$$\langle \xi_\alpha(t_1,\boldsymbol{r}_1)\xi_\beta(t_2,\boldsymbol{r}_2)\rangle = \frac{T\nu}{\rho}\delta(t_1-t_2)\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\left(q^2\delta_{\alpha\beta}-q_\alpha q_\beta\right)\exp(i\boldsymbol{qr})\,, \tag{12.3}$$

where $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$ and T is the temperature. Note, that for the incompressible case the pressure P is excluded from the set of dynamical variables. It is determined from the equation (12.1) and the incompressibility condition (12.2) which lead to the relation

$$\rho\nabla\left[(\boldsymbol{v}\nabla)\boldsymbol{v}\right] = -\nabla^2 P\,. \tag{12.4}$$

The same procedure as described in Section 10 enables one to reduce the problem of calculating fluctuational effects to the functional integral with the weight $\exp(i\mathcal{I})$ where $\mathcal{I}$ is the effective action constructed from the equation (12.1):

$$\mathcal{I} = \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left\{p_\alpha\partial_t v_\alpha + p_\alpha v_\beta\partial_\beta v_\alpha + \nu\partial_\alpha p_\beta\partial_\alpha v_\beta + i(T\nu/\rho)(\partial_\alpha p_\beta)^2\right\}\,. \tag{12.5}$$

Here $\boldsymbol{p}$ is an auxiliary field satisfying the condition $\nabla\boldsymbol{p}=0$, analogous to the incompressibility condition $\nabla\boldsymbol{v}=0$. Just due to the condition $\nabla\boldsymbol{p}=0$ the term with the pressure in the Navier-Stokes equation (12.1) does not contribute to the effective action (12.5). The last term in the effective action (12.5) appears after averaging over the statistics of the thermal noise in accordance with Eq. (12.3).

Now, say, the pair correlation function of the velocity is written as the following functional integral

$$F_{\alpha\beta}(t_1-t_2,\boldsymbol{r}_1-\boldsymbol{r}_2) = \langle v_\alpha(t_1,\boldsymbol{r}_1)v_\beta(t_2,\boldsymbol{r}_2)\rangle = \int \mathcal{D}v\,\mathcal{D}p\,\exp(i\mathcal{I})v_\alpha(t_1,\boldsymbol{r}_1)v_\beta(t_2,\boldsymbol{r}_2)\,. \tag{12.6}$$

It is worth to treat "cross" correlation functions, say, the pair correlation function

$$G_{\alpha\beta}(t_1-t_2,\boldsymbol{r}_1-\boldsymbol{r}_2) = \langle v_\alpha(t_1,\boldsymbol{r}_1)p_\beta(t_2,\boldsymbol{r}_2)\rangle = \int \mathcal{D}v\,\mathcal{D}p\,\exp(i\mathcal{I})v_\alpha(t_1,\boldsymbol{r}_1)p_\beta(t_2,\boldsymbol{r}_2)\,. \tag{12.7}$$

The pair correlation function (12.7) determines the susceptibility of the system. Suppose that the external force density f_α is applied to the system which has to be added to the right-hand side of the equation (12.1):

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = \nu\nabla^2\boldsymbol{v} - \nabla P/\rho + \boldsymbol{\xi} + \boldsymbol{f}/\rho\,. \tag{12.8}$$

Then an average velocity $\langle v_\alpha\rangle$ is generated which in the linear approximation is expressed as

$$\langle v_\alpha(t_1,\boldsymbol{r}_1)\rangle = -i\int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,G_{\alpha\beta}(t_1-t_2,\boldsymbol{r}_1-\boldsymbol{r}_2)f_\beta(t_2,\boldsymbol{r}_2)/\rho\,. \tag{12.9}$$

To prove the relation (12.9) one has to incorporate the external force $\boldsymbol{f}$ into the effective action (12.5), use the expression like Eqs. (12.6,12.7) for the average $\langle v_\alpha\rangle$, and expend $\exp(i\mathcal{I})$ up to the first order over $\boldsymbol{f}$. The pair correlation function $\langle p_\alpha(t_1,\boldsymbol{r}_1)p_\beta(t_2,\boldsymbol{r}_2)\rangle$ is zero (for proving see Section 10).

Now we are going to examine fluctuational effects in the system. As we will see, fluctuations lead to a logarithmic renormalization of the viscosity coefficient. Thus, the adequate way to examine fluctuations is the renorm-group technique.

As usual, we divide the fields $\boldsymbol{v}$ and $\boldsymbol{p}$ into "slow" and "fast" parts, designating the parts by prime and tilde correspondingly:

$$\boldsymbol{v} = \boldsymbol{v}' + \tilde{\boldsymbol{v}}\,, \qquad \boldsymbol{p} = \boldsymbol{p}' + \tilde{\boldsymbol{p}}\,. \tag{12.10}$$

Here $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{p}}$ contain Fourier harmonics with wave vectors q lying in the interval $\Lambda'<q<\Lambda$ where Λ is the ultraviolet cutoff. Substituting the decomposition (12.10) into the effective action (12.5) we get

$$\mathcal{I} = \mathcal{I}(\boldsymbol{v}',\boldsymbol{p}') + \mathcal{I}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{p}}) + \mathcal{I}_{\mathrm{i}nt}\,, \tag{12.11}$$

$$\mathcal{I}_{\mathrm{i}nt} = \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left\{-\partial_\beta p'_\alpha\tilde{v}_\beta\tilde{v}_\alpha + \tilde{p}_\alpha v'_\beta\partial_\beta\tilde{v}_\alpha + \tilde{p}_\alpha\tilde{v}_\beta\partial_\beta v'_\alpha\right\}\,. \tag{12.12}$$

Then from the definition

$$\exp(i\mathcal{I}') = \int \mathcal{D}\tilde{v}\,\mathcal{D}\tilde{p}\,\exp(i\mathcal{I})\,, \tag{12.13}$$

we get the correction

$$\Delta\mathcal{I}(\boldsymbol{v}',\boldsymbol{p}') \equiv \mathcal{I}'(\boldsymbol{v}',\boldsymbol{p}') - \mathcal{I}(\boldsymbol{v}',\boldsymbol{p}') = \langle\mathcal{I}_{int}\rangle + \frac{i}{2}\langle\langle\mathcal{I}_{int}^2\rangle\rangle - \frac{1}{6}\langle\langle\mathcal{I}_{int}^3\rangle\rangle + \dots\,, \tag{12.14}$$

where averaging is performed over "fast" fluctuations which is integration over $\tilde{v}$ and $\tilde{p}$ with the weight $\exp[i\mathcal{I}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{p}})]$, and double brackets designate irreducible averages (cumulants).

We examine the renorm-group equations in the one-loop approximation. Then one should keep the second-order terms in $\mathcal{I}(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{p}})$ and averaging in Eq. (12.14) is reduced to the Gaussian integrals. The integrals are expressed via the averages

$$\tilde{G}_{\alpha\beta} = -\int \frac{\mathrm{d}\omega\,\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\exp(-i\omega t + i\boldsymbol{q}\boldsymbol{r})\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right)\frac{1}{\omega + i\nu q^2}\,, \tag{12.15}$$

$$\tilde{F}_{\alpha\beta} = \int \frac{\mathrm{d}\omega\,\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\exp(-i\omega t + i\boldsymbol{q}\boldsymbol{r})\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right)\frac{2T\nu q^2}{\rho(\omega^2 + \nu^2 q^4)}\,. \tag{12.16}$$

The first term in the right-hand side of Eq. (12.14) is zero, the second and the third terms contain logarithmic terms (just the terms are of interest for us) and high-order terms can be neglected.

Let us first consider the contribution

$$\Delta_1\mathcal{I}(\boldsymbol{v}',\boldsymbol{p}') = \frac{i}{2}\langle\langle\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,p'_{1\alpha}\tilde{v}_{1\beta}\partial_\beta\tilde{v}_{1\alpha}p'_{2\mu}\tilde{v}_{2\nu}\partial_\nu\tilde{v}_{2\mu}\rangle\rangle\,, \tag{12.17}$$

originating from the second term in the right-hand side of Eq. (12.14). It can be rewritten as

$$\Delta_1\mathcal{I} = \frac{i}{2}\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,\partial_\beta p'_{1\alpha}\partial_\nu p'_{2\mu}\left[\tilde{F}_{\alpha\mu}(t,\boldsymbol{r})\tilde{F}_{\beta\nu}(t,\boldsymbol{r}) + \tilde{F}_{\alpha\nu}(t,\boldsymbol{r})\tilde{F}_{\beta\mu}(t,\boldsymbol{r})\right]\,, \tag{12.18}$$

where $t = t_1 - t_2$ and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. Since $\tilde{F}$ decreases fast for $r > \Lambda'^{-1}$ and p' is the "slow" field (slightly varying on the scale Λ'^{-1}), one can substitute in Eq. (12.18) in the main approximation p'_2 by p'_1. Then we get as an factor

$$\begin{aligned}&\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\left[\tilde{F}_{\alpha\mu}(t,\boldsymbol{r})\tilde{F}_{\beta\nu}(t,\boldsymbol{r}) + \tilde{F}_{\alpha\nu}(t,\boldsymbol{r})\tilde{F}_{\beta\mu}(t,\boldsymbol{r})\right]\\ &= \int \frac{\mathrm{d}\omega\,\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\frac{4T^2\nu^2q^4}{\rho^2(\omega^2+\nu^2q^4)^2}\left\{\left(\delta_{\alpha\mu} - \frac{q_\alpha q_\mu}{q^2}\right)\left(\delta_{\beta\nu} - \frac{q_\beta q_\nu}{q^2}\right) + \left(\delta_{\alpha\nu} - \frac{q_\alpha q_\nu}{q^2}\right)\left(\delta_{\beta\mu} - \frac{q_\beta q_\mu}{q^2}\right)\right\}\,,\end{aligned}$$

where we substituted the expression(12.16). The last integral is equal to

$$\frac{T^2}{8\pi\rho^2\nu}\ln\frac{\Lambda}{\Lambda'}\,(\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu})$$

Substituting this expression into Eq. (12.18) we get

$$\Delta_1\mathcal{I} = \frac{iT^2}{16\pi\rho^2\nu}\ln\frac{\Lambda}{\Lambda'}\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1\,\partial_\beta p'_{1\alpha}\partial_\beta p'_{2\alpha}\,, \tag{12.19}$$

where we have taken into account the conditions $\nabla\boldsymbol{v} = \nabla\boldsymbol{p} = 0$.

Let us now discuss other corrections produced by the term $\langle\langle\mathcal{I}_{int}^2\rangle\rangle$ in Eq. (12.14). First of all, it does not generate terms quadratic over v'. The formal reason for this property is that such corrections are proportional to the product $\tilde{G}(t_1 - t_2)\tilde{G}(t_2 - t_1)$ which is zero since $G(t)$ is zero for negative t due to causality. One could anticipate that terms quadratic over v' do not appear in the effective action since such terms violate the basic property $\langle pp\rangle = 0$. Therefore only the correction

$$\Delta_2\mathcal{I} = -i\left\langle\left\langle\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}t_2\,\mathrm{d}\boldsymbol{r}_2\,\partial_\beta p'_{1\alpha}\tilde{v}_{1\beta}\tilde{v}_{1\alpha}\left(\tilde{p}_{2\mu}v'_{2\nu}\partial_\nu\tilde{v}_{2\mu} + \tilde{p}_{2\mu}\tilde{v}_{2\nu}\partial_\nu v'_{2\mu}\right)\right\rangle\right\rangle\,, \tag{12.20}$$

has to be taken into account. Direct calculations including an expansion of $v'_{2\nu}$ upto the first order over $\boldsymbol{r}_1 - \boldsymbol{r}_2$ give

$$\Delta_2\mathcal{I} = \frac{T}{16\pi\rho\nu}\ln\frac{\Lambda}{\Lambda'}\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{r}_1\,\partial_\beta p'_{1\alpha}\partial_\beta v'_{1\alpha}\,. \tag{12.21}$$

Note that the term proportional to $\boldsymbol{p}'\partial_t\boldsymbol{v}'$ cannot appear in $\Delta_2\mathcal{I}$ since the expression in the right hand side of Eq. (12.20) contains the derivative $\partial_\beta p'_\alpha$.

Next, one should discuss the correction produced by the term $\langle\langle\mathcal{I}^3_{\mathrm{i}nt}\rangle\rangle$ in Eq. (12.14). This correction could renormalize the third-order contribution to the effective action (12.5). Nevertheless, direct calculations show, that the average $\langle\langle\mathcal{I}^3_{\mathrm{i}nt}\rangle\rangle$ does not generate any logarithmic term. The reason is in the Galilean invariance which reads that the "full derivative" $\partial_t + v_\alpha\partial_\alpha$ must enter the effective action. Since the term with $\boldsymbol{p}\partial_t\boldsymbol{v}$ in the effective is not renormalized, the third order term in the effective action (12.5) is not renormalized also.

Comparing the corrections (12.19,12.21) with the initial expression (12.5) we conclude that they are reduced to the unified correction to the viscosity coefficient

$$\Delta\nu = \frac{T}{16\pi\rho\nu}\ln\frac{\Lambda}{\Lambda'}\,. \tag{12.22}$$

This coincidence of the corrections to the viscosity coefficients in two different terms in Eq. (12.5) is a manifestation of the fluctuation-dissipation theorem, which has to be reproduced at the renorm-group procedure. The renorm-group equation corresponding to Eq. (12.22) is

$$\frac{\mathrm{d}g}{\mathrm{d}\xi} = -g^2\,, \qquad g = \frac{T}{8\pi\rho\nu^2}\,, \tag{12.23}$$

where $\xi = \ln(\Lambda/\Lambda')$ and g has the meaning of the invariant charge. We conclude from Eq. (12.23) that we encounter the "zero-charge" situation. Asymptotically at growing scale ξ increases, then $g \to \xi^{-1}$ and, consequently,

$$\nu \to \sqrt{\frac{T\xi}{8\pi\rho}}\,. \tag{12.24}$$

Thus, the viscosity coefficient increases with increasing scale. Note that its large-scale value is independent of its bare (small-scale) value.